The Langlands Parameters of Subquotients of Certain Derived Functor Modules

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Let $G$ be a noncompact, simple Lie group with finite center, let $K$ be a maximal compact subgroup, and let $g_0 = k_0 \oplus p_0$ be the corresponding decomposition of the Lie algebra. Suppose rank $G = \text{rank } K$, and let $t_0$ be a compact Cartan subalgebra and $b$ be a Borel subalgebra. Let $A_b(\lambda)$ be a derived functor module with infinitesimal character $\lambda + \delta$ which is nondominant with respect to a noncompact simple root. Suppose that $A = \lambda + 2\delta(p)$ is $K$ dominant so that the $K$ type, $\tau_A$, with highest weight $A$ occurs with multiplicity one in $A_b(\lambda)$. We develop conditions on the roots of $g$ under which a functor of cohomological induction maps a certain module of parabolic induction to another module of parabolic induction, extending a result due to Vogan. This allows us, in many cases, to determine the Langlands parameters of the subquotient of $A_b(\lambda)$ containing $\tau_A$, via a conjectured method of Knapp.

1. INTRODUCTION

Let $G$ be a linear, noncompact, simple Lie group with finite center, and $K$ be a maximal compact subgroup of $G$ corresponding to a global Cartan involution $\Theta$. Suppose that rank $G = \text{rank } K$, so that there is a maximal abelian subspace $t_0$ of $k_0$ that is also a Cartan subalgebra of $g_0$; Harish-Chandra showed that this equal-rank condition is a necessary and sufficient condition for $G$ to have discrete series representations. In analyzing a representation $(\pi, V)$ of $G$, one often examines the restriction of $\pi$ to $K$. The theorem of the highest weight parametrizes irreducible representations of $K$, and we call an equivalence class of irreducible representations of $K$ with highest weight $A$ a $K$ type, denoted $\tau_A$. A representation $\pi$ of $G$ is admissible if each $K$-type occurs with only finite multiplicity in $\pi|_K$. Work by Langlands [11], and subsequent work by Knapp and Zuckerman [10], parametrized irreducible admissible representations of $G$. The Langlands parameters of such a representation consist of a cuspidal parabolic subgroup MAN, a discrete series or a limit of discrete series on $M$, and a
complex-valued linear functional on the Lie algebra of $A$ satisfying a positivity condition.

When attempting to handle a representation $(\pi, V)$ algebraically, one often studies its underlying “$(\mathfrak{g}, K)$ module.” This is a vector space naturally associated with $V$ that carries compatibly both a $U(\mathfrak{g})$ module structure, where $U(\mathfrak{g})$ is the universal enveloping algebra of the complexified Lie algebra of $G$, and a representation of $K$ in which every vector lies in a finite-dimensional invariant subspace. General $(\mathfrak{g}, K)$ modules may also be defined, and theorems of Harish-Chandra, Lepowsky, and Rader show that every irreducible $(\mathfrak{g}, K)$ module is the underlying $(\mathfrak{g}, K)$ module of an irreducible admissible representation of $G$. Generally, the terminology of $G$-representations is transferred to $(\mathfrak{g}, K)$ modules. In particular, by the Langlands parameters of an irreducible $(\mathfrak{g}, K)$ module $V$, we mean those of an associated irreducible admissible representation of $G$.

Cohomological induction, introduced by Zuckerman in the late 1970s, is an algebraic technique used to construct admissible $(\mathfrak{g}, K)$ modules. Let us describe a functor of cohomological induction: Let $(\mathfrak{g}, K)$ be a reductive pair, let $q = l \oplus u$ be a stable parabolic subalgebra with Levi factor $l$ and nilpotent radical $u$, and let $\bar{q} = \bar{l} \oplus \bar{u}$ be the opposite parabolic of $q$. Let $Z$ be an $(l, L \cap K)$ module, and define $Z^\ast$ to be the $(\bar{l}, L \cap K)$ module $Z \otimes (\bigwedge^{\text{top}} u)$. Extend $Z^\ast$ to a $(\bar{q}, L \cap K)$ module $Z^\ast_{\bar{q}}$ by having $\bar{u}$ act as zero. Form the $(\mathfrak{g}, K)$ module

$$L_j(Z) = \Pi_j(U(\mathfrak{g}) \otimes_{U(\bar{q})} Z^\ast_{\bar{q}})$$

where $\Pi_j$ is the $j^{th}$ derived functor of the Bernstein functor $\Pi$. Let $S = \dim(u \cap l)$; this is the middle dimension among all degrees for which $\Pi_j$ can be nonzero. Zuckerman sketched an argument that $L_j(Z)$ is in the discrete series of $G$ if $L$ is compact, $Z$ is one-dimensional, and a certain translate $\lambda + \delta$ of the unique weight $\lambda$ of $Z$ has positive inner product with the roots of $u$ (in which case, we say that $\lambda$ or $Z$ is in the good zone).

One technique used to search for unusual unitary representations of $G$ is “to continue the discrete series analytically” by allowing a parameter to vary outside the range that produces discrete series. Wallach [16] in effect was one of the first to apply this approach, treating the case that $G/K$ is Hermitian symmetric, $Z$ is one-dimensional, and $u$ is built from all the noncompact positive roots ($L$ still being compact). In this case, $S = 0$. Wallach determined the exact range of the parameters in which $L_j(Z)$ is irreducible and infinitesimally unitary. Outside this range $L_j(Z)$ can be reducible, and Wallach determined exactly when the unique irreducible quotient of $L_j(Z)$ is infinitesimally unitary. Enright, Howe, and Wallach [1] and Jakobsen [4] independently extended Wallach’s results to $Z$ finite-dimensional. It is known that these unitary representations obtained...
via analytic continuation of discrete series play an important role in the classification of the unitary dual for certain groups $G$.

Enright, Parthasarathy, Wallach, and Wolf [2] considered a generalization in which $G/K$ is no longer Hermitian symmetric but $L$ is still compact. Again they considered “analytic continuations.” Their standing hypothesis was that a certain $K$-type parameter $A$ remained dominant for $K$; this condition had automatically been satisfied in the Hermitian case. Now, the parameter $S$ was no longer 0. The paper [2] was chiefly concerned with unitarizability, and the work was a predecessor of Vogan’s Unitarizability Theorem [14], which tidily extends the results of [2] by allowing $L$ non-compact and $Z$ infinite-dimensional.

Knapp undertook the task of determining for the setting of [2] the Langlands parameters of the unique irreducible subquotient of $L_{s}(Z)$ containing the $K$-type $\tau_{A}$. In [7], extrapolating from work of Wallach in the Hermitian case, he proposed a recursive process for doing so, and in certain cases Knapp proved via a combinatorial argument that his procedure worked. Roughly, the proof extracts the infinitesimal character and the minimal $K$-type and then shows that the Langlands parameters produced from the process are the only possible ones that can have these invariants.

We mention some features of the Knapp process. For a discrete series, the cuspidal parabolic subgroup is $G$ itself, and $A = 1$. As the parameter $\lambda + \delta$ moves outside the initial range (the good zone), the process increases the dimension of $A$ by 1 at each step, essentially projecting data to get the new $M$ and $A$ parameters.

Knapp’s combinatorial argument has a limited scope. It becomes more complicated for more complicated groups, and there appear to be cases not settled by Knapp where the infinitesimal character and the minimal $K$-type that it uses do not uniquely determine a set of Langlands parameters.

This paper provides a different, more representation-theoretic approach to the question of Langlands parameters and analytic continuations of discrete series. We start by exploiting some basic properties of representations of an $sl(2, \mathbb{R})$ subalgebra naturally embedded in $g$. Then we apply techniques of cohomological induction to produce a set of criteria on roots of $g$ that, when satisfied, allows us (essentially) to commute a functor of cohomological induction with a functor modelling ordinary parabolic induction [Th. 5.3]. As a result, we can construct a mapping, $\Phi$, which can be used to read off Langlands parameters.

The criteria given provide a true reduction of the problem, because they are simple and can be checked in a finite number of steps in any particular example. We then show that the cases handled by Knapp, as well as some other cases, are handled by the approach of this paper. Moreover, the approach taken here gives deeper insight into why Knapp’s process works, and suggests some lines of reasoning for how to proceed more generally.
2. PRELIMINARIES

2.1. Basic Notational Conventions

Let $G$ be a reductive Lie group with finite center and let $K$ be a maximal compact subgroup. We denote corresponding Lie algebras by the corresponding Gothic letters with subscripts 0, and we denote complexifications by dropping the subscripts. We let bar denote the conjugation of $g$ with respect to $g_0$. Let $\theta$ be the Cartan involution of $g_0$ corresponding to $K$ and let $g_0 = k_0 \oplus p_0$ be the associated Cartan decomposition.

Let $h_0$ be a Cartan subalgebra of $g_0$, and let $\mathcal{P} = \mathcal{P}(g, h)$ be the set of roots. We introduce in the usual way an inner product $\langle \cdot, \cdot \rangle$ and a norm squared $| \cdot |^2$ on the real linear span of the roots. We use a hat to denote a coroot; that is, if $\alpha$ then $\alpha ^\vee = |\alpha|^2 / \langle \alpha, \alpha \rangle$. If the Cartan subalgebra of $h_0$ lies in $k_0$ or in $p_0$, then each root vector lies in $k_0$ or in $p_0$, and the roots are called compact or noncompact accordingly. We denote the subset of compact roots by $\mathcal{P}_K$ and the subset of noncompact roots by $\mathcal{P}(p)$.

2.2. Functors of Cohomological Induction

The material in this section may also be found in [9] or [13]. Let $(g, K)$ be a reductive pair built from $G$ and $K$ as in [9, Section IV.3], and let $\mathcal{C}(g, K)$ be the category of all $(g, K)$ modules and $(g, K)$ maps. Let $q = l \oplus u$ a $\theta$ stable parabolic subalgebra containing a $\theta$ stable Cartan subalgebra of $h_0$ of $g_0$ and let $q = l \oplus u$ be the opposite parabolic of $q$. Let $L = N_G(q)$, which is connected and has Lie algebra $l_0 = l_0$. Let $Z$ be in $\mathcal{C}(l, L \cap K)$, and define

$$Z^* = Z \otimes \bigwedge^{\text{top}} u.$$

Since $u$ is an $(l, L \cap K)$ module, $\bigwedge^{\text{top}} u$ is a one-dimensional $(l, L \cap K)$ module with unique weight $2\delta(u)$ relative to $h$ so that $Z^*$ is in $\mathcal{C}(l, L \cap K)$. Let $\mathcal{L}(Z)$ and $\mathcal{M}(Z)$ be the members of $\mathcal{C}(g, K)$ given by

$$\mathcal{L}(Z) = (\Pi^b_{L \cap K})^j (\text{ind}_{L \cap K}^L (\mathcal{F}^b_{L \cap K}(Z^*)))$$

$$\mathcal{M}(Z) = (\Pi^c_{L \cap K})^j (\text{proj}_{L \cap K} (\mathcal{F}^c_{L \cap K}(Z^*)))$$

where $\Pi_j$ is the $j^{th}$ derived functor of the Bernstein functor $\Pi = \Pi^b_{L \cap K}$, $\Pi^j$ is the $j^{th}$ derived functor of the Zuckerman functor $\Pi = \Pi^c_{L \cap K}$, and $\mathcal{F}$ is the forgetful functor. The functors $\mathcal{L}$ and $\mathcal{M}$ are the functors of cohomological induction. We shall often abbreviate

$$Z^*_q = (\mathcal{F}^b_{L \cap K}(Z^*))$$

and

$$Z^*_q = (\mathcal{F}^c_{L \cap K}(Z^*))$$

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Let \( S = \dim (u \cap \mathfrak{t}) \). It is easy to see [9, Cor. 2.125b] that \( \Pi_1 \) and \( \Gamma' \) (and consequently \( \mathcal{L}' \) and \( \mathfrak{H}' \)) are 0 for \( j > 2S \). In fact, \( \mathcal{L}' \) and \( \mathfrak{H}' \) are 0 for \( j > S \) [9, Th. 5.35]. This dimension \( S \), called the middle dimension, is the one of primary interest.

