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The Boundary-value Problem in Domains with Very Rapidly Oscillating Boundary

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We study the asymptotic behavior of the solution to boundary-value problem for the second order elliptic equation in the bounded domain $\Omega_\varepsilon \subset R^n$ with a very rapidly oscillating locally periodic boundary. We assume that the Fourier boundary condition involving a small positive parameter ε is posed on the oscillating part of the boundary and that the $(n-1)$-dimensional volume of this part goes to infinity as $\varepsilon \to 0$. Under proper normalization conditions that homogenized problem is found and the estimates of the residual are obtained. Also, we construct an additional term of the asymptotics to improve the estimates of the residual. It is shown that the limiting problem can involve Dirichlet, Fourier or Neumann boundary conditions depending on the structure of the coefficient of the original problem.

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1. INTRODUCTION

The paper is devoted to the boundary-value problems in domains with rapidly oscillating boundaries. In recent years the interest in this kind of problem appears in connection with the development of technologies of porous, composite and other microinhomogeneous materials, and also as a result of various physical experiments. For example, the morphology of contacting surfaces plays an important role in the frictional behavior of deformable bodies. The roughness of the contact surface and the material properties near this surface are microcharacteristics which influence the large scale behavior. The mathematical analysis of these problems based on boundary homogenization was presented in [18], [5], [4], [17] and others. Different boundary homogenization problems were considered in [16], [1], [2], [13], [14], [7].

In the present paper the authors study an elliptic problem with the inhomogeneous Fourier boundary condition in domains with very rapidly oscillating locally periodic boundaries, depending on a small parameter. The peculiarity of the problems considered is the unlimited growth of the $(n-1)$-dimensional volume of the boundary as the small parameter tends to zero, while in the cases studied earlier [2], [4], [17], the $(n-1)$-dimensional volume of the oscillating part of the boundary remained uniformly bounded. Thus, in the case under consideration the height of the “cogs” forming the boundary is much greater than the size of their bases. This leads to the appearance of special normalizing factors both in the coefficients and in the right hand side of boundary operators. Depending on these normalizing factors, different limiting boundary conditions can be obtained.

Also, an additional term of the asymptotic expansion which allows precision of the limiting behavior of the solution and improvement of the estimate of the residual is constructed. This term is defined with the help of a family of auxiliary problems posed in a fixed domain, with nonoscillating boundary conditions. In this context we call it the smooth corrector. Earlier, in a different framework, the corrector and the asymptotic expansion questions were considered in [2], [13].

For simplicity we suppose the oscillating part of the boundary to lie in a small neighborhood of a hyperplane. Without loss of generality one can consider the coordinate hyperplane $\{x \mid x_n = 0\}$.

2. SETTING OF THE PROBLEM

Let Ω be a bounded domain in $\mathbf{R}^n$, $n \geq 2$, with a smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Ω lies in the half-space $x_n > 0$ and $\Gamma_1 \subset \{x : x_n = 0\}$. Given smooth nonpositive 1-periodic in the $\hat{\xi}$ function $F(\hat{x}, \hat{\xi})$, $\hat{x} =$

$(x_1, \dots, x_{n-1})$, $\hat{\xi} = (\xi_1, \dots, \xi_{n-1})$, define the domain Ω_ε as follows: $\partial\Omega_\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_2$, where we set $\Gamma_1^\varepsilon = \{x : \hat{x} \in \Gamma_1,\ x_n = \varepsilon F(\hat{x}, \hat{x}/\varepsilon^\alpha)\}$, $\alpha > 1$. Usually, we will assume $F(\hat{x}, \hat{\xi})$ to be compactly supported on Γ_1 uniformly in $\hat{\xi}$. Consider the following boundary-value problem:

$$\begin{cases} -\Delta u_\varepsilon = f(x) & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} + \varepsilon^\beta p\left(\hat{x}, \dfrac{\hat{x}}{\varepsilon^\alpha}\right) u_\varepsilon = \varepsilon^{\alpha-1} g\left(\hat{x}, \dfrac{\hat{x}}{\varepsilon^\alpha}\right) & \text{on } \Gamma_1^\varepsilon, \\ u_\varepsilon = 0 & \text{on } \Gamma_2, \end{cases} \tag{1}$$

where ν_ε is an outer normal on Γ_1^ε and $p(\hat{x}, \hat{\xi}), g(\hat{x}, \hat{\xi})$ are 1-periodic in $\hat{\xi}$.

Definition 1. The function $u_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_2)$ is a solution of problem (1) if the following integral identity

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon(x) \nabla v(x)\, dx + \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha}\right) u_\varepsilon(x) v(x)\, ds$$

$$= \int_{\Omega_\varepsilon} f(x) v(x)\, dx + \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha}\right) v(x)\, ds$$

holds for all $v \in H^1(\Omega_\varepsilon, \Gamma_2)$. Here we use the standard notation $H^1(\Omega_\varepsilon, \Gamma_2)$ for the closure by the $H^1(\Omega_\varepsilon)$ norm of the set $C^\infty(\overline{\Omega}_\varepsilon)$ of functions vanishing in a neighborhood of Γ_2.

In what follows we study the limiting behavior of solution u_ε of (1) as $\varepsilon \to 0$. It will be shown that the boundary condition of the limit problem depends on the relation between α and β. In particular, the Dirichlet boundary condition as well as the Neumann and the Fourier boundary conditions can be obtained.

3. THE CASE $\beta > \alpha - 1$

In this section we consider the case $\beta > \alpha - 1$. The main result here is the following theorem.

Theorem 1. *Let $\beta > \alpha - 1$, $f \in L_2(\mathbf{R}^n)$ and $F(\hat{x}, \hat{\xi})$, $g(\hat{x}, \hat{\xi})$, $p(\hat{x}, \hat{\xi})$ be periodic in $\hat{\xi}$ smooth functions. Suppose that $F(\hat{x}, \hat{\xi})$ is compactly supported in $x \in \Gamma_1$ uniformly in ξ. Then, for all sufficiently small ε, problem* (1) *does have unique solution and the inequality holds*

$$\|u_0 - u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le K_1\left(\max\left(\varepsilon^{\beta-\alpha+1}, \varepsilon^{\alpha-1}, \sqrt{\varepsilon}\right)\right), \tag{2}$$

where K_1 does not depend on ε, u_0 satisfies the following limiting equations

$$\begin{cases} -\Delta u_0 = f & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial x_n} = G(\hat{x}) & \text{on } \Gamma_1, \\ u_0 = 0 & \text{on } \Gamma_2, \end{cases} \tag{3}$$

and

$$G(\hat{x}) = \int_0^1 \int_0^1 \cdots \int_0^1 \sqrt{|\nabla_{\hat{\xi}} F(\hat{x}, \hat{\xi})|^2}\, g(\hat{x}, \hat{\xi})\, d\hat{\xi}.$$

Remark 1. The definition of solution to problem (3) is analogous to that given for problem (1).

Denote by V an n-dimensional open neighborhood of $(\operatorname{supp} F)$ such that $\operatorname{dist}((V \cap \Gamma_1), \partial\Gamma_1) > 0$; here $\partial\Gamma_1$ means the $(n-2)$-dimensional boundary of Γ_1. It is clear that for sufficiently small ε the oscillating part of Γ_1^ε belongs to V. Due to the smoothness of the boundary $\partial\Omega$ the solution u_0 belongs to $H^2(\Omega \cap V)$ [8], and, hence, can be continued on V to belong to $H^2(V)$ [12].

Preliminary lemma

LEMMA 1. *There exist such constants C_1, C_2 that for any $v \in H^1(\Omega_\varepsilon, \Gamma_2)$ the inequalities*

$$\left\| v\left(\hat{x}, \varepsilon F\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha}\right)\right) - v(\hat{x}, 0) \right\|_{L_2(\Gamma_1)} \le C_1 \sqrt{\varepsilon} \|v\|_{H^1(\Omega_\varepsilon)}, \tag{4}$$

$$\|v\|_{L_2(\Omega_\varepsilon \setminus \Omega)} \le C_2 \sqrt{\varepsilon} \|v\|_{H^1(\Omega_\varepsilon)} \tag{5}$$

take place.

