Identities of bilinear mappings and graded polynomial identities of matrices

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Abstract

We describe the polynomial identities of the bilinear mappings $V \otimes W \to K$ and $V_r \otimes V_r^* \to M_r(K)$, where $V$, $W$, $V_r$ are finite dimensional vector spaces over a field $K$ of characteristic 0, dim $V_r = r$ and $M_r(K)$ is the $r \times r$ matrix algebra. We show that these identities follow from the commutativity of the values in $K$ of the bilinear forms $V \otimes W \to K$ and $V_r^* \otimes V_r \to K$ and Capelli identities. We apply these results to the $G$-graded polynomial identities of the algebra $M_r(K)$ with the elementary $G$-grading associated with the $r$-tuple $(g, \ldots, g, h)$, where $G$ is an arbitrary group and $g, h \in G$ are such that $(g^{-1}h)^2 \neq e$. In particular, we describe the graded identities depending only on the variables $x_{i,h^{-1}g}x_{i,h^{-1}g^{-1}h}$, $i = 1, 2, \ldots$

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1. Introduction

There are various generalizations of the notion of polynomial identities, for example weak polynomial identities (or identities of representations), trace identities, etc. One of the purposes of this paper is to describe the identities of a bilinear form $\langle - , - \rangle : V \otimes W \to K$, where $V$ and $W$ are finite dimensional vector spaces over a field $K$ of characteristic 0. It turns out that they depend only on the rank of the form. We obtain that the identities follow from the commutativity

$\langle y_1, z_1 \rangle \langle y_2, z_2 \rangle = \langle y_2, z_2 \rangle \langle y_1, z_1 \rangle$

and the Capelli identity

$$\sum_{\sigma \in S_{r+1}} (\text{sign } \sigma) \langle y_{\sigma(1)}, z_1 \rangle \cdots \langle y_{\sigma(r+1)}, z_{r+1} \rangle = 0,$$

where $r$ is the rank of the form $\langle - , - \rangle$.

This result is equivalent to a well-known result in invariant theory called the second fundamental theorem of invariant theory of $GL_r(K)$ acting on $r$-vectors and $r$-covectors. It is established by De Concini and Procesi [3] in a characteristic free way, over a field of any characteristic. Their proof is based on standard Young tableaux and uses deep combinatorial results of Doubilet et al. [5]. Instead, we suggest a proof which works in characteristic 0 only but is based on easier arguments from representation theory of the general linear group in a way which has already shown its efficiency in the theory of algebras with polynomial identities.

We consider also the identities of the natural mapping $V \otimes V^* \to M_r(K)$, where $\dim(V) = r$ and $V^*$ is the dual space. Again, it has turned out that the identities follow from

$\langle y_1, z_1 \rangle \langle y_2, z_2 \rangle \langle y_3, z_3 \rangle = \langle y_1, z_2 \rangle \langle y_3, z_1 \rangle \langle y_2, z_3 \rangle$

which is a consequence of the commutativity law for the canonical bilinear form $V^* \otimes V \to K$ and the Capelli identities

$$\sum_{\sigma \in S_{r+1}} (\text{sign } \sigma) \langle y_{\sigma(1)}, z_1 \rangle \langle y_{\sigma(2)}, z_2 \rangle \cdots \langle y_{\sigma(r+1)}, z_{r+1} \rangle = 0,$$

$$\sum_{\tau \in S_{r+1}} (\text{sign } \tau) \langle y_1, z_{\tau(1)} \rangle \langle y_2, z_{\tau(2)} \rangle \cdots \langle y_{r+1}, z_{\tau(r+1)} \rangle = 0.$$

To the best of our knowledge, these are objects which have not yet appeared in the context of PI-algebras.

Our results are closely related to $G$-graded polynomial identities of matrix algebras. So far, the graded identities of $M_r(K)$ have been described only in very special cases, when $r = 2$ and $G = \mathbb{Z}_2$ [4]; $G = \mathbb{Z}$ and $G = \mathbb{Z}_r$, for any $r$ [8,9]; when the grading is fine or elementary with the identity component coinciding with the diagonal subalgebra and for the tensor products of these two types of gradings [2]. In this paper we consider an elementary $G$-grading of $M_r(K)$, for an arbitrary group...
G, induced by an \( r \)-tuple \( (g, \ldots, g, h) \), where \( g, h \in G \) are such that \((g^{-1} h)^2 \neq e\), and \( e \) is the identity element of \( G \). In this case \( M_r(K) \) may be considered as the algebra of graded endomorphisms of the vector space \( V = V_{r-1} \oplus V_1 \), where \( \dim V_{r-1} = r - 1 \), \( \dim V_1 = 1 \) and \( V_{r-1}, V_1 \) are the homogeneous components of \( V \) of degree \( g \) and \( h \), respectively. Then \( M_r(K) \) is divided into four blocks

\[
M_r(K) = \begin{pmatrix} M_{r-1}(K) & M_{(r-1)\times 1}(K) \\ M_{1\times (r-1)}(K) & K \end{pmatrix},
\]

where \( M_{p\times q}(K) \) is the vector space of \( p \times q \)-matrices. The homogeneous components of \( M_r(K) \) are

\[
(M_r(K))_e = \begin{pmatrix} M_{r-1}(K) & 0 \\ 0 & K \end{pmatrix},
\]

\[
(M_r(K))_{h^{-1}g} = \begin{pmatrix} 0 & 0 \\ M_{1\times (r-1)}(K) & 0 \end{pmatrix},
\]

\[
(M_r(K))_{g^{-1}h} = \begin{pmatrix} 0 & M_{1\times 1}(K) \\ 0 & 0 \end{pmatrix}.
\]

It is clear that the graded polynomial identities depend essentially on the ordinary polynomial identities of \( M_{r-1}(K) \) and the identities of bilinear mappings. Although our results on \( G \)-graded identities of \( M_r(K) \) are not in their final form, we give a complete description of the graded identities depending only on the variables corresponding to the homogeneous components \( (M_r(K))_{h^{-1}g} \) and \( (M_r(K))_{g^{-1}h} \). They are obtained by the natural translation of the identities of the bilinear mappings \( V_{r-1}^* \otimes V_{r-1} \to K \) and \( V_{r-1} \otimes V_{r-1}^* \to M_{r-1}(K) \).

