On rational B-splines with prescribed poles

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Abstract

Spaces of rational splines of maximal smoothness are considered which are constructed from certain rational functions with prescribed poles. For them a basis of splines having minimal compact supports was constructed in the literature. These functions which are called rational B-splines are obtained by solving certain linear equations with a block matrix depending on a parameter \( \varepsilon > 0 \). It is shown that in the limit \( \varepsilon \to 0 \) they tend to certain polynomial B-splines.

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1. ECT-systems

A thorough treatment of extended complete Chebycheff-systems (ECT-systems) can be found in [8]. Here we only introduce some notations used in the rest of the paper. Let \( I \) be a subinterval of the real line \( \mathbb{R} \) and let \( d \) be a natural number. A system of functions

\[ U = (u_1, \ldots, u_d) \in C^{d-1}(I; \mathbb{R}) \]

is called an ECT-system of order \( d \) on \( I \) provided

\[ U \begin{bmatrix} u_1, \ldots, u_k \\ t_1, \ldots, t_k \end{bmatrix} := \det(D^\nu u_i(t_j)) > 0 \]
for all \( t_1 \leq t_2 \leq \cdots \leq t_k \) in \( I \) and for all \( k = 1, \ldots, d \) where \( D := \frac{d}{dt} \) denotes the operator of ordinary differentiation and
\[
v_j := \max \{ l : t_j = t_{j-1} = \cdots = t_{j-l} \}.
\]

Then span \( U \) will be called an ECT-space of order \( d \) on \( I \).

It is well known (cf. [8, p. 376f]; [13, p. 364]) that the following assertions are equivalent:

(i) \((u_1, \ldots, u_d)\) is an ECT-system of order \( d \) on \( I \);
(ii) there exist positive weight functions \( w_j \in C^{d-j}(I; \mathbb{R}) \), \( j = 1, \ldots, d \), and for every \( c \in I \) coefficients \( c_{j,i} \in \mathbb{R} \) such that
\[
\begin{align*}
  u_j(x) &= w_1(x) \int_c^x w_2(t_2) \int_c^{t_2} \cdots \int_c^{t_{j-1}} w_j(t_j) \ dt_j \cdots \ dt_2 \\
  &\quad + \sum_{i=1}^{j-1} c_{j,i} \cdot u_i(x), \quad j = 1, \ldots, d, \quad x \in I.
\end{align*}
\]

Then the functions
\[
S(c) = (s_1(\cdot, c), \ldots, s_d(\cdot, c)),
\]
where
\[
s_j(x, c) := u_j(x) - \sum_{i=1}^{j-1} c_{j,i} \cdot u_i(x), \quad j = 1, \ldots, d
\]

satisfy
\[
\begin{align*}
  s_1(x, c) &= w_1(x), \\
  s_j(x, c) &= w_1(x) \cdot h_{j-1}(x, c; w_2, \ldots, w_j), \quad j = 2, \ldots, d. \tag{2}
\end{align*}
\]

Here for \( 1 \leq m \leq d \),
\[
\begin{align*}
  h_0(x, c) &:= 1, \\
  h_m(x, c; w_1, \ldots, w_m) &:= \int_c^x w_1(t) \cdot h_{m-1}(t, c; w_2, \ldots, w_m) \ dt. \tag{3}
\end{align*}
\]

System (2) forms a special basis of span \( U \) which we call an ECT-system in canonical form with respect to \( c \). Associated with an ECT-system (1) or (2) are the linear differential operators
\[
\begin{align*}
  D_0 u &= u, \\
  D_j u &= D \left( \frac{u}{w_j} \right), \quad j = 1, \ldots, d, \\
  \hat{L}_j u &= D_j \cdots D_0 u, \quad j = 0, \ldots, d, \\
  L_j u &= \frac{1}{w_{j+1}} \hat{L}_j u, \quad j = 0, \ldots, d - 1. \tag{4}
\end{align*}
\]
Obviously,
\[ \text{ker } \hat{L}_j = \text{span}\{u_1, \ldots, u_j\}, \quad j = 1, \ldots, d \]  
and
\begin{align*}
L_j s_{j+1}(x, c) &= 1, \quad j = 0, \ldots, d - 1, \quad (5) \\
L_j s_{l+1}(c, c) &= \delta_{j,l}, \quad j, l = 0, \ldots, d - 1. \quad (6)
\end{align*}

2. Cauchy–Vandermonde-systems

Cauchy–Vandermonde-systems (CV-systems for short) consist of rational functions with prescribed poles. Their interpolation properties are treated in [11]. When the poles are arranged in an appropriate order they generate ECT-systems. Let \([\alpha, \beta] \subset \mathbb{R}\) be a nontrivial compact interval of the real line and let \(P_2, \ldots, P_m\) be prescribed real poles outside \([\alpha, \beta]\). We assume the poles to be arranged allowing repetition according to
\[ (p_1, \ldots, p_d) := \left( \infty, \ldots, \infty, P_2, \ldots, P_2, \ldots, P_m, \ldots, P_m \right) \]  
with integers \(d, n_1, n_2, \ldots, n_m\) such that \(d \geq 1, n_1 + \cdots + n_m = d\), \(n_i \geq 1\) and for \(i = 2, \ldots, m\), \(n_i\) nonnegative. The condition \(n_1 \geq 1\) ensures that the constant function equal to one belongs to any system to be considered. We use the CV-system
\[ (u_1, \ldots, u_d) := \left( 1, x, \ldots, x^{n_1-1}, \frac{1}{x - P_2} \ldots, \frac{1}{(x - P_2)^{n_2}}, \ldots, \frac{1}{x - P_m} \ldots, \frac{1}{(x - P_m)^{n_m}} \right) \]  
generated by the system of poles (8) as a basis of the CV-space
\[ S := \text{span}\{u_1, \ldots, u_d\}. \]
CV-spaces allow interpolation in the sense of Hermite [11]: given interpolation points (also called “nodes”)
\[ \alpha \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_d \leq \beta \]
and corresponding real numbers \(\eta_1, \eta_2, \ldots, \eta_d\) with
\[ \nu_i := \#\{\tau_j : \tau_j = \tau_i, \ j = 1, \ldots, i - 1\}, \]
the multiplicity of \(\tau_i\) in its predecessors, the problem
\[ \text{—find } u \in S \text{ such that } D^{\nu_i} u(\tau_i) = \eta_i, \quad i = 1, \ldots, d— \]  
has a unique solution. In fact, it is possible to give an explicit representation of the determinant of the linear system for the coefficients of the \(u_j\) underlying (10) (cf. [4,5,11]):
\begin{align*}
\det(D^{\nu_i} u_i(\tau_j))_{j=1, \ldots, d} &= \prod_{k=1}^{d} (v_k!), \quad \prod_{k \geq j}^{d} (\tau_k - \tau_j)^{v_k} \prod_{k \geq j}^{d} (p_k - p_j)^{v_k}, \quad (11)
\end{align*}
where we use the shorthand notation for \( z \) in the extended complex plane
\[
    z^* := \begin{cases} 
        1 & \text{if } z = 0 \text{ or } z = \infty, \\
        z & \text{else.} 
    \end{cases}
\]

In particular, the Wronskians of every subsystem \((u_1, \ldots, u_j)\) of \((9)\)
\[
    W(u_1, \ldots, u_j)(x) := \det(D'(u_i(x)))_{i=0,\ldots,j-1} \neq 0
\]
have no zeros in the interval \([\alpha, \beta]\). Consequently, there are sign factors \(\sigma_1, \ldots, \sigma_d \in \{-1, 1\}\), such that all subsystems \((\tilde{u}_1, \ldots, \tilde{u}_j)\) of
\[
    (\tilde{u}_1, \ldots, \tilde{u}_d) := (\sigma_1 u_1, \ldots, \sigma_d u_d) \tag{12}
\]
have positive Wronskians in \([\alpha, \beta]\). Then the functions
\[
    v_1 := \tilde{u}_1, \\
    v_j := \tilde{u}_j - \text{LK}(\tilde{u}_1, \ldots, \tilde{u}_{j-1}), \quad j = 2, \ldots, d,
\]
where \(\text{LK}(\ldots)\) is a linear combination of \(\tilde{u}_1, \ldots, \tilde{u}_{j-1}\) that is uniquely determined such that \(v_j\) satisfies the initial conditions
\[
    D^l v_j(c) = 0, \quad l = 0, \ldots, j-2, \quad j = 2, \ldots, d \tag{13}
\]
with \(c \in [\alpha, \beta]\) form an ECT-system on \([\alpha, \beta]\) in canonical form with respect to \(c\). The sign factors and the weight functions are given in the following proposition which generalizes a result of [10].