Proof. Without loss of generality we assume $v \in C^\infty(\overline{\Omega}_\varepsilon)$. Note that

$$v(\hat{x}, y) - v(\hat{x}, 0) = \int_0^y \frac{\partial v(\hat{x}, t)}{\partial x_n}\, dt$$

for any $(\hat{x}, y) \in \overline{\Omega_\varepsilon \setminus \Omega}$. Then, we have

$$|v(\hat{x}, y) - v(\hat{x}, 0)|^2 = \left| \int_0^y \frac{\partial v(\hat{x}, t)}{\partial x_n}\, dt \right|^2 \le y \int_0^y \left| \frac{\partial v}{\partial x_n} \right|^2 dt.$$

Substituting $y = \varepsilon F(\hat{x}, \hat{x}/\varepsilon^{\alpha})$ and integrating over Γ_1, we obtain (4). Further, for $(\hat{x}, y) \in \overline{\Omega_\varepsilon \setminus \Omega}$ the following inequality

$$v^2(\hat{x}, y) \le 2v^2(\hat{x}, 0) + 2\left(\int_0^y \frac{\partial v(\hat{x}, t)}{\partial x_n}\, dt\right)^2$$
$$\le 2v^2(\hat{x}, 0) + 2\varepsilon F\left(\hat{x}, \frac{\hat{x}}{\varepsilon^{\alpha}}\right)\int_0^{\varepsilon F(\hat{x}, \hat{x}/\varepsilon^{\alpha})}\left|\frac{\partial v}{\partial x_n}\right|^2 dt$$

holds. Consequently,

$$\int_{\Gamma_1} d\hat{x} \int_0^{\varepsilon F(\hat{x}, \hat{x}/\varepsilon^{\alpha})} v^2(\hat{x}, y)\, dy \le 2\varepsilon \max F \|v\|^2_{L_2(\Gamma_1)}$$
$$+ \varepsilon^2 \max F^2 \int_{\Gamma_1} d\hat{x} \int_0^{\varepsilon F(\hat{x}, \hat{x}/\varepsilon^{\alpha})}\left|\frac{\partial v}{\partial x_n}\right|^2 dt.$$

This implies (5). The lemma is proved.

LEMMA 2. *Let* (ds) *be an element of the* $(n-1)$*-dimensional volume of* Γ_1^{ε}. *Then*

$$ds = \left(\sqrt{1 + \varepsilon^{2-2\alpha}|\nabla_{\hat{\xi}} F(\hat{x}, \hat{\xi})|^2}\,\Big|_{\hat{\xi} = \hat{x}/\varepsilon^{\alpha}}\right) d\hat{x}\,(1 + O(\varepsilon))$$
$$= \varepsilon^{1-\alpha}\left(\sqrt{|\nabla_{\hat{\xi}} F(\hat{x}, \hat{\xi})|^2}\,\Big|_{\hat{\xi} = \hat{x}/\varepsilon^{\alpha}} + O(\varepsilon^{\alpha-1})\right) d\hat{x}. \tag{6}$$

Proof. Due to our assumptions Γ_1^{ε} is defined by the equation

$$x_n - \varepsilon F\left(\hat{x}, \frac{\hat{x}}{\varepsilon^{\alpha}}\right) = 0.$$

Therefore,

$$ds = \sqrt{\left(\varepsilon F'_{x_1} + \varepsilon^{1-\alpha} F'_{\xi_1}\right)^2 + \cdots + \left(\varepsilon F'_{x_{n-1}} + \varepsilon^{1-\alpha} F'_{\xi_{n-1}}\right)^2 + 1}\; d\hat{x},$$

where $\hat{\xi} = \hat{x}/\varepsilon^{\alpha}$. The direct calculations show that

$$\sqrt{\varepsilon^2|\nabla_{\hat{x}} F|^2 + 2\varepsilon^{2-\alpha}\left(\nabla_{\hat{x}} F, \nabla_{\hat{\xi}} F\right) + \varepsilon^{2-2\alpha}|\nabla_{\hat{\xi}} F|^2 + 1} - \sqrt{1 + \varepsilon^{2-2\alpha}|\nabla_{\hat{\xi}} F|^2}$$
$$= \frac{\varepsilon^2|\nabla_{\hat{x}} F|^2 + 2\varepsilon^{2-\alpha}\left(\nabla_{\hat{x}} F, \nabla_{\hat{\xi}} F\right)}{\sqrt{\varepsilon^2|\nabla_{\hat{x}} F|^2 + 2\varepsilon^{2-\alpha}\left(\nabla_{\hat{x}} F, \nabla_{\hat{\xi}} F\right) + \varepsilon^{2-2\alpha}|\nabla_{\hat{\xi}} F|^2 + 1} + \sqrt{1 + \varepsilon^{2-2\alpha}|\nabla_{\hat{\xi}} F|^2}}$$
$$= O(\varepsilon)$$

and

$$\sqrt{\varepsilon^2|\nabla_{\hat{x}}F|^2+2\varepsilon^{2-\alpha}\left(\nabla_{\hat{x}}F,\nabla_{\hat{\xi}}F\right)+\varepsilon^{2-2\alpha}|\nabla_{\hat{\xi}}F|^2+1}-\varepsilon^{1-\alpha}\sqrt{|\nabla_{\hat{\xi}}F(\hat{x},\hat{\xi})|^2}$$
$$=\frac{\varepsilon^2|\nabla_{\hat{x}}F|^2+2\varepsilon^{2-\alpha}\left(\nabla_{\hat{x}}F,\nabla_{\hat{\xi}}F\right)+1}{\sqrt{\varepsilon^2|\nabla_{\hat{x}}F|^2+2\varepsilon^{2-\alpha}\left(\nabla_{\hat{x}}F,\nabla_{\hat{\xi}}F\right)+\varepsilon^{2-2\alpha}|\nabla_{\hat{\xi}}F|^2+1}+\varepsilon^{1-\alpha}\sqrt{|\nabla_{\hat{\xi}}F(\hat{x},\hat{\xi})|^2}}$$
$$\le C(1+O(\varepsilon))=\varepsilon^{1-\alpha}O(\varepsilon^{\alpha-1}),$$

where the constant C is independent of ε.

These inequalities imply (6). The lemma is proved.

PROPOSITION 3. *Uniformly in* $u, v \in H^{1/2}(\Gamma_1)$

$$\left|\int_{\Gamma_1} uv\,d\hat{x}\right| \le C_3\|u\|_{H^{1/2}(\Gamma_1)}\|v\|_{H^{1/2}(\Gamma_1)}. \tag{7}$$

Proof. This inequality is a direct consequence of the Cauchy–Schwartz–Bunyakovskii inequality and the compactness of embedding $L_2(\Gamma_1)$ in $H^{1/2}(\Gamma_1)$.

LEMMA 4. *There exists such a positive constant* M, *independent of* ε, *that*

$$\int_{\Omega_\varepsilon}|\nabla v|^2\,dx+\varepsilon^\beta\int_{\Gamma_1^\varepsilon}p\left(\hat{x},\frac{\hat{x}}{\varepsilon^\alpha}\right)v^2\,ds \ge M\|v\|^2_{H^1(\Omega_\varepsilon)}$$

for any $v \in H^1(\Omega_\varepsilon)$.