2. Preliminaries

We fix a field \( K \) of characteristic 0 and consider vector spaces and associative algebras over \( K \) only. Let \( Y = \{y_1, y_2, \ldots\} \) and \( Z = \{z_1, z_2, \ldots\} \) be two sets of variables and let \( KY \) and \( KZ \) be the vector spaces with bases \( Y \) and \( Z \), respectively. By analogy with generic matrices, and as in [3], we consider the generic bilinear form

\[
\langle -, - \rangle : KY \otimes KZ \to K[\xi_{ij} \mid i, j = 1, 2, \ldots],
\]

defined by \( \langle y_i, z_j \rangle = \xi_{ij} \), where \( \xi_{ij} \) are commuting variables. It is convenient to identify \( \langle y_i, z_j \rangle \) with its value \( \xi_{ij} \) and to consider the free algebra of bilinear forms \( F = K[\langle y_i, z_j \rangle \mid i, j = 1, 2, \ldots] \). Let \( V \) and \( W \) be two vector spaces with a bilinear form \( \phi(\cdot, \cdot) = \langle \cdot, \cdot \rangle : V \otimes W \to K \). We say that the expression

\[
f(y_1, \ldots, y_p; z_1, \ldots, z_q) = \sum \alpha_{ij} \langle y_{i_1}, z_{j_1} \rangle \cdots \langle y_{i_n}, z_{j_n} \rangle, \quad \alpha_{ij} \in K,
\]

is an identity for the form \( \phi \) if

\[
f(v_1, \ldots, v_p; w_1, \ldots, w_q) = \sum \alpha_{ij} \langle v_{i_1}, w_{j_1} \rangle \cdots \langle v_{i_n}, w_{j_n} \rangle = 0
\]
for all \( v_1, \ldots, v_p \in V \) and \( w_1, \ldots, w_q \in W \). Clearly, the set \( T(\phi) \) of all identities of \( \phi \) is an ideal of \( F \) which is invariant under replacement of the variables \( y_i \) and \( z_j \) with any elements of \( KY \) and \( KZ \), respectively. As in the case of associative or Lie algebras, or weak polynomial identities (see e.g. the books [1,6]), one can introduce consequences of a set of identities and bases (i.e. generating sets for the corresponding ideals of identities), varieties, relatively free algebras, etc. We call the factor algebra

\[
F(\phi) = F / T(\phi) = K[\{ y_i, z_j \mid i, j = 1, 2, \ldots \}] / T(\phi)
\]

the relatively free algebra of the variety of bilinear forms generated by \( \phi \). We shall denote the generators of \( F(\phi) \) by the same symbols \( y_i, z_j \) and shall consider the relatively free algebra \( F_m(\phi) \) of rank \( m \) generated as a subalgebra of \( F \) by \( \langle y_i, z_j \rangle \), \( i, j = 1, \ldots, m \). Since we work over a field of characteristic 0, all identities follow from the multilinear ones. As in the case of polynomial identities of algebras when one uses representation theory of the symmetric group \( S_n \) and the general linear group \( GL_m(K) \), one can use the representation theory of groups in the study of identities of bilinear forms (see [6, Chapter 12] for the ordinary polynomial identities). Since we have two sets of variables, we consider the action of \( S_n \times S_n \) on the vector space \( P_{2n} \) of multilinear identities of degree \( 2n \) by

\[
(\rho, \sigma) : \langle y_{i_1}, z_{j_1} \rangle \cdots \langle y_{i_n}, z_{j_n} \rangle \mapsto \langle y_{\rho(i_1)}, z_{\sigma(j_1)} \rangle \cdots \langle y_{\rho(i_n)}, z_{\sigma(j_n)} \rangle,
\]

\( (\rho, \sigma) \in S_n \times S_n \). Since \( T(\phi) \cap P_{2n} \) is an \( S_n \times S_n \)-submodule of \( P_{2n} \), this action is inherited by \( P_{2n}(\phi) = P_{2n} / (T(\phi) \cap P_{2n}) \). The irreducible \( S_n \times S_n \)-modules are tensor products of two irreducible \( S_n \)-modules and are indexed by the set of pairs \( (\lambda, \mu) \) of partitions of \( n \). We denote by \( \chi_\lambda \otimes \chi_\mu \) the corresponding irreducible character. In particular,

\[
\chi_{2n}(\phi) = \chi(P_{2n}(\phi)) = \sum_{\lambda, \mu \vdash n} m_{\lambda, \mu} \chi_\lambda \otimes \chi_\mu,
\]

is the cocharacter sequence of \( \phi \). Let \( GL_m = GL_m(K) \) and let \( W(\lambda) \) be the irreducible \( GL_m \)-module with highest weight \( \lambda = (\lambda_1, \ldots, \lambda_m) \). The group \( GL_m \times GL_m \) acts diagonally on \( F_m(\phi) \) by

\[
(\rho, \sigma) : \langle y_{i_1}, z_{j_1} \rangle \cdots \langle y_{i_n}, z_{j_n} \rangle \mapsto \langle g(y_{i_1}), h(z_{j_1}) \rangle \cdots \langle g(y_{i_n}), h(z_{j_n}) \rangle,
\]

\( (g, h) \in GL_m \times GL_m \). The \( GL_m \times GL_m \)-module \( F_m(\phi) \) is a direct sum of tensor products of irreducible \( GL_m \)-modules,

\[
F_m(\phi) = \bigoplus_{n \geq 0} \bigoplus_{\lambda, \mu \vdash n} m_{\lambda, \mu} W(\lambda) \otimes W(\mu)
\]

with the same multiplicities \( m_{\lambda, \mu} \) as in the cocharacter sequence. It is naturally multi-graded, counting the degree of each variable \( y_1, \ldots, y_m, z_1, \ldots, z_m \), and its Hilbert series is defined by

\[
H(F_m(\phi), t_1, \ldots, t_m; u_1, \ldots, u_m) = \sum \dim F_m^{(a, b)}(\phi) t_1^{a_1} \cdots t_m^{a_m} u_1^{b_1} \cdots u_m^{b_m},
\]
where $F_m^{(a,b)}(\phi)$ is the homogeneous component of degree $(a, b) = (a_1, \ldots, a_m; b_1, \ldots, b_m)$. It is a formal sum of products of Schur functions:

$$H(F_m(\phi), t_1, \ldots, t_m; u_1, \ldots, u_m) = \sum_{n \geq 0} \sum_{\lambda, \mu \vdash n} m_{\lambda,\mu} S_{\lambda}(t_1, \ldots, t_m) S_{\mu}(u_1, \ldots, u_m).$$

Every irreducible polynomial $GL_m \times GL_m$-module $W(\lambda) \otimes W(\mu)$ is generated by an element $w(y_1, \ldots, y_m; z_1, \ldots, z_m)$ of degree $\lambda_i$ in $y_i$ and $\mu_j$ in $z_j$, $i, j = 1, \ldots, m$. The element $w$ is unique up to a multiplicative constant and is called the highest weight vector of the module. It has the form $w_\lambda \otimes w_\mu \in W(\lambda) \otimes W(\mu)$, where $w_\lambda \in W(\lambda)$ and $w_\mu \in W(\mu)$ are the highest weight vectors in the corresponding factors of the tensor product; see e.g. [6, Chapter 12] for the explicit form of $w_\lambda$ and $w_\mu$.