**Proposition 2.1.** Suppose that
\[
    \cdots < P_4 < P_2 < \alpha \leq x \leq \beta < P_3 < P_5 < \cdots \tag{14}
\]
Then under assumptions \((8)\), the corresponding sign factors \(\sigma_j\) and the weight functions \(w_j, j = 1, \ldots, d\) are
\[
    \sigma_i = 1, \quad i = 1, \ldots, n_1, \\
    \sigma_{N_2 r - i + 1} = (-1)^{n_1 + n_3 + \cdots + n_{2r-1} + i-1}, \quad i = 1, \ldots, n_{2r}, \\
    \sigma_{N_2 r + i} = (-1)^{n_3 + n_5 + \cdots + n_{2r-1} + i}, \quad i = 1, \ldots, n_{2r+1},
\]
where for \(s = 1, \ldots, m\)
\[
    N_s := n_1 + n_2 + \cdots + n_s
\]
and
\[
    w_1(x) = 1, \\
    w_k(x) = k - 1, \quad k = 2, \ldots, n_1, \\
    w_{N_2 r - 1}(x) = N_{2r-1} \frac{\prod_{j=2}^{2r-2} (p_j - p_{2r})^{n_j}}{\prod_{j=2}^{2r-2} (p_{2r-1} - p_j)^{n_j}} \cdot \frac{(p_{2r-1} - x)^{n_{2r-1} - 1}}{(x - p_{2r})^{n_{2r-1} + 1}},
\]
\[ w_{N_{2r-1}+i}(x) = \left( N_{2r-1} + i - 1 \right) \frac{1}{(x - p_{2r})^2}, \quad i = 2, \ldots, n_{2r}, \]

\[ w_{N_{2r}+1}(x) = N_{2r} \cdot \prod_{j=2}^{2r} \frac{(p_{2r+1} - p_j)^{n_j}}{p_j - p_{2r}} \cdot \frac{(x - p_{2r})^{N_{2r}-1}}{(p_{2r+1} - x)^{N_{2r}+1}}, \]

\[ w_{N_{2r}+i}(x) = \left( N_{2r} + i - 1 \right) \cdot \frac{1}{(p_{2r+1} - x)^2}, \quad i = 2, \ldots, n_{2r+1}. \] (15)

The proof of Proposition 2.1 is rather technical. We only give the idea. According to a result of \[8, p.380\] the sign factors are uniquely determined by the conditions that all weight functions

\[ w_1(x) = \tilde{u}_1(x), \]

\[ w_2(x) = \frac{W(\tilde{u}_1, \tilde{u}_2)(x)}{[\tilde{u}_1(x)]^2}, \]

\[ w_k(x) = \frac{W(\tilde{u}_1, \ldots, \tilde{u}_{k-2})(x)W(\tilde{u}_1, \ldots, \tilde{u}_k)(x)}{[W(\tilde{u}_1, \ldots, \tilde{u}_{k-1})(x)]^2}, \quad k = 3, \ldots, d \]

are positive on \([\alpha, \beta]\). Since the Wronskians can be computed explicitly according to (11) with all nodes equal to \(x\), Proposition 2.1 can be proved by induction.

Notice that (14) covers all possible choices of poles outside of \([\alpha, \beta]\) since \(n_i\) can be taken equal to zero for \(i \geq 2\).

In the rest of the paper, we consider particular CV-systems of fixed order \(d \geq 4\). On an interval \([\alpha, \beta]\) such a system has the form

\[ V_{\alpha, \beta}^\varepsilon := (v_1^\varepsilon, \ldots, v_{d-2}^\varepsilon, v_{d-1}^\varepsilon, v_d^\varepsilon) \]

\[ = \left( 1, x - \alpha, \ldots, (x - \alpha)^{d-3}, \frac{1}{x - \alpha + \varepsilon}, \frac{1}{x - \beta - \varepsilon} \right). \] (16)

Observe, that only the last two functions do depend on \(\varepsilon\).

Let

\[ S_{\alpha, \beta}^\varepsilon(c) := (s_1^\varepsilon(\cdot, c), \ldots, s_{d-2}^\varepsilon(\cdot, c), s_{d-1}^\varepsilon(\cdot, c), s_d^\varepsilon(\cdot, c)) \] (17)

be the ECT-system in canonical form with respect to \(c \in [\alpha, \beta]\) which is a basis of span \(V_{\alpha, \beta}^\varepsilon\) on \([\alpha, \beta]\). The weights associated with (16) by some calculations are derived from (15)

\[ w_1(x) = 1, \]

\[ w_j(x) = j - 1, \quad j = 2, \ldots, d - 2, \]

\[ w_{d-1}(x) = \frac{d - 2}{(x - \alpha + \varepsilon)^{d-1}}, \]

\[ w_d(x) = \frac{(d - 1)(\beta - x + 2\varepsilon)(x - \alpha + \varepsilon)^{d-2}}{(\beta - x + \varepsilon)^d}. \] (18)
According to (2) and (3) by repeated integration (and some elementary but lengthy calculations) we derive
\[
\begin{align*}
    s_j^{\varepsilon}(x, c) &= (x - c)^{j-1}, \quad j = 2, \ldots, d - 2, \\
    s_{d-1}^{\varepsilon}(x, c) &= \frac{(x - c)^{d-2}}{(c - x + \varepsilon)^d(x - x + \varepsilon)}, \\
    s_d^{\varepsilon}(x, c) &= \frac{(x - c)^{d-1}(\beta - x + 2\varepsilon)}{(x - x + \varepsilon)(\beta - x + \varepsilon)(\beta + \varepsilon - c)^{d-1}}.
\end{align*}
\]
(19)
Alternatively, instead of repeated integration according to (3) for a proof of (19), conditions (5)–(7) may be verified, which is done more easily. Observe again that only the last two functions do depend on \(\varepsilon\).

3. Cauchy–Vandermonde-splines

We are going to construct rational splines with prescribed poles by pasting together linear combinations of different CV-systems in a way similar to the procedure polynomial splines are constructed. In fact, by putting all poles at \(\infty\), the construction to be described below will yield the polynomial B-splines normalized to have integrals equal to one.
Let
\[-\infty < x_0 < x_1 < \cdots < x_k < x_{k+1} < \infty\]
be a finite sequence \((x_i)_{i=0}^{k+1}\) of strictly increasing reals (“simple knots”). On each knot interval \(I_i := [x_i, x_{i+1}]\) consider the CV-space
\[
S_i^{\varepsilon} = \text{span} V_{x_i, x_{i+1}}^{\varepsilon},
\]
where \(V_{x_i, x_{i+1}}^{\varepsilon}\) denotes the CV-system
\[
V_{x_i, x_{i+1}}^{\varepsilon} := (v_{1,\varepsilon}^{x_i}, \ldots, v_{d-2,\varepsilon}^{x_i}, v_{d-1,\varepsilon}^{x_i}, v_d^{x_i}) := \left(1, (x - x_i), \ldots, (x - x_i)^{d-3}, \frac{1}{x - x_i + \varepsilon}, \frac{1}{x - x_{i+1} - \varepsilon}\right).
\]
(21)
Another basis of \(S_i^{\varepsilon}\) is
\[
S_{x_i, x_{i+1}}^{\varepsilon}(c) := (s_1^{\varepsilon}(\cdot, c), \ldots, s_{d-2}^{\varepsilon}(\cdot, c), s_{d-1}^{\varepsilon}(\cdot, c), s_d^{\varepsilon}(\cdot, c))
\]
(22)
with \(c \in [x_i, x_{i+1}]\) fixed where its functions are defined by (2), (3) and (18), accordingly. Using these CV-spaces we define the space of rational spline functions
\[
S_{[x_0, x_{k+1}]}^{\varepsilon} \ni \{s : s : [x_0, x_{k+1}] \rightarrow \mathbb{R}, s|_{I_j} \in S_j^{\varepsilon}, j = 0, \ldots, k; \}
\]
\[
D^l_+(s|_{I_j}) = D^l_+(s(x_i)), \quad l = 0, \ldots, d - 2, \quad i = 1, \ldots, k.
\]
(23)
Here \(D^l_-\) resp. \(D^l_+\) denote the ordinary left resp. right derivative of order \(l\) with respect to \(x\). Notice, that this space as well as any of its bases do depend on the parameter \(\varepsilon > 0\).
It is well known and easily seen that
\[ \dim S_{[x_0,x_{k+1}]}^{\max,\varepsilon} = d + k. \]