Proof. In view of (4) and Lemma 2

$$\begin{aligned}
&\varepsilon^\beta\int_{\Gamma_1^\varepsilon}p\left(\hat{x},\frac{\hat{x}}{\varepsilon^\alpha}\right)v^2\left(x,\varepsilon F\left(\hat{x},\frac{\hat{x}}{\varepsilon^\alpha}\right)\right)ds\\
&\qquad\le c\varepsilon^\beta\varepsilon^{\alpha-1}\int_{\Gamma_1^\varepsilon}v^2\left(x,\varepsilon F\left(\hat{x},\frac{\hat{x}}{\varepsilon^\alpha}\right)\right)d\hat{x}\\
&\qquad\le c'\left(\varepsilon^{\beta-\alpha+1}\|v\|^2_{L_2(\Gamma_1)}+\varepsilon^{\beta-\alpha+3/2}\|v\|^2_{H^1(\Omega_\varepsilon)}\right)\\
&\qquad\le c''\varepsilon^{\beta-\alpha+1}\|v\|^2_{H^1(\Omega_\varepsilon)}
\end{aligned}$$

Then, the assertion of the lemma is a consequence of the Friedrichs type inequality (see [10]).

LEMMA 5. *Let $h(\hat{x}, \hat{\xi})$ be* 1*-periodic in the ξ Lipschitz function such that*

$$\int_0^1 \int_0^1 \cdots \int_0^1 h(\hat{x}, \hat{\xi})\, d\hat{\xi} \equiv 0. \tag{8}$$

Then, an inequality

$$\left| \int_{\Gamma_1} h\left(\hat{x}, \frac{\hat{x}}{\varepsilon^{\alpha}} \right) u(\hat{x}) v(\hat{x})\, d\hat{x} \right| \le C_4 \sqrt{\varepsilon^{\alpha}} \|u\|_{H^{1/2}(\Gamma_1)} \|v\|_{H^{1/2}(\Gamma_1)} \tag{9}$$

is satisfied, C_4 depending only on the Lipschitz constant of h.

Remark 2. This statement for smooth $h(\hat{x}, \hat{x}/\varepsilon^{\alpha})$ and $n = 2$ was proved in [2] and, independently, in [3]. Here we use the technique developed in [2].

Proof. In view of (8) an equation

$$\Delta_{\hat{\xi}} \Psi(\hat{x}, \hat{\xi}) = h(\hat{x}, \hat{\xi})$$

is solvable in the space of 1-periodic in $\hat{\xi}$ functions. The solution is unique up to an additive constant.

Suppose, initially, that $v(\hat{x}, 0)$ and $u(\hat{x}, 0)$ belong to $\mathring{H}^1(\Gamma_1)$ and $h(\hat{x}, \hat{\xi})$ is a C^1 function. Then, substituting $h(\hat{x}, \hat{\xi}) = \Delta_{\hat{\xi}} \Psi(\hat{x}, \hat{\xi})$, integrating by parts, and using the Cauchy–Schwartz–Bunyakovskii inequality, we have

$$\begin{aligned}
&\left| \int_{\Gamma_1} h\left(\hat{x}, \frac{\hat{x}}{\varepsilon^{\alpha}} \right) u(\hat{x}) v(\hat{x})\, d\hat{x} \right| \\
&\quad = \left| \int_{\Gamma_1} \left(\Delta_{\hat{\xi}} \Psi(\hat{x}, \hat{\xi}) \right) \Big|_{\hat{\xi} = \hat{x}/\varepsilon^{\alpha}} u(\hat{x}) v(\hat{x})\, d\hat{x} \right| \\
&\quad = \left| \int_{\Gamma_1} \left(\varepsilon^{\alpha} \nabla_{\hat{x}} \left[\nabla_{\hat{\xi}} \Psi(\hat{x}, \hat{\xi}) \Big|_{\hat{\xi} = \hat{x}/\varepsilon^{\alpha}} \right] - \varepsilon^{\alpha} \left((\nabla_{\hat{x}}, \nabla_{\hat{\xi}}) \Psi(\hat{x}, \hat{\xi}) \right) \Big|_{\hat{\xi} = \hat{x}/\varepsilon^{\alpha}} \right) d\hat{x} \right| \\
&\quad \le \varepsilon^{\alpha} \left| \int_{\Gamma_1} \left(\nabla_{\hat{\xi}} \Psi(\hat{x}, \xi) \Big|_{\hat{\xi} = \hat{x}/\varepsilon^{\alpha}}, \nabla_{\hat{x}}(uv) \right) d\hat{x} \right| \\
&\qquad + \varepsilon^{\alpha} \left| \int_{\Gamma_1} \left((\nabla_{\hat{x}}, \nabla_{\hat{\xi}}) \Psi(\hat{x}, \hat{\xi}) \right) \Big|_{\hat{\xi} = \hat{x}/\varepsilon^{\alpha}} uv\, d\hat{x} \right| \\
&\quad \le C_6 \varepsilon^{\alpha} \|u\|_{H^1(\Gamma_1)} \|v\|_{H^1(\Gamma_1)}.
\end{aligned} \tag{10}$$

Now, approximating arbitrary Lipschitz function $h(\hat{x}, \hat{\xi})$ by a sequence of C^1 functions we extend the last inequality to the space of Lipschitz functions. Clearly, C_6 depends only on Lipschitz constant.

Let $p \in [0, 1]$. On the base of a bilinear form

$$\int_{\Gamma_1} h\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha}\right) u(\hat{x}) v(\hat{x})\, d\hat{x}, \quad u, v \in H^p(\Gamma_1)$$

we define an operator $T_\varepsilon : H^p(\Gamma_1) \to H^{-p}(\Gamma_1)$ by the formula

$$\langle T_\varepsilon u, v\rangle_p = \int_{\Gamma_1} h\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha}\right) u(\hat{x}, 0) v(\hat{x}, 0)\, d\hat{x}.$$

From (10) it follows

$$\|T_\varepsilon\|_{L(H^1(\partial\Omega), H^{-1}(\partial\Omega))} \le C_6 \varepsilon^\alpha.$$

Then, clearly

$$\|T_\varepsilon\|_{L(L_2(\partial\Omega), L_2(\partial\Omega))} \le C_7$$

with the constant, independent of ε.

Using the space interpolation technique ([9], [15]) we derive from the above estimates

$$\|T_\varepsilon\|_{L(H^p(\partial\Omega), H^{-p}(\partial\Omega))} \le C_7^{1-p} C_6^p \varepsilon^{p\alpha}. \tag{11}$$

Setting $p = \frac{1}{2}$ in (11), we obtain (9). The lemma is proved.

LEMMA 6. *There exists $C_8 > 0$, such that for all $v \in H^1(\Omega_\varepsilon)$*

$$\left| \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha-1} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha}\right) v\left(\hat{x}, \varepsilon F\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha}\right)\right) ds - \int_{\Gamma_1} G(\hat{x}) v(\hat{x}, 0)\, d\hat{x} \right|$$
$$\le C_8\left(\varepsilon^{\alpha-1} + \sqrt{\varepsilon}\right) \|v\|_{H^1(\Omega_\varepsilon)}. \tag{12}$$