Let \( \lambda \) be an analytically integral linear functional on \( \mathfrak{h} \) that is orthogonal to all members of \( \mathcal{A}(1) \), and let \( \mathbb{C} \) be the one-dimensional (\( L, L \cap K \)) module with highest weight \( \lambda \). Let \( A_q(\lambda) \) and \( A^q(\lambda) \) be the \((g, K)\) modules defined by

\[
A_q(\lambda) = \mathcal{L}_S(\mathbb{C}) \quad \text{and} \quad A^q(\lambda) = \mathfrak{H}^q(\mathbb{C}).
\]

The definition of functors of cohomological induction can be extended from one using a stable parabolic subalgebra and a twist by \( 2\theta(u) \) to one applicable to any germane parabolic subalgebra, in the sense of [9, Section IV.6]. If \( q = 1 \oplus u \) is a germane parabolic subalgebra we now allow \( L \) to be a subgroup of \( L \subseteq N_G(q) \cap N_G(\theta q) \) and define the "unnormalized" derived functors \((\mathcal{L}', \mathfrak{H}')\) from \( \mathcal{C}(l, L \cap K) \) to \( \mathcal{C}(g, K) \) by

\[
(\mathcal{L}'_q)_\Lambda(\lambda, L \cap K, \mathcal{C}) = (\Pi^{\mathcal{L}'_q}_q, L \cap K, \mathcal{C}) = \text{ind}_{\Lambda}(q, L \cap K) (\mathcal{F}_q, L \cap K, \mathcal{C})
\]

\[
(\mathfrak{H}'_q)_\Lambda(\lambda, L \cap K, \mathcal{C}) = (\Pi^{\mathfrak{H}'_q}_q, L \cap K, \mathcal{C}) = \text{ind}_{\Lambda}(q, L \cap K) (\mathcal{F}_q, L \cap K, \mathcal{C}).
\]

These unnormalized functors also arise naturally. According to Propositions 11.47 and 11.65 of [9], if \( \mathcal{L}'_\Lambda(\zeta, \nu) \) is a continuous-series representation and \( V^\mathcal{L}'_{K \cap M} \) is the underlying \((m, M \cap K)\) module of \( \zeta \), then the underlying \((g, K)\) module of \( \mathcal{L}'_{\Lambda M}(\zeta, \nu) \) is

\[
X^\mathcal{L}'_{K \cap M}(\zeta, \nu) \cong \mathcal{L}'_q_{\Lambda K \cap M} (V^\mathcal{L}'_{K \cap M} \otimes \mathbb{C}_{\nu, \rho}) \cong \mathcal{L}'_q_{\Lambda K \cap M} (V^\mathcal{L}'_{K \cap M} \otimes \mathbb{C}_{\nu, \rho}).
\]

(2.2.1)

where \( q = m \oplus a \oplus n \), \( m \) acts in \( V^\mathcal{L}'_{K \cap M} \) and \( a \) acts in \( \mathbb{C}_{\nu, \rho} \). We will use \( X^\mathcal{L}'_{K \cap M}(\zeta, \nu) \) to denote the obvious functor from \( \mathcal{C}(m, M \cap K) \) to \( \mathcal{C}(g, K) \).

Notice that this functor has an adjustment by \( \rho \). Modulo a technical matter involving double covers, we may define the "normalized" functors of cohomological induction, \( \mathfrak{H}^q \) and \( \mathcal{L}'_q \), to incorporate this shift (cf. [9, Section XI.7]).

2.3. The Bottom-Layer Map

All of the material in this section may be found in [9, Section V.6].

When analyzing a representation of \( G \), it is often helpful to study either the restriction of the representation to \( K \) or another type of \( K \) analog. We shall presently introduce the \((l, K)\) analogs, \( \mathcal{L}'_l(Z) \) and \( \mathfrak{H}'_l(Z) \), of the \((g, K)\) modules \( \mathcal{L}'(Z) \) and \( \mathfrak{H}'(Z) \). The bottom-layer map then provides a
link between the \((\mathfrak{f}, K)\) and \((q, K)\) modules. To obtain the nicest results, we assume that \((q, K)\) is a reductive pair, so that it arises from a reductive group, \(G\), [cf. 9, Prop. 4.31], and we assume that \(L\) meets every component of \(G\).

To start, we continue to write \(Z^*_q\) and \(Z^*_q\) for the \(K\) analogs

\[
\mathbb{F}_{\mathfrak{g}, L \cap K}^{(Z^*_q)} = \mathbb{F}_{\mathfrak{g}, L \cap K}^{(Z^*_q)},
\]

\[
\mathbb{F}_{\mathfrak{g}, L \cap K}^{\lambda \cdot L \cap K} = \mathbb{F}_{\mathfrak{g}, L \cap K}^{\lambda \cdot L \cap K}(Z^*_q),
\]

where the superscript \((\cdot)^\#\) continues to refer to the tensor product with \(\Lambda^u\). Define

\[
6_K^j = (P^k, L^K, L \cap K)^j \quad \text{and} \quad 1_J^j = (I^k, L^K, L \cap K)^j.
\]

The \(K\) analogs of \(\mathcal{L}_j(Z)\) and \(\mathcal{B}_j(Z)\) are the functors from \(\mathcal{E}(1 \cap \mathfrak{f}, L \cap K)\) to \(\mathcal{E}(\mathfrak{f}, K)\)

\[
\mathcal{L}_j^K(Z) = \Pi_j^K(P^k \cdot L \cap K(Z^*_q)),
\]

\[
\mathcal{B}_j^K(Z) = \Gamma_j^K(1_J^k \cdot L \cap K(Z^*_q)).
\]

The advantage of studying \(\mathcal{L}_j^K(Z)\) and \(\mathcal{B}_j^K(Z)\) is that we usually know exactly what they are. When \(G\) is connected and \(Z\) is an irreducible \((1 \cap \mathfrak{f}, L \cap K)\) module, the remarks following [9, Cor. 5.72] give that if \(\mu_L\) is the highest weight of \(Z\), then \(\mathcal{L}_j^K(Z) = 0\) unless the weight \(\mu_G = \mu_L + 2\delta(u \cap \mathfrak{p})\) is dominant for \(K\). When it is dominant for \(K\), \(\mathcal{L}_j^K(Z)\) is an irreducible representation of \(K\) with highest weight \(\mu_G\).

To take advantage of this concrete knowledge, we need to link \(\mathcal{L}_j^K(Z)\) and \(\mathcal{B}_j(Z)\). To do so, one sets \(m = \mathfrak{f}, m^* = \mathfrak{q}_q\), and \(Z = Z_q^\#\) in [9, Lemma 5.26] to obtain a one-one \((\mathfrak{f}, L \cap K)\) map

\[
\beta_Z^j: P^L \cap K \mathfrak{g}, L \cap K(Z_q^\#) \to P^L \cap K \mathfrak{g}, L \cap K(Z_q^\#),
\]

Because of the isomorphism from [9, Prop. 2.115],

\[
\Pi_j^K = \mathcal{F}_{\mathfrak{g}, L \cap K}^{\lambda \cdot L \cap K} \cong \mathcal{F}_{\mathfrak{g}, K}^{L \cap K} \Pi_j^K,
\]

it is meaningful to form \(\mathcal{B}_Z^j = \Pi_j^K(\beta_Z^j)\) from (2.3.1); the result is the map

\[
\mathcal{B}_Z^j: \mathcal{L}_j^K(Z) \to \mathcal{B}_j(Z).
\]

This map is called the bottom-layer map. The following theorem of Vogan exemplifies its usefulness.
Theorem 5.80(a) [9]. Suppose the center of $L \cap K$ acts by scalars in $Z$, and suppose $L$ meets every component of $G$. If $\tau$ is a $K$ type in $L^S(Z)$, then the bottom-layer map $\mathcal{A}^S_{\tau}$ maps the $\tau$ subspace one-one onto the $\tau$ subspace of $L^S(Z)$.

The $K$ types of $L^S(Z)$ that appear in $L^S(Z)$ are called the bottom-layer $K$ types of $L^S(Z)$.

2.4. Some Properties of $A_4(\lambda)$

In this subsection, we highlight some basic facts about the $A_4(\lambda)$ modules, due to Vogan and Zuckerman.

- If $Z$ is an $(l, L \cap K)$ module with infinitesimal character $\lambda$, then $L^S(Z)$ and $\mathcal{A}^S(l)$ have infinitesimal character $\lambda + \delta(u)$. In particular, $A_4(\lambda)$ has infinitesimal character $\lambda + \delta$ since in this case, the $(l, L \cap K)$ module $C_\lambda$ has infinitesimal character $\lambda + \delta(1)$ and $(\lambda + \delta(1)) + \delta(u) = \lambda + \delta$.

- If $A := \lambda + 2\delta(u \cap p)$ is $A^+(l, t)$ dominant, then $A_4(\lambda)$ contains the $K$ type $A$ with multiplicity one.

- If $L$ is compact, then $A_4(\lambda)$ is isomorphic to an $A_4(\lambda)$, where $b$ is a Borel subalgebra and $q \geq b$. (To see this, one can combine an algebraic Borel-Weil theorem with an induction-in-stages result.)

- If $\lambda$ is in the good zone, that is, the infinitesimal character $\lambda + \delta$ is strictly $A(u)$ dominant, then

  (a) $A$ is $A^+(l, t)$ dominant
  (b) $A_4(\lambda)$ is irreducible
  (c) $A_4(\lambda)$ is unitarizable if $\lambda$ is a unitary character of $L$

  (d) if also rank $G =$ rank $K$ and $t_0 \subset t_0$ is a Cartan subalgebra of $g_0$, then $A_4(\lambda)$ is a discrete series module. In fact, the good $A_4(\lambda)$’s, as $b$ varies, exhaust the discrete series of $G$.

The main result of this work concerns a natural subquotient $V$ of the $(g, K)$ module $A_4(\lambda)$ when $\lambda$ is no longer in the good zone. In particular, we are interested in determining a set of Langlands parameters of $V$.

