Multiple Node Splines with Boundary Conditions: The Fundamental Theorem of Algebra for Monosplines and Gaussian Quadrature Formulae for Splines

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INTRODUCTION

In this paper our main goal is to establish a “fundamental theorem of algebra” for monosplines having multiple knots and satisfying boundary conditions and to show the existence and uniqueness of multiple node Gaussian quadrature formulas for classes of splines where linear boundary functionals are included in the formulas.

The known results on the “fundamental theorem” are due to the following individuals. For simple knots and multiple zeros, the results were proven by Karlin and Schumaker [4]. In the multiple knot and simple zero setting Micchelli [9] established the results. For multiple knots and multiple zeros the theorems were developed by Barrar and Loeb [11]. For the simple knot, multiple zero, and the boundary condition case the results were derived by Karlin and Micchelli [9].

With respect to the quadrature formulas the principal investigators were Karlin [3], Karlin and Pinkus [7], Melkman [8], Michelli and Pinkus [10], and Karlin and Micchelli [5]. They developed the simple node case.

Our point of departure is the paper of Micchelli and Pinkus. They exhibited the duality properties of the monospline and quadrature problems. Using this linkage and the topological methods developed in [1] we will establish similar results for multiple nodes and multiple zeros.
II. Quadrature Problem

The basic question can be phrased in the following way. Given $r$ fixed knots $0 < \xi_1 < \cdots < \xi_r < 1$ and the positive integer $n$.

Consider the linear family, $S$, of spline functions of the form:

$$
\sum_{i=0}^{n-1} a_i x^i + \sum_{i=1}^{r} b_i \Phi_n(x, \xi_i),
$$

where $\{a_i, b_i\}$ are allowed to vary and $\Phi_n(x, \xi) = (x-\xi)^{n-1}$. Then for a given set of odd positive integers $\{m_i\}_{i=1}^r$ with $R = \max \{m_i\} \leq n-2$ and a set of $k$ linear functionals $\{C_i(\cdot)\}_{i=1}^k$ of the variety

\[
C_i(f) = \sum_{j=0}^{n-1} A_{ij} f^{(j)}(0) \quad (i = 1, \ldots, p)
\]

\[
C_{i+p}(f) = \sum_{j=0}^{n-1} B_{ij} f^{(j)}(1) \quad (i = 1, \ldots, q)
\]

with $k = p + q$ and $n + r + p + q = m + r$, we seek a quadrature formula of the type

$$
Q(f) = \sum_{j=1}^{k} b_j C_j(f) + \sum_{i=1}^{r} \sum_{j=0}^{m-1} a_{ij} f^{(j)}(y_j),
$$

with $0 < y_1 < y_2 < \cdots < y_r < 1$, so that

$$
Q(u) = \int_0^1 u(t) \, dt
$$

for all $u \in S$. The boundary problem associated with the functionals exhibited above is called a separable problem. Following Karlin [3], Karlin, Micchelli [5], Karlin, Pinkus [7], and Melkman [8], we impose a set of basic assumptions on our boundary functionals. Let $m = n + r - k$.

**Basic Hypothesis.** (a) $0 \leq p, q \leq n$.

(b) There exist $\{i_1, \ldots, i_p\}, \{j_1, \ldots, j_q\} \subseteq \{0, 1, \ldots, n-1\}$ satisfying $M_{r-1} + m = v$, $v = m + 1, \ldots, n$, where $M_v$ counts the number of terms in $\{i_1, \ldots, i_p, j_1, \ldots, j_q\}$ less than or equal to $v$ and

$$
\hat{A} \left( \begin{smallmatrix} 1, \ldots, p \end{smallmatrix} \right) B \left( \begin{smallmatrix} 1, \ldots, q \end{smallmatrix} \right) \neq 0
$$
with
\[ \hat{A} = \{ (-1)^j A_{ij} \}_{i=1}^{p} \}_{j=0}^{q-1}, \quad B = \{ B_{ij} \}_{i=1}^{q} \}_{j=0}^{r}. \]