Proof. According to Lemma 2 we have

$$
\begin{aligned}
&\left| \int_{\Gamma_1^{\varepsilon}} \varepsilon^{\alpha-1} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon^{\alpha}}\right) v\left(\hat{x}, \varepsilon F\left(\hat{x}, \frac{\hat{x}}{\varepsilon^{\alpha}}\right)\right) ds - \int_{\Gamma_1} G(\hat{x}) v(\hat{x}, 0)\, d\hat{x} \right| \\
&\quad = \left| \int_{\Gamma_1} \varepsilon^{\alpha-1} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon^{\alpha}}\right) v\left(\hat{x}, \varepsilon F\left(\hat{x}, \frac{\hat{x}}{\varepsilon^{\alpha}}\right)\right) \right. \\
&\qquad \times \varepsilon^{1-\alpha} \left(\sqrt{\left| \nabla_{\hat{\xi}} F\left(\hat{x}, \frac{\hat{x}}{\varepsilon^{\alpha}}\right) \right|^2} + O(\varepsilon^{\alpha-1}) \right) d\hat{x} \\
&\qquad \left. - \int_{\Gamma_1} G(\hat{x}) v(\hat{x}, 0)\, d\hat{x} \right| \\
&\quad \leq \left| \int_{\Gamma_1} \left(g\left(\hat{x}, \frac{\hat{x}}{\varepsilon^{\alpha}}\right) v\left(\hat{x}, \varepsilon F\left(\hat{x}, \frac{\hat{x}}{\varepsilon^{\alpha}}\right)\right) \sqrt{\left| \nabla_{\hat{\xi}} F\left(\hat{x}, \frac{\hat{x}}{\varepsilon^{\alpha}}\right) \right|^2} \right. \right. \\
&\qquad\qquad \left. \left. - g\left(\hat{x}, \frac{\hat{x}}{\varepsilon^{\alpha}}\right) v(\hat{x}, 0) \sqrt{\left| \nabla_{\hat{\xi}} F\left(\hat{x}, \frac{\hat{x}}{\varepsilon^{\alpha}}\right) \right|^2} \right) d\hat{x} \right| \\
&\qquad + \left| \int_{\Gamma_1} \left(g\left(\hat{x}, \frac{\hat{x}}{\varepsilon^{\alpha}}\right) v(\hat{x}, 0) \sqrt{\left| \nabla_{\hat{\xi}} F\left(\hat{x}, \frac{\hat{x}}{\varepsilon^{\alpha}}\right) \right|^2} - G(\hat{x}) v(\hat{x}, 0) \right) d\hat{x} \right| \\
&\qquad + C_9 \varepsilon^{\alpha-1} \|v\|_{H^1(\Omega_{\varepsilon})}. \qquad (13)
\end{aligned}
$$

Set $h(\hat{x}, \hat{\xi}) = \sqrt{|\nabla_{\hat{\xi}} F(\hat{x}, \hat{\xi})|^2}\, g(\hat{x}, \hat{\xi}) - G(\hat{x})$. Then, by the definition of $G(\hat{x})$

$$\int_0^1 \int_0^1 \cdots \int_0^1 h(\hat{x}, \hat{\xi})\, d\xi \equiv 0.$$

Hence, the inequality (12) follows if we estimate the first integral in the right-hand side by means of Lemma 1 and the second one by means of Lemma 5.

Proof of Theorem 1. Due to Lemma 4 the existence and the uniqueness of solution to problem (1) can be obtained on the base of the Lax-Milgram lemma [19].

Then, after simple transformations we find

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla(u_0 - u_\varepsilon)\nabla v\,dx + \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p(u_0 - u_\varepsilon)v\,ds \\
&\quad = \int_{\Omega_\varepsilon} \nabla u_0 \nabla v\,dx - \int_{\Omega_\varepsilon} fv\,dx - \varepsilon^{\alpha-1}\int_{\Gamma_1^\varepsilon} gv\,ds + \varepsilon^\beta \int_{\Gamma_1^\varepsilon} pu_0 v\,ds \\
&\quad = \int_{\Omega} \nabla u_0 \nabla v\,dx - \int_{\Omega_\varepsilon} fv\,dx - \varepsilon^{\alpha-1}\int_{\Gamma_1^\varepsilon} gv\,ds \\
&\qquad + \int_{\Omega_\varepsilon\setminus\Omega} \nabla u_0 \nabla v\,dx + \varepsilon^\beta \int_{\Gamma_1^\varepsilon} pu_0 v\,ds \\
&\quad = \int_{\Omega_\varepsilon\setminus\Omega} \nabla u_0 \nabla v\,dx - \varepsilon^{\alpha-1}\int_{\Gamma_1^\varepsilon} gv\,ds + \int_{\Gamma_1} Gv\,d\hat{x} \\
&\qquad - \int_{\Omega_\varepsilon\setminus\Omega} fv\,dx + \varepsilon^\beta \int_{\Gamma_1^\varepsilon} pu_0 v\,ds. \qquad (14)
\end{aligned}$$

According to Lemma 2 and Proposition 3 the last integral in the right-hand side of (14) is estimated as follows

$$\begin{aligned}
\varepsilon^\beta \left| \int_{\Gamma_1^\varepsilon} pu_0 v\,ds \right| &= \varepsilon^\beta \left| \int_{\Gamma_1} pu_0 v \left[\varepsilon^{1-\alpha} \left(\sqrt{\left| \nabla_{\hat{\xi}} F\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha} \right) \right|^2} + O(\varepsilon^{\alpha-1}) \right) \right] d\hat{x} \right| \\
&\le \varepsilon^{\beta-\alpha+1} C_{10} \left| \int_{\Gamma_1} pu_0 v\,d\hat{x} \right| \le \varepsilon^{\beta-\alpha+1} C_{10} \|v\|_{H^{1/2}(\Gamma_1)} \\
&\le \varepsilon^{\beta-\alpha+1} C_{11} \|v\|_{H^1(\Omega_\varepsilon)}.
\end{aligned}$$

By (5) considering the uniform boundedness of $\|u_0\|_{H^2(\Omega_\varepsilon)}$, we have

$$\left| \int_{\Omega_\varepsilon\setminus\Omega} \nabla u_0 \nabla v\,dx \right| \le \|\nabla u_0\|_{L_2(\Omega_\varepsilon\setminus\Omega)} \|v\|_{H^1(\Omega_\varepsilon)} \le C_2\sqrt{\varepsilon}\, \|u_0\|_{H^2(\Omega_\varepsilon)} \|v\|_{H^1(\Omega_\varepsilon)}$$

and

$$\left| \int_{\Omega_\varepsilon\setminus\Omega} fv\,dx \right| \le \|f\|_{L_2(\Omega_\varepsilon\setminus\Omega)} \|v\|_{L_2(\Omega_\varepsilon\setminus\Omega)} \le C_2\sqrt{\varepsilon}\, \|f\|_{L_2(\Omega_\varepsilon)} \|v\|_{H^1(\Omega_\varepsilon)}.$$

Then, Lemma 6 implies

$$\left| \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} gv\, ds - \int_{\Gamma_1} Gv\, d\hat{x} \right| \le C_8 \left(\varepsilon^{\alpha-1} + \sqrt{\varepsilon} \right) \|v\|_{H^1(\Omega_\varepsilon)}.$$

Combining these inequalities with (14) we deduce

$$\left| \int_{\Omega_\varepsilon} \nabla(u_0 - u_\varepsilon) \nabla v\, dx \right| + \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p(u_0 - u_\varepsilon) v\, ds$$
$$\le K_1 \|v\|_{H^1(\Omega_\varepsilon)} \left(\sqrt{\varepsilon} + \varepsilon^{\alpha-1} + \varepsilon^{\beta-\alpha+1} \right).$$

It remains to substitute $v = u_0 - u_\varepsilon$. Then, (2) follows from Lemma 4 and the Friedrichs type inequality (see, for example, [10]). The theorem is proved.

4. THE CASE $\beta = \alpha - 1$

In this section we study problem (1) in the case $\beta = \alpha - 1$. We show that the limit problem has an inhomogeneous Fourier boundary condition.

