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A DUALITY RELATION FOR ENTRANCE AND EXIT LAWS FOR MARKOV PROCESSES

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Markov processes X_t on (X, F_X) and Y_t on (Y, F_Y) are said to be dual with respect to the function $f(x, y)$ if $E_x f(X_t, y) = E_y f(x, Y_t)$ for all $x \in X$, $y \in Y$, $t \geq 0$. It is shown that this duality reverses the role of entrance and exit laws for the processes, and that two previously published results of the authors are dual in precisely this sense. The duality relation for the function $f(x, y) = 1_{\{x<y\}}$ is established for one-dimensional diffusions, and several new results on entrance and exit laws for diffusions, birth-death processes, and discrete time birth-death chains are obtained.

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1. Introduction

There are several notions of duality in the Markov process literature. In this paper we will be concerned with a type of duality that has proved quite useful in the study of certain interacting particle systems. We will show that this duality reverses the role of entrance and exit laws. Furthermore, we will show that Theorem 3.2 of [3] and Theorem 2.2 of [29] are 'dual' in precisely this sense. Several new results for entrance and exit laws for diffusions, birth-death processes, and birth-death chains are obtained, and the questions raised in [3] are resolved. A duality relation for one-dimensional diffusions is established which should be of independent interest. We begin with the definition of duality [25, p. 204].

Let X_t, $t \geq 0$ be a temporally homogeneous Markov process on the measure space (X, F_X) with transition function $p_t(x, du)$, and let Y_t, $t \geq 0$ be a temporally homogeneous Markov process on the measure space (Y, F_Y) with transition function $q_t(y, dv)$.

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Definition. Let f be a bounded measurable function $f: X \times Y \to [0, \infty)$. X_t and Y_t are called *dual with respect to* f if for each $x \in X$, $y \in Y$, $t \geq 0$,

$$E_x f(X_t, y) = E_y f(x, Y_t). \tag{1.1}$$

Many interacting particle systems have dual processes. For example, the basic contact process of Harris [15, 16] is a set valued Markov process which is self-dual for the function $f(x, y) = 1_{\{x \cap y \neq \emptyset\}}$ (1_B is the indicator function of B). Additional examples and applications of this duality can be found in [14, 17, 18]. For different choices of f see [18, 33], and many of the references in [14, 25].

For another source of dual processes consider the stochastically monotone processes (on the real line) of Daley [5]. X_t is said to be stochastically monotone if for each y and t, $P_x(X_t \leq y)$ decreases as x increases. Under mild continuity restrictions Siegmund [30] proves that a stochastically monotone process has a dual process (which is stochastically monotone) with respect to the function $f(x, y) = 1_{\{x < y\}}$.

To present our basic result concerning this duality we need two definitions of Dynkin [7]. For measures μ and functions g we will write

$$\mu p_t(\Gamma) = \int_X \mu(\mathrm{d}x) p_t(x, \Gamma)$$

and

$$p_t g(x) = \int_X p_t(x, \mathrm{d}u) g(u).$$

Unless otherwise noted a *measure* is either a sigma-finite positive measure or a finite signed measure.

Definition. An *entrance law* ν for X_t (or p_t) is a family $\nu = \{\nu_s\}_{s \in \mathbb{R}}$ of measures ν_s on (X, F_X) such that

$$\nu_s p_t = \nu_{s+t}, \quad s \in \mathbb{R}, t \geq 0. \tag{1.2}$$

Definition. An *exit law* h for X_t (or p_t) is a family $h = \{h_s\}_{s \in \mathbb{R}}$ of measurable functions $h_s : X \to [0, \infty)$ such that

$$p_t h_{s+t} = h_s, \quad s \in \mathbb{R}, t \geq 0. \tag{1.3}$$

An exit law h will be called *bounded* if $\sup_{s,x} h_s(x) < \infty$.

Of course it is possible to consider various modifications and generalizations of these definitions. For the most general case and applications of entrance and exit laws, the papers of Dynkin [7, 8, 9, 10], Föllmer [12], and Spitzer [32] should be consulted. For our purposes it will suffice to give two simple interpretations of entrance and exit laws.

If h is a bounded exit law for X_t, the formula $\tilde{h}(x,s)=h_{-s}(x)$ defines a bounded space time harmonic function h, and of course the converse is true. This means that exit laws give information about *tail fields* (see [27]). If ν is an entrance law of probability measures for p_t, then it is possible to construct a probability measure P^ν and a Markov process ξ_s with time parameter set $\mathbb{R}$ and transition function p_t (i.e. $P^\nu(\xi_{s+t}\in\Gamma\mid\xi_u,-\infty<u\leqslant s)=p_t(\xi_s,\Gamma)$ a.s. P^ν for $s\in\mathbb{R}$, $t\geqslant 0$). Conversely, given such a measure P^ν and process ξ_s, the formula $\nu_s(\Gamma)=P^\nu(\xi_s\in\Gamma)$ defines an entrance law.

The main result of this section shows that the duality equation (1.1) reverses the role of entrance and exit laws for dual processes. Before stating this result we present two more definitions.

Definition. A measure μ on (Y,F_Y) is *admissible for f* if, for all $x\in X$,

$$\int_Y \mu(\mathrm{d}y)f(x,y)\geqslant 0.$$

Definition. A nonnegative measurable function g on (X,F_X) is *representable by f* if there exists a unique measure μ on (Y,F_Y) such that

$$g(x)=\int_Y \mu(\mathrm{d}y)f(x,y)\quad \text{for all } x\in X.$$

An entrance law $\nu=\{\nu_s\}$ will be called admissible if each ν_s is admissible and an exit law $h=\{h_s\}$ will be called representable if each h_s is representable.

Our main result shows that the duality equation (1.1) reverses the role of entrance and exit laws for dual processes.

Theorem 1. *Let X_t and Y_t be dual with respect to f.*

(i) *Suppose ν is an admissible entrance law for Y_t and we define*

$$h_s(x)=\int_Y \nu_{-s}(\mathrm{d}y)f(x,y). \tag{1.4}$$

Then $h=\{h_s\}_{s\in\mathbb{R}}$ is an exit law for X_t.

