Journal of Computational and Applied Mathematics 235 (2011) 4796–4801

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

First-order three-point boundary value problems at resonance

Mesliza Mohamed [a,*], Bevan Thompson [b], Muhammad Sufian Jusoh [c]

[a] Fakulti Sains Komputer dan Matematik, Universiti Teknologi MARA (UiTM) Malaysia, 40450 Shah Alam, Selangor, Malaysia

[b] Department of Mathematics, The University of Queensland, Queensland 4072, Australia

[c] Fakulti Kejuruteraan Awam, Universiti Teknologi MARA (UiTM) Malaysia, 40450 Shah Alam, Selangor, Malaysia

ARTICLE INFO

Keywords:
First order
Three-point boundary value problem
Brouwer's Fixed Point Theorem
At resonance

ABSTRACT

We consider three-point boundary value problems for a system of first-order equations in perturbed systems of ordinary differential equations at resonance. We obtain new results for the above boundary value problems with nonlinear boundary conditions. The existence of solutions is established by applying a version of Brouwer's Fixed Point Theorem which is due to Miranda.

1. Introduction

In this paper we study three-point boundary value problems in perturbed systems of first-order ordinary differential equations at resonance. Consider

$$x' - A(t)x = H(t, x, \varepsilon) = \varepsilon F(t, x, \varepsilon) + E(t), \quad 0 \le t \le 1, \tag{1}$$

$$Mx(0) + Nx(\eta) + Rx(1) = \ell + \varepsilon g(x(0), x(\eta), x(1)), \tag{2}$$

where M, N and R are constant square matrices of order n, $A(t)$ is an $n \times n$ matrix with continuous entries, $E : [0, 1] \to \mathbb{R}$ is continuous, $F : [0, 1] \times \mathbb{R}^n \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ is a continuous function where $\varepsilon_0 > 0$, $\ell \in \mathbb{R}^n$, $\eta \in (0, 1)$ and $g : \mathbb{R}^{3n} \to \mathbb{R}^n$ is continuous. We see that if $M = 1$, $N = 0$, $\ell = 0$, $g(x(0), x(\eta), x(1)) \equiv 0$, and $R = -1$ or $R = +1$, then we have a periodic or anti-periodic BVP, respectively.

The work was motivated by [1,2] which considered the problem of finding periodic solutions of perturbed systems. We adapt that approach to study three-point boundary value problems in perturbed systems. We address the central question: If ε is sufficiently small do (1) and (2) have solutions? We turn (1) into an integral equation and use (2) to reformulate the problem to one of finding the zeroes of a nonlinear equation involving the initial condition $x(0) = c$. We use Brouwer's degree theory to compute the degree for $\varepsilon = 0$.

The three-point boundary value problem (1), (2) is called resonant or degenerate in the case that the rank of the matrix $\mathcal{L} = n - r$, $0 < n - r < n$, that is the matrix $\mathcal{L} = M + NY_0(\eta) + RY_0(1)$ is singular where M, N and R are the constant $n \times n$ matrices given in (1), and $Y(t)$ is a fundamental matrix of linear system $x' = A(t)x$ and $Y_0(t) = Y(t)Y^{-1}(0)$. In studying the resonant case, we will use a finite-dimensional version of the Lyapunov Schmidt procedure (see [2]). The resonance case for a system of first-order equations which is arising from a scalar second-order equation will appear in a forthcoming paper. There we study in particular, the nonlinear boundary conditions which extend the following BVPs with linear boundary conditions:

$$y'' = f(t, y, y') + e(t), \quad 0 \le t \le 1, \tag{3}$$

$$y'(0) = 0, \qquad y(1) = \alpha y(\eta), \tag{4}$$

* Corresponding author. Tel.: +60 134350131; fax: +60 49882526.
E-mail addresses: mesliza@perlis.uitm.edu.my (M. Mohamed), hbt@maths.uq.edu.au (B. Thompson), muhdpian@yahoo.com (M.S. Jusoh).