One can introduce identities for other bilinear mappings in the same way as above. We shall need the following. Let $V$ and $W$ be vector spaces and let $R$ be an algebra. Consider a bilinear mapping

$$\psi(-, -) : V \otimes W \to R.$$

The free object which serves as a source for the identities of $\psi$ is the free associative algebra

$$\mathcal{F} = K(\{y_i, z_j\} | i, j = 1, 2, \ldots),$$

where $\{\eta_{ij} = (y_i, z_j) | i, j = 1, 2, \ldots\}$ may be considered as noncommuting variables and $(-, -) : KY \otimes KZ \to \sum K \cdot \eta_{ij}$ is the associated bilinear mapping (where $Y = \{y_1, y_2, \ldots\}$ and $Z = \{z_1, z_2, \ldots\}$). The element of $\mathcal{F}$

$$f(y_1, \ldots, y_p; z_1, \ldots, z_q) = \sum \beta_{ij}(y_{i_1}, z_{j_1}) \cdots (y_{i_s}, z_{j_s}), \quad \beta_{ij} \in K,$$

is an identity for the mapping $\psi$ if

$$f(v_1, \ldots, v_p; w_1, \ldots, w_q) = \sum \beta_{ij}(v_{i_1}, w_{j_1}) \cdots (v_{i_s}, w_{j_s}) = 0$$

for all $v_1, \ldots, v_p \in V$ and $w_1, \ldots, w_q \in W$. Clearly, the set $\mathcal{F}(\psi)$ of all identities of $\psi$ is an ideal of $\mathcal{F}$, the factor algebra

$$\mathcal{F}(\psi) = \mathcal{F} / \mathcal{F}(\psi)$$

is the relatively free algebra and we may consider the action of the groups $S_n \times S_n$ and $GL_m \times GL_m$ on the corresponding subspaces of $\mathcal{F}$.

We shall be also interested in the elementary gradings of matrices and their graded polynomial identities. Let $G$ be an arbitrary group. A $G$-grading of an algebra $R$ is a vector space decomposition $R = \bigoplus_{g \in G} R_g$ such that $R_g R_h \subseteq R_{gh}$ for any $g, h \in G$. The subspace $R_g$ is called the homogeneous component of degree $g$ and its nonzero elements are called homogeneous of degree $g$. If $e$ is the identity element of $G$, then
$R_e$ is called the identity (or neutral) component of $R$. Let $g = (g_1, \ldots, g_r)$ be an $r$-tuple of elements of $G$. If any matrix unit $e_{ij} \in M_r(K)$ is homogeneous of degree $g = g_i g_j$, then the induced $G$-grading is called elementary. In this case the $G$-graded algebra $M_r(K)$ is isomorphic, as a graded algebra, to the graded algebra of endomorphisms of a $G$-graded vector space $V = \bigoplus_{g \in G} V_g$ with basis $v_1, \ldots, v_r$ such that $\deg(v_j) = g_j$, $j = 1, \ldots, r$. In this paper we consider an elementary $G$-grading of $M_r(K)$, for an arbitrary group $G$, induced by the $r$-tuple $(g, \ldots, g, h)$, where $g, h \in G$ are such that $(g^{-1} h)^2 \neq e$. In this case the corresponding graded vector space is $V = V_{r-1} \oplus V_1$, where $\dim V_{r-1} = r - 1$, $\dim V_1 = 1$ and $V_{r-1}$, $V_1$ are the homogeneous components of $V$ of degree $g$ and $h$, respectively. The homogeneous components of $M_r(K)$ are

$$(M_r(K))_e = \begin{pmatrix} M_{r-1}(K) & 0 \\ 0 & K \end{pmatrix},$$

$$(M_r(K))_{h^{-1} g} = \begin{pmatrix} 0 & 0 \\ M_{1 \times (r-1)}(K) & 0 \end{pmatrix},$$

$$(M_r(K))_{g^{-1} h} = \begin{pmatrix} 0 & M_{(r-1) \times 1}(K) \\ 0 & 0 \end{pmatrix}.$$ 

The $G$-algebra $M_r(K)$ satisfies the identities $x_1 = 0$ for all $k \in G$, $k \neq e$, $g^{-1} h$, $h^{-1} g$. Hence it is sufficient to consider only three kinds of generators of the free graded algebra, namely

$x_{i,e}, \quad x_{i,h^{-1} g}, \quad x_{i,g^{-1} h}, \quad i = 1, 2, \ldots$

Finally, we need the following identity of symmetric functions in two sets of commuting variables $\{t_1, \ldots, t_m\}$ and $\{u_1, \ldots, u_m\}$ (see [7, Chapter 1]):

$$\prod_{i,j=1}^{m} \frac{1}{1 - t_i u_j} = \sum S_\lambda(t_1, \ldots, t_m) S_\lambda(u_1, \ldots, u_m), \quad (2.1)$$

where the summation is taken over all partitions $\lambda = (\lambda_1, \ldots, \lambda_m).

3. Identities of bilinear forms

In this section we shall find a basis for the identities of an arbitrary bilinear form $\phi : V \otimes W \to K$ in terms of the rank of $\phi$. We keep the notations of the previous section. The description of the free object $F$ in the language of the representation theory of groups is given in the following proposition.

Proposition 3.1. Let $F$ be the polynomial algebra on the commuting variables $\langle y_i, z_j \rangle$, $i, j = 1, 2, \ldots$, and let $P_{2n}$ be the subset of $F$ consisting of all multilinear elements of degree $2n$ in the variables $y_1, \ldots, y_n; z_1, \ldots, z_n$. Then the character of the $S_n \times S_n$-module $P_{2n}$ is
\[ \chi_{2n} = \sum_{\lambda \vdash n} \chi_\lambda \otimes \chi_\lambda. \]

The highest weight vector of the \( GL_m \times GL_m \)-submodule \( W(\lambda) \otimes W(\lambda) \) of \( F_m \) is equal to the product

\[ w_\lambda(y_1, \ldots, y_m; z_1, \ldots, z_m) = \prod_{i=1}^k w_{p_i}(y_{1}, \ldots, y_{p_i}; z_1, \ldots, z_{p_i}), \]

where \( p_1, \ldots, p_k \) are the lengths of the columns of the diagram related to the partition \( \lambda \) and

\[ w_p = \sum_{\sigma \in S_p} (\text{sign } \sigma)(y_{\sigma(1)}, z_1) \cdots (y_{\sigma(p)}, z_p). \]

**Proof.** The Hilbert series of the polynomial algebra generated by \( p \) variables \( x_1, \ldots, x_p \), where \( x_i \) is associated with the “degree variable” \( \xi_i \), is

\[ H(K[x_1, \ldots, x_p], \xi_1, \ldots, \xi_p) = \prod_{i=1}^p \frac{1}{1 - \xi_i}. \]

Hence the Hilbert series of the polynomial algebra \( F_m \) generated by the \( m^2 \) “variables” \( x_{ij} = \langle y_i, z_j \rangle \), associated with the “degree variables” \( \xi_{ij} = t_i u_j \), is

\[ H(F_m, t_1, \ldots, t_m; u_1, \ldots, u_m) = \prod_{i,j=1}^m \frac{1}{1 - t_i u_j} = \sum S_\lambda(t_1, \ldots, t_m) S_\lambda(u_1, \ldots, u_m), \]

by (2.1). Since the \( GL_m \times GL_m \)-module structure of \( F_m \) is completely determined by its Hilbert series, we obtain that

\[ F_m \cong \sum W(\lambda) \otimes W(\lambda), \]

where the sum is on all partitions \( \lambda \) in not more than \( m \) parts. Using the correspondence between the \( GL_m \times GL_m \)-module structure of \( F_m \) and the \( S_m \times S_m \)-character of the multilinear elements of degree \( 2n \), we obtain that

\[ \chi_{2n} = \sum_{\lambda \vdash n} \chi_\lambda \otimes \chi_\lambda. \]