(ii) *Suppose h is a representable exit law for X_t, with each h_s having representation (1.4). Then $\nu=\{\nu_s\}_{s\in\mathbb{R}}$ is an entrance law for Y_t.*

Proof. For (i) it suffices to show that h_s defined by (1.4) satisfies $p_t h_{s+t}=h_s$. Using the hypothesis that ν is an entrance law and the duality equation (1.1) we obtain

$$p_t h_{s+t}(x)=\int_X p_t(x,\mathrm{d}u)\int_Y \nu_{-s-t}(\mathrm{d}y)f(u,y)$$

$$=\int_Y \nu_{-s-t}(\mathrm{d}y)\int_X p_t(x,\mathrm{d}u)f(u,y)$$

$$= \int_Y \nu_{-s-t}(\mathrm{d}y) \int_Y q_t(y, \mathrm{d}v) f(x, v)$$

$$= \int_Y \nu_{-s}(\mathrm{d}v) f(x, v) = h_s(x).$$

For (ii) we assume $p_t h_{s+t} = h_s$ and $\nu = \{\nu_s\}$ satisfies (1.4) so that the duality equation implies

$$\int_Y \nu_s q_t(\mathrm{d}y) f(x, y) = p_t h_{-s}(x) = h_{-s-t}(x)$$

$$= \int_Y \nu_{s+t}(\mathrm{d}y) f(x, y)$$

for $s \in \mathbb{R}$, $t \geq 0$. Since $h_{-s-t}(x)$ admits a *unique* representation of the form $\int_Y \mu(\mathrm{d}y) f(x, y)$, $\nu_s q_t = \nu_{s+t}$.

It is an interesting problem to determine whether or not a given Markov process has nonconstant bounded exit laws or nonconstant entrance laws of probability measures. In the following section a fairly complete discussion of this question is presented for the class of birth–death processes. In Section 3 some results on the structure of these laws are presented. The duality of monotone processes is considered for the class of one-dimensional diffusion processes with continuous speed functions in Section 4. Results for discrete time birth–death chains are given in Section 5.

2. Birth–death processes

A birth–death process on $X = \{0, 1, 2, \ldots\}$ or $X^* = \{-1, 0, 1, \ldots\}$ with birth rates λ_n, death rates μ_n makes transitions

$$n \to n+1 \quad \text{at rate } \lambda_n, \qquad n \to n-1 \quad \text{at rate } \mu_n. \tag{2.1}$$

We assume $\lambda_n > 0$ for $n \geq 0$ and $\mu_n > 0$ for $n \geq 1$. If $\mu_0 = 0$ (reflection at 0) the process has state space X; if $\mu_0 > 0$ we set $\mu_{-1} = \lambda_{-1} = 0$ (absorption at -1) and the process has state space X^*. As in [20] we define

$$\pi_0 = 1, \qquad \pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}, \quad n \geq 1.$$

If the rates satisfy the condition

$$C(\lambda_n, \mu_n): \quad \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{k=0}^{n} \pi_k = +\infty,$$

then there is a unique Markov process X_t which makes transitions (2.1). See [20] for a precise formulation and proof of this result. We will only consider rates which satisfy this condition.

We will now discuss the duality theory for birth–death processes. It will be most convenient to follow Van Doorn [36] instead of Siegmund [30]. Consider the transformation of rates

$$\lambda_n^* = \mu_{n+1}, \qquad \mu_n^* = \lambda_n \tag{2.2}$$

(and hence $\pi_n^* = \lambda_0/\lambda_n\pi_n$). If $C(\lambda_n^*, \mu_n^*)$ holds, the rates λ_n^*, μ_n^* determine a unique birth–death process X_t^*. Assuming $\mu_0 = 0$, so that X_t has state space X, then $\mu_0^* = \lambda_0 > 0$, so X_t^* has state space X^*. We introduce the condition

$$D(\lambda_n, \mu_n): \quad \sum_{n=1}^{\infty} \pi_n \sum_{k=0}^{n-1} \frac{1}{\lambda_k \pi_k} = +\infty,$$

and note that $C(\lambda_n, \mu_n)$ is equivalent to $D(\lambda_n^*, \mu_n^*)$, and $C(\lambda_n^*, \mu_n^*)$ is equivalent to $D(\lambda_n, \mu_n)$. As Van Doorn [36] points out, $+\infty$ is a *natural* boundary point if and only if both $C(\lambda_n, \mu_n)$ and $D(\lambda_n, \mu_n)$ hold.

Van Doorn proves ([36, Theorem 3.1] but see also Siegmund [30]) that if X_t is the birth–death process with rates λ_n, μ_n ($\mu_0 = 0$) which satisfy both $C(\lambda_n, \mu_n)$ and $D(\lambda_n, \mu_n)$, then

$$P_x(X_t \leq y) = P_y(X_t^* \geq x), \quad x \in X, y \in X^*. \tag{2.3}$$

That is, X_t and X_t^* are dual with respect to the function $f(x, y) = 1_{\{x \leq y\}}$. This duality equation does not seem to admit a 'path decomposition' type proof. One easy consequence of this equation is that X_t^* has positive probability of escaping to infinity if and only if X_t is positive recurrent.

Another form of this duality can be expressed in terms of hitting times. Fix $i < j$ and let λ_n, μ_n be birth–death rates with $\lambda_j = \mu_j = \mu_i = 0$ (i.e. i is reflecting, j absorbing) and λ_n, μ_n positive otherwise. The dual rates λ_n^*, μ_n^* defined by (2.2) satisfy $\mu_{i-1}^* = \lambda_{i-1}^* = \lambda_{j-1}^* = 0$ (i.e. $j-1$ is reflecting and $i-1$ absorbing.) Let τ_x (τ_x^*) be the first hitting time of x for X_t (X_t^*). Then a consequence of the duality equation (2.3) is

$$P_i(\tau_j \leq t) = P_{j-1}(\tau_{i-1}^* \leq t). \tag{2.4}$$

It should be noted that this equation depends on the assumptions that i is reflecting for X_t and $j-1$ is reflecting for X_t^*.