0377-0427/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2010.10.029

and

$$y'' = f(t, y, y') + e(t), \quad 0 \le t \le 1, \tag{5}$$

$$y(0) = 0, \qquad y(1) = \alpha y(\eta), \tag{6}$$

where $\eta \in (0, 1)$, $\alpha \in \mathbb{R}$, and $f : [0, 1] \times \mathbb{R}^2 \to \times\mathbb{R}$ is a continuous function, and $e : [0, 1] \to \mathbb{R}$ is a function in $L^1[0, 1]$. Feng and Webb [3] studied the existence of solutions of the BVPs (3), (4) and (5), (6). Both of the problems are resonance cases under the assumption $\alpha = 1$ for problem (3), (4) and $\alpha\eta = 1$ for problem (5), (6). In these cases $Ly = y''$ together with (4), (6) respectively, is noninvertible. This is the so-called resonance case, and the Leray–Schauder continuation theory cannot be applied directly, and hence they used the coincidence degree theory of Mawhin [4]. Their results allowed the assumption on linear and nonlinear growth on f. Gupta [5] studied the solvability of (3), (4) where $\alpha = 1$.

The existence of solutions to two-point, three-point, four-point or multipoint BVPs for ODEs at resonance have been studied by a number of authors (see, for example [6–9,3,10,5,11–19]). A great amount of work has been completed on the existence of solutions to BVPs for nonlinear systems of first-order ODEs at resonance which involve a small parameter (see, for example [20–23]). The resonance case for discrete systems of first-order difference equations has been considered by several authors (see for example [24–31]). In these cases, resonance happens when the associated linear homogeneous BVP admits nontrivial solutions.

Our existence theorem applies a version of Brouwer's Fixed Point Theorem which is due to Miranda (see Yang [32,33]).

Thus our results extend to three-point BVPs the approach to periodic solutions of perturbed systems of first-order equations at resonance considered in [20,1,2].

2. Preliminary results

Lemma 1. *Consider the system*

$$x' = A(t)x \tag{7}$$

where $A(t)$ is an $n \times n$ matrix with continuous entries on the interval $[0, 1]$. Let $Y(t)$ be a fundamental matrix of (7). Then the solution of (7) which satisfies the initial condition

$$x(0) = c \tag{8}$$

is $x(t) = Y(t)Y^{-1}(0)c$ where c is a constant n-vector. Abbreviate $Y(t)Y^{-1}(0)$ to $Y_0(t)$. Thus $x(t) = Y_0(t)c$.

Lemma 2. *Let $Y(t)$ be a fundamental matrix of (7). Then any solution of (1) and (7) can be written as*

$$x(t, c, \varepsilon) = x(t) = Y_0(t)c + \int_0^t Y(t)Y^{-1}(s)H(s, x(s), \varepsilon)\mathrm{d}s. \tag{9}$$

Solution (9) satisfies the boundary conditions (2) if and only if

$$\mathcal{L}c = \varepsilon\mathcal{N}(c, \alpha, \eta, \varepsilon) + d \tag{10}$$

where $\mathcal{L} = M + NY_0(\eta) + RY_0(1)$, $\mathcal{N}(c, \alpha, \eta, \varepsilon) = -(\int_0^\eta NY(\eta)Y^{-1}(s)F(s, x(s, c, \varepsilon), \varepsilon)\mathrm{d}s + \int_0^1 RY(1)Y^{-1}(s)F(s, x(s, c, \varepsilon), \varepsilon)\mathrm{d}s - g(c, x(\eta), x(1)))$, $d = -\big(\int_0^\eta NY(\eta)Y^{-1}(s)E(s)\mathrm{d}s + \int_0^1 RY(1)Y^{-1}(s)E(s)ds - \ell\big)$, and $x(t, c, \varepsilon)$ is the solution of (1) given $x(0) = c$.

Thus (10) is a system of n real equations in $\varepsilon, c_1, \ldots, c_n$ where $c_1, \ldots, c_n$ are the components of c. System (10) is sometimes called the branching equations.

Next we suppose that $\mathcal{L}$ is a singular matrix. This is sometimes called the resonance case or degenerate case. Now we consider the case rank $\mathcal{L} = n - r$, $0 < n - r < n$. Let E_r denote the null space of $\mathcal{L}$ and let E_{n-r} denote the complement in $\mathbb{R}^n$ of E_r, i.e.

$$\mathbb{R}^n = E_{n-r} \oplus E_r (\text{direct sum}).$$

Let $x_1, \ldots, x_n$ be a basis for $\mathbb{R}^n$ such that $x_1, \ldots, x_r$ is a basis for E_r, and $x_{r+1}, \ldots, x_n$ is a basis for E_{n-r}.