As in [6, Chapter 12], the element

\[ w'_p = \sum_{\sigma \in S_p} \sum_{\tau \in S_p} (\text{sign } \sigma) (\text{sign } \tau) \langle y_{\sigma(1)}, z_{\tau(1)} \rangle \cdots \langle y_{\sigma(p)}, z_{\tau(p)} \rangle, \]

if nonzero, is a highest weight vector of the submodule \( W(1^p) \otimes W(1^p) \) of \( F_m \). Since
in $F_m$, we obtain that the product

$$w_\lambda(y_1, \ldots, y_m; z_1, \ldots, z_m) = \prod_{i=1}^{k} w_{p_i}(y_1, \ldots, y_{p_i}; z_1, \ldots, z_{p_i}),$$

where $p_1, \ldots, p_k$ are the lengths of the columns of the diagram related to the partition $\lambda$, is a highest weight vector of the unique submodule $W(\lambda) \otimes W(\lambda)$ of $F_m$. □

Doubilet et al. [5] prove the straightening formula which gives that the polynomial algebra $F$ on the commuting variables $\langle y_i, z_j \rangle$, $i, j = 1, 2, \ldots$, has a basis which can be indexed by double standard tableaux. Since the $S_n \times S_n$-character $\chi_{2n}$ is determined by the $GL_m \times GL_m$-structure of $F_m$, the result in [5] implies our Proposition 3.1. Although our proof depends essentially on the characteristic, it has the advantage that we use only classical results of representation theory of the general linear group and the combinatorial formula (2.1). On the other hand, with some additional work, our approach based on (2.1) implies a new proof of the straightening formula in [5].

The following theorem describes the identities of a bilinear form, in terms of the rank of the form.

**Theorem 3.2.** Let $V$ and $W$ be finite dimensional vector spaces and let $\phi : V \otimes W \to K$ be a bilinear form of rank $r$. The identities of $\phi$ follow from the identity

$$w_{r+1} = \sum_{\sigma \in S_{r+1}} (\text{sign } \sigma) \langle y_{\sigma(1)}, z_1 \rangle \cdots \langle y_{\sigma(r+1)}, z_{r+1} \rangle = 0.$$ 

**Proof.** Since the rank of $\phi$ is equal to $r$, we may find suitable bases $\{e_1, \ldots, e_p\}$ of $V$ and $\{f_1, \ldots, f_q\}$ of $W$ such that

$$\phi(e_i, f_j) = \delta_{ij} \delta_{ij},$$

where $\delta_{ij}$ is the Kronecker symbol and $\delta_{ij} = 1$ for $i = 1, \ldots, r$, and $\delta_{ij} = 0$ for $i = r+1, \ldots, p$. Obviously, $w_{r+1} = 0$ is an identity for the form $\phi$ and all identities $w_k = 0$ are consequences of $w_{r+1} = 0$ for $k \geq r+1$. Hence, by Proposition 3.1, if $\lambda_{r+1} \neq 0$, then $w_3$ belongs to the ideal of identities $T(\phi)$ and the relatively free algebra $F_m(\phi)$, as a $GL_m \times GL_m$-module is isomorphic to a submodule of $\sum W(\lambda_1, \ldots, \lambda_r) \otimes W(\lambda_1, \ldots, \lambda_r)$. In order to complete the proof, it is sufficient to show that for $m \geq r$, the highest weight vectors $w_2$ are not equal to 0 in $F_m(\phi)$ for all $\lambda = (\lambda_1, \ldots, \lambda_r)$. Clearly, for $k \leq r$,

$$w_k(e_1, \ldots, e_k; f_1, \ldots, f_k) = \sum_{\sigma \in S_k} (\text{sign } \sigma) \langle e_{\sigma(1)}, f_1 \rangle \cdots \langle e_{\sigma(k)}, f_k \rangle = 1$$

and $w_2(e_1, \ldots, e_k; f_1, \ldots, f_k) = 1 \neq 0$. □
Theorem 3.2 is equivalent to the second fundamental theorem of invariant theory of $GL_r(K)$ (see [3]), but our proof does not use the straightening formula of Doubilet et al. [5].

Corollary 3.3. If $V$ and $W$ are infinite dimensional vector spaces and the bilinear form $\phi : V \otimes W \rightarrow K$ is not degenerate, then $\phi$ does not satisfy any identity.

Proof. By Proposition 3.1, if $\phi$ satisfies a nontrivial identity, then it satisfies also some $w_\lambda = 0$. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$. Since the form $\phi$ is not degenerate, we may choose subspaces $V' \subset V$ and $W' \subset W$ such that the restriction of $\phi$ on $V' \otimes W'$ is of rank $r$. By Theorem 3.2, $w_\lambda$ does not vanish on $V' \otimes W'$ which is a contradiction. Hence $\phi$ does not satisfy a nontrivial identity. □

4. Bilinear mappings to matrix algebras

Let $V$ be an $r$-dimensional vector space with basis $\{e_1, \ldots, e_r\}$ and let $V^*$ be the dual space of $V$ with basis $\{e_1^*, \ldots, e_r^*\}$ such that the linear forms $e_i^*$ are defined by $e_i^*(e_j) = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker symbol. There is a natural mapping $\psi : V \otimes V^* \rightarrow M_r(K)$ defined by

$$\psi(e_i, e_j^*) = (e_i, e_j^*) = e_{ij}, \quad i, j = 1, \ldots, r.$$  

The main purpose of this section is to give a basis for the identities of the mapping $\psi$. We shall keep the notation of Section 2. In particular, we shall work in the algebra $\mathcal{F}$, the free associative algebra with free generators $(y_i, z_j)$, $i, j = 1, 2, \ldots$.

Lemma 4.1. The natural mapping $\psi : V \otimes V^* \rightarrow M_r(K)$ satisfies the identities

$$\begin{align*}
(y_1, z_1)(y_2, z_2)(y_3, z_3) &= (y_1, z_2)(y_3, z_1)(y_2, z_3), \\
\sum_{\sigma \in S_{r+1}} (\text{sign } \sigma)(y_{\sigma(1)}, z_1)(y_{\sigma(2)}, z_2) \cdots (y_{\sigma(r+1)}, z_{r+1}) &= 0, \\
\sum_{\tau \in S_{r+1}} (\text{sign } \tau)(y_1, z_{\tau(1)})(y_2, z_{\tau(2)}) \cdots (y_{r+1}, z_{\tau(r+1)}) &= 0.
\end{align*}$$  

Proof. Since the identities (4.1)–(4.3) are multilinear, it is sufficient to evaluate them on the basis elements of $V$ and $V^*$. The identity (4.1) expresses the fact that the canonical bilinear form $(-, -) : V^* \otimes V \rightarrow K$ satisfies the commutative law: If $v_1, v_2, v_3 \in V$, $w_1, w_2, w_3 \in V^*$, then

$$(w_1, v_2)(w_2, v_3) = (w_2, v_3)(w_1, v_2)$$
Lemma 4.2. Modulo the identity (4.1) every element of \( \mathcal{E}_r \) is a linear combination of the products

\[
(y_{1}, z_{j_{1}})(y_{2}, z_{j_{2}}) \cdots (y_{n-1}, z_{j_{n-1}})(y_{n}, z_{j_{n}}),
\]

(4.4)

where \( i_{p}, j_{q} = 1, \ldots, r, j_{1} \leq j_{2} \leq \cdots \leq j_{n-1}, \) and if \( j_{q} = j_{q+1} \) for some \( q = 1, \ldots, n - 2, \) then \( i_{q+1} \leq i_{q+2} \). The elements (4.4) are linearly independent modulo the identities of the natural mapping \( \psi : V \otimes V^{*} \to M_{r}(K) \).