In order to construct a basis for the space of rational splines consisting of functions having minimal compact supports (rational “B-splines”) we need an extension of the knot sequence \((x_i)_{i=0}^{k+1}\). We assume that \((x_i)_{i=-d+1}^{k+1}\) is a strictly increasing sequence of reals and that for \(j = -d + 1, \ldots, k + d - 1\) on \(I_j\) the CV-space \(S_j^\varepsilon\) is used where \(S_j^\varepsilon = \text{span} V_{x_j,x_{j+1}}^\varepsilon\) and \(V_{x_j,x_{j+1}}^\varepsilon\) is defined by (16). Again, the basis \(V_{x_j,x_{j+1}}^\varepsilon\) sometimes is replaced by a basis \((22)\) \(S_j^\varepsilon(x), c \in [x_j,x_{j+1}]\).

According to Theorem 4.2 of [3] there exist rational B-splines
\[ r_i^\varepsilon \in C^{d-2}(\mathbb{R}), \quad i = -d + 1, \ldots, k \]
with \(r_i^\varepsilon\) having minimal support \([x_i,x_i+d]\) which are normalized to have integral 1 over the real axis and which form a basis of the spline space \(S_{[x_0,x_{k+1}]}^{\max,\varepsilon}\).

According to Theorem 4.4(i)(d) of [3]
\[ r_i^\varepsilon(x) = \sum_{l=i}^{i+d-1} \sum_{j=1}^{d} c_j^{i+\varepsilon} \cdot v_j^{i+\varepsilon}(x), \quad (24) \]
where the stack vector
\[ c^{i+\varepsilon} = (c_1^{i+\varepsilon}, \ldots, c_d^{i+\varepsilon}, c_1^{i+1+\varepsilon}, \ldots, c_d^{i+1+\varepsilon}, \ldots, c_1^{i+d-1+\varepsilon}, \ldots, c_d^{i+d-1+\varepsilon}) \quad (25) \]
is the unique solution of the linear system
\[ \varphi^{i+\varepsilon}_j c^{i+\varepsilon} = (0, \ldots, 0, 1)^T, \quad (26) \]
whose matrix is with \(j := i + d - 1\)
\[ \varphi^{i+\varepsilon}_j := \left( \begin{array}{cccccc}
V_+^{i+\varepsilon} & 0 & \ldots & 0 \\
V_-^{i+1,\varepsilon} & -V_+^{i+1,\varepsilon} & 0 \\
0 & V_-^{i+2,\varepsilon} & -V_+^{i+2,\varepsilon} & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & V_-^{j-1,\varepsilon} & -V_+^{j-1,\varepsilon} & 0 \\
0 & V_-^{j,\varepsilon} & -V_+^{j,\varepsilon} \\
J_j^{\varepsilon} & J_1^{\varepsilon} & \ldots & J_{j-1}^{\varepsilon} & J_{j,\varepsilon}
\end{array} \right). \quad (27) \]

Its entries in the last row are
\[ J_j^{\varepsilon} := \left( \int_{x_k}^{x_{k+1}} v_1^{k,\varepsilon}(x) \, dx, \ldots, \int_{x_k}^{x_{k+1}} v_d^{k,\varepsilon}(x) \, dx \right) \]
and the other entries are

\[ V^k_{-} := \left( D^l_+ v^k_{-}(x_l) \right)_{l=0,\ldots,d-2}^{m=1,\ldots,d}, \quad V^k_{+} := \left( D^l_+ v^k_{+}(x_l) \right)_{l=0,\ldots,d-2}^{m=1,\ldots,d}. \]  

The matrices \( V^k_{-} \) and \( V^k_{+} \) are of type \((d-1) \times d\) having full rank \(d-1\). Hence, \( V^k_{-} \) is of type \( d^2 \times d^2 \). By Theorem 4.4 of [3] this matrix is regular. It should be observed that the coefficients \( c_{i}^{k_{-}} \) for \( k = i, \ldots, i + d - 1 \), \( l = 1, \ldots, d \) do depend also on the local bases of \( S^k_{\varepsilon} \) for \( k = i, \ldots, i + d - 1 \) chosen, a fact not reflected by the notation \( V^k_{-} \) for the matrix (27).

Remark.

(i) Existence and uniqueness of the rational B-splines \( r^\varepsilon_{i} \) have been derived from the linear system (26). For small dimensions \( d \), it can be solved exactly with no difficulties using computer algebra. For \( d = 4, 5, 6 \) this has been done in [2] using Maple 6.

(ii) It is an open problem if there exists a recurrence relation computing the B-splines constructed by pasting together linear combinations of different local CV-systems of dimension \( d \) on the knot intervals from those constructed from CV-systems of lower dimensions.

(iii) For Chebyshevian B-splines constructed by pasting together restrictions to the knot intervals of linear combinations of one global Chebyshev-system, Lyche [9] has proved a three-term recurrence relation. Its practical use is limited for in each step its coefficients are quotients of differences of generalized divided differences with respect to the dual ECT-system of the basic global ECT-system.

(iv) When dealing with Chebyshevian splines constructed from a global CV-system (9) with prescribed poles (8) all outside \([a,b]\) Gresbrand [7] has found a three-term recurrence relation for its B-splines. Even worse, in each step its coefficients are quotients of permanents which cannot be computed recursively.

(v) Nevertheless, there are fast algorithms solving the linear system (26). CV-matrices \( V \) which are the blocks of the matrix \( V^k_{-} \) have been shown to have displacement structure of Sylvester’s type [14]. As a consequence, there exists a fast algorithm due to Gohberg et al. [6] solving the linear system with a nonsingular CV-matrix. The matrix \( V^k_{-} \) of the linear system (26) having block structure with blocks that are CV-matrices can be shown to have displacement structure too [1]. Hence, also in this case a fast algorithm solving (26) exists which for large dimensions \( d \) actually proves to be faster than Gaussian elimination with partial pivoting [1].

(vi) There is a different approach to generalized B-splines [12]. It is possible to paste together linear combinations of different local ECT-systems. The pasting conditions at each knot \( x_k \) involve the natural linear differential operators associated with the neighbored local ECT-systems and a connection matrix \( A^{[i]} \) which is nonsingular, lower triangular and totally positive. This approach applies in particular to local CV-systems which may be completely different apart from the condition that the constant function equal to one belongs to the spline space generated. Under these assumptions, a basis of continuous B-splines can be constructed that is normalized to form a positive partition of unity.
4. A limit theorem for certain Cauchy–Vandermonde B-splines

The main result of this paper (Theorem 4.7) needs some preparation.

**Lemma 4.1.** For \( l \in \{0, \ldots, d - 2\} \) let

\[
p_{d-1,l} : \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad \varepsilon \mapsto D^l s_{d-1}^\varepsilon(x_{i+1}, x_i),
\]

\[
p_{d,l} : \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad \varepsilon \mapsto D^l s_d^\varepsilon(x_{i+1}, x_i),
\]

where \( D := \partial / \partial x \) denotes the derivative with respect to the first variable. Then

(i) \( p_{d-1,l} \) as a function of the parameter \( \varepsilon \) is a rational function having a pole of order \( d \) at the point \( \varepsilon = 0 \). Moreover, \( p_{d-1,l}(\varepsilon) = (-1)^l \varepsilon^l \), \( \varepsilon \to 0 \), where \( \oplus \) denotes a rational function of \( \varepsilon \) which tends to a positive limit for \( \varepsilon \to 0 \).