(c) If \( m \) is replaced by \( m + 1 \), then for all \( \{i_1, \ldots, i_p\}, \{j_1, \ldots, j_q\} \) satisfying (b),
\[ \hat{A} \left( \begin{array}{c} 1, \ldots, p \\ i_1, \ldots, i_p \end{array} \right) B \left( \begin{array}{c} 1, \ldots, q \\ j_1, \ldots, j_q \end{array} \right) \]
is of one fixed sign.

\[ \hat{A} \left( \begin{array}{c} 1, \ldots, p \\ i_1, \ldots, i_p \end{array} \right) \]
is the determinant formed from the \( p \) rows of \( \hat{A} \) and the columns \( \{i_1, \ldots, i_p\} \) of \( \hat{A} \).

### III. THE TOTAL POSITIVITY OF \( S(C) \)

Let \( u_i(x) = x^{i-1} \) (\( i = 1, \ldots, n \)) and \( u_{i+n}(x) = \Phi_n(x, \xi) \) (\( i = 1, \ldots, r \)). Define
\[ S(C) = \{ u \in S: C_i(u) = 0, i = 1, \ldots, k \} \]
and
\[ U \left( \begin{array}{c} i_1, \ldots, i_s \\ C_1, \ldots, C_k, \ x_1, \ldots, x_{l-k} \end{array} \right) \]
\[ = \begin{vmatrix} C_1(u_{i_1}) & \cdots & C_k(u_{i_1}) & u_{i_1}(x_1) & \cdots & u_{i_1}(x_{l-k}) \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ C_1(u_{i_s}) & \cdots & C_k(u_{i_s}) & u_{i_s}(x_1) & \cdots & u_{i_s}(x_{l-k}) \\ C_1(\Phi_n(\cdot, y_1)) & \cdots & C_k(\Phi_n(\cdot, y_1)) & \Phi_n(x_1, y_1) & \cdots & \Phi_n(x_{l-k}, y_1) \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ C_1(\Phi_n(\cdot, y_{l-s})) & \cdots & C_k(\Phi_n(\cdot, y_{l-s})) & \Phi_n(x_1, y_{l-s}) & \cdots & \Phi_n(x_{l-k}, y_{l-s}) \end{vmatrix} \]
for integers \( \{i_j: 1 \leq i_j \leq n+r\}_{j=1}^{s}, 0 < y_1 < \cdots < y_{l-s} < 1, \) and \( 0 < x_1 < \cdots < x_{l-k} < 1. \) When some of the \( x_j \)'s are equal, the usual definition involving derivatives [2] is employed with the restriction that at most \( n-1 \) \( x \)'s can coincide.
In [10] (Section 3 and Theorem 3.1) it was noted that under the Basic Hypothesis there exist integers, \(1 < i_1 < i_2 < \cdots < i_k \leq n + r\), so that

\[
d := U \left( \begin{array}{c} i_1, \ldots, i_k \\ C_1, \ldots, C_k \end{array} \right) \neq 0.
\] (1)

Following [10], the collection of functions

\[
v_l(t) = U \left( \begin{array}{c} i_1, \ldots, i_k, l_j \\ C_1, \ldots, C_k, t \end{array} \right), \quad l = 1, \ldots, n + r - k,
\] (2)

with \(\{i'_1, \ldots, i'_{n+r-k}\}\) the ordered set which is complementary to \(\{i_1, \ldots, i_k\}\) in \(\{1, \ldots, n + r\}\), forms a basis for \(S(C)\) (see Theorem 1). Then for each set, \(0 < x_1 < \cdots < x_{n+r-k} < 1\), by Sylvester's determinant lemma, [2],

\[
\begin{vmatrix}
  v_1(x_1) & \cdots & v_1(x_{n+r-k}) \\
  \vdots & & \vdots \\
  v_{n+r-k}(x_1) & \cdots & v_{n+r-k}(x_{n+r-k})
\end{vmatrix} = d^{n+r-k-1} U \left( \begin{array}{c} 1, \ldots, n + r \\ C_1, \ldots, C_k, x_1, \ldots, x_{n+r-k} \end{array} \right).
\] (3)