Proof. The first statement of the lemma is obvious because (4.1) allows to change the places of the pairs \((z_{j_{q}}, y_{i_{q}+1})\) and \((z_{j_{q}}, y_{i_{q}+2})\) in \((y_{1}, z_{j_{1}}) \cdots (y_{n}, z_{j_{n}})\). Hence we may assume that \( j_{1} \leq j_{2} \leq \cdots \leq j_{n-1} \) and, if \( j_{q} = j_{q+1} \), we may arrange \( i_{q+1} \leq i_{q+2} \). For the proof of the second statement, we shall replace the variables \( y_{i} \) and \( z_{j} \), \( i, j = 1, \ldots, r \), respectively, by the “generic” elements

\[
v_{i} = \omega_{i_{1}}e_{1} + \cdots + \omega_{i_{r}}e_{r},
\]

where \( \omega_{i_{p}}, i, p = 1, \ldots, r, \) are commuting variables, and by the elements \( e_{j}^{*} \). Let

\[
f(y_{1}, \ldots, y_{r}; z_{1}, \ldots, z_{r}) = \sum_{i,j} a_{ij}(y_{i_{1}}, z_{j_{1}})(y_{i_{2}}, z_{j_{2}}) \cdots (y_{i_{n-1}}, z_{j_{n-1}})(y_{i_{n}}, z_{j_{n}})
\]

be a linear combination of elements of the form (4.4), where \( a_{ij} \in K \) depends on the indices \((i_{1}, i_{2}, \ldots, i_{n})\) and \((j_{1}, \ldots, j_{n-1}, j_{n})\) and let \( f(v_{1}, \ldots, v_{r}; w_{1}, \ldots, w_{r}) = 0 \). Direct calculations show that
Let us denote
\[ f^{(p)}_{i_1 j_n} = \sum_{i,j} \alpha_{ij} \omega_{pi_1} \omega_{j_1} \omega_{j_2} \cdots \omega_{j_{n-1}} \omega_{jn} \]
where the sum is on all \(n\)-tuples \((i_1, i_2, \ldots, i_n)\) and \((j_1, \ldots, j_{n-1}, j_n)\) with fixed \(i_1\) and \(j_n\). Then (4.5) has the form
\[ f(v_1, \ldots, v_r; w_1, \ldots, w_r) = \sum_{p=1}^{r} \sum_{j_n=1}^{r} \left( \sum_{i_1=1}^{r} \omega_{pi_1} f^{(p)}_{i_1 j_n} \right) e_{pj_n} = 0. \tag{4.6} \]
Comparing the entries in the matrix equation (4.6) we obtain that
\[ \sum_{i_1=1}^{r} \omega_{pi_1} f^{(p)}_{i_1 j_n} = 0, \quad p = 1, \ldots, r. \]
Viewing these \(r\) equalities as a linear homogeneous system with unknowns \(f^{(p)}_{i_1 j_n}\), \(i_1 = 1, \ldots, r\), and bearing in mind that the entries of its matrix
\[
\begin{pmatrix}
\omega_{11} & \omega_{12} & \cdots & \omega_{1r} \\
\omega_{21} & \omega_{22} & \cdots & \omega_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
\omega_{r1} & \omega_{r2} & \cdots & \omega_{rr}
\end{pmatrix}
\]
are algebraically independent commuting variables, we obtain that the determinant of the matrix is different from zero and the only solution of the system is \(f^{(p)}_{i_1 j_n} = 0, \quad p = 1, \ldots, r\). Hence all coefficients \(\alpha_{ij}\) of \(f(y_1, \ldots, y_r; z_1, \ldots, z_r)\) are equal to 0. \(\square\)

The following theorem describes the identities of the natural mapping \(V \otimes V^* \to M_r(K)\).

**Theorem 4.3.** Let \(\dim V = r\). The identities (4.1)–(4.3) form a basis for the identities of the natural mapping \(\psi : V \otimes V^* \to M_r(K)\).

**Proof.** Let \(\mathcal{F}\) be the ideal of identities of the algebra \(\mathcal{F}\) generated by the polynomials
\[
(y_1, z_1)(y_2, z_2)(y_3, z_3) - (y_1, z_2)(y_3, z_1)(y_2, z_3),
\]
\[
\sum_{\sigma \in S_{r+1}} (\text{sign } \sigma)(y_{\sigma(1)}, z_1)(y_{\sigma(2)}, z_2) \cdots (y_{\sigma(r+1)}, z_{r+1}),
\]
\[
\sum_{\tau \in S_{r+1}} (\text{sign } \tau)(y_1, z_{\tau(1)})(y_2, z_{\tau(2)}) \cdots (y_{r+1}, z_{\tau(r+1)}),
\]

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corresponding to the identities (4.1)–(4.3), and let \( F/\mathcal{T} \) and \( F_m/(F_m \cap \mathcal{T}) \) be the corresponding relatively free algebras of infinite rank and rank \( m \), respectively. Since, by Lemma 4.1, the natural mapping \( \psi : V \otimes V^* \to M_r(K) \) satisfies the identities (4.1)–(4.3), we obtain that the relatively free algebra \( F(\psi) \) is a homomorphic image of \( F/\mathcal{T} \) and the \( 2n \)th cocharacter of \( \mathcal{T}(\psi) \) is a subcharacter of the \( 2n \)th cocharacter of \( \mathcal{T} \). Let the \( 2n \)th cocharacters of \( \mathcal{T} \) and \( \mathcal{T}(\psi) \) be respectively

\[
\chi_{2n}(\mathcal{T}) = \sum_{\lambda, \mu \vdash n} m_{\lambda \mu} \chi_\lambda \otimes \chi_\mu,
\]

\[
\chi_{2n}(\mathcal{T}(\psi)) = \sum_{\lambda, \mu \vdash n} n_{\lambda \mu} \chi_\lambda \otimes \chi_\mu,
\]

where \( n_{\lambda \mu} \leq m_{\lambda \mu} \) for all pairs of partitions \( \lambda \) and \( \mu \) of \( n \). The translation of these decompositions into the language of \( GL_m \times GL_m \)-modules is that

\[
F_m/(F_m \cap \mathcal{T}) = \sum_{n \geq 0} \sum_{\lambda, \mu \vdash n} m_{\lambda \mu} W(\lambda) \otimes W(\mu),
\]