The following result of Rösler [29] and Fristedt and Orey [13] was proved for diffusions, but of course holds for birth–death processes. The symbol $\xrightarrow{w}$ denotes weak convergence.

Theorem 2. *Let X_t be a birth–death process with rates λ_n, μ_n ($\mu_0 = 0$) which satisfy $C(\lambda_n, \mu_n)$. Then the following conditions are equivalent.*

(a) *X_t has a nonconstant bounded exit law.*

(b) *There are numbers* $t_k \to \infty$ *and a probability measure* ϕ *such that*

$$P_0(\tau_k - t_k \in \mathrm{d}u) \xrightarrow{w} \phi(\mathrm{d}u) \quad \text{as } k \to \infty.$$

(c) $$\lim_{k\to\infty} E_0(\tau_k) = +\infty, \qquad \lim_{k\to\infty} \mathrm{Var}_0(\tau_k) < \infty.$$

(d) $$\sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{k=0}^{n} \pi_k = +\infty, \qquad \sum_{n=0}^{\infty} \pi_n \sum_{k=0}^{n} \pi_k \Big(\sum_{j=n}^{\infty} \frac{1}{\lambda_j \pi_j} \Big)^2 < \infty.$$

Several comments are in order here. Each of these conditions is equivalent to X_t having a nontrivial tail field (see [13] and [29]). In fact, if (b) holds, then $\tau_k - t_k \to$ a.e. to a random variable which generates the tail field. Since recurrent birth–death processes have trivial tail fields, Theorem 2 is a theorem about *transient* processes.

The equivalence of (c) and (d) is a calculation which can be made using difference equations or the technique of Theorem 2 of [13]. The relation (c) ⇒ (b) follows from a standard result about sums of independent random variables. The relation (b) ⇒ (c) was (apparently) first proved by Karlin and McGregor [21]. The key fact is their observation (see also [22]) that for each k there are positive numbers $a_j^{(k)}$, $j = 1, 2, \ldots, k$, depending only on λ_n, μ_n, for $n \leq k$, such that

$$E_0(\mathrm{e}^{\mathrm{i}t\tau_k}) = \prod_{j=1}^{k} \frac{a_j^{(k)}}{a_j^{(k)} - \mathrm{i}t}, \qquad t \in R. \tag{2.5}$$

That is, under P_0, τ_k is equal in distribution to the sum of k independent exponential random variables. This depends on the fact that 0 is reflecting. Using this fact it is not hard to show that for $t \neq 0$, $|E_0(\mathrm{e}^{\mathrm{i}t\tau_k})| \to 0$ if $\mathrm{Var}_0(\tau_k) \to +\infty$. This proves (b) ⇒ (c).

The corresponding result for entrance laws is remarkably similar.

Theorem 3. *Let X_t be a birth–death process with rates λ_n, μ_n ($\mu_0 \geq 0$) which satisfy $C(\lambda_n, \mu_n)$. Then the following conditions are equivalent.*

(a) *X_t has a nonconstant entrance law of probability measures.*

(b) *There are numbers $t_k \to \infty$ and a probability measure ϕ such that*

$$P_k(\tau_0 - t_k \in \mathrm{d}u) \xrightarrow{w} \phi(\mathrm{d}u) \text{ as } k \to \infty.$$

(c) $$\lim_{k\to\infty} E_k(\tau_0) = +\infty, \qquad \lim_{k\to\infty} \mathrm{Var}_k(\tau_0) < \infty.$$

(d) $$\sum_{n=1}^{\infty} \pi_n \sum_{k=0}^{n-1} \frac{1}{\lambda_k \pi_k} = +\infty, \qquad \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{k=0}^{n} \frac{1}{\lambda_k \pi_k} \Big(\sum_{j=n+1}^{\infty} \pi_j \Big)^2 < \infty.$$

Note that a process which satisfies (b) must be *recurrent*. The relations (a) ⇔ (b), (c) ⇔ (d), and (c) ⇒ (b) can be found in [3] (the birth–death version of Theorem

3.2), [2, Lemma 4.3], and [2, Theorem 4.2]. The *new result* is (b) ⇒ (c), which resolves the two questions raised at the end of [4]. The difficulty lies in the fact that τ_0, under P_k, has no known simple representation like (2.5).

Proof of (b) ⇒ (c). The Chebyshev estimate

$$P_k(|\tau_0 - t_k| \le M) \le P_k(\tau_0 \ge t_k - M) \le E_k(\tau_0)/t_k - M$$

and the fact that $t_k \to \infty$ imply that (b) cannot hold unless $E_k(\tau_0) \to \infty$. The argument for $\mathrm{Var}_k(\tau_0)$ is more involved.

Let X_t^x denote the birth–death process with the same rates as X_t, except that reflection occurs at x ($\lambda_x^x = 0$), and let τ_0^x be the first hitting time of 0 for X_t^x. We will show that a consequence of (b) is

$$P_k(\tau_0^k - t_k \in \mathrm{d}u) \xrightarrow{w} \phi(\mathrm{d}u) \quad \text{as } k \to \infty. \tag{2.6}$$

Once this is done, the rest is easy. By the remark after Theorem 2, especially equation (2.5), (2.6) implies $\lim_{k\to\infty} \mathrm{Var}_k(\tau_0^k) < \infty$. By calculating this limit in terms of birth–death rates, one finds that the second sum in (d) must be finite, and hence $\lim_{k\to\infty} \mathrm{Var}_k(\tau_0) < \infty$, which is the desired result (since (d) ⇔ (c)).