Further the result is valid if we allow some of the \(x_i's\) to coincide using the appropriate derivatives in the formulas. Thus if one can show for some fixed \(\sigma = \pm 1\) and all \(0 < x_1 < x_2 < \cdots < x_{n+r-k} < 1\),

\[
\sigma U \left( \begin{array}{c} 1, \ldots, n + r \\ C_1, \ldots, C_k, x_1, \ldots, x_{n+r-k} \end{array} \right) \geq 0
\]

then the \(\{v_{i_j}^{n+r-k}\}_{j=1}^k\) form a weak Tchebycheff system, i.e., for all sets \(0 < x_1 < \cdots < x_{n+r-k} < 1\), (3) does not change sign. Indeed Melkman [8] proved:

**Theorem 1.** Let the conditions of the Basic Hypothesis be satisfied. If \(n + r = k + m\), then

\[
\sigma U \left( \begin{array}{c} 1, \ldots, n + r \\ C_1, \ldots, C_k, x_1, \ldots, x_m \end{array} \right) \geq 0,
\] (4)

for all \(0 < x_1 \leq x_2 \leq \cdots \leq x_m < 1\), where at most \(n\) of the \(x_i's\) coincide. Here \(\sigma = \pm 1\) is fixed independent of all choices of the \(x_i's\). Moreover, strict inequality prevails in (4) iff

\[
\xi_{\mu+p-n} < x_\mu < \xi_{\mu+p}; \quad \mu = 1, \ldots, m.
\]
Define the convexity cone $K(S(C))$ by

$$K(S(C)) := \{ f \in C^R[0, 1]: 0 < t_1 < \cdots < t_{n+k} < 1 \}$$

$$\Rightarrow \begin{bmatrix} v_1(t_1) & \cdots & v_1(t_{n+k}) \\ \vdots & & \vdots \\ v_{n+k}(t_1) & \cdots & v_{n+k}(t_{n+k}) \\ f(t_1) & \cdots & f(t_{n+k}) \end{bmatrix} \geq 0$$

Consider $\Phi_n(x, y)$ for any $y \in (0, 1)$ and let

$$f_y(t) = U \left( \begin{array}{c} i_1, \ldots, i_k, y \\ C_1, \ldots, C_k, t \end{array} \right)$$

By (3), (4) and the Basic Hypothesis condition (c), $(\sigma f_y) \in K(S(C))$ for some $\sigma = \pm 1$.

**Lemma 1.** For each set $0 < x_1 \leq x_2 \leq \cdots \leq x_m < 1$ (with the usual restriction on the number of equal $x_i$’s) there exist points $0 < y_1 < \cdots < y_m < 1$ so that

$$U \left( \begin{array}{c} i_1, \ldots, i_k, y_1, \ldots, y_m \\ C_1, \ldots, C_k, x_1, \ldots, x_m \end{array} \right) \neq 0,$$

where

$$U \left( \begin{array}{c} i_1, \ldots, i_k \\ C_1, \ldots, C_k \end{array} \right) \neq 0.$$