Let P_r be the matrix projection onto Ker $\mathcal{L} = E_r$, and $P_{n-r} = I - P_r$, where I is the identity matrix. Thus P_{n-r} is a projection onto the complementary space E_{n-r} of E_r, and

$$P_r^2 = P_r, \qquad P_{n-r}^2 = P_{n-r} \quad \text{and} \quad P_{n-r}P_r = P_rP_{n-r} = 0. \tag{11}$$

Without loss of generality, we may assume that

$$P_rc = (c_1, \ldots, c_r, 0, \ldots, 0) \quad \text{and} \quad P_{n-r}c = (0, \ldots, 0, c_{r+1}, \ldots, c_n). \tag{12}$$

We will identify P_rc with $c^r = (c_1, \ldots, c_r)$ and $P_{n-r}c$ with $c^{n-r} = (c_{r+1}, \ldots, c_n)$ whenever it is convenient to do so.

Let H be a nonsingular $n \times n$ matrix satisfying

$$\mathcal{L} = P_{n-r}. \tag{13}$$

Matrix H can be computed easily. The nature of the solutions of the branching equations depends heavily on the rank of the matrix $\mathcal{L}$.

Lemma 3. *The matrix $\mathcal{L}$ has rank $n-r$ if and only if the three-point BVP (7) and $Mx(0) + Nx(\eta) + Rx(1) = 0$ have exactly r linearly independent solutions.*

Proof. Assume that rank $\mathcal{L} = n - r$. Then there exist r linearly independent solutions $h_1, \ldots, h_r$ satisfying

$$\mathcal{L}h_i = 0, \quad 1 \le i \le r.$$

Let $x_i(t) = Y(t)h_i$ where $Y(t)$ is the fundamental matrix. Clearly $x_i(0) = h_i$ and $x_i(t)$ are linearly independent solutions of (7) and $Mx(0) + Nx(\eta) + Rx(1) = 0$. Conversely as $x' = A(t)x$ and $Mx(0) + Nx(\eta) + Rx(1) = 0$ has r linearly independent solutions $x_i(t)$, $1 \le i \le r$. Thus $x_i(t)$ are r linearly independent and satisfy $Mx(0) + Nx(\eta) + Rx(1) = 0$. So $\mathcal{L}$ has rank less than or equal to $n - r$. The result follows. □

Next we give a necessary and sufficient condition for the existence of solutions of $x(t, c, \varepsilon)$ of three-point BVPs for $\varepsilon > 0$ such that the solution satisfies $x(0) = c$ where $c = c(\varepsilon)$ for suitable $c(\varepsilon)$.

3. Existence results

We need to solve (10) for c when ε is sufficiently small. The problem of finding solutions to (1) and (2) is reduced to that of solving the branching equations (10) for c as function of ε for $|\varepsilon| < \varepsilon_0$. So consider (10) which is equivalent to

$$\mathcal{L}(P_r + P_{n-r})c = \varepsilon \mathcal{N}((P_r + P_{n-r})c, \alpha, \eta, \varepsilon) + d.$$

Multiplying (10) by the matrix H and using (13), we have

$$P_{n-r}c = \varepsilon H \mathcal{N}((P_r + P_{n-r})c, \alpha, \eta, \varepsilon) + Hd, \tag{14}$$

where $H\mathcal{N}((P_r + P_{n-r})c, \alpha, \eta, \varepsilon) = -H(\int_0^\eta NY(\eta)Y^{-1}(s)F(s, x(s, c, \varepsilon), \varepsilon)\mathrm{d}s + \int_0^1 RY(1)Y^{-1}(s)F(s, x(s, c, \varepsilon), \varepsilon)\mathrm{d}s - g(c, x(\eta), x(1)))$ and

$$Hd = -H\left(\int_0^\eta NY(\eta)Y^{-1}(s)E(s)\mathrm{d}s + \int_0^1 RY(1)Y^{-1}(s)E(s)\mathrm{d}s - \ell\right).$$

Since the matrix H is nonsingular, solving (10) for c is equivalent to solving (14) for c. The following theorem due to Cronin [1,2] gives a necessary condition for the existence of solutions to the BVP (1) and (2).