3. $SL(2, \mathbb{R})$ AND SOME OF ITS REPRESENTATIONS

In this paper, we use some basic relationships among the representations of $SL(2, \mathbb{R})$. Let us now recall some of the relevant facts—all of which may be found in [5].
Let \( G = SL(2, \mathbb{R}) \), \( K = SO(2) \), and \( S = MAN \) the Langlands decomposition of the upper triangular subgroup of \( G \). Then \( M = \{ \pm I \} \), and let \( + \) (resp. \( - \)) denote the trivial (resp. nontrivial) representation of \( M \). Fix \( v \in \mathbb{C}^* \) and identify \( v \in \mathfrak{a}^* \) with \( iv \) by
\[
v \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} = ivt.
\]
Now form the principal series representation \( \mathcal{P}^{\pm,v} = \text{ind}^G_{MAN}( \pm \exp v \otimes 1 ) \). If we call
\[
\mathcal{P}^{\pm,v} = \text{ind}^G_{MAN}( \exp v \otimes 1 )
\]
the \( n^{th} \) type, then the \( K \) types of \( \mathcal{P}^{\pm,v} \) (resp. \( \mathcal{P}^{-,v} \)) are given by the even (resp. odd) integers, and all occur with multiplicity one.

Further, let \( n \geq 2 \) be an integer, and denote by \( \mathcal{D}^+_n \) (resp. \( \mathcal{D}^-_n \)) the discrete series representation with \( K \) types \( n + 2m \) (resp. \( -(n + 2m) \)) where \( m \) is a non-negative integer. Then we have the following reducibility:
\[
\mathcal{D}^+_n \oplus \mathcal{D}^-_n \subseteq \begin{cases} 
\mathcal{P}^{+,n-1} & \text{if } n \text{ even} \\
\mathcal{P}^{-,n+1} & \text{if } n \text{ odd}
\end{cases}
\]

4. SETTING THE STAGE

4.1. Langlands Parameters

As previously mentioned, theorems of Harish-Chandra, Lepowsky, and Rader show that every irreducible \((g,K)\) module globalizes to an irreducible admissible representation of \( G \). Therefore it is reasonable to transfer the language of \( G \) representations to that of \((g,K)\) modules. The Langlands classification of irreducible admissible representations of \( G \) is well known (see, for example, [5, Th. 14.91]) and by the Langlands parameters of an irreducible \((g,K)\) module \( V \) we mean a triple \((MAN, \_\_, \&\&)\) such that

(i) \( MAN \) is a cuspidal parabolic subgroup of \( G \);

(ii) \( \_\_ \) is a discrete series or limit of discrete series on \( M \) with infinitesimal character \( \lambda_\_ \);

(iii) \( \&\& \) is a complex-valued linear functional on the Lie algebra \( \mathfrak{a}_0 \) of \( A \) with \( \text{Re } \&\& \) in the closed positive Weyl chamber;

(iv) the induced representation \( \text{ind}^G_{MAN}(\sigma \otimes e^\gamma \otimes 1) \) has a unique irreducible quotient, called the Langlands quotient and denoted \( J(MAN; \sigma, v) \);
V is equivalent with the underlying \((g, K)\) module of 
\(J(M \cdot AN, \sigma, \nu)\).

As stated above, the Langlands parameters are not unique.

4.2. Knapp’s Conjectural Method

The setting for this section is as follows: \(G\) is a linear, simple noncompact Lie group with finite center, \(K\) a maximal compact subgroup, and rank \(G = \text{rank } K\). Let \(T \subseteq K\) be a Cartan subgroup, and let \(A = A(g, t)\) be the set of roots. Fix a positive system \(A^+ = A^+(g, t)\) and assume that there is exactly one noncompact simple root and this root has multiplicity at most two in the highest root. Define \(A^+_L, \delta, \delta_K\) as usual. Let \(q = 1 \oplus u\) be the \(\theta\) stable parabolic subalgebra with \(l\) formed from the compact simple roots, and \(u\) formed from the remaining positive roots. Let \(\lambda\) be an analytically integral form on \(t\) that is orthogonal to the roots of \(A_L\) and let \(A = \lambda + 2\delta(u \cap p) = \lambda + 2\delta(p) = (\lambda + \delta) + (\delta - 2\delta_K)\). Assume that \(A\) is \(K\) dominant, and consider \(A_q(\lambda)\).

In [7], Knapp outlined a recursive procedure which he conjectured would produce the Langlands parameters of the irreducible subquotient \(V\) of \(A_q(\lambda)\) containing the \(K\) type \(A\). Using combinatorial arguments, he proved that this method gives the correct parameters in some cases. In this paper, we approach this problem from another point of view which will allow us to reduce the question about the success of Knapp’s method to a simple question about dominance properties of a finite set of roots. As a result, we will be able to prove that the procedure works for a much wider class of \(A_q(\lambda)\)-modules, not just those which are isomorphic to the \(A_q(\lambda)\)-types described above.

Let us describe, with slight modifications, the Conjectural Method of [7]. Assume that

1. \(\langle \lambda + \delta, \beta \rangle > 0\) for all compact simple roots, \(\beta\), of \(A^+(g, t)\), and
2. \(A\) is \(A^+_L\) dominant.

Roughly, if the infinitesimal character \(\lambda + \delta\) of \(A_q(\lambda)\) is nondominant versus a noncompact root \(\alpha\), then split, by the Cayley transform relative to \(\alpha\), the Cartan subalgebra of \(t\) into \(t' \oplus \alpha'\). Project the infinitesimal character onto the dual of each of these pieces, but negate the projection onto the \(\alpha'\) piece. Label these projections \(\lambda_{\alpha'}\) and \(v\). Form \(M' = Z_{G}(d')\) and the roots \(A^+(m', t')\), which may be identified with the roots of \(A^+(g)\) orthogonal to \(\alpha\). As shown in [7], the functional \(\lambda_{\alpha'}\) will be dominant versus the compact simple roots of \(A^+(m')\) and the corresponding weight \(A'\) will be \(M' \cap K\) dominant. Thus, one may continue this process on \(M'\) and the corresponding \(A_q(\lambda')\) with infinitesimal character \(\lambda_{\alpha'}\), increasing the dimension of \(\alpha\) at each step until you produce a discrete series module on a subsequent \(M'\).
More precisely, set $M_0 = G$, $A_0 = \{ I \}$, $t_0^0 = t_0$, $a_0^0 = 0$, $h_0^0 = 1_{0}^0 \oplus a_0^0$, $\lambda_0 = \lambda$, $\delta_0 = \delta$, $\lambda_{a_0} = \lambda_0 + \delta_0$, $v_0 = 0$, $A_0 = \lambda_0 = (\delta_0 - 2\delta_{0,K})$. Suppose $M_j$, $A_j$, $t_j$, $a_j^0$, $h_j^0$, $\lambda_j$, $\delta_j$, $\lambda_{a_j}$, $A_j$ and $v_j$ are given with $\dim A_j = j$ and with $\lambda_j$ dominant nonsingular with respect to all simple roots of $M_j$ that are $M_j$ compact.

There are now two cases:

(a) If $\langle \lambda_j, \alpha \rangle > 0$ for all simple roots $\alpha$ of $M_j$ that are $M_j$ noncompact, the recursive construction ends. Define $M = M^*_j$, $A = A^*_j$, $\lambda_{a} = \lambda_j$, and $v = v_0 + \cdots + v_j$. Define $N$ so that $v$ is dominant relative to $N$. Then $M^* A N \lambda_{a}$, and $v$ are the cuspidal parabolic subgroup, the infinitesimal character of the $M$ representation, and the parameter on $a_0$ of a set of Langlands parameters for the irreducible subquotient of $A_4(\lambda)$ containing the $K$ type $A$.

(b) Otherwise, of the $M_j$ noncompact simple roots $\alpha$ with $\langle \lambda_j, \alpha \rangle < 0$, set $\sigma_{j+1}$ to be the one for which $-\langle \lambda_j, \alpha \rangle/|\alpha|^2$ is greatest. Further, set

$$c_{j+1} = -\frac{\langle \lambda_j, \sigma_{j+1} \rangle}{|\sigma_{j+1}|^2} = \frac{\langle s_{\sigma_0}(\lambda_j), \sigma_{j+1} \rangle}{|\sigma_{j+1}|^2}$$

(4.2.1)

where $s_{\sigma_0}$ is the Weyl group reflection corresponding to $\sigma_{j+1}$. Applying the Cayley transform relative to $\sigma_{j+1}$ [8, Section VI.7], we write $h_0^* = t_0^j \oplus a_0^{j+1}$ for the transformed version of $h_0^j$ and let $A_{j+1} = \exp(a_0^{j+1})$ with $\dim A_{j+1} = j + 1$. Identifying $\sigma_{j+1}$ with its Cayley transform, set

$$v_{j+1} = c_{j+1} \sigma_{j+1} \quad \text{and} \quad \nu^{j+1} = \nu^j + v_{j+1}.$$ 

(4.2.2)

Define $N_{j+1}$ so that $\nu^{j+1}$ is dominant relative to $N_{j+1}$. Let $M_{j+1} A_{j+1} = Z_G(A_{j+1})$, and we identify $A(m_{j+1}, t^{j+1})$ with the subset of $A(m_{j}, t^{j})$ orthogonal to $\sigma_{j+1}$. Set $A_{j+1}' = A_{j+1} \cap A_{j}$. Let $\delta_{j+1}$ be half the sum of the positive roots and $\delta_{j+1, K}$ be half the sum of the positive $M_{j+1}$ compact roots. Define $\lambda_{\sigma_{j+1}}$ to be the projection of $\lambda_{\sigma}$ orthogonal to $\sigma_{j+1}$, so that

$$\lambda_{\sigma_{j+1}} = \lambda_{\sigma} - \frac{\langle \lambda_{\sigma}, \sigma_{j+1} \rangle}{|\sigma_{j+1}|^2} \sigma_{j+1}$$

$$= \lambda_{\sigma} + \frac{\langle s_{\sigma_0}(\lambda_{\sigma}), \sigma_{j+1} \rangle}{|\sigma_{j+1}|^2} \sigma_{j+1} = \lambda_{\sigma} + c_{j+1} \sigma_{j+1} = \lambda_{\sigma} + v_{j+1}.$$ 

(4.2.3)

We also define $\lambda_{j+1}$ so that $\lambda_{\sigma_{j+1}} = \lambda_{j+1} + \delta_{j+1}$ and set $A_{j+1} = \lambda_{j+1} + (\delta_{j+1} - 2\delta_{j+1, K})$. Then $\lambda_{\sigma_{j+1}}$ is dominant nonsingular relative to the $M_{j+1}$ compact simple roots, and the recursive construction continues.
From these definitions, we also note that repeated iterations of (4.2.3) yield
\[
\lambda_{j+1} = \lambda_j + v_1 + \cdots + v_{j+1} = (\lambda + \delta) + v^{j+1}.
\] (4.2.4)

Moreover, (4.2.3) shows that
\[
\lambda_{j+1} = s_{\eta_{j+1}}(\lambda_j)
\]
and
\[
\lambda_{j+1} + v_{j+1} = s_{\eta_{j+1}}(\lambda_j).
\]

Proposition 10 of [7] shows that the Conjectural Method runs into no obstruction in finding parameters $MAN$, $\lambda_j$, and $\psi$. In fact, the hypotheses used in Proposition 10 are relaxed from those initially described in the setting of this section. In particular, we still take $q = l \oplus u$ to be a parabolic subalgebra formed from the compact simple roots of $A^+(q, t)$ and $A(u) \subseteq A^+$. But, we no longer assume that $\lambda$ is orthogonal to the roots of $l$, only that $\lambda$ is dominant for $A^+ = A^+ \cap A_K$. Further, we do not make any assumption on the number of noncompact simple roots. Proposition 10 then shows that

1. $\lambda_j$ is analytically integral,
2. $A_j$ is $A_{M_j, \lambda}^+$ dominant,
3. $\lambda_j$ is dominant for the compact simple roots of $A_{M_j}^+$.