**Proof.** By (1) the result is valid for $m = 0$; so assume it is true for $m \geq 0$. To simplify the notation we let the $x_i$’s be distinct. Then for $y \in (0, 1)$,

$$U \left( \begin{array}{c} i_1, \ldots, i_k, y_1, \ldots, y_m, y \\ C_1, \ldots, C_k, x_1, \ldots, x_m, x_{m+1} \end{array} \right)$$

$$= \sum_{i=p+1}^k d_i C_i(\Phi_n(\cdot, y)) + \sum_{i=1}^m e_i (x_i - y)^{n-1}_+ + e (x_{m+1} - y)^{n-1}_+$$

for some numbers $d_i, e_i, e$, where by induction $e \neq 0$. Note that

$$C_i(\Phi_n(\cdot, y)) = \sum_{j=0}^{n-1} B_{i-p,j} \frac{(n-1)!}{(n-1-j)!} (1-y)^{n-1-j} (i = p + 1, \ldots, k).$$
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Clearly then in \((0,1)\) \((x_{m+1} - y)^{n-1}\) is not in the linear span of the remaining functions. The result follows.

**Lemma 2.** For each set: \(0 < x_1 < \ldots < x_s < 1\), and the given odd multiplicities \(m_i\) \((i = 1, \ldots, s)\) (where recall that \(m_i \leq n - 2\)), there exists points \(0 < y_1 < \ldots < y_m < 1\) so that

\[
W:\left( f_{y_1}, \ldots, f_{y_m} \right)_{\left( x_1, \ldots, x_s \right)} := \left| \begin{array}{cccc}
 f_{y_1}(x_1) & \cdots & f_{y_1}^{(m_1)}(x_1) & f_{y_1}(x_2) & \cdots & f_{y_1}^{(m_2)}(x_2) & \cdots & f_{y_1}(x_s) & \cdots & f_{y_1}^{(m_s)}(x_s) \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 f_{y_m}(x_1) & \cdots & f_{y_m}^{(m_1)}(x_1) & f_{y_m}(x_2) & \cdots & f_{y_m}^{(m_2)}(x_2) & \cdots & f_{y_m}(x_s) & \cdots & f_{y_m}^{(m_s)}(x_s)
\end{array} \right| \neq 0.
\]

(6)

**Proof.** By Lemma 1 there is a set \(0 < y_1 < \ldots < y_m < 1\), so that

\[
0 \neq U:\left( l_{i_1}, \ldots, l_{i_k}, y_1, \ldots, y_m \right)_{\left( C_1, \ldots, C_k, x_1, \ldots, x_s, x_{s+1}, \ldots, x_{s+1} \right)}
\]

(7)

By (3), the determinants in (6) and (7) differ by a non-zero factor.

**IV. Existence and Uniqueness of Quadrature Formulas**

To proceed we add the following assumption to Lemma 2 in [1]: For some \(0 < \xi_1 < \ldots < \xi_s < 1\), the determinant \(D(\xi_1, \ldots, \xi_s) \neq 0\). This will insure that \(C_\varepsilon\) is bounded in equation (3) of [1]. This assumption is met in all applications of this lemma in [1] and in the present paper. We say that \(S(C)\) satisfies the “cone condition” if for each set \(0 < x_1 < \cdots < x_s < 1\) there is a collection \(\{f_i\}_{i=1}^m \subset K(S(C))\) so that

\[
W:\left( f_1, \ldots, f_m \right)_{\left( x_1, \ldots, x_s \right)} \neq 0.
\]

With this proviso the main theorem of [1] phrased in the notation of this paper is:
THEOREM 2. If $S(C)$ satisfies the "cone condition" then there is a unique quadrature formula of the form

$$Q_0(f) = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} a_{ij} f^{(j)}(x_i), \quad (8)$$

with the properties: $Q_0(v_i) = \int_0^1 v_i(x) \, dx$, $i = 1, \ldots, m$. Further the desired formula has the features: $0 < x_1 < \cdots < x_s < 1$ and $a_{i,m_i-1} > 0$ ($i = 1, \ldots, s$).