Theorem 1. *A necessary condition that (14) can be solved for c, with $|\varepsilon| < \varepsilon_0$, for some $\varepsilon_0 > 0$ is $P_rHd = 0$.*

Proof. Multiplying (14) by P_{n-r} and P_r, and using the notation in (11), we obtain

$$P_{n-r}c = \varepsilon P_{n-r}H\mathcal{N}((P_r + P_{n-r})c, \alpha, \eta, \varepsilon) + P_{n-r}Hd, \tag{15}$$

$$0 = \varepsilon P_rH\mathcal{N}((P_r + P_{n-r})c, \alpha, \eta, \varepsilon) + P_rHd. \tag{16}$$

Solving (14) is equivalent to solving (15) and (16) simultaneously for $P_{n-r}c$ and P_rc. If we can solve for c for each fixed ε such that $|\varepsilon| < \varepsilon_0$, then setting $\varepsilon = 0$ in (16) we have

$$0 = P_rHd.$$

So the theorem follows. □

Remark 1. Now we solve (14) for c in terms of ε. Using $P_rHd = 0$ which is a necessary condition of Theorem 1, we thus obtain

$$P_{n-r}c = \varepsilon P_{n-r}H\mathcal{N}((P_r + P_{n-r})c, \alpha, \eta, \varepsilon) + P_{n-r}Hd, \tag{17}$$

$$0 = \varepsilon P_rH\mathcal{N}((P_r + P_{n-r})c, \alpha, \eta, \varepsilon). \tag{18}$$

Let (17) have initial solution

$$\varepsilon = 0,$$
$$c^r = c_0^r,$$
$$P_{n-r}c = P_{n-r}Hd,$$

where c_0^r is given a constant vector. Let $P_r c = c^r$ and $P_{n-r} c = c^{n-r}$. Let

$$G(c^r, c^{n-r}, \alpha, \eta, \varepsilon) = c^{n-r} - \varepsilon P_{n-r} H \mathcal{N}(c^r \oplus c^{n-r}, \alpha, \eta, \varepsilon) - P_{n-r} Hd.$$

If

$$\det \left. \frac{\partial G(c^r, c^{n-r}, \alpha, \eta, \varepsilon)}{\partial c^{n-r}} \right|_{(\varepsilon=0, c^r=c_0^r, c^{n-r}=P_{n-r}Hd)} \neq 0,$$

and $G(c_0^r, P_{n-r}Hd, \alpha, \eta, 0) = 0$, then by the implicit function theorem (see [34, p. 222]), we can solve $G(c^r, c^{n-r}, \alpha, \eta, \varepsilon) = 0$ for c^{n-r} in terms of c^r and ε in a neighbourhood of $(c^r, c^{n-r}, \alpha, \eta, 0) = (c_0^r, P_{n-r}Hd, \alpha, \eta, 0)$ such that there exists a unique solution

$$c^{n-r} = c^{n-r}(c^r, \varepsilon). \tag{19}$$

Moreover $c^{n-r}(c^r, 0) = P_{n-r}Hd$ since $G(c^r, c^{n-r}, \alpha, \eta, 0) \equiv c^{n-r} - P_{n-r}Hd$ when $\varepsilon = 0$. Moreover $c^{n-r}(c^r, \varepsilon)$ is a differentiable function of c^r and ε. Substituting (19) into (18) we reduce problem (17), (18) to that of solving

$$0 = \varepsilon P_r H \mathcal{N}(c^r \oplus c^{n-r}(c^r, \varepsilon), \alpha, \eta, \varepsilon). \tag{20}$$

Next we divide (20) by ε and define a continuous mapping $\Phi_\varepsilon : \mathbb{R}^r \to \mathbb{R}^r$, given by

$$\Phi_\varepsilon(c_1, \dots, c_r) = P_r H \mathcal{N}(c^r \oplus c^{n-r}(c^r, \varepsilon), \alpha, \eta, \varepsilon). \tag{21}$$

We set $\varepsilon = 0$. Thus define a mapping $\Phi_0 : \mathbb{R}^r \to \mathbb{R}^r$ such that

$$\Phi_0(c_1, \dots, c_r) = P_r H \mathcal{N}(c^r \oplus P_{n-r}Hd, \alpha, \eta, 0).$$