The proof of (2.6) is based on the two equations

$$\tau(x,0) - t(x,0) \overset{\mathscr{D}}{=} l(x,y) + \bar\tau(x,y) - t(x,y) + \tau(y,0) - t(y,0) \tag{2.7}$$

and

$$\tau^x(x,0) - t(x,0) \overset{\mathscr{D}}{=} l^x(x,y) + \bar\tau^x(x,y) - t(x,y) + \tau^x(y,0) - t(y,0), \tag{2.8}$$

where $0 < y < x$, $\mathscr{D}$ denotes equality in distribution, $t(x,y) = t_x - t_y$, the superscript x refers to the process X_t^x, and the random variables $l(x,y)$, $\tau(x,y)$, $\bar\tau(x,y)$ and $\tau(y,0)$ are defined as follows. For the process starting at x, let $\tau(x,y) = \tau_y$, let

$$l(x,y) = \sup\{t \ge 0 \colon X_t = x \text{ and } \tau_y < t\},$$

and let $\bar\tau(x,y) = \tau(x,y) - l(x,y)$. Thus $\tau(x,y)$ is a copy of the time it takes the process to go from x to y, $l(x,y)$ is the *last* time the process is at x before hitting y, and $\bar\tau(x,y)$ is a copy of the time it takes the process to go from x to y conditional on not returning to x before hitting y. Finally, $\tau(y,0)$ is a copy of the time it takes the process to go from y to 0, and is taken to be independent of all other random variables. Equations (2.7) and (2.8) follow from the strong Markov property.

To exploit (2.7) and (2.8) effectively we need the following.

Lemma. *For $\varepsilon > 0$, $\delta > 0$ there exists a finite $M = M(\varepsilon, \delta)$ such that:*

(i) $P(|\tau(x,y) - t(x,y)| \ge \delta) \le \varepsilon$ *if* $M \le y \le x$.

(ii) $P(l(x,y) \ge \delta) \le \varepsilon$ *if* $M \le y \le x$.

(iii) $P(l^x(x, y) \geq \delta) \leq P(l(x, y) \geq \delta)$.

(iv) $\bar{\tau}(x, y) \overset{\mathcal{L}}{=} \bar{\tau}^x(x, y)$.

(v) $P(\tau^x(y, 0) \in \mathrm{d}u) \overset{w}{\to} P(\tau(y, 0) \in \mathrm{d}u)$ *as* $x \to \infty$.

Proof. (i) By the strong Markov property, for $0 \leq y \leq x$,

$$\tau(x, 0) - t(x, 0) \overset{\mathcal{L}}{=} \tau(x, y) - t(x, y) + \tau(y, 0) - t(y, 0)$$

where $\tau(x, y)$ and $\tau(y, 0)$ are independent. Property (i) follows from this decomposition and the convergence of $\tau_k - t_k$.

(ii) Fix $\delta > 0$ and define $\tau_\delta(x) = \inf\{t \geq \delta : X_t = x\}$. Then

$$\begin{aligned} P_x(\tau_y - t(x, y) \geq \tfrac{1}{2}\delta) &\geq P_x(\tau_y - t(x, y) \geq \tfrac{1}{2}\delta, \tau_\delta(x) < \tau_y) \\ &= \int_\delta^x P_x(u < \tau_y, \tau_\delta(x) \in \mathrm{d}u) P_x(\tau_y - t(x, y) + u \geq \tfrac{1}{2}\delta) \\ &\geq \int_\delta^\infty P_x(u < \tau_y, \tau_\delta(x) \in \mathrm{d}u) P_x(\tau_y - t(x, y) \geq -\tfrac{1}{2}\delta) \\ &= P_x(\tau_\delta(x) < \tau_y) P_x(\tau_y - t(x, y) \geq -\tfrac{1}{2}\delta). \end{aligned}$$

That is,

$$P(|\tau(x, y) - t(x, y)| \geq \tfrac{1}{2}\delta) \geq P_x(\tau_\delta(x) < \tau_y) P(\tau(x, y) - t(x, y) \geq -\tfrac{1}{2}\delta).$$

In view of (i), it is clear that for large x and y, $P_x(\tau_\delta(x) < \tau_y)$ must be small. This implies (ii) since $P(l(x, y) \geq \delta) \leq P_x(\tau_\delta(x) < \tau_y)$.

(iii) & (v) It is possible to construct a *coupling* of the processes X_t and X_t^x, say (X_t, X_t^x), such that if (z, z) is the initial state for the coupled system, and $z \leq x$,

$$P_{(z,z)}(X_t^x \leq X_t \text{ for all } t \geq 0, \text{ with } X_t^x = X_t \text{ for } 0 \leq t \leq \tau_x) = 1.$$

If we take $(z, z) = (x, x)$, then $l^x(x, y) \leq l(x, y)$ a.e. and (iii) is immediate. If we take $(z, z) = (y, y)$ then $\tau_0 = \tau_0^x$ on $\{\tau_0 < \tau_x\}$. Thus

$$P(\tau^x(y, 0) \neq \tau(y, 0)) \leq P_y(\tau_x < \tau_0)$$

and this tends to zero as $x \to \infty$, proving (v).

(iv) This is obvious, since X_t and X_t^x, conditional on not hitting x, are the same process.

Finally, the proof of (2.6) is as follows. By the lemma, for x and y sufficiently large, $l(x, y)$ and $l^x(x, y)$ can be made arbitrarily small, and

$$\bar{\tau}(x, y) - t(x, y) \overset{\mathcal{L}}{=} \bar{\tau}^x(x, y) - t(x, y).$$

Thus in (2.7) and (2.8), the difference between $\tau(x, 0) - t(x, 0)$ and $\tau^x(x, 0) - t(x, 0)$ is the difference between $\tau(y, 0) - t(y, 0)$ and $\tau^x(y, 0) - t(y, 0)$. By the Lemma, this difference goes to zero as $x \to \infty$. This proves (2.6).

A completely different approach to proving (b) ⇒ (c) of Theorem 3 can be based on the following proposition and duality arguments. All that is needed is Theorem 2 and (d) ⇒ (c) ⇒ (b) of Theorem 3.