For any $u \in S$ let

$$f_u(t) = U\left( i_1, \ldots, i_k, u \right). \quad (9)$$

Noting that $f_u \in S(C)$, it follows from (9), Theorem 2 and Lemma 2 that there is exactly one quadrature formula of the type

$$Q(f) = \sum_{i=1}^{k} b_i C_i(f) + \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} a_{ij} f^{(j)}(x_i), \quad (10)$$

with the property,

$$Q(u) = \int u \, dx \quad \forall u \in S. \quad (11)$$

It is given by the expression

$$Q(f) = \sum_{j=1}^{k} d^{-1}(e_j - b_j) C_j(f) + Q_0(f), \quad (12)$$

where if $a_j(t)$ is the cofactor of $C_j(f)$ on the right-hand side of (9),

$$e_j = Q_0(a_j(\cdot)), \quad j = 1, \ldots, k$$

$$b_j = \int_0^1 a_j(t) \, dt,$$

and $Q_0$ is the quadrature formula of Theorem 2. So in summary,

THEOREM 3. The is exactly one formula, $Q(f)$ of the form (10) with the property (11). It is given by (12).
In this section we consider the existence question for $C[0, 1]$ (i.e., we require only that $m_i \leq n - 1$). The analysis of [1] can be readily modified to establish the desired result. We sketch the proof.

For any $f \in C[0, 1]$ and any $\varepsilon > 0$, let

$$
(T_\varepsilon f)(x) = \frac{2}{\varepsilon \sqrt{2\pi}} \int_0^1 \exp \left( -\frac{1}{2} \left( \frac{y - x}{\varepsilon} \right)^2 \right) f(y) \, dy,
$$

i.e., $T_\varepsilon f$ is the Gaussian transform of $f$.

Then as noted in [1] it is known, [6, 11], that there exists a unique quadrature formula of the form (8), $Q_\varepsilon(f)$, with knots:

$$
0 < x_1(\varepsilon) < \cdots < x_s(\varepsilon) < 1,
$$

so that $Q_\varepsilon(v) = \int_0^1 v \, dx$ for all

$$
v \in S_\varepsilon(C) := \{ w \in S(C) : w \in S(C) \}.
$$

Then by going to a subsequence one can assume $\lim_{\varepsilon \downarrow 0} x_i(\varepsilon) = x_i$, where $0 \leq x_1 \leq \cdots \leq x_s \leq 1$. As in [1]:

**Lemma 3.** The limit nodes satisfy $0 < x_1 < \cdots < x_s < 1$.

For these limit nodes and $v \in S(C)$, let

$$
\hat{v} = (v(x_1), \ldots, v^{(m-1)}(x_1), v(x_2), \ldots, v^{(m-1)}(x_{s-1}), v(x_s), \ldots, v^{(m-1)}(x_s))
$$

and form the $m \times (m - s)$ matrix,

$$
M = \begin{pmatrix}
\hat{v}_1 \\
\vdots \\
\hat{v}_m
\end{pmatrix},
$$

where $\{v_1, \ldots, v_m\}$ is a basis for $S(C)$ (see (2)).

A direct application of the techniques of [1] is:

**Lemma 4.** The columns of $M$ are linearly independent.

Using a standard compactness argument based on Lemma 4, and the convergence properties of the Gaussian transform yields the fact that coefficients of the quadrature formulas are uniformly bounded for $0 \leq \varepsilon < \varepsilon_0$. Thus by going to another subsequence one can be assured that each
coefficient \( a_y(\varepsilon) \to a_y \) as \( \varepsilon \downarrow 0 \). Again by the convergence properties of the transform:

\[
Q_0(v) := \sum_{i=1}^{s} \sum_{i=0}^{m_i-1} a_{ij} v^{(j)}(x_i) = \int_0^1 v \, dx
\]

for all \( v \in S(C) \). Applying (12) yields a formula which is exact for \( S \).