THEOREM 2. *Let* $\beta = \alpha - 1$, f *belong to* $L_2(\mathbf{R}^n)$ *and* $F(\hat{x}, \hat{\xi})$, $p(\hat{x}, \hat{\xi})$, $g(\hat{x}, \hat{\xi})$ *be periodic in* ξ *smooth functions. Suppose that* $F(\hat{x}, \hat{\xi})$ *is compactly supported in* $x \in \Gamma_1$ *uniformly in* ξ *and that the function*

$$P(\hat{x}) = \int_0^1 \int_0^1 \cdots \int_0^1 \sqrt{|\nabla_{\hat{\xi}} F(\hat{x}, \hat{\xi})|^2}\, p(\hat{x}, \hat{\xi})\, d\hat{\xi}$$

is non-negative. Then, for all sufficiently small $\varepsilon > 0$ *the existence and the uniqueness of solution to problem* (1) *follow and*

$$\|u_0 - u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le K_2 \left(\sqrt{\varepsilon} + \varepsilon^{\alpha-1} \right), \tag{15}$$

where K_2 *does not depend on* ε, $u_0(x)$ *satisfies the following limiting equation*

$$\begin{cases} -\Delta u_0 = f & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial x_n} + P(\hat{x}) u_0 = G(\hat{x}) & \text{on } \Gamma_1, \\ u_0 = 0 & \text{on } \Gamma_2, \end{cases} \tag{16}$$

and $G(\hat{x})$ *was defined in Theorem* 1.

Remark 3. The definition of solution to problem (16) is analogous to that given for problem (1). Problem (16) is well-posed because of the assumptions of the theorem.

Due to the smoothness of the boundary $\partial\Omega$, the solution u_0 belongs to $H^2(\Omega \cap V)$ [8], and, hence, can be continued on V to belong to $H^2(V)$ [12].

LEMMA 7. *There exists* $C_{12} > 0$, *such that*

$$\left| \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha-1} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha}\right) u_0\left(\hat{x}, \varepsilon F\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha}\right)\right) v\left(\hat{x}, \varepsilon F\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha}\right)\right) ds - \int_{\Gamma_1} P(\hat{x}) u_0(\hat{x}, 0) v(\hat{x}, 0)\, d\hat{x} \right|$$

$$\leq C_{12}\left(\varepsilon^{\alpha-1} + \sqrt{\varepsilon}\right) \|u_0\|_{H^1(\Omega_\varepsilon)} \|v\|_{H^1(\Omega_\varepsilon)}. \tag{17}$$

for all $u, v \in H^1(\Omega_\varepsilon)$.

The proof is similar to that of Lemma 6.

Proof of Theorem 2. The existence and uniqueness of u_ε are due to the positiveness of $P(\hat{x})$, Lemma 7 and the Lax–Milgram lemma.

Then, according to (1) and (16)

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla(u_0 - u_\varepsilon)\nabla v\, dx + \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} p(u_0 - u_\varepsilon) v\, ds \\
&\quad = \int_{\Omega_\varepsilon} \nabla u_0 \nabla v\, dx - \int_{\Omega_\varepsilon} fv\, dx - \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} gv\, ds + \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} pu_0 v\, ds \\
&\quad = \int_{\Omega} \nabla u_0 \nabla v\, dx - \int_{\Omega_\varepsilon} fv\, dx - \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} gv\, ds \\
&\qquad + \int_{\Omega_\varepsilon \setminus \Omega} \nabla u_0 \nabla v\, dx + \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} pu_0 v\, ds \\
&\quad = \int_{\Omega_\varepsilon \setminus \Omega} \nabla u_0 \nabla v\, dx - \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} gv\, ds + \int_{\Gamma_1} G(\hat{x}) v\, d\hat{x} - \int_{\Omega_\varepsilon \setminus \Omega} fv\, dx \\
&\qquad + \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} pu_0 v\, ds - \int_{\Gamma_1} P(\hat{x}) u_0 v\, d\hat{x}.
\end{aligned}$$

Let us estimate all the terms in the right hand side of the last relation. By (5) considering the smoothness of u_0, we have

$$\left| \int_{\Omega_\varepsilon \setminus \Omega} \nabla u_0 \nabla v\, dx \right| \leq \|\nabla u_0\|_{L^2(\Omega_\varepsilon \setminus \Omega)} \|v\|_{H^1(\Omega_\varepsilon)} \leq C_2 \sqrt{\varepsilon} \|u_0\|_{H^2(\Omega_\varepsilon)} \|v\|_{H^1(\Omega_\varepsilon)}$$

and

$$\left|\int_{\Omega_\varepsilon\setminus\Omega} fv\,dx\right| \le \|f\|_{L_2(\Omega_\varepsilon\setminus\Omega)}\|v\|_{L_2(\Omega_\varepsilon\setminus\Omega)} \le C_2\sqrt{\varepsilon}\|f\|_{L_2(\Omega_\varepsilon)}\|v\|_{H^1(\Omega_\varepsilon)}.$$

Then, according to Lemma 6 and 7 the inequalities

$$\left|\varepsilon^{\alpha-1}\int_{\Gamma_1^\varepsilon} gv\,ds - \int_{\Gamma_1} Gv\,d\hat{x}\right| \le C_8\left(\varepsilon^{\alpha-1}+\sqrt{\varepsilon}\right)\|v\|_{H^1(\Omega_\varepsilon)}$$

and

$$\left|\varepsilon^{\alpha-1}\int_{\Gamma_1^\varepsilon} pu_0v\,ds - \int_{\Gamma_1} Pu_0v\,d\hat{x}\right| \le C_{12}\left(\varepsilon^{\alpha-1}+\sqrt{\varepsilon}\right)\|u_0\|_{H^1(\Omega_\varepsilon)}\|v\|_{H^1(\Omega_\varepsilon)}$$

hold. With the help of these inequalities we obtain

$$\left|\int_{\Omega_\varepsilon}\nabla(u_0-u_\varepsilon)\nabla v\,dx + \int_{\Gamma_1^\varepsilon} p(u_0-u_\varepsilon)v\,ds\right| \le K_2\|v\|_{H^1(\Omega_\varepsilon)}\left(\sqrt{\varepsilon}+\varepsilon^{\alpha-1}\right).$$

Substituting $v = u_0 - u_\varepsilon$ and using Lemma 7 and the Friedrichs type inequality (see [10]), we obtain (15). The theorem is proved.

5. THE CASE $\beta < \alpha - 1$

This section is devoted to the case of the Dirichlet limiting problem. Here we suppose the function $P(\hat{x})$ to be uniformly positive on Γ_1. This means, in particular, that $F(\hat{x},\hat{\xi})$ is not compactly supported in $x \in \Gamma_1$.

THEOREM 3. *Let $\beta < \alpha - 1$, f belong to $L_2(\mathbf{R}^n)$ and $F(\hat{x},\hat{\xi})$, $p(\hat{x},\hat{\xi})$, $g(\hat{x},\hat{\xi})$ be periodic in ξ smooth functions. Assume $P(\hat{x}) \ge C_0 > 0$ on Γ_1 and, also, at least one of the following conditions, $p(\hat{x},\hat{\xi}) \ge 0$ or $\beta > \alpha - 2$, to be satisfied. Then, for each sufficiently small $\varepsilon > 0$, problem* (1) *does have a unique solution. The family of solutions is uniformly bounded in the $H^1(\Omega_\varepsilon)$-norm and an estimate*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon)} \le K_3\left(\varepsilon^{\alpha-\beta-1}+\sqrt{\varepsilon}\right)^{1/2} \tag{18}$$

holds, where u_0 satisfies the following Dirichlet problem

$$\begin{cases} -\Delta u_0 = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{19}$$

If $\beta > \alpha - 2$ *then, in addition,*

$$\|u_\varepsilon - u_0\|_{H^1(\Omega_\varepsilon)} \le K_3\left(\varepsilon^{(\beta-\alpha+2)/2} + \varepsilon^{(\alpha-\beta-1)/2} + \varepsilon^{1/4}\right)^{1/2}, \quad (20)$$

In both inequalities above the constant K_3 *is independent of* ε.

Remark 4. In general one has only weak convergence of u_ε in $H^1(\Omega_\varepsilon)$.
Due to the smoothness of the boundary $\partial\Omega$ the solution u_0 belongs to $H^2(\Omega)$ [8], and, hence, can be continued on $\mathbf{R}^n$ to belong to $H^2(\mathbf{R}^n)$ [12].