Proposition. *Let X_t be a birth–death process with rates λ_n, μ_n ($\mu_0 = 0$) which satisfy $C(\lambda_n, \mu_n)$ and $D(\lambda_n, \mu_n)$. Let X_t^* be the process with rates λ_n^*, μ_n^* defined in (2.2). Then X_t has a nonconstant bounded exit law if and only if X_t^* has a nonconstant entrance law of probability measures.*

Proof. Suppose $\nu^* = \{\nu_s^*\}_{s\in\mathbb{R}}$ is a nonconstant entrance law of probability measures for X_t^*. Equation (1.4) becomes

$$h_s(x) = \sum_{y=-1}^{x-1} \nu_{-s}^*(\{y\}), \quad x \in X, \tag{2.9}$$

and by Theorem 1, $\{h_s\}$ is a bounded exit law for X_t and is clearly nonconstant.

Now suppose that $h = \{h_s\}_{s\in\mathbb{R}}$ is a nonconstant exit law for X_t. It is possible to define $\nu_{-s}(\{y\})$ by (2.9), but there is no guarantee that ν_{-s} is positive, so we proceed as follows. By Theorem 2, (d) must hold. Using the transformation (2.2) we see this is equivalent to

$$\sum_{n=1}^{\infty} \pi_n^* \sum_{k=0}^{n-1} \frac{1}{\lambda_k^* \pi_k^*} = +\infty, \qquad \sum_{n=0}^{\infty} \frac{1}{\lambda_n^* \pi_n^*} \sum_{k=0}^{n} \frac{1}{\lambda_k^* \pi_k^*} \left(\sum_{j=n+1}^{\infty} \pi_j^* \right)^2 < \infty.$$

Thus (d) of Theorem 3 holds for the rates λ_n^*, μ_n^*. Since (d) ⇒ (a) we conclude X_t^* has a nonconstant entrance law of probability measures.

3. Exit laws

In this section X_t will denote a birth–death process with strictly positive rates λ_n, μ_n (except $\mu_0 = 0$) which satisfy $C(\lambda_n, \mu_n)$. Let H be the set of bounded exit laws $h = \{h_s\}$ for X_t and let H_m be the set of those $\{h_s\}$ in H such that each h_s is a monotone nonincreasing function. We have shown that H_m can be identified with the set of entrance laws of finite measures for the dual process X_t^*. In this section we will examine in more detail the structure of H and H_m.

Let B be the set of bounded Borel functions $g : \mathbb{R} \to [0, \infty)$ and let B_m be the set of those functions in B which are right continuous and monotone nondecreasing. We will identify functions which agree a.e. (Lebesgue measure).

If X_t has a nonconstant bounded exit law, then it has a nontrivial tail field generated by $T = \lim_{z\to\infty} \tau_z - E_0\tau_z$ (see [13]). The random variable T is tail field measurable and the distribution function of T has a Lebesgue derivative which is strictly positive on $\mathbb{R}$.

Theorem 4. *Let X_t be a birth–death process with strictly positive rates λ_n, μ_n (except $\mu_0 = 0$) which satisfy $C(\lambda_n, \mu_n)$ and condition (d) of Theorem 2. Then there is a 1 : 1* *onto correspondence ϕ from B to H given by*

$$(\phi g)_s(x) = E_x g(T-s), \qquad x \in X, s \in \mathbb{R}. \tag{3.1}$$

Furthermore, if ϕ_m is the restriction of ϕ to B_m, then ϕ_m is a 1 : 1 *onto correspondence from B_m to H_m.*

Proof. Suppose that $g \in B$. An application of the Markov property, using the fact that T is tail field measurable, shows that ϕg defined in (4.1) is an exit law. Now suppose $h \in H$. The Markov property and the definition of an exit law show that $h_t(X_t)$ is a martingale. Since $h_t(X_t)$ is bounded, and is right continuous with left limits (this follows from the sample path properties of X_t and the exit law definition which implies that $h_s(x)$ is continuous in s for fixed x), the martingale convergence theorem applies. There is a random variable $H \geq 0$ such that $h_t(X_t) \to H$ a.s. and in L' as $t \to \infty$. Since H must be tail field measurable there exists some $g \in B$ with $H = g(T)$ a.s., and it is easy to show $h_s(x) = E_x g(T-s)$.

Suppose $g \in B_m$. Then g has a unique representation

$$g(y) = \int_{[-\infty,\infty)} 1_{\{y \geq u\}}\, \nu(\mathrm{d}u)$$

where ν is a (finite) Borel measure $[-\infty, \infty)$. By Fubini

$$(\phi_m g)_s(x) = \int_{[-\infty,\infty)} E_x 1_{\{T-s \geq u\}} \nu(\mathrm{d}u).$$

To show $\phi_m g \in H_m$ it suffices to show that $E_x 1_{\{T-s \geq u\}}$ is nonincreasing in x for fixed s and u. But this is simple since

$$P_x(T-s \geq u) = \lim_{z \to x} P_x(\tau_z \geq s+u+E_0\tau_z)$$

$$\geq \lim_{z \to x} P_y(\tau_z \geq s+u+E_0\tau_z) = P_y(T-s \geq u)$$

if $y > x$.

Now suppose $h \in H_m$. As before $h_t(X_t) \to g(T)$ a.s. as $t \to \infty$ for some $g \in B$, and $h_s(x) = E_x g(T-s)$. To show g is nondecreasing let $\Omega_0 = \{\omega | h_t(X_t) \to g(T)\}$ and suppose $\omega_1, \omega_2 \in \Omega_0$, with $T\omega_1 < T\omega_2$. If $t_n = \tau_n(\omega_1)$, then $X_{t_n}(\omega_1) \geq X_{t_n}(\omega_2)$ for all n sufficiently large, and hence $h_{t_n}(X_{t_n}(\omega_1)) \leq h_{t_n}(X_{t_n}(\omega_2))$ for all n sufficiently large. Taking limits we obtain $g(T(\omega_1)) \leq g(T(\omega_2))$. Since the density of T is everywhere positive, g is nondecreasing. Finally, we take the unique version of g in $\mathcal{B}_m$. This completes the proof.