Later, we shall restrict ourselves to the situation in which $A^+(q)$ contains only one noncompact simple root. This characteristic, however, is not necessarily retained by the subsequent $A_{M_j}^+$. We will handle this possibility later.

4.3. The Approach

Knapp proved via combinatorial arguments that his Method does produce the Langlands parameters in some cases. We now approach the problem from another, more representation-theoretic, point of view. This new approach will ultimately reduce the problem of verifying the Method to checking a finite number of computations. First, we sketch the approach.

We start by forming a $\theta$-stable parabolic subalgebra $q_j = l_j \oplus u_j$ of $m_{j-1}$ with $(\mathfrak{l})_l \cong l_j \oplus \mathfrak{sl}(2, \mathbb{R})$ where the $\mathfrak{sl}(2, \mathbb{R})$ is built from $\mathfrak{z}_j$, and $\mathfrak{u}_j$ is built from the remaining positive root spaces of $m_{j-1}$. Then, at the level of $(L_j, L_j \cap K)$ modules—think $\mathfrak{sl}(2, \mathbb{R})$—we have a short exact sequence, roughly described as

\[
0 \rightarrow \begin{pmatrix} \text{discrete} \\ \text{series} \end{pmatrix} \rightarrow \begin{pmatrix} \text{principal} \\ \text{series} \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{sl}_{2, \mathbb{R}-1} & (L_j, L_j \cap K) \end{pmatrix} \rightarrow 0
\] (4.3.1)
where the principal series has \( v_j \) as its \( a_j \) parameter. We then form a long exact sequence of \((m_j, M_j \cap K)\) derived functor modules from (4.3.1) and the covariant, right exact \( \mathcal{L} \) functor of cohomological induction. Focusing on the resulting

\[
\to (\mathcal{L})_S \left\langle \text{principal series module} \right\rangle \to \to (\mathcal{L})_S \mathcal{L}_{(b_j, v_j \cap K)}(C_\lambda) \to (4.3.2)
\]

section of the long exact sequence, where \( S_j \) is the so-called middle dimension, \( \dim(u_j \cap l) \), we

(1) show, via a “bottom layer”-type argument, that the \( M_{j-1} \cap K \) type with highest weight \( A_{j-1} \), which is nonzero in the right-hand side, is in the image of the map,

(2) use induction-in-stages to write the right-hand side of (4.3.2) as a single functor of cohomological induction,

(3) show that the left-hand side of (4.3.2) may be regarded as a principal series \((m_j, M_j \cap K)\) module, and then

(4) induce this principal series \((m_j, M_j \cap K)\) module up to a \((g, K)\) module by applying a covariant, exact functor modelling ordinary parabolic induction with \( A_{j-1} \) parameter \( v_1 + \cdots + v_{j-1} \). Then, using double induction and a Frobenius reciprocity argument, we show that the \( K \) type with highest weight \( A \) behaves as desired.

5. REDUCTION OF THE PROBLEM

5.1. Steps (1) and (2) of the Approach

In this section, we carry out steps (1) and (2) of the Approach (see Section 4.3). To do so, we will prove the following theorem.

**Theorem 5.1.** Let \( G \) be a linear, noncompact Lie group, \( K \) a maximal compact subgroup, and suppose \( \text{rank } G = \text{rank } K \). Let \( T \subseteq K \) be a Cartan subgroup, and let \( \Delta = \Delta(g, t) \) be the set of roots. Fix a positive system \( \Delta^+ = \Delta^+(g, t) \) and let \( b \) be the Borel subalgebra formed by the positive roots. Define \( A^+_K \), \( \delta \), and \( \delta(p) \) in the usual way. Let \( \lambda \) be an analytically integral form on \( t \). Assume

(1) \( \langle \lambda + \delta, \beta \rangle > 0 \) for all compact simple roots, \( \beta \), of \( \Delta^+(g, t) \), and

(2) \( \lambda = \lambda + 2\delta(p) \) is \( A^+_K \) dominant.
Suppose \( x_1 \) is a noncompact simple root such that \( \langle \lambda + \delta, x_1 \rangle < 0 \). Then form the \( \theta \) stable parabolic subalgebra \( q = l \oplus u \) by building \( l \) from \( x_1 \) and \( u \) from the remaining positive root spaces. Let \( b_l = t \oplus g_{x_1} \) be a Borel subalgebra of \( L = N_G(q) \cap N_G(\theta q) \) and \( S = \dim(u \wedge l) \).

Apply the Cayley transform [8, Section VI.7] relative to \( x_1 \) and write \( t_0 = \exp(g_{x_1}) \) for the transformed version of \( t_0 \). Identify \( x_1 \) with its image under the Cayley transform, and define \( \lambda_{x_1} \) to be the projection of \( \lambda + \delta \) orthogonal to \( x_1 \). Build \( A = \exp(\alpha) \) and form the minimal parabolic subgroup of \( L \) with Langlands decomposition \( M_L \cdot AN_L \) in the usual way, taking \( N_L \) to be formed from the transformed \( g_{x_1} \). Since \( L \) is split modulo center, we have that \( M_L = T' \).

Consider the parabolically induced \((l, L \cap K)\) module \( X_{L \cap K}(\xi_L, v) \) (see (2.2.1)) where, in the notation of [9],

(i) \( \xi_L = \hat{Z}_L \) is the irreducible \((\hat{t}', \hat{T}_L)\) module of infinitesimal character \( \lambda_{x_1} - \delta(\hat{u}) \) (which we can regard as a \((\hat{t}', \hat{T}_L)\) module), that matches the action of \( T' \) in \( \mathcal{D}_{\xi_L}(C_\mathbb{C}) \), and

(ii) \( v = -\frac{\langle \lambda + \delta, x_1 \rangle}{|x_1|^2} \in (a')^* \).

Then there exists a \((g, K)\) module map,

\[
\mathcal{D}_s(\varphi): \mathcal{D}_s(X_{L \cap K}(\xi_L, v)) \to A_b(\lambda) \tag{5.1.1}
\]

whose image contains the multiplicity–one \( K \) type with highest weight \( A \).

Note. For the rest this paper, any numerical label beginning with 11 refers to Chapter XI of [9].

Proof. Let \( x_1 \) be the noncompact simple root with \( \langle \lambda + \delta, x_1 \rangle < 0 \). Build \( l \) from \( x_1 \), and \( u \) from the remaining positive root spaces so that \( q = l \oplus u \) is \( \theta \) stable and \( x_1 \) is imaginary. We have \( l = t \oplus u \oplus g_{x_1} \)\( = t \oplus \mathfrak{sl}(2, \mathbb{C}) \).

First, we will form a short exact sequence of \((l, L \cap K)\) modules. Form \( b_l = t \oplus g_{x_1} \), a Borel subalgebra of \( l \). Since \( t \) is \( \theta \) stable and \( x_1 \) is imaginary, we have \( b_l = (t \oplus g_{-x_1}) \). Then \( \mathcal{D}_{b_l}(\hat{t}, \hat{T}_L) \) is an upside down Verma module for \( \hat{t} \oplus \mathfrak{sl}(2, \mathbb{C}) \) with infinitesimal character \( \lambda + \delta(t) \) and lowest weight \( \lambda + 2\delta(t) \) relative to the Cartan subalgebra \( t_0 \).

Next we consider \( X_{L \cap K}(\xi_L, v) \) (see (2.2.1)) as the Harish-Chandra module of a principal series representation of \( L \). This has a positive \((a')^*\) parameter, and according to [9, Prop. 11.43] its infinitesimal character is
\[
\lambda_{\eta_j} - \delta(u) + v = \lambda_{\eta_j} + v - \delta(u) \\
= s_u(\lambda + \delta) - \delta(u) \\
= s_u(\lambda + \delta - \delta(u)) \quad \text{since} \quad z_1 \perp \delta(u) \ [9, \text{Prop. 4.69}].
\]

Accordingly, as \( L \) is locally isomorphic to \( T' \times SL(2, \mathbb{R}) \) we know from Chapter 2 that \( X_{L \cap K}(\xi_L, v) \) contains the underlying \((g, K)\) module of a representation we can call a

\[
(\text{matching character on } T') \times (SL(2, \mathbb{R}) \text{ discrete series}), \text{ with infinitesimal character } (\lambda + \delta) - \delta(u)
\]
as a submodule. By matching infinitesimal characters, \( L \cap K \) types and \( l \) actions, we see that the resulting quotient of \( X_{L \cap K}(\xi_L, v) \) by this discrete series module is \( \mathcal{O}^{l, L \cap K}_{b_\delta}(C_\lambda) \). Therefore, we have the short exact sequence

\[
0 \rightarrow \left( \begin{array}{c}
\text{discrete series} \\
\text{module}
\end{array} \right) \rightarrow X_{L \cap K}(\xi_L, v) \rightarrow \mathcal{O}^{l, L \cap K}_{b_\delta}(C_\lambda) \rightarrow 0
\]

We let

\[
\varphi: X_{L \cap K}(\xi_L, v) \rightarrow \mathcal{O}^{l, L \cap K}_{b_\delta}(C_\lambda)
\]

be the quotient map.

To continue the proof, we use an argument not unlike that on page 765 of [9], with \( \varphi \) above replacing the \( \varphi \) in [9]. Accordingly, we form the diagram

\[
\begin{array}{ccc}
U(\mathfrak{g}) \otimes \mathbb{C}[\mathfrak{g}] & \xrightarrow{\text{id}} & U(\mathfrak{g}) \otimes \mathbb{C}[\mathfrak{g}] \\
\beta_X \downarrow & & \downarrow \beta_d \\
U(\mathfrak{g}) \otimes \mathbb{C}[\mathfrak{g}] & \xrightarrow{\text{id}} & U(\mathfrak{g}) \otimes \mathbb{C}[\mathfrak{g}]
\end{array}
\]

(5.1.2)

where the maps \( \beta_X \) and \( \beta_d \) are the one-one \((1, L \cap K)\) maps given by (2.3.1). This diagram commutes since effectively \( \beta_X \) and \( \beta_d \) act in the respective first factors while \( \text{id} \) and \( \beta_d \) act in the respective second factors. Applying \( \Pi_K^S \) to this diagram, where \( S = \dim (u \cap 1) \) is the appropriate middle dimension, we obtain the commutative diagram
\[ \mathcal{D}_{\delta}^K(X_{L \cap K}(\xi_L, v)) \overset{\mathcal{D}_{\delta}^K(\phi)}{\longrightarrow} \mathcal{D}_{\delta}^L(\mathcal{D}_{b_0 T}^L(X_{b_0 T}^L(\xi_L, v))) \]
\[ \mathcal{D}_{\delta}^L(X_{L \cap K}(\xi_L, v)) \overset{\mathcal{D}_{\delta}^L(\phi)}{\longrightarrow} \mathcal{D}_{\delta}^S(\mathcal{D}_{b_0 T}^{L \cap K}(C_{b_0 T})) \]

in which \( \mathcal{D}_{\delta}^K \) and \( \mathcal{D}_{\delta}^L \) are bottom-layer maps as in (2.3.2).