(4.7)

\[
F_m(\psi) = \sum_{n \geq 0} \sum_{\lambda, \mu \vdash n} n_{\lambda \mu} W(\lambda) \otimes W(\mu).
\]

(4.8)

Let \( W(\lambda) \otimes W(\mu) \) be a submodule of \( F_m/(F_m \cap \mathcal{T}) \) which vanishes in \( F_m(\psi) \) and let the element \( w(y_1, \ldots, y_p; z_1, \ldots, z_q) \) be its highest weight vector, where \( \lambda = (\lambda_1, \ldots, \lambda_p) \), \( \mu = (\mu_1, \ldots, \mu_q) \) (and \( m \) is sufficiently large, e.g. \( m \geq p, q \)). Hence \( w(y_1, \ldots, y_p; z_1, \ldots, z_q) = 0 \) is an identity for \( \psi \). If \( p, q \leq r \), this is impossible because Lemma 4.2 gives that all identities in \( r \) variables for \( \psi \) follow from (4.1)–(4.3) and hence belong to \( \mathcal{T} \). In this way, the multiplicities \( m_{\lambda \mu} \) and \( n_{\lambda \mu} \) are equal if \( p, q \leq r \). So, we may assume that \( p > r \) and \( \lambda_{r+1} > 0 \) or \( q > r \) and \( \mu_{r+1} > 0 \). First, let \( p > r \) and let us write explicitly the highest weight vector \( w(y_1, \ldots, y_p; z_1, \ldots, z_q) \), as prescribed in [6, Chapter 12]. It is a linear combination of the alternating sums

\[
\sum_{\pi \in S_p} (\text{sign } \pi)(y_{a_1}, z_{b_1}) \cdots (y_{a_k}, z_{b_k})(y_{\pi(1)}, z_{j_1}) (y_{a_{k+1}}, z_{b_{k+1}}) \cdots \times (y_{a_l}, z_{b_l})(y_{\pi(2)}, z_{j_2}) \cdots (y_{\pi(p)}, z_{j_p}) \cdots (y_{a_t}, z_{b_t}),
\]

(4.9)

where the \( p \) elements \( (y_{\pi(i)}, z_{j_i}) \) are separated by products of some \( (y_{a_l}, z_{b_l}) \). Since \( \dim V = r < p \), we obtain that all sums (4.9) vanish on the pair \( V \otimes V^* \) and this means that \( n_{\lambda \mu} = 0 \). Now, we shall work modulo the identities (4.1)–(4.3). If the sum (4.9) starts with \( y_{a_1} \), we may use (4.1) and rearrange \( y_{a_2}, \ldots, y_{a_t}, y_{\pi(1)}, \ldots, y_{\pi(p)} \) and to present (4.9) in the form

\[
\sum_{\pi \in S_p} (\text{sign } \pi)(y_{a_1}, z_{c_1})(y_{\pi(1)}, z_{d_1}) \cdots (y_{\pi(p)}, z_{d_p})(y_{a_2}, z_{c_2}) \cdots (y_{a_t}, z_{c_t}),
\]
for some indices $c_1, \ldots, c_s, d_1, \ldots, d_p$, and this is 0 by (4.2). If the sum (4.9) starts with $y_π(1)$, then again (4.1) gives that (4.9) is of the form

$$\sum_{π \in S_p} (\text{sign} \ π)(y_π(1), z_{d_1}) \cdots (y_π(p), z_{d_p})(y_{a_1}, z_{c_1}) \cdots (y_{a_s}, z_{c_s}),$$

and we use (4.2). In both the cases $w(y_1, \ldots, y_p; z_1, \ldots, z_q) = 0$ in $F_m/(F_m \cap T)$ and we derive that $m_{λμ} = 0$. The case $q > r$ is similar. The skew-symmetry arguments and dim $V^* = r < q$ give that the highest weight vector $w$ is equal to 0 in $F_m(ψ)$ and we use (4.1) and (4.3) to show that $w = 0$ in $F_m/(F_m \cap T)$. As a result, in (4.7) and (4.8) the multiplicities $m_{λμ}$ and $n_{λμ}$ are the same if $p, q \leq r$ and both are equal to 0 if $λ r + 1 ≠ 0$ or $μ r + 1 ≠ 0$. Hence the cocharacter sequences of $T$ and $T(ψ)$ are equal and this means that all identities of the mapping $ψ$ are consequences of (4.1)–(4.3). □

**Remark 4.4.** The statement of Theorem 4.3 is a translation of Theorem 3.2 in terms of the mapping $ψ : V \otimes V^* \rightarrow M_r(K)$. Although inspired by Theorem 3.2, our proof does not use it. It is interesting to show that Theorem 4.3 is a straightforward consequence of Theorem 3.2. A possible way is to use a standard trick from the theory of trace identities. Namely, the polynomial $f(x_1, \ldots, x_n)$ from the free algebra $K \langle X \rangle$ is a polynomial identity for $M_r(K)$ if and only if $tr(f(x_1, \ldots, x_n)x_{n+1}) = 0$ is a trace identity. It is easy to see that for any $v_1, v_2, \ldots, v_n \in V$ and $w_1, \ldots, w_n \in V^*$,

$$tr(ψ(v_1, w_1)ψ(v_2, w_2) \cdots ψ(v_n, w_n))$$

$$= tr(ψ(v_1, w_1)) tr(ψ(v_2, w_2)) \cdots tr(ψ(v_n, w_n))$$

$$= ⟨w_1, v_1⟩⟨w_2, v_2⟩ \cdots ⟨w_n, v_n⟩.$$

It seems possible to use this relation and to derive directly Theorem 4.3 from Theorem 3.2.

5. Graded polynomial identities of matrices

In this section we shall fix the elementary $G$-grading of the $r \times r$ matrix algebra with homogeneous components

$$(M_r(K))_e = \begin{pmatrix} M_{r-1}(K) & 0 \\ 0 & K \end{pmatrix},$$

$$(M_r(K))_{h^{-1}g} = \begin{pmatrix} 0 & 0 \\ M_{1 \times (r-1)}(K) & 0 \end{pmatrix},$$

$$(M_r(K))_{g^{-1}h} = \begin{pmatrix} 0 & M_{(r-1) \times 1}(K) \\ 0 & 0 \end{pmatrix}. $$
where $G$ is an arbitrary group and $g, h \in G$ are such that $(g^{-1}h)^2 \neq e$. Our purpose is the complete description of the $G$-graded polynomial identities of $M_r(K)$ depending only on the two kinds of variables $t_i = x_{i,h^{-1}g}$ and $u_i = x_{i,g^{-1}h}$.