Definition 1. Let E_r denote the null space of $\mathcal{L}$ and let E_{n-r} denote the complement in $\mathbb{R}^n$ of E_r given in Section 2. Let P_r be the matrix projection onto Ker $\mathcal{L} = E_r$, and $P_{n-r} = I - P_r$, where I is the identity matrix. Thus P_{n-r} is a projection onto the complementary space E_{n-r} of E_r. If E_{n-r} is properly contained in $\mathbb{R}^n$ then E_r is an r-dimensional vector space where $0 < r < n$. If $c = (c_1, \dots, c_n)$, let $P_r c = c^r$ and $P_{n-r} = c^{n-r}$, then define a continuous mapping $\Phi_\varepsilon : \mathbb{R}^r \to \mathbb{R}^r$, given by

$$\Phi_\varepsilon(c_1, \dots, c_r) = P_r H \mathcal{N}(c^r \oplus c^{n-r}(c^r, \varepsilon), \alpha, \eta, \varepsilon). \tag{22}$$

By abuse of notation we will identify $P_r c$ and c^r when convenient and where the meaning is clear from the context so that in defining Φ_ε above from the context we interpreted $P_r H \mathcal{N}$ as $(H\mathcal{N}_1, \dots, H\mathcal{N}_r)$. Similarly we will sometimes identify $P_{n-r}c$ and c^{n-r}. Setting $\varepsilon = 0$, we have

$$\Phi_0(c_1, \dots, c_r) = P_r H \mathcal{N}(c^r \oplus P_{n-r}Hd, \alpha, \eta, 0)$$

where $c^{n-r}(c^r, 0) = P_{n-r}Hd$; note that from the context $c^{n-r}(c^r, 0) = P_{n-r}Hd$ is interpreted as $c^{n-r}(c^r, 0) = (Hd_{r+1}, \dots, Hd_n)$.

If $E_r = \mathbb{R}^n$ and $P_r = I$, then $P_{n-r} = 0$. Since $P_{n-r} = 0$ it follows that the matrix H is the identity matrix. Thus define a continuous mapping $\Phi_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$, given by $\Phi_\varepsilon(c) = \mathcal{N}(c, \alpha, \eta, \varepsilon)$. Setting $\varepsilon = 0$, we have $\Phi_0(c) = \mathcal{N}(c, \alpha, \eta, 0)$.

We establish an existence theorem by applying a version of Brouwer's Fixed Point Theorem which is due to Miranda (see, [32], [33, p. 214–215] and [35, p. 171–172]).

Lemma 4 (*Lemma 3, Yang [32]*). *Suppose there exist n pairs of numbers $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ such that $a_i < b_i$, $i = 1, 2, \dots, n$. Let*

$$\Phi_0(c) = \left(\Phi_0^1(c_1, \dots, c_n), \dots, \Phi_0^n(c_1, \dots, c_n)\right),$$

be n continuous functions defined in the box $B : a_i \le c_i \le b_i$, $i = 1, \dots, n$. If each Φ_0^i has constant sign on each of the faces $c_i = a_i$, $c_i = b_i$ of B and these two signs are opposite, then the functions $\Phi_0^1, \dots, \Phi_0^n$ have at least one common zero in B.

Both, Corollary 1 and Remark 2 below, are due to Corollary 5 of [32] and [35, p. 171–172].

Corollary 1 (*Generalized Intermediate Value Theorem*). *Suppose there exist $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ such that $a_i < b_i$, $i = 1, \dots, n$. Let $P = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Let $\Phi_0(c) = \Phi_0^1(c_1, \dots, c_n), \dots, \Phi_0^n(c_1, \dots, c_n)$, be n continuous functions defined in the box $B : a_i \le c_i \le b_i$, $i = 1, \dots, n$. Denote*

$$R_i = \{c \in P : c_1, \dots, c_{i-1}, a_i, c_{i+1}, \dots, c_n\},$$
$$S_i = \{c \in P : c_1, \dots, c_{i-1}, b_i, c_{i+1}, \dots, c_n\}, \quad i = 1, \dots, n.$$

If $\Phi_0^i(b)\Phi_0^i(e) \leq 0$, $b \in R_i$, $e \in S_i$, $i = 1, 2, \dots, n$, then either $\Phi_0(c) = 0$ for some $c \in \partial P$ or $d(\Phi_0, P^0, 0) \neq 0$, and $\Phi_0(c) = 0$ for some $c \in P^0$. Note that if

$$\Phi_0^i(b)\Phi_0^i(e) < 0, \tag{23}$$

then

$$d(\Phi_0, P^0, 0) = \prod_{1\leq i\leq n} (sign_{e\in S_i} \Phi_0^i(c)) = \pm 1,$$

where $P^0 = (a_1, b_1) \times \cdots \times (a_n, b_n)$ is the interior of P.