(ii) \( p_{d,l} \) as a function of the parameter \( \varepsilon \) is a rational function having a pole of order \( l + 1 \) at the point \( \varepsilon = 0 \). Moreover, \( p_{d,l}(\varepsilon) > 0 \) for \( \varepsilon > 0 \).

**Proof.** (i) By developing \( s_{d-1}^\varepsilon(x, x_i) \) into a power series around \( x = x_{i+1} \) one finds with \( h := x_{i+1} - x_i \)

\[
s_{d-1}^\varepsilon(x, x_i) = \frac{1}{\varepsilon^d (h + \varepsilon)} \frac{[h + (x - x_{i+1})]^d}{1 + (x - x_{i+1})/(h + \varepsilon)}
\]

\[
= \frac{1}{\varepsilon^d (h + \varepsilon)} \sum_{l=0}^{\infty} \left( \sum_{a=0}^{d-2} \binom{d-2}{a} h^{d-2-a} (-1)^{l-a} (h + \varepsilon)^{l-a} \right) (x - x_{i+1})^l
\]

\[
= \sum_{l=0}^{\infty} a_l (x - x_{i+1})^l
\]

with

\[
a_l = \frac{(-1)^l}{\varepsilon^d (h + \varepsilon)^{l+1}} \sum_{a=0}^{d-2} \binom{d-2}{a} h^{d-2-a} (-1)^{l-a} (h + \varepsilon)^{l-a}
\]

\[
= (-1)^l \varepsilon^d, \varepsilon \to 0 \quad (l = 0, \ldots, d - 2).
\]

(ii) The positivity of \( p_{d,l} \) follows at once from the fact that \( x \mapsto s_d^\varepsilon(x, x_i) \) and all its derivatives (with respect to \( x \)) are positive in \((x_i, x_{i+d}]\). In fact, by Leibniz’ rule for every nonnegative integer \( l \in \mathbb{N}_0 \)

\[
D^l s_d^\varepsilon(x, x_i) = \frac{(h + 2\varepsilon)}{(h + \varepsilon)^{d-l}} D^l \left( \frac{(x - x_i)^{d-1}}{(x - x_i + \varepsilon)(x_{i+1} + \varepsilon - x)} \right)
\]

\[
= \frac{1}{(h + \varepsilon)^{d-l}} \sum_{j=m_l}^{l} \lambda_j (x - x_i)^{d-1-l+j} \left( \frac{(-1)^j}{(x - x_i)^{j+1}} + \frac{1}{(x_{i+1} + \varepsilon - x)^{j+1}} \right)
\]
with $m_l := \max\{l - d + 1, 0\}$ and $\lambda_j = (l!/(l - j)!)((d - 1)!/(d - 1 - l + j)! > 0$ since

$$D^j \frac{1}{(x - x')(\beta - x)} = \frac{j!}{\beta - x} \left( \frac{(-1)^j}{(x - x')^{j+1}} + \frac{1}{(\beta - x)^{j+1}} \right)$$

and

$$D^{l+j} (x - x')^{d-1} = \frac{(d - 1)!}{(d - 1 - l + j)!} (x - x')^{d-1 - l + j}.$$  

From this

$$p_{d,l}(\varepsilon) = \frac{1}{(h + \varepsilon)^{d-1}} \sum_{j=m_l}^l \lambda_j \mu_{d-l-j} \left( \frac{(-1)^j}{(h + \varepsilon)^{j+1}} + \frac{1}{\varepsilon^{j+1}} \right)$$

obtains. Clearly, $p_{d,l}$ as function of the parameter $\varepsilon$ is a rational function. For $\varepsilon \to 0$ the dominant term is $\varepsilon^{-(l+1)}$ with a positive coefficient, i.e. $p_{d,l}$ has a pole of order $l + 1$ at $\varepsilon = 0$. Moreover, $p_{d,l}(\varepsilon) > 0$ for $\varepsilon > 0$ is evident. \( \square \)

**Lemma 4.2.** For $l \in \mathbb{N}_0$ let

$$q_{d-l}(x) : \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad \varepsilon \mapsto (-D)^j s_{d-1}^{j+d-1, \varepsilon}(x_{i+d-1}, x_{i+d})$$

$$q_{d,l}(x) : \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad \varepsilon \mapsto (-D)^j s_{d}^{j+d-1, \varepsilon}(x_{i+d-1}, x_{i+d})$$

where again $D = \partial / \partial x$ denotes the derivative with respect to the first variable. Then

(i) $q_{d-l}$ as a function of the parameter $\varepsilon$ is a rational function having a pole of order $l + 1$ at the point $\varepsilon = 0$. Moreover, $q_{d-l}(\varepsilon) = (-1)^d q_{d+1}(\varepsilon) \to 0$, where $\oplus$ denotes a rational function of $\varepsilon$ which tends to a positive limit for $\varepsilon \to 0$.

(ii) $q_{d,l}$ as a function of the parameter $\varepsilon$ is a rational function having a pole of order $d + 1$ at the point $\varepsilon = 0$. Moreover, $q_{d,l}(\varepsilon) > 0$ for $\varepsilon > 0$.

**Proof.** (i) By developing $s_{d-1}^{j+d-1, \varepsilon}(x, x_{i+d})$ into a power series around $x = x_{i+d-1}$ one finds with $h := x_{i+d} - x_{i+d-1}$

$$s_{d-1}^{j+d-1, \varepsilon}(x, x_{i+d}) = \frac{[ - h + (x - x_{i+d-1})]^{d-2}}{(h + \varepsilon)^d(1 + (x - x_{i+d-1})/\varepsilon)}$$

$$= \frac{1}{\varepsilon(h + \varepsilon)^d} \sum_{l=0}^\infty \sum_{z=0}^l \binom{d-2}{z} (-1)^{d-2-z} \frac{h^{d-2-z}}{\varepsilon^{l-z}} (x - x_{i+d-1})^l$$

$$= \sum_{l=0}^\infty m_l(x - x_{i+d-1})^l.$$
with
\[
m_{l} = \frac{(-h)^{d-2}}{\varepsilon(h + \varepsilon)^{d}(-\varepsilon)^{l}} \sum_{x=0}^{l} \binom{d - 2}{x} \left(\frac{\varepsilon}{h}\right)^{x}
\]
\[= (-1)^{d-l} \sum_{\varepsilon^{l+1}}^{l+1}, \quad \varepsilon \to 0, \quad l = 0, \ldots, d - 2.\]

(ii) By Leibniz’ rule for every \(l \in \mathbb{N}_0\) as in the proof of Lemma 4.1 we have
\[
(-D)^{l} s_d^{i+d-1, \varepsilon}(x, x_{i+d})
\]
\[= \frac{h + 2\varepsilon}{\varepsilon^{d-1}} (-D)^{l} \left(\frac{(x_{i+d} - x)^{d-1}}{(x - x_{i+d-1} + \varepsilon)(x_{i+d} + \varepsilon - x)}\right)
\]
\[= \frac{1}{\varepsilon^{d-1}} \sum_{j=m_{l}}^{l} \lambda_{j}(x_{i+d} - x)^{d-1-l+j} \left(\frac{1}{(x - x_{i+d-1} + \varepsilon)^{j+1}} + \frac{(-1)^{j}}{(x_{i+d} + \varepsilon - x)^{j+1}}\right).
\]

From this
\[
q_{d,l}(\varepsilon) = \frac{1}{\varepsilon^{d-1}} \sum_{j=m_{l}}^{l} \lambda_{j} \theta_{d-1-l+j} \left(\frac{1}{\varepsilon^{j+1}} + \frac{(-1)^{j}}{(h + \varepsilon)^{j+1}}\right)
\]
results with \(\lambda_{j}\) as in the proof of Lemma 4.1. Obviously, \(q_{d,l}\) as a function of \(\varepsilon\) is rational. For \(\varepsilon \to 0\) the dominant term is \(\varepsilon^{-(l+d)}\) with a positive coefficient, i.e. \(q_{d,l}\) has a pole of order \(d + l\) at \(\varepsilon = 0\). Clearly, \(q_{d,l}(\varepsilon) > 0\) for \(\varepsilon > 0\). \(\square\)

**Lemma 4.3.** For \(k = i, \ldots, d + i - 1\) and \(l = 1, \ldots, d\) let
\[
C_{i}^{k}: \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad \varepsilon \mapsto c_{i}^{k, \varepsilon},
\]
where \(c_{i}^{k, \varepsilon}\) are the entries of the solution vector (25) of the linear system (26).