**Lemma 5.1.** In the above notation, the $r \times r$ matrix algebra $M_r(K)$ satisfies the graded identities

\begin{align*}
  t_1 t_2 & = 0, \quad u_1 u_2 = 0, \quad (5.1) \\
  (t_1 u_1)(t_2 u_2) & = (t_2 u_2)(t_1 u_1), \quad (5.2) \\
  \sum_{\sigma \in S_r} (\text{sign } \sigma)(u_{\sigma(1)} t_1) \cdots (u_{\sigma(r-1)} t_{r-1}) u_{\sigma(r)} & = 0, \quad (5.3) \\
  \sum_{\tau \in S_r} (\text{sign } \tau)(t_{\tau(1)} u_1) \cdots (t_{\tau(r-1)} u_{r-1}) t_{\tau(r)} & = 0. \quad (5.4)
\end{align*}

**Proof.** Obviously, $((M_r(K))_{g^{-1}h})^2 = ((M_r(K))_{h^{-1}g})^2 = 0$ which gives (5.1). The identity (5.2) follows from the fact that $(M_r(K))_{h^{-1}g} (M_r(K))_{g^{-1}h} = K e_{rr} \cong K$ and the field $K$ satisfies the commutative law. The identities (5.3) and (5.4) express the fact that $\dim (M_r(K))_{g^{-1}h} = \dim (M_r(K))_{h^{-1}g} = r - 1$. □

There are natural isomorphisms $\theta$ between the vector space $V$ with basis $\{e_1, \ldots, e_{r-1}\}$ and the homogeneous component $(M_r(K))_{e^{-1}h} = Ke_{rr} + Ke_{rr} + \cdots + Ke_{r-1,r}$ and $\theta^*$ between the dual space $V^*$ of $V$ with basis $\{e^*_1, \ldots, e^*_{r-1}\}$ and $(M_r(K))_{h^{-1}g} = Ke_{r1} + Ke_{r2} + \cdots + Ke_{r,r-1}$ defined by

\begin{align*}
  \theta(e_i) = e_{ir}, \quad \theta^*(e^*_i) = e_{ri}, \quad i = 1, \ldots, r-1. \quad (5.5)
\end{align*}

Clearly, these isomorphisms agree with the natural mappings $\phi : V^* \otimes V \to K$ and $\psi : V \otimes V^* \to M_{r-1}(K)$ considered in Sections 3 and 4 in the following sense. If we identify $(M_r(K))_e$ with $M_{r-1}(K) \oplus K$, then, with respect to the usual multiplication of $M_r(K)$,

\begin{align*}
  \phi(v^* \otimes w) = \theta^*(v*) \theta(w), \quad \psi(w \otimes v^*) = \theta(w) \theta^*(v^*), \quad v^* \in V^*, \quad w \in V. \quad (5.6)
\end{align*}

We shall use this property of $\theta$ and $\theta^*$ and shall apply the results of Sections 3 and 4 to the graded identities of $M_r(K)$.

**Lemma 5.2.** The following elements depending on the $2(r-1)$ variables $t_1, \ldots, t_{r-1}, u_1, \ldots, u_{r-1}$ only

\begin{align*}
  w_{11} & = (u_{j_1} t_{i_1}) \cdots (u_{j_k} t_{i_k}), \quad w_{22} = (t_{i_1} u_{j_1}) \cdots (t_{i_k} u_{j_k}), \\
  w_{12} & = u_{j_0} (t_{i_1} u_{j_1}) \cdots (t_{i_k} u_{j_k}), \quad w_{21} = (t_{i_1} u_{j_1}) \cdots (t_{i_k} u_{j_k}) t_{i_{k+1}},
\end{align*}
is, j_s = 1, ..., r − 1, are linearly independent modulo the graded polynomial identities of \( M_r(K) \).

**Proof.** Since the evaluations of \( w_{11}, w_{22}, w_{12}, w_{21} \) on \( M_r(K) \) belong respectively to the subspaces of block matrices

\[
\begin{pmatrix}
M_{r-1}(K) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & M_{r-1\times 1}(K) & 0 \\
M_{1\times (r-1)}(K) & 0 & 0 & 0 \\
\end{pmatrix}
\]

we obtain that all monomials \( w_{11}, w_{22}, w_{12}, w_{21} \) are linearly independent modulo the graded identities of \( M_r(K) \) if and only if the monomials of each group are linearly independent. Since \( t_1 w_{12} \) and \( u_1 w_{21} \) are of the form \( w_{11} \), it is sufficient to show that the monomials of the form \( w_{11} \) are linearly independent. We may consider the multiplication \( (M_r(K))_{g^{-1}h} \cdot (M_r(K))_{h^{-1}g} \) as a bilinear mapping \( (M_r(K))_{g^{-1}h} \otimes (M_r(K))_{h^{-1}g} \rightarrow M_{r-1}(K) \). In this way, by (5.6), the vector space isomorphisms \( \theta \) and \( \theta^* \) from (5.5) induce an isomorphism \( V \otimes V^* \cong (M_r(K))_{g^{-1}h} \otimes (M_r(K))_{h^{-1}g} \) which preserves the bilinear mappings \( V \otimes V^* \rightarrow M_{r-1}(K) \) and

\[
(M_r(K))_{h^{-1}g} \otimes (M_r(K))_{g^{-1}h} \rightarrow \begin{pmatrix}
M_{r-1}(K) & 0 \\
0 & 0 \\
\end{pmatrix}.
\]

Now the proof follows from the second part of Lemma 4.2. □

**Theorem 5.3.** Let \( G \) be any group, let \( g, h \in G \) be such that \((g^{-1}h)^2 \neq e\) and let the \( r \times r \) matrix algebra \( M_r(K) \) be equipped with the elementary grading induced by the \( r \)-tuple \((g, ..., g, h)\). The \( G \)-graded polynomial identities of \( M_r(K) \) which depend on the variables \( t_i = x_{i,h^{-1}g} \) and \( u_i = x_{i,g^{-1}h} \) only are consequences of the graded identities (5.1)–(5.4).

**Proof.** We shall work in the factor algebra of the free \( G \)-graded algebra

\[K\langle x_{i,e}, x_{i,h^{-1}g}, x_{i,g^{-1}h} \mid i = 1, 2, \ldots \rangle\]

modulo the \( G \)-graded identities (5.1)–(5.4). We shall consider the subalgebras \( F(T, U) \) and \( F_m(T, U) \) of this factor algebra generated respectively by the sets of variables

\[
\{t_i = x_{i,h^{-1}g}, u_i = x_{i,g^{-1}h} \mid i = 1, 2, \ldots \}, \quad \{t_i, u_i \mid i = 1, \ldots, m\}.
\]

Let us assume that \( M_r(K) \) satisfies some graded identity

\[f(t_1, \ldots, t_p; u_1, \ldots, u_q) = 0 \quad (5.7)\]

which does not follow from (5.1)–(5.4), and hence \( f \neq 0 \) in \( F(T, U) \). We may assume that \( f \) is homogeneous in each variable. By (5.1), we may assume that the monomials of \( f \) in (5.7) do not contain products \( t_i t_j \) and \( u_i u_j \) of two consecutive variables of the same grading. Hence \( f = f_{11} + f_{12} + f_{21} + f_{22} \), where
\[ f_{11} = \sum \alpha_{ij}(u_j t_1) \cdots (u_n t_n), \quad \alpha_{ij} \in K, \]
\[ f_{22} = \sum \beta_{ij}(t_1 u_j) \cdots (t_n u_n), \quad \beta_{ij} \in K, \]
\[ f_{12} = \sum \gamma_{ij} u_j (t_1 u_j) \cdots (t_n u_n), \quad \gamma_{ij} \in K, \]
\[ f_{21} = \sum \delta_{ij}(t_1 u_j) \cdots (t_n u_n) t_{n+1}, \quad \delta_{ij} \in K. \]