Proof of Theorem 3. In the case of positive $p(\hat{x}, \hat{\xi})$ the existence, uniqueness and uniform boundedness of u_ε follow directly from standard energy estimates and the Lax–Milgram Lemma (see [11]). Otherwise, they are based on the following statement. Denote

$$P_1(\varepsilon, \hat{x}) = \int_0^1 \int_0^1 \cdots \int_0^1 \sqrt{\varepsilon^{2\alpha-2} + |\nabla_{\hat{\xi}} F(\hat{x}, \hat{\xi})|^2}\, p(\hat{x}, \hat{\xi})\, d\hat{\xi}.$$

LEMMA 8. *For all* $u, v \in H^1(\Omega_\varepsilon)$

$$\begin{aligned}
&\left| \int_{\Gamma_1} P_1(\varepsilon, \hat{x}) u(x) v(x)\, d\hat{x} - \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha}\right) u(x) v(x)\, ds \right| \\
&\qquad \le C_1 \sqrt{\varepsilon} \left(\|u\|_{H^1(\Omega_\varepsilon)} \|v\|_{L_2(\Gamma_1)} + \|u\|_{L_2(\Gamma_1)} \|v\|_{H^1(\Omega_\varepsilon)} \right) \\
&\qquad\quad + \varepsilon \|u\|_{H^1(\Omega_\varepsilon)} \|v\|_{H^1(\Omega_\varepsilon)} \\
&\qquad \le C_1 \sqrt{\varepsilon} \|u\|_{H^1(\Omega_\varepsilon)} \|v\|_{H^1(\Omega_\varepsilon)}
\end{aligned}$$

where C_1 *does not depend on* ε.

Proof. The lemma can be proved in the same way as Lemma 6.
Now, let us transform the integral identity for the solution of problem (1) as follows

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla u_\varepsilon(x) \nabla v(x)\, dx + \varepsilon^{\beta-\alpha+1} \int_{\Gamma_1} P_1(\varepsilon, \hat{x}) u_\varepsilon(x) v(x)\, d\hat{x} \\
&\qquad = \int_{\Omega_\varepsilon} f(x) v(x)\, dx + \left(\varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha}\right) v(x)\, ds - \int_{\Gamma_1} G(\hat{x}) v(x)\, d\hat{x} \right) \\
&\qquad\quad + \int_{\Gamma_1} G(\hat{x}) v(x)\, d\hat{x} + \left(\varepsilon^{\beta-\alpha+1} \int_{\Gamma_1} P_1(\varepsilon, \hat{x}) u_\varepsilon(x) v(x)\, d\hat{x} \right. \\
&\qquad\qquad \left. - \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha}\right) u_\varepsilon(x) v(x)\, ds \right). \quad (21)
\end{aligned}$$

Substituting here $v = u_\varepsilon$ and estimating all the terms in the right hand side with the help of Lemma 6, Lemma 8, and the Cauchy–Schwartz–Bunyakovskii inequality we find

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^2\,dx + \varepsilon^{\beta-\alpha+1}\int_{\Gamma_1} P_1(\varepsilon,\hat{x})u_\varepsilon^2(x)\,d\hat{x}$$
$$\le \|f\|_{L_2(\Omega_\varepsilon)}\|u_\varepsilon\|_{L_2(\Omega_\varepsilon)}$$
$$+ C\Big(\big(\varepsilon^{\alpha-1} + \sqrt{\varepsilon}\big)\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} + \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}$$
$$+\varepsilon^{\beta-\alpha+3/2}\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}\|u_\varepsilon\|_{L_2(T_1)} + \varepsilon^{\beta-\alpha+2}\|u_\varepsilon\|^2_{H^1(\Omega_\varepsilon)}\Big). \quad (22)$$

Next, considering an evident estimate $|P_1(\varepsilon,\hat{x}) - P(\hat{x})| \le C\varepsilon^{\alpha-1}$ it is easy to see that under conditions of the theorem

$$P_1(\varepsilon,\hat{x}) \ge C_0' > 0. \quad (23)$$

Thus, under the assumption $\beta > \alpha - 2$ we have for all sufficiently small ε

$$C\Big(\varepsilon^{\beta-\alpha+3/2}\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}\|u_\varepsilon\|_{L_2(\Gamma_1)} + \varepsilon^{\beta-\alpha+2}\|u_\varepsilon\|^2_{H^1(\Omega_\varepsilon)}\Big)$$
$$\le \frac{1}{2}\int_{\Omega_\varepsilon}|\nabla u_\varepsilon(x)|^2\,dx + \varepsilon^{\beta-\alpha+1}\int_{\Gamma_1} P_1(\varepsilon,\hat{x})u_\varepsilon^2(x)\,d\hat{x};$$

the constant C here is taken from (22). Then, the existence and the uniqueness of the solution to problem (1) as well as its uniform boundedness in the $H^1(\Omega_\varepsilon)$-norm follow from (21), (22) and (23) by standard arguments [11].

In order to prove (18) let us divide (22) by $\varepsilon^{\beta-\alpha+1}$:

$$\varepsilon^{\alpha-\beta-1}\int_{\Omega_\varepsilon}|\nabla u_\varepsilon(x)|^2\,dx + \int_{\Gamma_1} P_1(\hat{x})u_\varepsilon^2(x)\,d\hat{x}$$
$$\le \varepsilon^{\alpha-\beta-1}\|f\|_{L_2(\Omega_\varepsilon)}\|u_\varepsilon\|_{L_2(\Omega_\varepsilon)}$$
$$+ C\Big(\big(\varepsilon^{2\alpha-\beta-2} + \varepsilon^{\alpha-\beta-1/2}\big)\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}$$
$$+\varepsilon^{\alpha-\beta-1}\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} + \sqrt{\varepsilon}\|u_\varepsilon\|^2_{H^1(\Omega_\varepsilon)}\Big). \quad (24)$$

From this relation taking into account (23) and the uniform boundedness of $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}$, we derive

$$\|u_\varepsilon\|_{L_2(\Gamma_1)} \le \big(\varepsilon^{\alpha-\beta-1} + \sqrt{\varepsilon}\big)^{1/2}. \quad (25)$$

Since the family $\{u_\varepsilon\}$ is uniformly bounded in $H^1(\Omega_\varepsilon)$, it is compact in $H^{1/2}(\Omega)$. Consider arbitrary convergent subsequence u_{ε_k}, $\varepsilon_k \to 0$. It is evident that the limiting function $u'(x)$ satisfies the equation $\Delta u' = f$ and, in view of (25), the boundary condition $u'|_{\Gamma_1} = 0$. Hence, $u'(x) = u_0(x)$. Then, the whole family $\{u_\varepsilon\}$ does converge and the estimate (18) holds.

The estimate (20) is based on the following transformations

$$
\begin{aligned}
\int_{\Omega_\varepsilon} &|\nabla(u_\varepsilon - u_0)|^2\,dx \\
&= \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla(u_\varepsilon - u_0)\,dx \\
&\quad - \int_{\Omega} \nabla u_0 \nabla(u_\varepsilon - u_0)\,dx - \int_{\Omega_\varepsilon\setminus\Omega} \nabla u_0 \nabla(u_\varepsilon - u_0)\,dx \\
&= \int_{\Omega_\varepsilon} f(u_\varepsilon - u_0)\,dx - \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p\left(x, \frac{\hat{x}}{\varepsilon^\alpha}\right) u_\varepsilon (u_\varepsilon - u_0)\,ds \\
&\quad - \int_{\Omega} f(u_\varepsilon - u_0)\,dx + \int_{\Gamma_1} \frac{\partial}{\partial x_n} u_0 (u_\varepsilon - u_0)\,d\hat{x} \\
&\quad - \int_{\Omega_\varepsilon\setminus\Omega} \nabla u_0 \nabla(u_\varepsilon - u_0)\,dx + \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} g\left(x, \frac{\hat{x}}{\varepsilon^\alpha}\right)(u_\varepsilon - u_0)\,ds \\
&= \int_{\Omega_\varepsilon\setminus\Omega} f(u_\varepsilon - u_0)\,dx - \int_{\Omega_\varepsilon\setminus\Omega} \nabla u_0 \nabla(u_\varepsilon - u_0)\,dx \\
&\quad - \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p\left(x, \frac{\hat{x}}{\varepsilon^\alpha}\right) u_\varepsilon^2\,ds + \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} g\left(x, \frac{\hat{x}}{\varepsilon^\alpha}\right)(u_\varepsilon - u_0)\,ds \\
&\quad + \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p\left(x, \frac{\hat{x}}{\varepsilon^\alpha}\right) u_\varepsilon u_0\,ds + \int_{\Gamma_1} u_\varepsilon \frac{\partial}{\partial x_n} u_0\,d\hat{x}. \qquad (26)
\end{aligned}
$$

In further considerations we use the following statement that can be proved in the same way as Lemma 1.