The \( L \cap K \) type with highest weight \( \lambda + 2\delta(1) \) occurs with multiplicity–one in the the \( \mathfrak{sl}(2, \mathbb{R}) \) principal series type module, \( X_{L \cap K}(\xi_L, v) \), and \( \phi \) is one–one on that \( L \cap K \) type. Since \( A = (\lambda + 2\delta(1)) + 2\delta(u \cap v) = \lambda + 2\delta(p) \) is assumed to be \( K \) dominant, the \( K \) type \( \tau_d \) with highest weight \( A \) occurs with multiplicity–one in \( \mathcal{D}_{\delta}^K(X_{L \cap K}(\xi_L, v)) \) and \( \mathcal{D}_{\delta}^L(\phi) \) is one–one on \( \tau_d \). By [9, Th. 5.80a, see Section 2.3], \( \mathcal{D}_{\delta} \) is one–one onto for the \( K \) type with highest weight \( A \). Consequently, \( \mathcal{D}_{\delta}^L(\phi) \colon \mathcal{D}_{\delta}^L(\xi_L, v) \) maps onto the multiplicity–one \( K \) type of \( \mathcal{D}_{\delta}^L(\mathcal{D}_{b_0 T}^{L \cap K}(C_{b_0 T})) \) with highest weight \( A \).

Thus, by the commutativity of the diagram, the same thing must be true of \( \mathcal{D}_{\delta}(\phi) \circ \mathcal{D}_{\delta}, \) and we conclude that the image of \( \mathcal{D}_{\delta}(\phi) \) contains the multiplicity–one \( K \) type with highest weight \( A \).

To complete the proof, we must identify the range representation of the map \( \mathcal{D}_{\delta}(\phi) \) as \( A_b(\lambda) \). Since \( \lambda \) is a noncompact root, we have that the appropriate middle dimension of \( \mathcal{D}_{\delta}^L(X_{L \cap K}(\xi_L, v)) \) is 0. Because the functors of cohomological induction vanish above the middle dimension, we have \( \mathcal{D}_{\delta}^L(X_{L \cap K}(\xi_L, v)) \) is nonvanishing in only one degree. Moreover, the 0th derived functor is nothing more than \( \mathcal{D}_{\delta}^L(\mathcal{D}_{b_0 T}^{L \cap K}(C_{b_0 T})) \). Therefore the double induction result in [9, Cor. 11.86a] is applicable. When combined with a supplementary argument to take \( \wedge^{n_{\mathfrak{a}}} \) into account (cf. [9, Section XI.7]), it gives \( \mathcal{D}_{\delta}^L(\mathcal{D}_{b_0 T}^{L \cap K}(C_{b_0 T})) \equiv \mathcal{D}_{\delta}^L(\mathcal{D}_{b_0 T}^{L \cap K}(C_{b_0 T})) \equiv A_b(\lambda) \) where \( b \) is the natural Borel subalgebra formed from \( A^{-1}(q, l) \).

Let us now specialize the above theorem to produce a corollary which will be used in proving Knapp’s Conjectural Method for \( A_b(\lambda) \). Set \( G = \mathcal{M}_{j-1}, \eta_1 = I_{\mathfrak{g}} \oplus u \), the parabolic subalgebra of \( m_{j-1} \) with \( l_j \) formed from \( \xi_j \) and \( u \), from the remaining positive root spaces, \( \xi_j \) to be the appropriate character on \( T_1 \), and abbreviate \( \mathcal{D}_{\delta}^L(X_{L_j \cap K}(\xi_L, v)) \) by \( \mathcal{D}_{\delta}^L \), so that we obtain the corollary below.

**Corollary 5.2.** In the setting of Section 4.2 there exists an \((m_{j-1}, M_{j-1}) \cap K \) module map

\[ \mathcal{D}_{\delta}^L(\phi) \colon \mathcal{D}_{\delta}^L(X_{L_j \cap K}(\xi_L, v)) \to A_{m_{j-1}}(\lambda_{j-1}) \]

whose image contains the multiplicity one \((M_{j-1} \cap K) \) type with highest weight \( A_{j-1} \).
5.2. Reduction of Step (3) to Calculable Conditions

At this point, we would like to rewrite the domain space of \( \mathcal{L}_S(\varphi_j) \) in (5.1.4) as an \((m, M_j \cap K) \) principal series module as follows:

\[
\mathcal{L}_S(X_{L_{j \cap K}}(\varphi_j), v_j) \cong X_{M_{j \cap K}}(\varphi_j, v_j),
\]

(5.2.1)

where \( \varphi_j^k \) is an \( M_j \) representation with underlying \((m, M_j \cap K) \) module \( A_{\lambda_j} \). This amounts to a change of polarization and is the subject of Theorem 11.225 of [9]. Unfortunately, the dominance condition (11.220) required to apply Theorem 11.225 will not usually be satisfied. However, in our situation, since \( A' \) is one dimensional, we are able to run through the proof of Theorem 11.225 to extract weaker conditions under which the isomorphism (5.2.1) holds.

**Theorem 5.3.** Assume the same setup as in Theorem 5.1. Let \( MA = Z_G(a_0) \), \( A^+(m) = A^+(g) \cap A(m, V) \), \( \delta(m) \) be half the sum of the positive roots of \( m \), and \( b_m \) the Borel subalgebra formed from \( A^+(m) \). Let \( \lambda_1 = \lambda_{\alpha_1} - \delta(m) \). Let

\[
C = \{ \gamma \in A^+(g) - \{ x_1 \} | \langle \gamma, x_1 \rangle > 0, \langle s_{\alpha_1}(\lambda + \delta), \gamma \rangle \in \mathbb{Z} - \{ 0 \} \}.
\]

(5.2.2)

If

\[
\langle s_{\alpha_1}(\lambda + \delta), \gamma \rangle > 0 \quad \text{for all} \quad \gamma \in C
\]

(5.2.3)

then the domain space of \( \mathcal{L}_S(\varphi_j) \) in (5.1.1) is a \((g, K) \) principal series module. Specifically

\[
\mathcal{L}_S(X_{L_{j \cap K}}(\varphi_j), v_j) \cong X_K(\varphi_j^k, v_j)
\]

(5.2.4)

where \( \varphi_j^k \) has \( A_{\lambda_1} \) as its underlying \((m, M \cap K) \) module.

**Remark.** The proof of this theorem follows the lines of the proof of Theorem 11.225 [9], except that we replace condition (11.220) of that theorem, with condition (5.2.3) above.

**Proof.** We restate the setup of [9, Th. 11.225] in our notation as it applies to our situation, dropping the hypothesis (11.220) on the functional \( \lambda_{\alpha_1} \) in (iv) below. We start with

(i) the \( \theta \) stable parabolic subalgebra \( q = l \oplus u \),

(ii) the Levi subgroup \( L = N_G(q) \cap N_G(\theta q) \) for \( l \);

(iii) the Cartan pair \( (b', T') \) for both \((l, L \cap K) \) and \((q, K) \) with \( h_0' = l_0' \oplus a_0' \) and \( T' = Z_L(\varphi_j^k) = Z_K(\varphi_j^k) \),

(iv) the functional \( \lambda_{\alpha_i} \in i h_0^* \).
Let $A_L = A = \exp (a_0)$ and consider two continuous-series representations

$$I_{MAN}^L (\zeta, \nu) \quad \text{and} \quad I_{MLANL}^L (\zeta_L, \nu) \quad (5.2.5)$$

and their underlying modules subject to the following conditions:

$$MA = Z_G (a_0) \quad \text{and} \quad M_L A = Z_L (a_0),$$

$$N \geq N_L,$$

$$< \text{Re} \, v, \beta > \geq 0 \quad \text{for every positive } \alpha \text{-root } \beta \text{ of } g,$$

$$b_{m_L} = \text{Borel subalgebra of } m_L,$$

$$b_m = \text{Borel subalgebra of } m,$$

$$b_m \supseteq b_{m_L} \quad \text{and} \quad b_m \supseteq m_{\beta} \quad \text{if } m_{\beta} \subseteq u, \quad (5.2.6)$$

$Z_L = \text{irreducible } (t', \tilde{T}_L)$ module of infinitesimal character $\lambda_{L, \delta (u)},$

$\tilde{Z} = \text{irreducible } (t', \tilde{T}^')$ module of infinitesimal character $\lambda_{\nu},$

$$\zeta_L = \zeta (Z_L, b_{m_L}) \text{ as a representation of } M_L,$$

$$\zeta = \zeta (\tilde{Z}, b_m) \text{ as a representation of } M.$$

Accordingly, the underlying $(m, M \cap K)$ module of $\zeta$ is $(\beta)_{\nu}^{\text{\scriptsize B}\, (\nu, M \cap K)} (\tilde{Z}).$ By $\zeta$ we shall mean the representation of $M$ with underlying $(m, M \cap K)$ module $(\beta)_{\nu}^{\text{\scriptsize B}\, (\nu, M \cap K)} (\tilde{Z}),$ which after tracing through the definitions is $A_{\nu} (\lambda),$ apart from technicalities involving double covers. Also, $\zeta^\nu_L \cong \zeta_L$ since $M_L = T.$ Further, we remark that $b_{m_L}$ is nothing more than $t'$ because $L$ is split modulo center. Also, we identify the positive roots as follows:

$$A_{\text{\scriptsize imag}} (g, h') = \text{members of } A (m, t') \text{ contributing to } b_m,$$

$$A_{\text{\scriptsize real}} (g, h') = \text{real roots contributing to } n,$$

$$A_{\text{\scriptsize imag}} (l, h') = \text{members of } A (m_L, t') \text{ contributing to } b_{m_L} = \emptyset, \quad (5.2.7)$$

$$A_{\text{\scriptsize real}} (l, h') = \text{real roots contributing to } n_L = x_1.$$

In this setup, we argue as in the proof of Theorem 11.225 by letting $X_K^L (\zeta, \nu)$ and $X_{L\cap K}^L (\zeta_L, \nu)$ be the underlying Harish-Chandra modules for $I_{MAN}^L (\zeta, \nu)$ and $I_{MLANL}^L (\zeta_L, \nu),$ respectively. We shall prove, in a moment, that

$$\beta^S (X_{L\cap K}^L (\zeta_L, \nu)) \cong X_K^L (\zeta, \nu). \quad (5.2.8)$$
Assuming (5.2.8), we have, just as in the proof of Theorem 11.225,
\[ \mathcal{L}(X_{L \cap K}([\zeta_L, v])^b) \cong \mathcal{L}(X_{L \cap K}([\zeta_L^b, v])^b) \quad \text{since } \zeta_L \text{ is unitary} \]
\[ \cong \mathcal{R}(X_{L \cap K}([\zeta_L^b, v])^b) \quad \text{by [9, (6.24), (6.21a)]} \]
\[ \cong \mathcal{R}(X_{L \cap K}([\zeta_L^b, (v)^b])) \quad \text{by [9, Cor. 11.59]} \]
\[ \cong \mathcal{R}(X_{L \cap K}((v)\zeta_L, -v)) \quad \text{by admissibility of } \zeta_L \]
\[ \cong X_K((v)\zeta_L, (v)^b) \quad \text{by admissibility of } \zeta \]
\[ \cong (X_K((v)\zeta_L, v))^b \quad \text{by [9, Cor. 11.59].} \]

Setting \( j = S \), taking \((-)^b\) of both sides and again using admissibility, we see that (5.2.8) implies the result of the theorem.