As in the proof of Lemma 5.2, the evaluations of \( f_{11}, f_{22}, f_{12}, f_{21} \) on \( M_r(K) \) belong to different blocks of \( M_r(K) \), and since \( f = 0 \) is a graded identity for \( M_r(K) \), we obtain that all components \( f_{ij} = 0 \) are also graded identities. We shall repeat the main steps of the proof of Theorem 4.3. We fix an integer \( m \geq p, q \). Without loss of generality we may assume that each \( f_{ij} \) is a highest weight vector and generates an irreducible \( GL_m \times GL_m \)-module isomorphic to \( W(\lambda) \otimes W(\mu) \) where \( \lambda = (\lambda_1, \ldots, \lambda_p) \) and \( \mu = (\mu_1, \ldots, \mu_q) \) depend on the indices \( i, j \). The case \( p, q \leq r - 1 \) is impossible because implies that, in the notation of Lemma 5.2, some elements of the form \( w_{ij} \) are linearly dependent in \( F_{r-1}(T, U) \), and hence modulo the graded identities of \( M_r(K) \). This contradicts Lemma 5.2. Hence \( \lambda_r \neq 0 \) or \( \mu_r \neq 0 \). First, let \( \lambda_r = 0 \). Hence \( f_{ij} \) is a linear combination of alternating sums
\[ \sum_{\pi \in \lambda_p} \left( \begin{array}{c} \text{sign} \pi \end{array} \right) f_0(t_{\pi(1)} u_{\pi(j)}) f_1(t_{\pi(2)} u_{\pi(j)}) f_2 \cdots f_{p-2}(t_{\pi(p-1)} u_{\pi(j)}) f_{p-1} t_{\pi(p)} f_p, \]
(5.8)

where \( p \geq r \) and all \( f_1, \ldots, f_{p-1} \) are products of some \( t_i u_j \). Using the identity (5.2), we bring (5.8) to the form
\[ f_0 f_1 \cdots f_{p-1} \left( \sum_{\pi \in \lambda_p} \left( \begin{array}{c} \text{sign} \pi \end{array} \right)(t_{\pi(1)} u_{\pi(j)}) (t_{\pi(2)} u_{\pi(j)}) \cdots (t_{\pi(p-1)} u_{\pi(j)}) t_{\pi(p)} \right) f_p, \]
which is equal to zero modulo (5.4). Similarly, if \( \mu_r \neq 0 \), then \( f_{ij} \) is a linear combination of sums of the form
\[ \sum_{\rho \in \lambda_q} \left( \begin{array}{c} \text{sign} \rho \end{array} \right) f_0 u_{\rho(1)} f_1(t_{\rho(2)} u_{\rho(q)}) f_2 \cdots f_{q-1}(t_{\rho(q)} u_{\rho(q)}) f_q, \]
(5.9)

where \( q \geq r \) and \( f_1, \ldots, f_{q-1} \) are products of \( t_i, u_j \). Using again (5.2), we present (5.9) in the form
\[ f_0 \left( \sum_{\rho \in \lambda_q} \left( \begin{array}{c} \text{sign} \rho \end{array} \right) u_{\rho(1)} (t_{\rho(2)} u_{\rho(q)}) \cdots (t_{\rho(q)} u_{\rho(q)}) \right) f_1 \cdots f_{q-1} f_q, \]
which vanishes modulo (5.3). In all the cases the polynomial \( f \) is identically zero in \( F(T, U) \) which is a contradiction. Hence all graded identities of \( M_r(K) \) depending only on the variables \( t_i, u_j, i, j = 1, 2, \ldots \), are consequences of (5.1)–(5.4). \( \square \)
Remark 5.4. The fact that the components \( f_{11} \) and \( f_{22} \) in the proof of Theorem 5.3 are identically zero in \( F(T, U) \) follows also directly from Theorems 3.2 and 4.3. For the proof, let \( f_{22} = 0 \) be a graded identity for \( M_r(K) \). Considering the multiplication \( (M_r(K))_{h-1} \otimes (M_r(K))_{g-1} \) as a bilinear form \( (M_r(K))_{h-1} \otimes (M_r(K))_{g-1} \rightarrow K \), the natural isomorphism \( V^* \otimes V \cong (M_r(K))_{h-1} \otimes (M_r(K))_{g-1} \) preserves the bilinear forms. Since the form \( (M_r(K))_{h-1} \otimes (M_r(K))_{g-1} \rightarrow K \) satisfies the commutative law (5.2), we obtain that

\[
g_{22} = \sum \hat{h}_{ij}(y_{i1}, z_{j1}) \cdots (y_{i_n}, z_{j_n}) = 0
\]

is an identity for the bilinear mapping \( V^* \otimes V \rightarrow K \). By Theorem 3.2 the polynomial \( g_{22} \) belongs to the ideal of \( K[\{y_i, z_j | \ i, j = 1, 2, \ldots \}] \) generated by all polynomials

\[
\sum_{\sigma \in S_r} (\text{sign } \sigma)(y_{\sigma(p_1)}, z_{q_1}) \cdots (y_{\sigma(p_r)}, z_{q_r}),
\]

where the symmetric group \( S_r \) acts on the set \( \{p_1, \ldots, p_r\} \). Hence \( f_{22} \) belongs to the ideal generated by

\[
\sum_{\sigma \in S_r} (\text{sign } \sigma)(t_{\sigma(p_1)}u_{q_1}) \cdots (t_{\sigma(p_r)}u_{q_r})
\]

in the commutative subalgebra generated by all products \( t_iu_j, i, j = 1, 2, \ldots \) in the factor algebra of \( K \{t_i, u_j | \ i, j = 1, 2, \ldots \} \) modulo the identities (5.1) and (5.2). This implies that \( f_{22} = 0 \) is a consequence of (5.1), (5.2) and (5.4). The considerations for the component \( f_{11} \) are similar. We consider the natural isomorphism \( V \otimes V^* \cong (M_r(K))_{g-1} \otimes (M_r(K))_{h-1} \) which preserves the bilinear mappings \( V^* \otimes V \rightarrow M_r-1(K) \) and

\[
(M_r(K))_{h-1} \otimes (M_r(K))_{g-1} \rightarrow \begin{pmatrix} M_r-1(K) & 0 \\ 0 & 0 \end{pmatrix}
\]

The graded identity

\[
(u_1t_1)(u_2t_2)(u_3t_3) = (u_1t_2)(u_3t_1)(u_2t_3)
\]

is a consequence of (5.2) and, by Theorem 4.3, the identity \( f_{11} = 0 \) follows from (5.2)–(5.4) in the same way as

\[
g_{11} = \sum a_{ij}(y_{j1}, z_{i1}) \cdots (y_{jn}, z_{in}) = 0
\]

follows from (4.1)–(4.3).

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