PROPOSITION 9. *Let $u(x)$ belong to $H^2(\mathbf{R}^n)$. Then,*

$$
\left\| u\left(\hat{x}, \varepsilon F\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha}\right)\right) - u(\hat{x}, 0) \right\|_{L_2(\Gamma_1)} \le C_5 \varepsilon \|u\|_{H^2(\Omega_\varepsilon)}.
$$

Now, in case of non-negative $p(\hat{x}, \hat{\xi})$ we deduce from (26) by means of (18), Lemma 1, Lemma 2 and Proposition 9

$$\int_{\Omega_\varepsilon} |\nabla(u_\varepsilon - u_0)|^2\, dx + \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p\left(x, \frac{\hat{x}}{\varepsilon^\alpha}\right) u_\varepsilon^2\, ds$$
$$\le C\Big(\sqrt{\varepsilon}\|f\|_{L_2(\Omega_\varepsilon)}\|u_\varepsilon - u_0\|_{H^1(\Omega_\varepsilon)} + \sqrt{\varepsilon}\|u_0\|_{H^2(\mathbf{R}^n)}\|u_\varepsilon - u_0\|_{H^1(\Omega_\varepsilon)}$$
$$+ \varepsilon^{\beta-\alpha+1}\varepsilon\|u_0\|_{H^2(\mathbf{R}^n)}\big(\varepsilon^{(\alpha-\beta-1)/2} + \varepsilon^{1/4}\big)\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}$$
$$+ \big(\varepsilon^{(\alpha-\beta-1)/2} + \varepsilon^{1/4}\big)\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} + \varepsilon\|u_0\|_{H^2(\mathbf{R}^n)}\Big).$$

Since $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ is uniformly bounded, this implies

$$\int_{\Omega_\varepsilon} |\nabla(u_\varepsilon - u_0)|^2\, dx \le C\big(\varepsilon^{1/4} + \varepsilon^{(\alpha-\beta-1)/2} + \varepsilon^{(\beta-\alpha+3)/2} - \varepsilon^{\beta-\alpha+9/4}\big)$$

and (20) follows.

In the case $\beta > \alpha - 2$ we start with the following identity

$$\int_{\Omega_\varepsilon} |\nabla(u_0 - u_\varepsilon)|^2\, dx + \varepsilon^{\beta-\alpha+1} \int_{\Gamma_1} P_1(\varepsilon, \hat{x})(u_0 - u_\varepsilon)^2\, d\hat{x}$$
$$= \int_{\Omega_\varepsilon} \nabla u_0 \nabla(u_0 - u_\varepsilon)\, dx - \int_{\Omega_\varepsilon} f(u_0 - u_\varepsilon)\, dx$$
$$- \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} g(u_0 - u_\varepsilon)\, ds + \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p u_0 (u_0 - u_\varepsilon)\, ds$$
$$+ \varepsilon^{\beta-\alpha+1} \int_{\Gamma_1} P_1(\varepsilon, \hat{x})(u_0 - u_\varepsilon)^2\, d\hat{x} - \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p(u_0 - u_\varepsilon)^2\, ds. \tag{27}$$

Then, taking into account the assumption $\beta > \alpha - 2$ we obtain by (23) and Lemma 8

$$\left|\varepsilon^{\beta-\alpha+1} \int_{\Gamma_1} P_1(\varepsilon, \hat{x})(u_0 - u_\varepsilon)^2\, d\hat{x} - \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p(u_0 - u_\varepsilon)^2\, ds\right|$$
$$\le \varepsilon^{\beta-\alpha+1} C\Big(\sqrt{\varepsilon}\|u_\varepsilon - u_0\|_{L_2(\Gamma_1)}\|u_\varepsilon - u_0\|_{H^1(\Omega_\varepsilon)} + \varepsilon\|u_\varepsilon - u_0\|^2_{H^1(\Omega_\varepsilon)}\Big)$$
$$\le \frac{1}{2}\left(\int_{\Omega_\varepsilon} |\nabla(u_0 - u_\varepsilon)|^2\, dx + \varepsilon^{\beta-\alpha+1} \int_{\Gamma_1} P_1(\varepsilon, \hat{x})(u_0 - u_\varepsilon)^2\, d\hat{x}\right)$$

for all sufficiently small $\varepsilon > 0$. Combining this with (27) we find

$$\int_{\Omega_\varepsilon} |\nabla(u_0 - u_\varepsilon)|^2 \, dx + \varepsilon^{\beta-\alpha+1} \int_{\Gamma_1} P_1(\varepsilon, \hat{x})(u_0 - u_\varepsilon)^2 \, d\hat{x}$$
$$\leq 2 \left| \int_{\Omega_\varepsilon} \nabla u_0 \nabla (u_0 - u_\varepsilon) \, dx - \int_{\Omega_\varepsilon} f(u_0 - u_\varepsilon) \, dx \right.$$
$$\left. - \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} g(u_0 - u_\varepsilon) \, ds + \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p u_0 (u_0 - u_\varepsilon) \, ds \right|$$

Now, (20) can be obtained in the same way as above. The theorem is proved.

6. THE "SMOOTH" CORRECTOR

In this section we find more precise asymptotics of u^ε all over the domain Ω_ε in the case $\beta \geq \alpha - 1$. For this aim we introduce a family of auxiliary problems posed in the fixed domain Ω, with nonoscillating boundary conditions depending on ε. With the help of these auxiliary problems we define the corrector term which is regular in ε and allows us to improve the estimate of the residual. For simplicity we suppose $p(\hat{x}, \hat{\xi}) \equiv 0$. One can easily generalize the results to the case of arbitrarily smooth $p(\hat{x}, \hat{\xi})$ and $\beta \geq \alpha - 1$.

THEOREM 4. *Let all the conditions of Theorem* 1 *be satisfied and* $p(\hat{x}, \hat{\xi}) \equiv 0$. *Then,*

$$\|U_0 - u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq K_4 \sqrt{\varepsilon} \tag{28}$$

where K_4 *does not depend on* ε, $U_0(\varepsilon, x)$ *is the solution of the problem*

$$\begin{cases} -\Delta U_0 = f & \text{in } \Omega, \\ \dfrac{\partial U_0}{\partial x_n} = G_1(\varepsilon, \hat{x}) & \text{on } \Gamma_1, \\ U_0 = 0 & \text{on } \Gamma_2, \end{cases} \tag{29}$$

and

$$G_1(\varepsilon, \hat{x}) = \int_0^1 \int_0^1 \cdots \int_0^1 \sqrt{\varepsilon^{2\alpha-2} + |\nabla_{\hat{\xi}} F(\hat{x}, \hat{\xi})|^2} \, g(\hat{x}, \hat{\xi}) \, d\hat{\xi}.$$

The relation

$$\|U_0(\varepsilon, x) - u_0(x)\|_{H^1(\Omega)} = O(\varepsilon^{\alpha-1}) \tag{30}$$

holds.