(i) For any local basis of \(S_{i}^{k}\), \(C_{i}^{k}\) is a rational function of the parameter \(\varepsilon\).
(ii) If \(\eta > 0\) is sufficiently small then \(C_{i}^{k}|_{[0, \eta]}\) is a monotonic function of \(\varepsilon\).

**Proof.** For any choice of a local basis of \(S_{i}^{k}\) the entries of matrix (27) are rational functions of \(\varepsilon\). Hence, also the entries of its inverse are rational functions of \(\varepsilon\). From this (i) follows since the right-hand side of system (26) has constant entries.

To prove (ii) observe that the set of zeros or poles of the first derivative of \(\varepsilon \to C_{i}^{k}(\varepsilon) = c_{i}^{k, \varepsilon}\) is finite since this function also is rational. Choose \(\eta\) to be the smallest positive element of this set. \(\square\)
Theorem 4.4. Assume that the knots \((x_i)^{k+d}_{i=-d+1}\) are chosen equidistant with distance \(h > 0\):

\[ x_i = x_0 + ih, \quad i = -d + 1, \ldots, k + d. \]

Then

(i) the B-spline \(r^\varepsilon_{i+1}\) is the translate of \(r^\varepsilon_i\) by \(h\), i.e.,

\[ r^\varepsilon_{i+1}(x) = r^\varepsilon_i(x - h), \quad x \in \mathbb{R} \]

for \(i = -d + 1, \ldots, k - 1\).

(ii) For every \(\varepsilon > 0\) \(r^\varepsilon_i\) is unimodal, i.e. \(r^\varepsilon_i\) has precisely one maximum.

(iii) The splines \(r^\varepsilon_i\) are uniformly bounded for \(i \in \{-d + 1, \ldots, k\}\) and \(\varepsilon > 0\), i.e. there exists a constant \(A > 0\) such that for every \(i\)

\[ |r^\varepsilon_i(x)| \leq A \quad \text{for all} \ x \in \mathbb{R} \ \text{and all} \ \varepsilon > 0. \]

Proof. (i) Let \(i \in \{-d + 1, \ldots, k + d - 2\}\) be arbitrary. Choose in the knot interval \([x_i, x_{i+1}]\) as basis of \(S^\varepsilon_i\) the CV-systems (22) \(S^\varepsilon_{x_i, x_{i+1}}(x_i)\) with \(\alpha = x_i, \beta = x_{i+1}\) and \(c = x_i\) and in the next knot interval \([x_{i+1}, x_{i+2}]\) as basis of \(S^\varepsilon_{i+1}\) the CV-systems (22) \(S^\varepsilon_{x_{i+1}, x_{i+2}}(x_{i+1})\) with \(\alpha = x_{i+1}, \beta = x_{i+2}\) and \(c = x_{i+1}\). According to (19) then the latter is the \(h\)-translate of the former. Since the matrices \(\gamma^\varepsilon_{i, j}\) and \(\gamma^\varepsilon_{i+1, j}\) are identical and the right-hand side is independent of \(i\), so are the solutions of the linear system (26). This proves (i).

To show (ii) observe that the spline \(r^\varepsilon_i\) is a positive \(C^{d-1}\)-function on \(\mathbb{R}\) having compact support \([x_i, x_{i+d}]\). Consequently, it has there a positive maximum. On the other side the space spanned by \(Dr^\varepsilon_i\) \((i = -d + 1, \ldots, k)\) is also an interpolation space in the sense of [3] of dimension \(d - 1 + k\) of maximal smoothness \(C^{d-2}\). The spline \(Dr^\varepsilon_i\) has support \([x_i, x_{i+d}]\). According to the zero count (7) of [3] it has at most \(d(d - 1) - (d + 1)(d - 2) - 1 = 1\) zeros in \([x_i, x_{i+d}]\). This proves (ii).

We prove (iii) by contradiction. Suppose that the set of functions \(\{r^\varepsilon_i : \varepsilon > 0\}\) is not uniformly bounded on \([x_i, x_{i+d}]\). Let \(x^\varepsilon_i\) be the point where \(r^\varepsilon_i\) takes its maximum in this interval. Consider the net \((r^\varepsilon_i(x^\varepsilon_i))_{\varepsilon > 0}\). By assumption, \(\limsup_{\varepsilon \to 0} r^\varepsilon_i(x^\varepsilon_i) = \infty\). Since \([x_i, x_{i+d}]\) is compact, there exists a subnet \((x^\varepsilon_i)\) of \((x^\varepsilon_i)\) that converges, say \(\lim_{\eta \to 0} x^\varepsilon_i = : x^\eta\). For simplicity, we denote this subnet again by \((x^\varepsilon_i)\). If \(x^\eta \in [x_p, x_{p+1}]\) then take \(V^\varepsilon_{x_p, x_{p+1}}\) as basis of \(S^\varepsilon_p\) and consider

\[ \phi^\varepsilon := \sum_{j=1}^{d} c^{p, \varepsilon}_j v^{p, \varepsilon}_j \in S^\varepsilon_p, \]

where \(c^{p, \varepsilon}_1, \ldots, c^{p, \varepsilon}_d\) is the \(p\)th part of the solution \(c^{p, \varepsilon}\) of the linear system (26). For each \(\varepsilon > 0\) the continuous function \(\phi^\varepsilon\) is a positive rational function in \([x_p, x_{p+1}]\) having precisely two poles of order one each in \(x_p - \varepsilon\) and \(x_{p+1} + \varepsilon\). It is either monotonic or unimodular on \([x_p, x_{p+1}]\) and satisfies

\[ \int_{x_p}^{x_{p+1}} \phi^\varepsilon(x) \, dx \leq 1. \] (29)
Since \( \phi_\varepsilon \) for \( \varepsilon \to 0 \) becomes arbitrary large near \( x' \) for every \( y \in [x_p, x_{p+1}] \) distinct from \( x' \) we must have
\[
\phi_\varepsilon(y) = \mathcal{O}(1), \quad \varepsilon \to 0
\]
(30)
for otherwise we would have a contradiction to (29).

Choose \( d \) points \( y_1, \ldots, y_d \in (x_p, x_{p+d}) \) pairwise distinct and distinct from \( x' \). Let \( U^\varepsilon \) be the generalized Vandermonde matrix
\[
U^\varepsilon = U^\varepsilon \left( \begin{array}{cccc}
1 & y_1 - x_p & \cdots & (y_1 - x_p)^{d-3} \\
1 & y_2 - x_p & \cdots & (y_2 - x_p)^{d-3} \\
\vdots & \vdots & \ddots & \vdots \\
1 & y_d - x_p & \cdots & (y_d - x_p)^{d-3}
\end{array} \right)
\]
\[
= \left( \begin{array}{cccc}
1 & y_1 - x_p & \cdots & (y_1 - x_p)^{d-3} \\
1 & y_2 - x_p & \cdots & (y_2 - x_p)^{d-3} \\
\vdots & \vdots & \ddots & \vdots \\
1 & y_d - x_p & \cdots & (y_d - x_p)^{d-3}
\end{array} \right).
\]

Since \( V_p^\varepsilon \) is an ECT-system there is a positive constant \( \gamma \) such that for all \( \varepsilon \) with \( 1 \geq \varepsilon > 0 \)
\[
|\det U^\varepsilon| \geq \gamma > 0,
\]
since \( |\det U^\varepsilon| \) is a positive continuous function of \( \varepsilon \) on the compact interval \( 0 \leq \varepsilon \leq 1 \). From this and from (30) we conclude that there is a positive constant \( C \) such that for all \( \varepsilon \), \( 0 \leq \varepsilon \leq 1 \),
\[
\|(U^\varepsilon)^{-1}\|_\infty \leq C \quad \text{where} \quad \| \cdot \|_\infty \text{ denotes the matrix norm induced by the vector norm } \| \cdot \|_\infty.
\]
From
\[
U^\varepsilon \cdot (c_1^{p,\varepsilon}, \ldots, c_d^{p,\varepsilon})^T = (\phi_\varepsilon(y_1), \ldots, \phi_\varepsilon(y_d))^T,
\]
it follows
\[
\|(c_1^{p,\varepsilon}, \ldots, c_d^{p,\varepsilon})\|_\infty \leq \|(U^\varepsilon)^{-1}\|_\infty \|((\phi_\varepsilon(y_1), \ldots, \phi_\varepsilon(y_d))\|_\infty = \mathcal{O}(1), \quad \varepsilon \to 0.
\]
We do know that \( c_j^{p,\varepsilon} \) finally (for \( \varepsilon \to 0 \)) are monotonic functions of \( \varepsilon \), for every \( j = 1, \ldots, d \). Whence \( c_j^{p,\varepsilon} = \mathcal{O}(1), \varepsilon \to 0 \) for all \( j = 1, \ldots, d \). Next we are going to show that for \( j = d - 1, d \) \( c_j^{p,\varepsilon} = \mathcal{O}(\varepsilon) \).