To begin the proof of (5.2.8) we start by writing, via (11.210),
\[ X_{L \cap K}([\zeta_L, v]) = (\mathcal{R}^L_{b + n_m} \mathcal{T}_{n}) (\mathcal{Z}_L \otimes \mathcal{C}_v). \]

Since \( A^+ (L, b') \) does not contain any imaginary roots, the index \( p = 0 \). Therefore, a Mackey isomorphism and an induction-in-stages result give
\[ \mathcal{R}(X_{L \cap K}([\zeta_L, v])^b) = (\mathcal{R}^L_{b + n_m} \mathcal{T}_{n}) (\mathcal{Z}_L \otimes \mathcal{C}_v). \]

\[ \mathcal{R}^L_{b + n_m} \mathcal{T}_{n} = (\mathcal{R}^L_{b + n_m} \mathcal{T}_{n}) \mathcal{R}(\mathcal{Z}_L \otimes \mathcal{C}_v). \]

At this point in the proof, we would like to change the Borel subalgebra \( b' + u + n_m \rightarrow \) to the Borel subalgebra \( b' + n' + n_m \) where \( n_m \) is the nilpotent radical of \( b_m \). In order to make this change, we shall apply Lemma 11.128 of [9], which we restate here:

**Lemma 11.128 [9]** Let \( \{(b, T), \lambda, A^\text{imag}, A^\text{real}, \{Z(b)\}\} \) be a set of data for standard \((q, K)\) modules satisfying (i) and either (ii) or ((ii')) in (11.110), and let \( b = b' \oplus n \) and \( b' = b' \oplus n' \) be two Borel subalgebras satisfying the conditions

(i) \( A^\text{imag} \subseteq A(n) \cap A(n') \);

(ii) \( A^\text{real} \subseteq A(n) \cap A(n') \);

These conditions ensure that the change of subalgebras can be made without altering the admissibility properties of the modules involved.
whenever $x$ is a complex root with $x \in \mathcal{A}(n')$ but not in $\mathcal{A}(n)$ and is such that $\langle \lambda, \hat{x} \rangle$ is a nonzero integer, then the integer is positive, and $\theta x$ is in $\mathcal{A}(n) \cap \mathcal{A}(n')$.

If $p = \dim(n \cap k)$ and $p' = \dim(n' \cap k)$, then for all $q$

\[
\begin{align*}
(\sigma b^+_{\mathcal{K}})^{p+q} (Z(b)) &\cong (\sigma b^+_{\mathcal{K}})^{p'} (Z(b')) \\
(\sigma b^-_{\mathcal{K}})^{p+q} (Z(b)) &\cong (\sigma b^-_{\mathcal{K}})^{p'} (Z(b')).
\end{align*}
\]

In our application of this lemma, $b' + u + n^{-1}_L$ will play the role of $b'$ and $b' + n^{-1} + n_{im}$ will play the role of $b$. Condition (11.110) is used only in the definition of “standard module” [cf. 9, Section XI.6] and has no role in the proof.

In this case, conditions (i) and (ii) in the lemma are immediate from (5.2.7). Therefore, we only need to show that condition (iii) is satisfied. First, let us be a bit more explicit about what needs to be shown.

We set

\[
\begin{align*}
C &= \{ \gamma \in \mathcal{A}_{cplx}(u + n^{-1}_L) \mid \gamma \notin \mathcal{A}(n^{-1} + n_{im}) \text{ and } \langle \lambda_{n_{im}} + v, \hat{\gamma} \rangle \in \mathbb{Z} - \{0\} \}.
\end{align*}
\]

We are to show that if $\gamma \in C$ then

(a) $\theta \gamma \in \mathcal{A}(u + n^{-1}_L) \cap \mathcal{A}(n^{-1} + n_{im})$ and

(b) $\langle \lambda_{n_{im}} + v, \hat{\gamma} \rangle > 0$.

Let $\gamma \in C$. If $\gamma \notin \mathcal{A}(n^{-1}_L)$, then $\gamma \in \mathcal{A}(n^{-1}) \subseteq \mathcal{A}(n^{-1} + n_{im})$. So when $\gamma \in C$ we must have $\gamma \in \mathcal{A}(u)$. Since $\mathcal{A}(u)$ is $\theta$ stable $\theta \gamma \in u$ and hence also in $\mathcal{A}(u + n^{-1}_L)$. Moreover, $\gamma \in \mathcal{A}(n^{-1})$ and the fact that $\gamma$ is a complex root gives that $-\gamma \in \mathcal{A}(n^{-1})$ which is closed under conjugation. Therefore $-\gamma = \theta \gamma$ is in $\mathcal{A}(n^{-1})$ and also in $\mathcal{A}(n^{-1} + n_{im})$. Hence we have (a), i.e., $\theta \gamma \in \mathcal{A}(u + n^{-1}_L) \cap \mathcal{A}(n^{-1} + n_{im})$.

For (b), using the information in the above paragraph we can reexpress $C$ as

\[
C = \{ \gamma \in \mathcal{A}_{cplx}(u) \mid -\gamma \notin \mathcal{A}(n^{-1}) \text{ and } \langle \lambda_{n_{im}} + v, \hat{\gamma} \rangle \in \mathbb{Z} - \{0\} \}.
\]

Further, because $A$ is one dimensional, we have the following equivalences

\[
-\gamma \in \mathcal{A}(n^{-1}) \iff \langle -\gamma, v \rangle < 0
\]

\[
\iff \langle -\gamma, x_1 \rangle < 0 \quad \text{since} \quad v = cx_1 \text{ with } c > 0
\]

\[
\iff \langle \gamma, x_1 \rangle > 0
\]

which allow us to write

\[
C = \{ \gamma \in \mathcal{A}_{cplx}(u) \mid \langle \gamma, x_1 \rangle > 0 \text{ and } \langle \lambda_{n_{im}} + v, \hat{\gamma} \rangle \in \mathbb{Z} - \{0\} \}.
\]
If we apply the inverse Cayley transform with respect to $\alpha_1$ to $h_0$ so that we once again have a compact Cartan subalgebra of, and identify the roots with their image under this transformation we then may write

$$C = \{ \gamma \in D(\mathfrak{u}) \mid \langle \gamma, \alpha_1 \rangle > 0, \langle s_{\gamma}(\lambda + \delta), \bar{\gamma} \rangle \in \mathbb{Z} - \{0\} \}$$

$$= \{ \gamma \in D^+(\mathfrak{g}) \mid \langle \gamma, \alpha_1 \rangle > 0, \langle s_{\gamma}(\lambda + \delta), \bar{\gamma} \rangle \in \mathbb{Z} - \{0\} \}.$$

Hence, by hypothesis (5.2.3), we have that (b) holds as well.

Therefore, Lemma 11.128 is applicable. So, in the last line of (5.2.9) we change the Borel subalgebra $h' + u + n_L$ to the Borel subalgebra $h' + n^+ + n_m$. At this point, the arguments given with [9, Th. 11.216] complete the proof of (5.2.8) and therefore the proof of Theorem 5.3.

We again specialize to the setup of the Conjectural Method and apply Theorem 5.3 to the various steps of Knapp’s Conjectural Method to obtain the following corollaries.

**Corollary 5.4.** Using the notation of the Conjectural Method, let

$$C_j = \{ \gamma \in D^+(m_{j-1}) - \{x_j\} \mid \langle \gamma, x_j \rangle > 0, \langle s_{\gamma}(\lambda_{m_{j-1}}), \bar{\gamma} \rangle \in \mathbb{Z} - \{0\} \}$$

If

$$\langle s_{\gamma}(\lambda_{m_{j-1}}), \bar{\gamma} \rangle > 0 \text{ for all } \gamma \in C_j$$

then (5.2.1) holds; that is, the domain space of $L_{S_j}(\varphi_j)$ in (5.1.4) is a $(m_{j-1}, M_{j-1} \cap K)$ principal series module, which has been written in Section 5.2 as

$$L_{S_j}(X_{L_j \cap K}(\xi L_j, \nu_j)) \cong X_{M_{j-1} \cap K}(\xi^h L_j, \nu_j),$$

where $\xi^h$ is an $M_j$ representation with underlying $(m_j, M_j \cap K)$ module $A_{b_{m_j}}(\lambda_j)$.

Furthermore, Corollary 5.4 combined with Corollary 5.2 gives

**Corollary 5.5.** Assuming (5.2.10), there exists an $(m_{j-1}, M_{j-1} \cap K)$ module map which has been written in Section 5.1 as

$$L_{S_j}(\varphi_j): X_{M_{j-1} \cap K}(\xi^h L_j, \nu_j) \rightarrow A_{b_{m_{j-1}}}(\lambda_{j-1})$$

whose image contains the $(M_{j-1} \cap K)$ type with highest weight $A_{j-1}$. 

5.3. Step (4) of the Approach

We conclude this section with the final reduction:

**Corollary 5.6.** Using the notation of the Conjectural Method, let

\[ C_j = \{ \gamma \in A^+(m_{j-1}) - \{ x_j \} \mid \langle \gamma, x_j \rangle > 0, \langle x_j, \lambda_{n_j} \rangle, \gamma \rangle \in \mathbb{Z} - \{0\} \} . \]

Suppose the recursive process of the Conjectural Method stops after \( n \) steps. If (5.2.10) holds for all \( j, 1 \leq j \leq n \), then there exists a \((g, K)\) map from a standard continuous series module to \( A_\lambda(\lambda) \),

\[ \Phi: X_K(\xi^h_{n}, v^n) \to A_\lambda(\lambda), \]

(5.2.11)

whose image contains the \( K \) type with highest weight \( A \).