One consequence of Theorem 4 is that each $h \in H_m$ has a unique representation of the form

$$h_s(x) = \int_{[-\infty,\infty)} E_x 1_{\{T-s \geq u\}}\, \nu(\mathrm{d}u)$$

where ν is a finite Borel measure on $[-\infty, \infty)$. It is clear from this representation that if H_m^1 is the set of $h \in H_m$ with $\sup_{s,x} h_s(x) = 1$, then the set of extreme points of H'_m is $\{h^u, -\infty \le u < \infty | h_s^u(x) = P_x(T > s+u)\}$. We interpret $h^{-\infty}$ as $h_s^{-\infty}(x) = 1$ for all s, x, the constant exit law. Observe that $h_s^u = h_{s+u}^0$, and hence that except for translation, there is only one nonconstant extremal exit law in H_m^1. This type of result for entrance laws has been pointed out in [4].

4. Diffusions

The results of the previous sections can be extended to one-dimensional diffusions. For simplicity, we will only consider diffusions on $[0, \infty)$ with $+\infty$ inaccessible and 0 either absorbing or reflecting. The scale function is S, the speed measure is m, and we will call $M(x) = m((0, x])$ the speed function. Unless otherwise noted, all facts quoted about diffusions can be found in Chapter 16 of Breiman's book [1]. In this section we will examine the scale and speed functions for birth-death process, convert the duality transformation (2.2) to a scale and speed transformation, and then apply Stone's work on weak convergence to obtain the duality result for diffusions.

Let λ_n, μ_n ($\mu_0 = 0$) satisfy $C(\lambda_n, \mu_n)$ and $D(\lambda_n, \mu_n)$, with λ_n^*, μ_n^* defined by (2.2). Following Feller [11] we define

$$V(x) = \begin{cases} 0, & x = 0, \\ \sum_{k=0}^{x-1} \frac{\lambda_0}{\lambda_k \pi_k}, & x \ge 1, \end{cases} \tag{4.1a}$$

$$V^*(x) = \begin{cases} 0, & x = -1, \\ \frac{1}{\mu_0^*}, & x = 0, \\ \frac{1}{\mu_0^*} + \sum_{k=0}^{x-1} \frac{1}{\lambda_k^* \pi_k^*}, & x \ge 1, \end{cases} \tag{4.1b}$$

and

$$U(x) = \frac{1}{\lambda_0} \sum_{k=0}^{x} \pi_k, \quad x \ge 0, \qquad U^*(x) = \sum_{k=0}^{x} \pi_k^*, \quad x \ge 0. \tag{4.2}$$

The functions V and U (and V^* and U^*) act as scale and speed functions for birth–death processes, exactly as S and M act for diffusion processes. Using (2.2) we see that

$$V^*(x) = U(x), \quad U^*(x) = V(x+1), \qquad x \ge 0. \tag{4.3}$$

This shows that (2.2) is really an interchange of scale and speed, and motivates the following definitions.

For increasing functions S and M we define

$$S^*(x) = M(x), \qquad M^*(x) = S(x) \tag{4.4}$$

and the conditions $\bar{C}(S, M)$, $\bar{D}(S, M)$ by

$$\bar{C}(S,M): \quad \int_{(0,\infty)} M(x)\,\mathrm{d}S(x) = +\infty,$$

$$\bar{D}(S,M): \quad \int_{(0,\infty)} S(x)\,\mathrm{d}M(x) = +\infty.$$

Note that $C(\lambda_n, \mu_n)$ is equivalent to $\bar{C}(V, U)$ and $D(\lambda_n, \mu_n)$ is equivalent to $\bar{D}(V, U)$. As in the birth–death case, $C(S, M)$ and $D(S, M)$ imply that $+\infty$ is a natural boundary point.

To state the duality result we need some additional notation. Let C^2 be the set of bounded real value functions with bounded continuous second derivatives on $[0, \infty)$, and define

$$Af = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}M}\frac{\mathrm{d}}{\mathrm{d}S}f, \qquad f \in D(A) = \left\{f \in C^2 : \frac{\mathrm{d}f}{\mathrm{d}S}(0) = 0\right\},$$

$$A^*f = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}M^*}\frac{\mathrm{d}}{\mathrm{d}S^*}f, \quad f \in D(A^*) = \{f \in C^2 : A^*f(0) = 0\}$$

(see [26]). A diffusion with generator A has 0 as an instantaneous reflecting boundary, while a diffusion with generator A^* has 0 as an absorbing boundary.

Theorem 5. *Let S and M be positive, continuous, strictly increasing functions on $[0, \infty)$, which satisfy $\bar{C}(S, M)$ and $\bar{D}(S, M)$. Let X_t be the diffusion with generator A and let X_t^* be the diffusion with generator A^*. Then for all $x, y \geq 0$, $t \geq 0$,*

$$P_x(X_t < y) = P_y(X_t^* > x). \tag{4.5}$$

That is, X_t and X_t^* are dual with respect to the function $f(x, y) = 1_{\{x<y\}}$. Observe that we assume the speed measure has no atoms; this is necessary for S^* to be continuous.

Sketch of proof. Fix $x, y > 0$ and construct a sequence of birth–death processes $X_t^{(n)}$ with state space $X^{(n)}$, scale and speed functions $V^{(n)}$ and $U^{(n)}$, and corresponding dual processes $X_t^{(n)*}$ on $X^{(n)*}$ with scale and speed functions $V^{(n)*}$ and $U^{(n)*}$, such that:

(i) $X^{(n)}$ and $X^{(n)*}$ become dense in $[0, \infty)$.
(ii) $V^{(n)} \to S$, $U^{(n)} \to M$, $V^{(n)*} \to S^*$, $U^{(n)*} \to M^*$ pointwise.
(iii) $\bar{C}(V^{(n)}, U^{(n)})$ and $\bar{D}(V^{(n)}, U^{(n)})$ hold.