If \( x' \neq x_{p+1} \) then \( c_d^{p,\varepsilon} = \mathcal{O}(\varepsilon) \), since \( v_d^{p,\varepsilon}(x_{p+1}) = -1/\varepsilon \) in view of \( \phi_\varepsilon(x_{p+1}) = \mathcal{O}(1), \varepsilon \to 0 \). If \( x' \neq x_p \), then \( c_{d-1}^{p,\varepsilon} = \mathcal{O}(\varepsilon), \varepsilon \to 0 \), since \( v_{d-1}^{p,\varepsilon}(x_p) = -1/\varepsilon \) in view of \( \phi_\varepsilon(x_p) = \mathcal{O}(1), \varepsilon \to 0 \).

If \( x' = x_{p+1} \) is a knot, we claim \( c_d^{p,\varepsilon} = \mathcal{O}(\varepsilon^2), \varepsilon \to 0 \). Assume to the contrary \( c_d^{p,\varepsilon} \neq \mathcal{O}(\varepsilon^2), \varepsilon \to 0 \), then there exists a sequence \((\varepsilon_v), \varepsilon_v \to 0 \), such that for a positive constant \( m \)
\[
|D^-_\phi_\varepsilon(x_{p+1})| \geq m \quad \text{for all} \quad v \in \mathbb{N},
\]
since
\[
D \left( \sum_{j=1}^{d-1} c_j^{p,\varepsilon} v_j^{p,\varepsilon}(x_{p+1}) \right) = \mathcal{O}(1), \quad v \to \infty.
\]
and
\[
\begin{vmatrix}
\frac{\partial P_{\varepsilon}^d}{\partial x} (x_{p+1})
\end{vmatrix}
\rightarrow \infty, \quad v \rightarrow \infty
\]
by assumption. But the derivatives of \( \phi_{\varepsilon} \) at \( x = x_{p+1} \) must tend to zero for \( v \rightarrow \infty \) since \( x' = x_{p+1} \) is the limit of points where \( \phi_{\varepsilon} \) takes its maximum. Similarly, if \( x' = x_p \) then \( c_{d-1}^{P_{\varepsilon}} = \mathcal{O}(\varepsilon^2), \varepsilon \rightarrow 0 \) can be proved. Since in any case we have got a contradiction, Theorem 4.4 is proved.

Next, we analyse the asymptotics of the coefficients of the B-spline \( r_j^\varepsilon \) in more details. We do this with respect to the following local bases of the CV-spaces involved. For the knot interval \([x_i, x_{i+1}]\) we choose for \( S_{i+1}^\varepsilon \) the basis \( S_{i+1,j}^\varepsilon (x_j) \) which is defined by (22), for \([x_j, x_{j+1}] (j = i + 1, \ldots, i + d - 2) \) and for \( S_j^\varepsilon \) we choose the basis \( V_{x_j}^\varepsilon (x_j) \) which is defined by (21), and for \([x_{i+d-1}, x_{i+d}]\) we choose for \( S_{i+d}^\varepsilon \) the basis \( S_{i+d,j}^\varepsilon (x_{i+d}) \) which is defined by (22). Next, we analyse how the corresponding submatrices of the block coefficient matrix (27) \( V_j^\varepsilon \) for this choice of bases do depend on \( \varepsilon \). The entries are the matrices
\[
V_{i,j}^\varepsilon = (D_{i,j}^\varepsilon m(x_i, x_j))_{m=1,\ldots,d},
\]
\[
V_{i+1,j}^\varepsilon = (D_{i,j+1}^\varepsilon m(x_i+1, x_j))_{m=1,\ldots,d},
\]
\[
V_{i,j}^1 = (D_{i,j}^1 m(x_j))_{m=1,\ldots,d},
\]
\[
V_{i,j}^{d+1} = (D_{i,j}^{d+1} m(x_i+d, x_j))_{m=1,\ldots,d-2},
\]
\[
V_{i+1,j}^{d+1} = (D_{i+1,j}^{d+1} m(x_i+d, x_j))_{m=1,\ldots,d-2}.
\]

By \( \oplus \) we denote a positive entry depending on its position in the matrix and (if \( \oplus \) is an element of the last two columns) on \( \varepsilon \) such that \( \oplus = \mathcal{O}(1), \varepsilon \rightarrow 0 \). By \( * \) we denote an entry possibly depending on \( \varepsilon \) such that \( * = \mathcal{O}(1), \varepsilon \rightarrow 0 \). The entries of the block matrix (27) \( V_j^\varepsilon \) for this choice of bases are easily computed

\[
V_{i,j}^\varepsilon =
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 2! & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & (d - 3)! & 0 & 0 \\
0 & 0 & 0 & \ldots & (d - 2)! & \frac{(d - 2)!}{\varepsilon^{d-1}} & 0 \\
0 & 0 & 0 & \ldots & \frac{(d - 2)!}{\varepsilon^{d-1}} & \frac{(d - 2)!}{\varepsilon^{d-1}} & \frac{(d - 2)!}{\varepsilon^{d-1}}
\end{pmatrix},
\]
\[ V_{+1, \varepsilon} = \begin{pmatrix}
1 & \oplus & \ldots & \oplus & (-1)^0 & \frac{1}{\varepsilon} & \varepsilon \\
0 & 1 & \oplus & \ldots & \oplus & (-1)^1 & \frac{1}{\varepsilon^2} \\
0 & 0 & 2! & \oplus & \ldots & \oplus & (-1)^2 & \frac{1}{\varepsilon^3} \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & \ldots & (d-3)! & (-1)^{d-3} & \frac{1}{\varepsilon^{d-2}} & \\
0 & 0 & 0 & \ldots & 0 & (-1)^{d-2} & \frac{1}{\varepsilon^{d-1}} & 
\end{pmatrix}, \\
\]

\[ V_{+1, \varepsilon} = \begin{pmatrix}
1 & \ast & \ldots & \ast & \frac{1}{\varepsilon} & -\oplus \\
0 & 1 & \ast & \ldots & \ast & (-1) & \frac{1}{\varepsilon^2} & -\oplus \\
0 & 0 & 2! & \ast & \ldots & \ast & (-1)^2 & \frac{1}{\varepsilon^3} & -\oplus \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & \ldots & (d-3)! & (-1)^{d-3} & \frac{(d-3)!}{\varepsilon^{d-2}} & -\oplus \\
0 & 0 & 0 & \ldots & 0 & (-1)^{d-2} & \frac{(d-2)!}{\varepsilon^{d-1}} & -\oplus 
\end{pmatrix}, \\
\]

\[ V_{+1, \varepsilon} = \begin{pmatrix}
1 & \ast & \ldots & \ast & (-1)^0 & \frac{1}{\varepsilon} \\
0 & 1 & \ast & \ldots & \ast & (-1)^1 & -\frac{1}{\varepsilon^2} \\
0 & 0 & 2! & \ast & \ldots & \ast & (-1)^2 & -\frac{1}{\varepsilon^3} \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & \ldots & (d-3)! & (-1)^{d-3} & -\frac{(d-3)!}{\varepsilon^{d-2}} \\
0 & 0 & 0 & \ldots & 0 & (-1)^{d-2} & -\frac{(d-2)!}{\varepsilon^{d-1}} 
\end{pmatrix}, \\
\]
Here the entries of the last two columns of the submatrix \( V_{i+1,d-1}^{+,e} \) resp. of \( V_{i+d-1,d}^{-,e} \) are from Lemma 4.1 resp. from Lemma 4.2. All other entries are obtained by simple calculations.