**Proof.** Apply the functor

\[ X_K(\cdot, v^{l-1}): \mathfrak{g}(m_{j-1}, M_{j-1} \cap K) \to \mathfrak{g}(g, K) \]

to the mapping (5.1.4) of Corollary 5.5. Denote the resulting \((g, K)\) map by \( \phi_j \), and use double induction [9, p. 740] to combine the resulting domain space as \( X_K(\xi^h_j, v_j + v^{l-1}) \). Since \( v_j + v^{l-1} = v^l \) and \( A_{bn-1}(\tilde{\lambda}_{j-1}) \) can be identified with \( \xi^h_j \), we have the \((g, K)\) maps

\[ \phi_j: X_K(\xi^h_j, v^j) \to X_K(\xi^h_{j-1}, v^{l-1}). \]

By Corollary 5.5 and Frobenius reciprocity, the \( K \) type with highest weight \( A \) lies in the image of each \( \phi_j \). Therefore, if the Conjectural Method terminates after \( n \) steps, we can compose the \( \phi_j \) maps to create a \((g, K)\) map

\[ \Phi: X_K(\xi^h_{n}, v^n) \to X_K(\xi^h_0, v^0) \]

whose image contains the \( K \) type with highest weight \( A \). We have \( M_0 = G \) and \( A_0 = I \) so that the range representation of \( \Phi \) is \( \xi^h_0 = A_\lambda(\lambda) \):

\[ \Phi: X_K(\xi^h_{n}, v^n) \to A_\lambda(\lambda) \]

Since the Conjectural Method stops when the infinitesimal character of \( \xi^h_n \), namely \( \lambda_{n_j} \), is \( A^+(m_{j-1}) \) dominant, \( \xi^h_n \) is a discrete series (or limit of discrete series) module.

**Remark.** It is the mapping \( \Phi \) that allows one to read off the Langlands parameters of the submodule \( V \) of \( A_\lambda(\lambda) \) generated by the \( K \) type \( A \), provided that (5.2.10) holds for all \( j \).
6. APPLICATION OF THE RESULTS

In order to use Corollary 5.6, we need a situation in which condition (5.2.10) holds at each step, \( j \), of the Conjectural Method. In this chapter, we provide such a situation, culminating with Theorem 6.9 and Corollary 6.10.

6.1. Main Hypotheses

The setup for this chapter will be similar to that in Section 4.2. \( G \) is a linear, simple, noncompact Lie group with finite center, \( K \) a maximal compact subgroup, and \( \text{rank } G = \text{rank } K \). Let \( T \subseteq K \) be a Cartan subgroup, and let \( \mathcal{A} = \mathcal{A}(g, t) \) be the set of roots. Fix a positive system \( \mathcal{A}^+ = \mathcal{A}^+(g, 1) \). To obtain the best results we will assume

\[ (*) \quad \text{there is exactly one noncompact simple root of } \mathcal{A}^+(g), \text{ and this root has multiplicity at most two in the highest root of } \mathcal{A}^+(g). \]

Define \( A_K^+, \delta, \delta_K, \) and \( \delta(p) \) as usual. Let \( \lambda \) be an analytically integral form on \( t \), and let

\[ A = \lambda + 2\delta(p) = (\lambda + \delta) + (\delta - 2\delta_K). \]

**Main Hypotheses.** The hypotheses that we will invoke this chapter are

1. (ia) \( \langle \lambda, \beta \rangle \geq 0 \) for all simple roots \( \beta \), except for one noncompact simple root, \( \alpha_1 \), for which \( \langle \lambda + \delta, \delta_1 \rangle \) is a negative integer, or
2. (ib) \( \langle \lambda + \delta, \beta \rangle \geq 0 \) for all simple roots \( \beta \), except for one noncompact simple root, \( \alpha_1 \), for which \( \langle \lambda + \delta, \delta_1 \rangle \) is a negative integer, and
3. \( A := \lambda + 2\delta(p) = (\lambda + \delta) + (\delta - 2\delta_K) \) is \( A_K^+ \) dominant.

Note. Since \( \langle \delta, \beta \rangle > 0 \) for simple roots \( \beta \), if condition (ia) holds, then so does condition (ib).

6.2. Satisfying the Reduced Conditions when \( j = 1 \)

Let us recall what needs to be shown. Fix \( j \geq 1 \). Let

\[ C_j = \{ \gamma \in \mathcal{A}^+ + (m_{j-1}) - \{ \alpha_j \} \mid \langle \gamma, \alpha_j \rangle > 0, \langle s_{\alpha_j}(\lambda_{j-1}), \gamma \rangle \in \mathbb{Z} - \{0\} \}. \]

We need to show that

\[ \text{if } \gamma \in C_j \text{ then } \langle s_\alpha(\lambda_{j-1}), \gamma \rangle > 0. \quad (6.2.1) \]

To start the proof of (6.2.1), we let \( j = 1 \) so that

\[ C_1 = \{ \gamma \in \mathcal{A}^+(g) - \{ \alpha_1 \} \mid \langle \gamma, \alpha_1 \rangle > 0, \langle s_{\alpha_1}(\lambda + \delta), \gamma \rangle \in \mathbb{Z} - \{0\} \}. \]
First, note that if $\gamma \in C_1$ then the condition $\langle \gamma, \alpha_1 \rangle > 0$ forces the coefficient of $\alpha_1$ in the $A^+(g)$-simple expansion of $\gamma$ to be $\geq 1$.

**Proposition 6.1.** Let $\gamma \in C_1$ and suppose that the coefficient of $\alpha_1$ in the $A^+(g)$-simple expansion of $\gamma$ is one. Assuming (ib), we have $\langle s_\alpha(\lambda + \delta), \gamma \rangle > 0$.

**Remark.** We are imposing no compactness/noncompactness restrictions on the remaining simple roots of $A^+(g)$. In particular, we are not assuming (*). Further, we are not assuming (ii).

**Proof.** We write $\gamma$ in its $A^+(g)$-simple expansion as

$$\gamma = \sum a_i \alpha_i + \sum b_i \beta_i + ax_1,$$

where the first sum is over the simple roots non-adjacent to $\alpha_1$, the second sum is over the simple roots adjacent to $\alpha_1$, $a_i \geq 0$, $b_i \geq 0$ with some $b_i > 0$.

Then $0 < \langle \gamma, \beta_i \rangle = \sum b_i \langle \beta_i, \beta_i \rangle + 2a$. If we set $\kappa = \sum b_i \langle \beta_i, \beta_i \rangle$, then $\kappa$ is a strictly negative integer. Therefore, $0 < -\kappa < 2a$ with each inequality strict. So, if $a = 1$, then $\kappa = -1$, $\kappa + a = 0$, and $\langle \gamma, \beta_1 \rangle = \kappa + 2a = 1$. We compute

$$\langle s_\alpha(\lambda + \delta), \gamma \rangle = \langle \lambda + \delta, s_\alpha \gamma \rangle$$

$$= \langle \lambda + \delta, \gamma - \langle \gamma, \beta_1 \rangle \alpha_1 \rangle$$

$$= \langle \lambda + \delta, \gamma - \alpha_1 \rangle$$

$$= \left\langle \lambda + \delta, \sum a_i \alpha_i + \sum b_i \beta_i \right\rangle$$

$$> 0, \quad \text{by (ib)}.$$

Further, since $\gamma \in C_1$, we in fact have $\langle s_\alpha(\lambda + \delta), \gamma \rangle > 0$.

**Proposition 6.2.** Assume (ia) and (ii). Let $\gamma \in C_1$ and suppose that the coefficient of $\alpha_1$ in the $A^+(g)$-simple expansion of $\gamma$ is two. Assume further that $\gamma \in A_K$ and that $|\gamma|^2 \geq |\alpha_1|^2$. Then $\langle s_\alpha(\lambda + \delta), \gamma \rangle > 0$.

**Proof.** We have

$$\langle s_\alpha(\lambda + \delta), \gamma \rangle = \langle \lambda + \delta, \gamma \rangle - \langle \lambda + \delta, \beta_1 \rangle \langle \alpha_1, \gamma \rangle,$$

and therefore

$$\langle s_\alpha(\lambda + \delta), \gamma \rangle = \langle A, \gamma \rangle + \langle 2\delta_K - \delta, \gamma \rangle - \langle \lambda + \delta, \beta_1 \rangle \langle \alpha_1, \gamma \rangle.$$
Summing, we get
\[ 2 \langle s_n(\lambda + \delta), \gamma \rangle = \langle A, \gamma \rangle + \left[ \langle 2\delta - \delta, \gamma \rangle - 2 \langle \lambda + \delta, \delta_1 \rangle \right] + \langle x_1, \gamma \rangle + \langle \lambda + \delta, \gamma \rangle \]
\[ = \langle A, \gamma \rangle + \left[ \langle 2\delta, \gamma \rangle - 2 \langle \lambda + \delta, \delta_1 \rangle + \langle \lambda, \gamma \rangle \right] \quad \text{since} \quad \langle x_1, \gamma \rangle = 1 \]
\[ \geq \langle A, \gamma \rangle + [2 - 2 \langle \lambda, \delta_1 \rangle - 2 + \langle \lambda, \gamma \rangle] \quad \text{since} \quad \langle 2\delta, \gamma \rangle \geq 2 \]
\[ \quad \quad \quad \quad \text{and} \quad \langle \delta, \delta_1 \rangle = 1 \]
\[ = \langle A, \gamma \rangle + \langle \lambda, \gamma \rangle - 2\delta_1 \quad \text{by (ii)}. \]
\[ \geq \langle \lambda, \gamma \rangle - 2\delta_1 \quad \text{by (ii)}. \]

Now, \[ \gamma = (2/|\gamma|^2)(\gamma - (2/|x_1|^2)(2x_1) = (2/|\gamma|^2)(\gamma - (2/|x_1|^2)(2x_1)), \]
where \(|\gamma|^2/|x_1|^2 \geq 1\). We let \(c = |\gamma|^2/|x_1|^2 \in \{1, 2, 3\}\). Writing \(\gamma = \sum b_i \beta_i + ax_1\) as a sum of simple roots of \(A_+^+(g)\) so that each \(b_i \geq 0\) and \(a = 2\), we get
\[ \gamma = (2/|\gamma|^2)(\gamma - 2x_1) = (2/|\gamma|^2)[\sum b_i \beta_i + (a - 2c)x_1]. \]
So,
\[ 2 \langle s_n(\lambda + \delta), \gamma \rangle = \langle \lambda, \gamma \rangle - 2\delta_1 \geq \langle \lambda, \gamma \rangle - 2\delta_1. \]

By (ia), each \(\langle \lambda, \beta_i \rangle \geq 0\), and
\[ \langle \lambda, x_1 \rangle = \langle \lambda, \delta_1 \rangle \cdot \frac{|x_1|^2}{2} \]
\[ = \langle \lambda + \delta, \delta_1 \rangle \cdot \frac{|x_1|^2}{2} \quad \text{since} \quad \langle \delta, \delta_1 \rangle = 1 \]
\[ \leq -|x_1|^2 \quad \text{by (ia)} \]
\[ < 0. \]

Hence, since \(a = 2, c \in \{1, 2, 3\}\) gives \((a - 2c) \leq 0\), we have from (6.2.2) that
\[ 2 \langle s_n(\lambda + \delta), \gamma \rangle \geq 0. \]
Finally, since \(\gamma \in C_1\), we get that this is a strict inequality.