By Stone's results in [34] and [35] we conclude that for each fixed t, $X_t(n) \xrightarrow{w} X_t$ and $X_t^{(n)*} \xrightarrow{w} X_t^*$. Since M and M^* are (by hypothesis) continuous, $P_x(X_t = y) =$

$P_y(X_t^* = x) = 0$ for $x > 0$. By taking limits of the duality equation

$$P_x(X_t^{(n)} \leqslant y) = P_y(X_t^{(n)*} \geqslant x),$$

we have

$$P_x(X_t \leqslant y) = P_y(X_t^* \geqslant x),$$

or

$$P_x(X_t < y) = P_y(X_t^* > x), \quad x, y > 0.$$

This equation (unlike the previous one) holds if one or both of x, y are zero. This completes the argument.

We will now state the entrance law result for diffusions, and will omit the proof, which parallels the proof of Theorem 3. The exit law result can be found in [13] or [29].

Theorem 6. *Let X_t be a diffusion on $[0, \infty)$ with $+\infty$ inaccessible and 0 accessible. Suppose X_t has scale function S and speed function M. Then the following are equivalent.*

(a) *X_t has a nonconstant entrance law of probability measures.*

(b) *There are numbers $t_x \to \infty$ and a probability measure ϕ such that*

$$P_x(\tau_0 - t_x \in \mathrm{d}u) \xrightarrow{w} \phi(\mathrm{d}u) \text{ as } x \to \infty.$$

(c) $$\lim_{x \to \infty} E_x(\tau_0) = +\infty, \qquad \lim_{x \to \infty} \mathrm{Var}_0(\tau_0) < \infty.$$

(d) $$\int_0^\infty S(x)\, \mathrm{d}M(x) = +\infty, \qquad \int_0^\infty (M(\infty) - Mx))^2 S(x)\, \mathrm{d}S(x) < \infty.$$

As with Theorem 3, the new result is (b) $\Rightarrow$ (c) (and an alternate proof of this fact can be based on the duality theory, at least in the case when the speed function M is continuous).

We would like to present two examples of dual diffusions. The first example was well known to Levy (see [30]). If X_t is Brownian motion reflected at 0 and X_t^* is Brownian motion absorbed at 0, a direct calculation shows (3.5) is satisfied. However, neither X_t nor X_t^* possess nonconstant exit or entrance laws.

For the second example it is helpful to rewrite the generators in terms of infinitesimal drift and variance coefficients. If

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}M} \frac{\mathrm{d}}{\mathrm{d}S} f(x) = \tfrac{1}{2}\sigma^2(x) f''(x) + \mu(x) f'(x), \quad x > 0,$$

then the formulas in [26, Section 1.4] and (3.4) imply (see also [4, p. 924])

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}M^*}\frac{\mathrm{d}}{\mathrm{d}S^*}f(x)=\tfrac{1}{2}\sigma^2(x)f''(x)+\left[\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\sigma^2(x)-\mu(x)\right]f'(x),\quad x>0.$$

Consider the pair of dual generators

$$Af(x)=\tfrac{1}{2}f''(x)+xf'(x),\qquad x>0, f'(0)=0,$$

$$A^*f(x)=\tfrac{1}{2}f''(x)-xf'(x),\quad x>0, f''(0)=0.$$

The diffusion X_t^* with generator A^* is the Ornstein–Uhlenbeck process with absorption at 0. The dual diffusion X_t with generator A has a drift away from 0 instead of towards 0. X_t^* has nonconstant entrance laws (see [3]) and therefore X_t has nonconstant exit laws (and a nontrivial tail field).

5. Discrete time birth–death chains

Once again let $\mathcal{X}=\{0,1,2,\ldots\}$ and consider the discrete time birth–death chain X_n, which is a Markov chain with transition function

$$p(x,y)=\begin{cases}p(x), & y=x+1,\\ r(x), & y=x,\\ q(x), & y=x-1,\\ 0, & \text{otherwise.}\end{cases}$$

Here p, r and q are strictly positive (except $q(0)=0$) and $p+q+r\equiv 1$. It is rather curious that, unlike the continuous time chains in Section 2, not every chain X_n has a dual X_n^* which satisfies

$$P_x(X_n\leqslant y)=P_y(X_n^*\geqslant x). \tag{5.1}$$

It is not difficult to establish that (5.1) holds if and only if

$$P_y(X_1^*=x)=P_{x-1}(X_1\leqslant y)-P_x(X_1\leqslant y)\geqslant 0.$$

A simple calculation shows this is equivalent to

$$p(x)\leqslant p(x+1)+r(x+1),\quad x\geqslant 0.$$

Unless this rather stringent condition is satisfied, the duality equation (5.1) is unavailable.

A duality *will* exist for the case $r\equiv 0$ if we consider not X_n but $Y_n=X_{2n}$. It can be shown that a dual process Y_n^* exists and satisfies the basic duality equation. In view of this fact it is not surprising that the analogues of Theorems 2 and 3 are valid. The result for exist laws can be found in [28]. We will not use the duality, but instead will state and give a direct proof of part of the entrance law result, assuming $r\equiv 0$.

Theorem 7. *Let X_n be a birth–death chain with transition function given in* (5.1), $r \equiv 0$, $q(0) = 0$ *and* $0 < p(x) < 1$ *for all* $x \geq 1$. *Then the following are equivalent*

(a) *X_n has a nonconstant entrance law of probability measures.*

(b) *There are constants $t_n \to \infty$ and a probability measure ϕ on $\mathbb{Z}$ such that*

$$P_n(\tau_0 - t_n \in \mathrm{d}u) \xrightarrow{w} \phi(\mathrm{d}u) \text{ as } n \to \infty.$$

(c) $\lim_{n \to \infty} \mathrm{Var}_n(\tau_0) < \infty$.

(d) $\sum_{n=1}^{\infty} p(n) < \infty$.