**Lemma 4.5.** Let \((a_n)\) and \((b_n)\) be two sequences of real numbers. Assume that for a positive integer \(m\) for \(n \to \infty\)

\[
a_n n^{m-1} + b_n n^{m-1} = \mathcal{O}(1) \quad \text{and} \quad a_n n^m - b_n n^m = \mathcal{O}(1).
\]

Then \(a_n = \mathcal{O}(n^{-m+1})\) and \(b_n = \mathcal{O}(n^{-m+1})\). Moreover, \(a_n = o(n^{-m+1})\) if \(b_n = o(n^{-m+1})\).

**Proof.** Clearly, by elimination we find from the assumptions that \(a_n n^m = \mathcal{O}(n)\) and \(b_n n^m = \mathcal{O}(n)\). Suppose now that \(b_n = o(n^{-m+1})\). Then from the second equation assumed it follows \(a_n n^m = \mathcal{O}(1) + o(n) = o(n)\), hence \(a_n = o(n^{-m+1})\). If \(a_n = o(n^{-m+1})\), similarly \(b_n = o(n^{-m+1})\) is derived. \(\square\)

**Lemma 4.6.** Let \(c_l^i = (c_1^{i,e}, \ldots, c_d^{i,e}, c_1^{i+1,e}, \ldots, c_d^{i+1,e}, \ldots, c_1^{i+d-1,e}, \ldots, c_d^{i+d-1,e})^T\) be the solution (25) of the linear system (26), where the bases of the local spaces \(S_l^i\) are chosen as described above. Then

(i) \(c_l^{i,e} = \mathcal{O}(1), \varepsilon \to 0, j = i, \ldots, i + d - 1, l = 1, \ldots, d,\)
(ii) \(c_l^{i,e} = 0\) and \(c_l^{i+d-1,e} = 0, l = 1, \ldots, d - 1,\)
(iii) \(c_l^{i,e} = \mathcal{O}(\varepsilon^{d-2}), \varepsilon \to 0, j = i, \ldots, i + d - 2, l = d - 1, d,\)
(iv)  
\[ c_{d+1}^{i+2} = \mathcal{O}(\varepsilon^{d-2}), \quad \varepsilon \to 0, \]  
and  
\[ c_{d}^{i+1} = \mathcal{O}(\varepsilon^{2d-3}), \quad \varepsilon \to 0, \]

(v)  
\[ c_{d-1}^{j} \neq \mathcal{O}(\varepsilon^{d-2}), \quad \varepsilon \to 0, \quad j = i + 1, \ldots, i + d - 2, \]

(vi)  
\[ c_{d}^{i+1} \neq \mathcal{O}(\varepsilon^{d-2}), \quad \varepsilon \to 0, \quad j = i, \ldots, i + d - 1, \]

Proof. (i) has been already shown in the proof of Theorem 4.4(iii), where it has been established that  
\[ \|c_{d}^{i+1}\|_{\infty} = \mathcal{O}(1), \quad \varepsilon \to 0. \]

(ii) The first \( d - 1 \) equations of (26) are  
\[ \sum_{l=0}^{d-2} c_{l}^{i+1} = 0, \quad l = 1, \ldots, d - 2, \quad \sum_{l=1}^{d-1} c_{l}^{i+1} = 0. \]

The \( d - 1 \) equations labelled \( d^2 - d \) to \( d^2 - 2 \) are  
\[ \sum_{l=0}^{d-2} c_{l+d}^{i+1} = 0, \quad l = 1, \ldots, d - 2, \quad \sum_{l=1}^{d-1} c_{l+d}^{i+1} = 0. \]

Consequently,  
\[ c_{l}^{i+1} = 0, \quad l = 1, \ldots, d - 1, \]

and  
\[ c_{l}^{i+1} = 0, \quad l = 1, \ldots, d - 1. \]

(iii) In view of (i), (ii) yields for the equations labelled \( 2d - 3 \) and \( 2d - 2 \)  
\[ c_{d}^{i+1} \sum_{l=0}^{d-3} - c_{d-1}^{i+1} (-1)^{d-1} \sum_{l=0}^{d-2} = \mathcal{O}(1), \quad \varepsilon \to 0, \]

\[ c_{d}^{i+1} \sum_{l=0}^{d-1} - c_{d-1}^{i+1} (-1)^{d} \sum_{l=0}^{d-1} = \mathcal{O}(1), \quad \varepsilon \to 0. \]

According to Lemma 4.5 this implies  
\[ c_{d}^{i+1} = \mathcal{O}(\varepsilon^{d-2}), \quad \varepsilon \to 0 \quad \text{and} \quad c_{d-1}^{i+1} = \mathcal{O}(\varepsilon^{d-2}), \quad \varepsilon \to 0. \]

Consider next the equations labelled \( 3d - 4 \) and \( 3d - 3 \):  
\[ -c_{d}^{i+1} \sum_{l=0}^{d-3} - c_{d-1}^{i+2} (-1)^{d-2} \sum_{l=0}^{d-2} = \mathcal{O}(1), \quad \varepsilon \to 0, \]

\[ -c_{d}^{i+1} \sum_{l=0}^{d-1} - c_{d-1}^{i+2} (-1)^{d-2} \sum_{l=0}^{d-1} = \mathcal{O}(1), \quad \varepsilon \to 0. \]

By Lemma 4.5 this implies  
\[ c_{d}^{i+1} = \mathcal{O}(\varepsilon^{d-2}), \quad \varepsilon \to 0 \quad \text{and} \quad c_{d-1}^{i+2} = \mathcal{O}(\varepsilon^{d-2}), \quad \varepsilon \to 0. \]

Similarly, the same reasoning applied to the equations \( kd - k - 1 \) and \( kd - k \) for \( k = 4, \ldots, d - 1 \) gives  
\[ c_{d}^{i+k-2} = \mathcal{O}(\varepsilon^{d-2}), \quad \varepsilon \to 0 \quad \text{and} \quad c_{d-1}^{i+k-1} = \mathcal{O}(\varepsilon^{d-2}), \quad \varepsilon \to 0. \]

(iv) The equations labelled \( d^2 - d - 1 \) and \( d^2 - d \) give  
\[ -c_{d}^{i+2} \sum_{l=0}^{d-2} - c_{d}^{i+1} (-1)^{d-3} \sum_{l=0}^{d-3} = \mathcal{O}(1), \quad \varepsilon \to 0, \]
As in the proof of Lemma 4.5 from this we infer for sufficiently small \( \varepsilon \)

\[
-c_d^{i+d-2,\varepsilon} \frac{1}{\varepsilon d-1} - c_d^{i+d-1,\varepsilon} (-1)^{d-2} \frac{1}{\varepsilon^{2d-2}} = O(1), \quad \varepsilon \to 0.
\]

This proves (iv).

(v) To prove (v) assume to the contrary that for some \( j = i + 1, \ldots, i + d - 2 \)

\[
c_{d-1}^{j,\varepsilon} = o(\varepsilon^{d-2}), \quad \varepsilon \to 0
\]

or that for some \( j = i, \ldots, i + d - 2 \)

\[
c_{d}^{j,\varepsilon} = o(\varepsilon^{d-2}), \quad \varepsilon \to 0.
\]

Then by the structure of the matrix \( T_j^d \) according to Lemma 4.5 it would follow that all coefficients

\[
c_{d-1}^{j,\varepsilon} = o(\varepsilon^{d-2}), \quad \varepsilon \to 0, \quad j = i, \ldots, i + d - 1,
\]

\[
c_{d}^{j,\varepsilon} = o(\varepsilon^{d-2}), \quad \varepsilon \to 0, \quad j = i, \ldots, i + d - 2,
\]

\[
c_{d}^{i+d-1,\varepsilon} = o(\varepsilon^{d-3}), \quad \varepsilon \to 0
\]

and all remaining coefficients

\[
c_{l}^{j,\varepsilon} = o(1), \varepsilon \to 0, \quad j = i, \ldots, i + d - 1, \quad l = 1, \ldots, d - 2.
\]

As a consequence, all coefficients of polynomials tend to zero for \( \varepsilon \to 0 \). Therefore, in view of (iii), for sufficiently small \( \varepsilon > 0 \), \( \|r_i^\varepsilon\|_\infty < 1/(x_{i+d} - x_i) \) which contradicts the last equation of (26) which ensures that this function has integral equal to one over \([x_i, x_{i+d}]\). (vi) is proved similarly. \( \square \)

Now we are ready to prove our main result.