The results can be easily extended to the setting where

\[
s = 1 c_{x_1} + 1 b_{i_j} D_{x_1} (x, t_j) i = 0, i = I, i = 0, 5, \text{ the positive integers } \{d_i\}_{i=1}^r \text{ are fixed.}
\]

Setting \( D = \max_{1 < i < r} d_i \), we have with \( r = \sum_{i=1}^{I} d_i \):

**THEOREM 4.** Under the Basic Hypothesis if \( D + R \leq n \), then there is a quadrature formula \( Q(u) \) which is exact for \( S \); that is, \( u \in S \Rightarrow \)

\[
Q(u) = \int_0^1 u \, dx.
\]

If in addition, \( D + R \leq n - 1 \), then \( Q \) is unique. Further these results extend to Tchebycheffian splines [4].

**V. FUNDAMENTAL THEOREM OF ALGEBRA FOR MONOSPLINES WITH MULTIPLE KNOTS WHICH SATISFY BOUNDARY CONDITIONS**

Consider the set of linear functionals of the form

\[
G_i(f) = \sum_{j=0}^{n-1} E_{ij} f^{(j)}(0), \quad i = 1, \ldots, p
\]

\[
G_{i+p}(f) = \sum_{j=0}^{n-1} F_{ij} f^{(j)}(1), \quad i = 1, \ldots, q,
\]

where \( p + q = 2n - k \).

We examine all monosplines of the type

\[
M(x) = x^n + \sum_{i=0}^{n-1} a_i x^i + \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} a_{ij} \frac{\partial^j}{\partial y^j} \Phi_n(x, y) |_{y=y_i},
\]

(13)
with $0 < y_1 < \cdots < y_s < 1$, which satisfy the boundary conditions:

$$G_i(M) = 0, \quad i = 1, \ldots, 2n - k. \quad (14)$$

Let $m = \sum_{i=1}^s (m_i + 1)$ and $r = k + m - n$. We assume that

(a) $0 \leq p, q \leq n$;

(b) There exists $\{i_1, \ldots, i_p\}, \{j_1, \ldots, j_q\} \subseteq \{0, 1, \ldots, n-1\}$ satisfying $M_{r-1} + r \geq v, v = r + 1, \ldots, n$, and

$$\hat{E}\left(\begin{array}{c} 1, \ldots, p \\ i_1, \ldots, i_p \end{array}\right) F\left(\begin{array}{c} 1, \ldots, q \\ j_1, \ldots, j_q \end{array}\right) \neq 0 \quad (15)$$

with

$$\hat{E} = ((-1)^{p+q+n+1} E_{ij})_{i=1, j=0}^{p, n-1}$$

$$F = (F_{ij})_{i=1, j=0}^{q, n-1};$$

(c) For all $\{i_1, \ldots, i_p\}, \{j_1, \ldots, j_q\}$ satisfying (c),

$$\hat{E}\left(\begin{array}{c} 1, \ldots, p \\ i_1, \ldots, i_p \end{array}\right) F\left(\begin{array}{c} 1, \ldots, q \\ j_1, \ldots, j_q \end{array}\right)$$

is of one sign for "$r" replaced by "r + 1."

Further, consider a collection of positive integers $\{d_i\}_{i=1}^t$, where $D = \max\{d_i\}$, with $r = \sum_{i=1}^t d_i$, and fixed points, $0 < \xi_1 < \cdots < \xi_t < 1$.

**Theorem 5.** If $D + R \leq n$, and $G$ satisfies (15)(a)–(d) for both "$r" and "r + 1" then there is a monospline $M^*$ of the form (13) satisfying

$$G_i(M^*) = 0, \quad i = 1, \ldots, 2n - k$$

$$M^*(\xi_j) = 0, \quad i = 0, 1, \ldots, d_j - 1, j = 1, \ldots, t. \quad (16)$$

If in addition $D + R \leq n - 1$, $M^*$ is unique.

**Proof.** The proof follows immediately from Theorem 4 and the duality result of [10, Theorem 4.1] involving monosplines and quadrature formulas.