Proof. The only difficult part of this result not in [2] is (b) ⇒ (c), which is the new result. The proof is as follows. On an appropriate probability space define independent random variables σ_n, $n \geq 1$, where σ_n is a copy of the time it takes the chain to go from n to $n-1$. The convergence in (b) implies that the sum $\sum \sigma_n - (t_n - t_{n-1})$ must converge a.s. Therefore by Borel–Cantelli for any $\varepsilon > 0$, $\sum_{n=1}^{\infty} P(|\sigma_n - (t_n - t_{n-1})| > \varepsilon) < \infty$. According to results of [28], $P(\sigma_n = j)$ is maximized by taking $j = 1$. So, if $\varepsilon < \frac{1}{2}$,

$$\sum_{n=1}^{\infty} P(|\sigma_n - (t_n - t_{n-1})| > \varepsilon) \geq \sum_{n=1}^{\infty} P(|\sigma_n - 1| > \varepsilon) = \sum_{n=1}^{\infty} 1 - P(\sigma_n = 1)$$

$$= \sum_{n=1}^{\infty} 1 - q(n) = \sum_{n=1}^{\infty} p(n).$$

Since $\sum_{n=1}^{\infty} P(|\sigma_n - (t_n - t_{n-1})| > \varepsilon) < \infty$, (d) holds.

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References

[1] L. Breiman, Probability (Addison Wesley, MA, 1968).

[2] J.T. Cox, Entrance laws for Markov chains, Ann. Probab. 5 (1977) 533–549.

[3] J.T. Cox, On one-dimensional diffusions with time parameter set (−∞, ∞), Ann. Probab. 5 (1977) 807–813.

[4] J.T. Cox, Further results on one-dimensional diffusions with time parameter set (−∞, ∞), Ann. Probab. 7 (1979) 537–542.

[5] D.J. Daley, Stochastically monotone Markov chains, Z. Wahrsch. Verw. Gebiete 10 (1968) 305–317.

[6] J.L. Doob, Stochastic Processes (Wiley, New York, 1953).
[7] E.B. Dynkin, Entrance and exit spaces for a Markov process, Actes Congress Int. Math., t. 2 (1970) 507–512.
[8] E.B. Dynkin, The initial and final behavior of trajectories of a Markov process, Russian Math. Surveys 26 (1971) 165–182.
[9] E.B. Dynkin, Integral representation of excessive measures and functions, Russian Math. Surveys 27 (1972) 43–84.
[10] E.B. Dynkin, Duality for Markov processes, in: A Friedman and M. Pinsky, eds.: Stochastic Analysis (Academic Press, New York, 1978).
[11] W. Feller, The birth and death processes as diffusion processes, Journ de Math. 38 (1959) 301–345.
[12] H. Föllmer, On the potential theory of stochastic fields, Bull. Inst. Internat. Statist. 46 (1975) 362–370, 405–408.
[13] B. Fristedt and S. Orey (1978). The tail σ-field of one-dimensional diffusions, in: A. Friedman and M. Pinsky, eds., Stochastic Analysis (Academic Press, New York, 1978).
[14] D. Griffeath, Additive and cancelative interacting particle systems, Lecture Notes in Mathematics 724 (Springer, New York, 1979).
[15] T. E. Harris, Contact interactions on a lattice, Ann. Probab. 2 (1974) 969–988.
[16] T.E. Harris, On a class of set-valued Markov processes, Ann. Probab. 4 (1976) 175–194.
[17] T.E. Harris, Additive set-valued Markov processes and percolation methods, Ann. Probability 6 (1978) 355–378.
[18] R. Holley and D. Stroock, Dual processes and their application to infinite particle systems, Adv. Math. (1976).
[19] S. Karlin, Total Positivity I (Stanford University Press, Stanford, CA., 1968).
[20] S. Karlin and J.L. McGregor, The differential equations of birth-and-death processes and the Stieltjes moment problem, Trans. Amer. Math. Soc. 85 (1957) 489–546.
[21] S. Karlin and J. L. McGregor, Coincidence properties of birth-and-death processes, Pacific J. Math. 9 (1959) 1109–1140.
[22] J. Keilson, Log-concavity and log-convexity in passage time densities of diffusion and birth–death processes, J. Appl. Prob. 8 (1971) 391–398.
[23] J.G. Kemeny, J.L. Snell and A.W. Knapp, Denumerable Markov Chains (Springer, New York, 1976).
[24] H. Kunita and T. Watanabe, Markov processes and Martin boundaries, Part I, Illinois J. Math. 9 (1965) 485–526.
[25] T. Liggett, The stochastic evolution of interacting particles, Lecture Notes in Mathematics 598 (Springer, New York, 1977) pp. 187–248.
[26] P. Mandl, Analytical Treatment of One-Dimensional Markov Processes (Springer, New York, 1968).
[27] S. Orey, Limit Theorems for Markov Chain Transition Probabilities (Van Nostrand Reinhold, London, 1971).
[28] U. Rösler, Das 0-1 Gesetz der terminalen σ-Algebra Harrisirrfahrten, Z. Wahrsch. Verw. Gebiete 37 (1977) 227–242.
[29] U. Rossler, The tail σ-field of time homogeneous one-dimensional diffusion processes, Ann. Probability 7 (1979) 847–857.
[30] D. Siegmund, The equivalence of absorbing and reflecting barrier problems for stochastical monotone Markov processes, Ann. Probability 4 (1976) 914–924.
[31] F.L. Spitzer, Interaction of Markov processes, Adv. Math. 5 (1970) 246–290.
[32] F.L. Spitzer, Phase transition in one-dimensional nearest-neighbor systems, J. Func. Anal. 20 (1975) 240–255.
[33] F.L. Spitzer, Infinite systems with locally interacting components, Ann. Probability (1980), to appear.
[34] C.J. Stone, Limit theorems for birth-and-death processes and diffusion processes, Ph.D. thesis, Stanford University, CA, 1961).
[35] C.J. Stone, Limit theorems for random walks, birth-and-death processes, and diffusion processes. Illinois J. Math. 7 (1963) 638–660.
[36] E.A. Van Doorn, Stochastic monotonicity of birth–death processes, Adv. Appl. Probab. 12 (1980) 59–80.