**Theorem 4.7.** Let \( d \geq 4 \) and denote by \( r_i^\varepsilon \), \( i = -d + 1, \ldots, k \) the B-splines of the space \( S_{[x_0, x_{i+d}]}^{\max, \varepsilon} \) having support \([x_i, x_{i+d}]\) that are normalized to have integral 1 over the real axis. By \( M_{i+1}^{d-3} \) we denote the polynomial B-spline of degree \( d - 3 \) with support interval \([x_{i+1}, x_{i+d-1}]\) of maximal smoothness \( C_{d-4} \) that is normalized to have integral 1 over \( \mathbb{R} \). Then

\[
\lim_{\varepsilon \to 0} \|D^m r_i^\varepsilon - D^m M_{i+1}^{d-3}\|_\infty = 0, \quad m = 0, \ldots, d - 4.
\]

**Proof.** Let \( j \in \{i, \ldots, i + d - 1\} \) and \( l \in \{1, \ldots, d\} \) and consider the coefficient \( c_{j}^{l,\varepsilon} \). According to Lemma 4.6, it is bounded for \( \varepsilon \to 0 \) and according to Lemma 4.3 it is finally monotonic. Hence, \( c_{j}^{l,0} = \lim_{\varepsilon \to 0} c_{j}^{l,\varepsilon} \) exists and is a real number.

First, we consider \( l \in \{1, \ldots, d - 2\} \). Since \( v_i^\varepsilon \) does not depend on \( \varepsilon \), i.e. for all \( \varepsilon \) it is the same polynomial \( v_i^l \), we infer that for every nonnegative integer \( m \) as \( \varepsilon \to 0 \) \( D^m c_{j}^{l,\varepsilon} v_i^\varepsilon \to c_{j}^{l,0} D^m v_i^l \).

In particular, \( c_{j}^{l,0} = c_{j+1}^{l+1,0} \) and \( D^m c_{j}^{l,\varepsilon} v_i^\varepsilon \to c_{j}^{l,0} D^m v_i^l \).

In particular, \( c_{j}^{l,0} = c_{j+1}^{l+1,0} \) and \( D^m c_{j}^{l,\varepsilon} v_i^\varepsilon \to c_{j}^{l,0} D^m v_i^l \).
is the polynomial B-spline of degree \( d - 3 \) of maximal smoothness \( C^{d-4} \) having support \([x_{i+1}, x_{i+d-1}]\) that is normalized to have integral equal to one over the real axis.

Consider now \( l \in \{d - 1, d\} \). According to Lemma 4.6 then

\[
\left\| D^m \psi_j^{l, \varepsilon} \right\|_\infty = \| \psi_j^{l, \varepsilon} \| : \left\| D^m v_j^{l, \varepsilon} \right\|_\infty = \mathcal{O}(\varepsilon), \quad \varepsilon \to 0,
\]

hence

\[
\left\| D^m v_j^{l, \varepsilon} \right\|_\infty = \mathcal{O}(\varepsilon^{-m-1}), \quad \varepsilon \to 0, \quad m = 0, \ldots, d - 2.
\]

Accordingly, for \( m = 0, \ldots, d - 4 \) (31) obtains. For \( j = i \) we have \( c_l^{i, \varepsilon} = 0 \) for \( l = 1, \ldots, d - 1 \) and

\[
\left\| D^m c_d^{i, \varepsilon} s_d^{l, \varepsilon} (\cdot, x_i) \right\|_\infty \to \mathcal{O}(\varepsilon^{d-2})\mathcal{O}(\varepsilon_{-m-1})
\]

\[
= \mathcal{O}(\varepsilon) \quad \varepsilon \to 0
\]

for \( m = 0, \ldots, d - 4 \). For \( j = i + d - 1 \) we have \( c_l^{i+d-1, \varepsilon} = 0 \) for \( l = 1, \ldots, d - 1 \) and

\[
\left\| D^m c_d^{i+d-1, \varepsilon} s_d^{l, \varepsilon} (\cdot, x_{i+d}) \right\|_\infty \to \mathcal{O}(\varepsilon^{2d-3})\mathcal{O}(\varepsilon_{-m-1}) = \mathcal{O}(\varepsilon)
\]

for \( m = 0, \ldots, d - 4 \).

Since in the intervals \([x_i, x_{i+1}]\) and \([x_{i+d-1}, x_{i+d}]\) the coefficients of the polynomials in the bases \( S_d^{l, \varepsilon}(x_i) \) resp. \( S_d^{i+d-1, \varepsilon}(x_{i+d}) \) vanish, we arrive at

\[
\lim_{\varepsilon \to 0} \left\| D^m r_{i}^{\varepsilon} - D^m \left( \sum_{j=i+1}^{i+d-1} \sum_{l=1}^{d-2} c_l^{j, 0} (-x_j)^{l-1} \right) \right\|_\infty = 0, \quad m = 0, \ldots, d - 4.
\]

Since the convergence is uniformly in \( x \) also

\[
\int_{x_{i+1}}^{x_{i+d-1}} \sum_{j=i+1}^{i+d-1} \sum_{l=1}^{d-2} c_l^{j, 0} (x - x_j)^{l-1} \, dx
\]

\[
= \int_{x_{i+1}}^{x_{i+d-1}} \lim_{\varepsilon \to 0} r_{i}^{\varepsilon} (x) \, dx = \lim_{\varepsilon \to 0} \int_{x_{i+1}}^{x_{i+d-1}} r_{i}^{\varepsilon} (x) \, dx = 1.
\]

This completes the proof of Theorem 4.7. \( \Box \)

In particular, for \( d = 4 \) we have established: the rational B-spline \( r_{i}^{\varepsilon} \) with compact support \([x_i, x_{i+d}]\) and with prescribed simple poles outside and in distance \( \varepsilon \) to each knot interval converges uniformly to the hat function with compact support \([x_{i+1}, x_{i+3}]\). This is illustrated by Fig. 1.
It is not hard to see that its first derivative converges pointwise to the derivative of the hat function where the latter exists. The second derivative diverges at every knot of the support interval of the hat function.

Remark.

(i) It is easily checked that Theorem 4.7 holds true even if the knots are not equidistant. For instance, if we use \( d = 4 \) and \( x_i = 0, x_{i+1} = 1, x_{i+2} = 2, x_{i+3} = 4, x_{i+4} = 5 \), Fig. 2 shows the convergence in case \( d = 4 \) for \( D^j r_i^\varepsilon \) for \( j = 0, 1, 2 \) in a neighborhood of \( x_{i+2} = 2 \). Here \( \varepsilon = \frac{1}{100} \) for the pictures in the first row and \( \varepsilon = \frac{1}{1000} \) for the pictures in the last row.

(ii) If the first function of each local ECT-system is the constant function equal to one and the local ECT-systems are derived from a global CV-system by cutting it into pieces, and the last
two functions on \([x_i, x_{i+1}]\) as before are the rational functions
\[
\frac{1}{x - x_i + \varepsilon}, \quad \frac{1}{x - x_{i+1} - \varepsilon},
\]
then Theorem 4.7 can be generalized to this case. Cf. [2, p. 60] for this generalization.

(iii) If instead of (32) the local CV-systems are augmented by two rational functions
\[
\frac{1}{x - x_i + \varepsilon}, \quad \frac{1}{(x - x_i + \varepsilon)^2},
\]
then Theorem 4.7 no longer is true. For \(d = 4\), the \(r_i^d\) then converge to a constant function on \([x_{i+1}, x_{i+3}]\) having integral equal to one, cf. [2, p. 62].

References