**Remarks.** Theorem 5 is an extension of the Fundamental Theorem of Algebra for Monosplines [1, 4, 5, 9, 10]. It follows directly from the duality principle that any monospline, $M$, of the type (13) which satisfies our basic boundary conditions, $G_i(M) = 0, i = 1, \ldots, 2n - k$, has at most $r$ zeros in $(0, 1)$. 
Consider the boundary linear functionals of the variety:

\[ C_i(f) = \sum_{j=0}^{n-1} A_{ij} f^{(j)}(0) + \sum_{j=0}^{n-1} B_{ij} f^{(j)}(1), \quad i = 1, \ldots, k. \]  

(17)

Then let

\[ A = \{ A_{ij} \}_{i=1}^{k} \begin{pmatrix} \frac{n-1}{2} \end{pmatrix}, \quad B = \{ B_{ij} \}_{i=1}^{k} \begin{pmatrix} \frac{n-1}{2} \end{pmatrix} \]

and \( C \) be the \( k \times 2n \) matrix of the form \( C = (A, B) \). If

\[ \text{rank } C = \text{rank } A + \text{rank } B \]  

(18)

then the linear functionals can be written in "separable form" without changing its determinant properties. A similar situation holds for monosplines.

All of the separable examples of [10] satisfy our basic hypothesis.

VI. NON-SEPARABLE PROBLEM

Noting the last remarks of the previous section, we call linear functionals of the type depicted in (17) non-separable iff Eq. (18) is violated. Consider the \((k \times 2n)\) matrix,

\[ C = \{ C_{ij} \}_{i=1}^{k} \begin{pmatrix} \frac{2n-1}{2} \end{pmatrix}, \]

where

\[ C_{ij} = \begin{cases} (-1)^{i+j+n+r} A_{ij}, & i = 1, \ldots, k, \ j = 0, 1, \ldots, n-1, \\ B_{i,2n-1-j}, & i = 1, \ldots, k, \ j = n, \ldots, 2n-1 \end{cases} \]

We assume that \( C \) satisfies:

**Basic Hypothesis.**

(i) \( 0 \leq k \leq \min\{2n, n+r\} \).

(ii) There exist \( \{i_1, \ldots, i_l, j_1, \ldots, j_{k-1}\} \subseteq \{0, 1, \ldots, 2n-1\} \), where \( 0 \leq i_1 < \cdots < i_l \leq n-1 < j_1 < \cdots < j_{k-1} \leq 2n-1 \), satisfying \( M_{v-1} + m \geq v \), \( v = m+1, \ldots, n \), where \( M_v \) is the number of terms in \( \{i_1, \ldots, i_t, 2n-1-j_1, \ldots, 2n-1-j_{k-1}\} \) less than or equal to \( v \) and

\[ C\begin{pmatrix} 1, & \ldots, & k \end{pmatrix}^{T} \begin{pmatrix} i_1, \ldots, i_l, j_1, \ldots, j_{k-1} \end{pmatrix} \neq 0. \]
(iii) For all \( \{i_1, \ldots, i_l, j_1, \ldots, j_{k-1}\} \) satisfying (ii),

\[
C \begin{pmatrix} 1, & \ldots, & k \\
\vdots & \ddots & \vdots \\
i_1, & \ldots, & j_l, \ldots, j_{k-1} \end{pmatrix} \neq 0,
\]

is of one fixed sign.

(iv) When replacing "r" by "r + 1" in the definition of the matrix \( C \), (iii) is valid with "m" replaced by "m + 1."

Using the techniques of the previous sections it can be easily shown that the theorems for the separable case are also valid for the non-separable case.

An example where the Basic Hypothesis is valid is given by the class of matrices of the type:

\[
C = (e_1, 0, \ldots, 0, e_2, \ldots, e_{k-m}, e_{k-m+1}, \ldots, e_k, \ldots, (-1)^{m+1} e_1, 0, \ldots, 0).
\]

Here \( k = n \) and \( n > 2m + 1 \) and \((e_1, \ldots, e_k)\) is the natural basis for \( R^k \).

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