Proof. The proof is similar to that of [35] and so is omitted. □

Remark 2. It is clear that $d(\Phi_0, P^0, 0) \neq 0$ exists as a consequence of (23), $\Phi_0(c) \neq 0$, for $c \in \partial P = \cup_{1\leq i\leq n}(R_i \cup Si)$. For any one-dimensional mapping $H : \mathbb{R} \to \mathbb{R}$,

$$d(H, (a, b), 0) = 0, \quad \text{if } H(a)H(b) > 0, \tag{24}$$

$$d(H, (a, b), 0) = \pm 1, \quad \text{if } H(a)H(b) < 0. \tag{25}$$

Compare the following theorem with Theorem 3.8, p. 69 of Cronin [2].

Theorem 2. *If $r = n$, a necessary condition in order that (14) has a solution for each ε with $|\varepsilon| < \varepsilon_0$ is $d = 0$, that is*

$$\int_0^\eta NY(\eta)Y^{-1}(s)E(s)\mathrm{d}s + \int_0^1 RY(1)Y^{-1}(s)E(s)\mathrm{d}s = \ell.$$

Theorem 3. *If $r = n$, $d = 0$ and*

$$\Phi_\varepsilon(c) = -\int_0^\eta NY(\eta)Y^{-1}(s)F(s, x(s, c, \varepsilon), \varepsilon)\mathrm{d}s - \int_0^1 RY(1)Y^{-1}(s)F(s, x(s, c, \varepsilon), \varepsilon)\mathrm{d}s + g(c, x(\eta), x(1)) \tag{26}$$

satisfies

$$d(\Phi_0, B_k, 0) \neq 0$$

for some $k > 0$, then (1), (2) has a solution $x(t, c, \varepsilon)$ with $x(0, c, \varepsilon) = c$ where $c \in B_k \subset \mathbb{R}^n$ and $|\varepsilon| < \varepsilon_0$ for some $\varepsilon_0 > 0$.