Proof of Theorem 4. We start with the following statement that can be proved in the same way as Lemma 6.

PROPOSITION 10. *For any* $v \in H^1(\Omega_\varepsilon)$

$$\left| \int_{\Gamma_1^\varepsilon} \varepsilon^{\alpha-1} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha}\right) v\left(\hat{x}, \varepsilon F\left(\hat{x}, \frac{\hat{x}}{\varepsilon^\alpha}\right)\right) ds - \int_{\Gamma_1} G_1(\varepsilon, \hat{x}) v(\hat{x}, 0)\, d\hat{x} \right|$$
$$\leq C\sqrt{\varepsilon}\|v\|_{H^1(\Omega_\varepsilon)}.$$

Due to the smoothness of the boundary $\partial\Omega$ the solution U_0 belongs to $H^2(\Omega \cap V)$ [8], and, hence, can be continued on V to belong to $H^2(V)$ [12].

Let us write the integral identity for problem (29)

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla(U_0 - u_\varepsilon)\nabla v\, dx \\
&\quad = \int_{\Omega_\varepsilon} \nabla U_0 \nabla v\, dx - \int_{\Omega_\varepsilon} fv\, dx - \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} gv\, ds \\
&\quad = \int_{\Omega} \nabla U_0 \nabla v\, dx - \int_{\Omega_\varepsilon} fv\, dx - \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} gv\, ds + \int_{\Omega_\varepsilon\setminus\Omega} \nabla U_0 \nabla v\, dx \\
&\quad = \int_{\Omega_\varepsilon\setminus\Omega} \nabla U_0 \nabla v\, dx - \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} gv\, ds + \int_{\Gamma_1} G_1 v\, d\hat{x} - \int_{\Omega_\varepsilon\setminus\Omega} fv\, dx.
\end{aligned}$$

By Lemma 1

$$\left| \int_{\Omega_\varepsilon\setminus\Omega} \nabla U_0 \nabla v\, dx \right| \leq \|\nabla U_0\|_{L_2(\Omega_\varepsilon\setminus\Omega)} \|v\|_{H^1(\Omega_\varepsilon)} \leq C_2\sqrt{\varepsilon}\|U_0\|_{H^2(\Omega_\varepsilon)} \|v\|_{H^1(\Omega_\varepsilon)}$$

and

$$\left| \int_{\Omega_\varepsilon\setminus\Omega} fv\, dx \right| \leq \|f\|_{L_2(\Omega_\varepsilon\setminus\Omega)} \|v\|_{L_2(\Omega_\varepsilon\setminus\Omega)} \leq C_2\sqrt{\varepsilon}\|f\|_{L_2(\Omega_\varepsilon)} \|v\|_{H^1(\Omega_\varepsilon)}.$$

Then, due to Proposition 10

$$\left| \varepsilon^{\alpha-1} \int_{\Gamma_1^\varepsilon} gv\, ds - \int_{\Gamma_1} G_1 v\, d\hat{x} \right| \leq C\sqrt{\varepsilon} \|v\|_{H^1(\Omega_\varepsilon)}.$$

From these inequalities we deduce

$$\left| \int_{\Omega_\varepsilon} \nabla(U_0 - u_\varepsilon) \nabla v\, dx \right| \leq K_4 \sqrt{\varepsilon} \|v\|_{H^1(\Omega_\varepsilon)}.$$

Substituting $v = U_0 - u_\varepsilon$ and applying the Friedrichs type inequality, we obtain (28). The theorem is proved.

In the remaining part of this section we analyze in more detail the asymptotics of corrector term $(u_0 - U_0)$ depending on the structure of function $F(\hat{x}, \hat{\xi})$. It is clear that the principal term of these asymptotics depends only on the behavior of F in a small neighborhood of the singular points of F (zero points of ∇F). For simplicity we consider the case $n = 2$. The multidimensional case can be studied in an analogous way.

The following assertion takes place.

PROPOSITION 11. *Suppose that $n = 2$, $(\alpha - 1)$ is sufficiently small and*

$$\operatorname{dist}\big(\operatorname{supp}_x g(\hat{x}, \hat{\xi}), \Gamma_1 \setminus \operatorname{supp}_x F(\hat{x}, \hat{\xi})\big) > 0.$$

Then,

(i) *If F is a Morse function of ξ uniformly in $x \in \operatorname{supp}_x g$ and all its singular points are quadratically nondegenerate, then*

$$\|u_0 - U_0\|_{H^1(\Omega)} \leq M_1 \varepsilon^{2(\alpha-1)} |\ln \varepsilon|.$$

(ii) *If F has a finite number of singular points and the degeneracy of $|\nabla F|^2$ at these points is of order $2k$, then*

$$\|u_0 - U_0\|_{H^1(\Omega)} \leq M_2 \varepsilon^{(1+1/k)(\alpha-1)}.$$

Remark 6. In general estimate (30) cannot be improved.

Proof. Let us introduce a new small parameter $\omega = \varepsilon^{\alpha-1}$ and estimate the derivative in ω of the function $G_1(\omega, \hat{x})$ for small ω. We have

$$\frac{\partial G_1(\omega, \hat{x})}{\partial \omega} = \int_0^1 \frac{g(\hat{x}, \hat{\xi})\omega\, d\xi}{\sqrt{\omega^2 + |\nabla_\xi F|^2}}.$$

Now,

(i) If $F(\hat{x}, \hat{\xi})$ is a Morse function one can easily verify that the main contribution to the derivative $\partial G_1(\omega, \hat{x})/\partial\omega|_{\omega=0}$ is given by the small neighborhood of the singular points. Let us estimate the contribution of one of them. We denote it by ξ_0. By our assumption $|\nabla_\xi F|^2 = C|\xi - \xi_0|^2 \cdot (1 + o(1))$ as $\xi - \xi_0 \to 0$; therefore,

$$\left| \int_{\xi_0-\delta}^{\xi_0+\delta} \frac{g(\omega, \hat{x})\,\omega\, d\xi}{\sqrt{\omega^2 + |\nabla_\xi F|^2}} \right| \le C_1 \left| \int_{\xi_0-\delta}^{\xi_0+\delta} \frac{g(\omega, \hat{x})\,\omega\, d\xi}{\sqrt{\omega^2 + C|\xi - \xi_0|^2}} \right| + O(\omega)$$

$$\le C_2 \int_0^{\delta/\omega} \frac{\omega\, ds}{\sqrt{1 + Cs^2}} \le C_3\,\omega\,|\ln \omega|.$$

Hence,

$$|G_1(\omega, \hat{x}) - G(\hat{x})| \le C_4\,\omega^2\,|\ln \omega|,$$

and by the standard energy estimate

$$\|u_0 - U_0\|_{H^1(\Omega)} \le M_1 \varepsilon^{2(\alpha-1)}\,|\ln \varepsilon|.$$

(ii) By the same reason as in 1 we will estimate the contribution of one of the singular points. If $|\nabla_\xi F|^2 = C|\xi - \xi_0|^{2k}(1 + o(1))$ as $\xi - \xi_0 \to 0$, then

$$\left| \int_{\xi_0-\delta}^{\xi_0+\delta} \frac{g(\omega, \hat{x})\,\omega\, d\xi}{\sqrt{\omega^2 + |\nabla_\xi F|^2}} \right| \le C_1 \int_{\xi_0-\delta}^{\xi_0+\delta} \frac{\omega\, d\xi}{\sqrt{\omega^2 + C|\xi - \xi_0|^{2k}}}$$

$$\le C_2 \int_0^{\delta/\omega^{1/k}} \frac{\omega^{1/k}\, ds}{\sqrt{1 + Cs^{2k}}} \le C_3\,\omega^{1/k}.$$

Thus, in this case

$$|G_1(\omega, \hat{x}) - G(\hat{x})| \le C_4\,\omega^{1+1/k}.$$

The proposition is proved.

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