References

[1] J. Cronin, Fixed Points and Topological Degree in Nonlinear Analysis, in: Math. Surveys, vol. 11, Amer. Math. Soc, Providence, Rhode Island, 1964.
[2] J. Cronin, Differential Equations: Introduction and Qualitative Theory, 2nd ed., Marcel Dekker, Inc., New York, 1994.
[3] W. Feng, J.R.L. Webb, Solvability of three point boudary value problems at resonance, Nonlinear Anal. TMA 30 (1997) 3227–3238.
[4] J. Mawhin, Topological degree and boundary value problems for nonlinear differential equations, in: P.M. Fitzpatrick, M. Martelli, J. Mawhin, R. Nussbaum (Eds.), Topological Methods for Ordinary Differential Equations, in: Lecture Notes in Mathematics, vol. 1537, Springer, New York, 1997, pp. 74–142.
[5] C.P. Gupta, Solvability of a multi-point boundary value problem at resonance, Results Math. 28 (3–4) (1995) 270–276.
[6] Z. Bai, W. Li, W. Ge, Existence and multiplicity of solutions for four-point boundary value problems at resonance, Nonlinear Anal. 60 (2005) 1151–1162.
[7] Z. Du, G. Cai, W. Ge, A class of third order multipoint boundary value problem, Taiwanese J. Math. 9 (1) (2005) 81–94.
[8] Z. Du, X. Lin, W. Ge, Some higher-order multi-point boundary value problem at resonance, J. Comput. Appl. Math. 177 (2005) 55–65.
[9] W. Feng, J.R.L. Webb, Solvability of M-point boudary value problems with nonlinear growth, J. Math. Appl. 212 (1997) 467–480.
[10] C.P. Gupta, Existence theorems for a second order M-point boundary value problem at resonance, Internat. J. Math. Sci. 18 (1995) 705–710.
[11] C.P. Gupta, On a third-order three-point boundary value problem at resonance, Differential Integral Equations 2 (1989) 1–12.
[12] B. Liu, J.S. Yu, Solvability of multi-point boundary value problems at resonance (I), Indian J. Pure Appl. Math. 34 (2002) 475–494.
[13] B. Liu, J.S. Yu, Solvability of multi-point boundary value problems at resonance (II), Appl. Math. Comput. 136 (2003) 353–373.
[14] B. Liu, J.S. Yu, Solvability of multi-point boundary value problems at resonance (III), Appl. Math. Comput. 129 (2002) 119–143.
[15] B. Liu, J.S. Yu, Solvability of multi-point boundary value problems at resonance (IV), Appl. Math. Comput. 143 (2003) 275–299.
[16] R. Ma, Multiplicity results for a M-point boundary value problem at resonance, Indian J. Math. 47 (1) (2005) 15–31.
[17] X. Ni, W. Ge, Multi-point boundary value problems for the p-laplacian at resonance, Electron. J. Differential Equations 112 (2003) 1–7.
[18] B. Przeradzki, R. Stanczy, Solvability of a multi-point boundary value problem at resonance, J. Math. Anal. Appl. 264 (2001) 253–261.
[19] C. Xue, Z. Du, W. Ge, Solutions to M-point boundary value problems of third order ordinary differential equations at resonance, J. Appl. Math. Comput. 17 (1–2) (2004) 229–244.
[20] E.A. Coddington, N. Levinson, Perturbations of Linear Systems with Constants Coefficients Possesing Periodis Solutions, in: Contributions to the Theory of Nonlinear Oscillations, vol. II, Princeton University Press, Princeton, 1950, pp. 19–35.
[21] J. Mawhin, Degre topologique et solutions periodique des systems differentiels non lineaires, Bull. R. Soc. Liege 38 (1969) 308–398.
[22] J. Mawhin, Landesman-Lazer's type problems for nonlinear equations, Conference Seminario Matematica Dell Universita di Bari 147 (1977).
[23] M. Urabe, An existence theorem for multipoint boundary value problems, Funkcial. Ekvac. 9 (1966) 43–60.
[24] R.P. Agarwal, On multipoint boundary value problem for discrete equations, J. Math. Anal. Appl. 96 (2) (1985) 520–534.
[25] R.P. Agarwal, D. O'Regan, Multipoint boundary value problems for general discrete systems: the degenerate case, Comm. Appl. Anal. 1 (3) (1997) 269–288.
[26] R.P. Agarwal, M. Sambandham, Multipoint boundary value problems for general discrete systems, Dynam. Systems Appl. 6 (4) (1997) 469–492.

[27] D.L. Etheridge, J. Rodriguez, Scalar discrete nonlinear two-point boundary value problems, J. Difference Equ. Appl. 4 (1998) 127–144.
[28] J. Rodriguez, Resonance in nonlinear discrete systems with nonlinear constraints, in: Proceedings of 24th Conference on Decision and Control, IEEE, Ft. Lauderdale, FL, 1985.
[29] J. Rodriguez, On perturbed discrete boundary value problems, Journal of Difference Equations and Applications 8 (5) (2002) 447–466.
[30] J. Rodriguez, On resonant discrete boundary value problems, Appl. Anal. 19 (1985) 265–274.
[31] M. Urabe, The degenerate case of boundary value problems associated with weakly nonlinear differential systems, Publ. Res. Inst. Math. Sci. Kyoto Univ. 4 (1968) 545–584.
[32] X. Yang, Upper and lower solutions for periodic problems, Appl. Math. Comput. 137 (2003) 413–422.
[33] L.C. Piccinini, G. Stampacchia, G. Vidossich, Ordinary differential equations in R^n (A. LoBello Trans.), in: Problems and Methods, Springer-Verlag, New York, 1984.
[34] W.L. Voxman, R.H. Goetschel, Advanced Calculus, An Introduction to Modern Analysis, Marcel Dekker, Inc., New York, 1981.
[35] N. Rouche, J. Mawhin, Ordinary Differential Equations, Stability and Periodic Solutions, Pitman Advanced Publishing Program, Boston, 1980.