**Proposition 6.3.** Suppose the Dynkin diagram of \(A_+^+(g)\) is of type \(B_n\). Assume (ia), (ii), and that \(\alpha_1\) is a long simple root. If \(\gamma \in C_1 \cap A_+^\perp\), then \(\langle s_n(\lambda + \delta), \gamma \rangle > 0\).

**Proof.** First suppose \(\gamma\) is a long root, so that \(|\gamma|^2 = |x_1|^2\). Moreover, the coefficient of \(x_1\) in \(\gamma\) is \(\leq 2\). Therefore, by the above proposition, we have
\[ \langle s_\lambda(\lambda + \delta), \gamma \rangle > 0. \] Next, suppose \( \gamma \) is a short root. In this case, the coefficient of \( \lambda_1 \) in \( \gamma \) is one. Therefore, Proposition 6.1 gives the result.

Putting these three propositions together, along with a further assumption that \( \lambda_1 \) is the lone noncompact simple root of \( \mathcal{A}^+(g) \) yields the following conclusion:

**Theorem 6.4.** Assume conditions (\( \ast \)), (iia) and (iib). Then for all \( \gamma \in C_1 \), we have \( \langle s_\lambda(\lambda + \delta), \gamma \rangle > 0. \)

**Remark.** The lone-noncompact simple root hypothesis of (\( \ast \)) placed on \( \lambda_1 \) allows us to characterize \( \mathcal{A}^+_{\lambda} \) as the set of roots which contain \( \lambda_1 \) an even number of times in its \( \mathcal{A}^+(g) \)-simple expansion.

**Proof.** Let \( \gamma \in C_1 \). If the coefficient of \( \lambda_1 \) in \( \gamma \) is one, then we get the result from Proposition 6.1—without using condition (ii). So assume that the coefficient of \( \lambda_1 \) in \( \gamma \) is two. Since \( \lambda_1 \) is the lone noncompact simple root of \( \mathcal{A}^+(g) \), \( \gamma \) is compact. Now we just run through the allowable cases.

If the Dynkin diagram is a single line diagram, then Proposition 6.2 gives the result.

If the Dynkin diagram of \( \mathcal{A}^+(g) \) is of type \( B_n \) and \( \lambda_1 \) is a short root, then Proposition 6.2 gives the result. If \( \lambda_1 \) is a long root, then Proposition 6.3 gives the result.

If the Dynkin diagram of \( \mathcal{A}^+(g) \) is of type \( C_n \), then \( \lambda_1 \) must be a short root by the coefficient two assumption on \( \gamma \). Therefore, Proposition 6.2 gives the result.

If the Dynkin diagram of \( \mathcal{A}^+(g) \) is of type \( F_4 \), then \( \lambda_1 \) can be either node. If \( \lambda_1 \) is the short root node, then Proposition 6.2 gives the result. If \( \lambda_1 \) is the long root node, then the coefficient two assumption on \( \gamma \) forces \( \gamma \) to be the highest root of \( F_4 \), which is a long root. So then \( |\gamma|^2 = |\lambda_1|^2 \) and Proposition 6.2 applies.

If the Dynkin diagram of \( \mathcal{A}^+(g) \) is of type \( G_2 \), then \( \lambda_1 \) is the long simple root. Again, the coefficient two assumption on \( \gamma \) forces \( \gamma \) to be the highest root of \( G_2 \), which is a long root. So then \( |\gamma|^2 = |\lambda_1|^2 \) and Proposition 6.2 applies.

6.3. Satisfying the Reduced Conditions when \( j > 1 \)

As previously noted, from [7] we know that

1. \( \lambda_j \) is analytically integral,
2. \( \lambda_j \) is \( \mathcal{A}^+_{\lambda_j} \)-dominant, and
3. \( \lambda_j \) is dominant for the compact simple roots of \( \mathcal{A}^+_{\lambda_j} \).

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so that the Conjectural Method runs into no obstacles. However, our Approach does hit a slight snag; part of the Main Hypotheses on \( \lambda \) and \( A^+ (g) \) may not be inherited by \( \lambda_j \) and \( A^+ (m_j) \). In particular, although \( \lambda \) is nondominant versus only one simple root of \( A^+ (g) \), when we apply the Conjectural Method, this may no longer hold for \( \lambda_{j-1} \) and \( A^+ (m_{j-1}) \) when \( j > 1 \). As a result, we need to supplement some of the arguments of Section 6.2 to handle this possibility.

Recall what the goal is: Let \( j > 1 \) and
\[
C_j = \{ \gamma \in A^+(m_{j-1}) - \{ x_j \} \mid \langle \gamma, x_j \rangle > 0, \langle s_{x_j}(\lambda_{j-1}), \gamma \rangle \in \mathbb{Z} - \{ 0 \} \}.
\]
We are to prove, as in Section 6.2, that
\[
\text{if } \gamma \in C_j, \text{ then } \langle s_{x_j}(\lambda_{j-1}), \gamma \rangle > 0. \tag{6.2.1}
\]

We start by investigating the noncompact roots of \( A^+(m_{j-1}) \). To do so, we shall assume (\( \ast \)) throughout this section:

(\( \ast \)) there is exactly one noncompact simple root of \( A^+(g) \), and this root has multiplicity at most two in the highest root of \( A^+(g) \).

**Proposition 6.5.** Assuming (\( \ast \)), every \( A^+(m_j) \) noncompact root is \( A^+(g) \) noncompact.

**Note.** First we recall why this proposition is not entirely obvious. The property of an imaginary root being compact or noncompact is not always preserved by the Cayley transform [cf. 9, Prop. 11.249]. If \( \beta \) is a compact (resp. noncompact) root in \( A^+(g, t^0) \) that is strongly orthogonal to \( x_1 \), then it remains compact (resp. noncompact) as a root in \( A^+(m_1, t^0) \). But if \( \beta \) is orthogonal to \( x_1 \) but not strongly orthogonal, then \( \beta \) is noncompact (resp. compact) as a root in \( A^+(m_1, t^0) \).

**Proof.** The proof proceeds by induction on \( i \). Let \( i = 1 \). Let \( \beta \in A^+(m_1, t^0) \) be noncompact. Then either

(a) \( \beta \) is noncompact in \( A^+(g) \) and \( \beta \) and \( x_1 \) are strongly orthogonal, or

(b) \( \beta \) is compact in \( A^+(g) \), but \( \beta \) and \( x_1 \) are not strongly orthogonal.

We show that case (b) cannot exist.

Suppose (b) holds. Since \( \beta \) and \( x_1 \) are not strongly orthogonal, and \( x_1 \) is simple, both \( \beta \pm x_1 \) are positive roots. Moreover, since \( \beta \) is \( A^+(g) \) compact, (\( \ast \)) implies that \( \beta \) contains \( x_1 \) with coefficient 0 or 2 in its \( A^+(g) \) simple expansion. If the coefficient of \( x_1 \) in \( \beta \) is 0, then \( \beta - x_1 \) has a negative \( x_1 \) coefficient while the coefficient of another simple \( A^+(g) \) root is
positive. This is not possible [5, Prop. 4.6], so we must have that the coefficient of \( \alpha_1 \) in \( \beta \) is 2. But in this case the root \( \beta + \alpha_1 \) has \( \alpha_1 \) coefficient 3, which contradicts assumption (*). Hence any root with \( \alpha_1 \) coefficient 0 or 2 which is also orthogonal to \( \alpha_1 \) must be strongly orthogonal to \( \alpha_1 \). Therefore, case (b) never occurs, and we have the base step of the induction.

Next suppose the proposition holds for \( i = j - 1 \), and let \( \beta \) be a noncompact root of \( A^+(m_i) \). Suppose \( \beta \) is compact as a root of \( A^+(m_{j-1}) \). Then \( \beta \pm \alpha_j \) are also roots of \( A^+(m_{j-1}) \). In fact, since \( \alpha_j \) is a root of \( A^+(m_{j-1}) \) noncompact, both \( \beta \pm \alpha_j \) are \( A^+(m_{j-1}) \) noncompact. Hence, by the inductive hypothesis, both \( \beta \pm \alpha_j \) are noncompact and therefore each contains \( \alpha_1 \) with coefficient one in its \( A^+(g) \) simple expansion. This forces the \( \alpha_1 \) coefficient of \( \alpha_j \) in its \( A^+(g) \) simple expansion to be 0, which is a contradiction. Therefore \( \beta \) cannot be a compact root of \( A^+(m_{j-1}) \).

On the other hand, if \( \beta \) is noncompact as a root of \( A^+(m_{j-1}) \) then the inductive hypothesis gives the result.

This proposition and the assumption that \( \alpha_1 \) is the lone noncompact simple root of \( A^+(g) \) combine to give the following:

**Corollary 6.6.** Assuming (*), any noncompact root of \( A^+(m_i) \) has \( \alpha_1 \) coefficient one in its \( A^+(g) \) simple expansion.

Recall that we are trying to handle the situation in which \( \lambda_{\alpha_{j-1}} \) is nondominant with respect to some noncompact simple roots of \( A^+(m_{j-1}) \), and \( j > 1 \). If \( \gamma \in C_j \), then \( \langle \gamma, \alpha_j \rangle \neq 0 \) so that \( \gamma \) is a root in the same connected component of the Dynkin diagram of \( A^+(m_{j-1}) \) as \( \alpha_j \). Therefore, if there is at most one noncompact simple root per connected component of the Dynkin diagram of \( A^+(m_{j-1}) \), then our Main Hypotheses do indeed pass to this next stage, and we have no difficulties applying the propositions of the previous sections.

**Proposition 6.7.** Assume (ib), and that the coefficient of \( \alpha_1 \) in the highest root of \( A^+(g) \) is one. Then, for all \( j \), there is at most one noncompact simple root of \( A^+(m_{j-1}) \) against which \( \lambda_{\alpha_{j-1}} \) is nondominant. Therefore, for all \( j \), (6.2.1) holds.

**Remark.** We are making no compactness/noncompactness assumptions on the remaining simple roots of \( A^+(g) \). Further, we are not assuming (ii).

**Proof.** By (ib), \( \lambda_{\alpha_{j-1}} \) can be nondominant versus a root only if that root contains \( \alpha_1 \) in its \( A^+(g) \) simple expansion. Further, since the sum of
all simple roots of $A^+(m_{j-1})$ is also a root, the coefficient one restriction implies that there can be at most one simple root per component of $A^+(m_{j-1})$ containing $x_1$. By changing appropriate indices, Proposition 6.1 then implies (6.2.1).

Let us now consider the case in which there is more than one noncompact simple root in some connected component of the Dynkin diagram for $A^+(m_{j-1})$ against which $\lambda_{\gamma_{j-1}}$ is nondominant. First, under our Main Hypotheses, $\lambda_{\gamma_{j-1}}$ can be nondominant versus a root only if that root contains $x_1$ in its $A^+(g)$ simple expansion. Second, since the sum of the simple roots of a Dynkin diagram is also a root, and since, assuming ($*$) so that $x_1$ occurs with coefficient $\leq 2$ in the $A^+(g)$ highest root, there can be at most two simple roots in any component of $A^+(m_{j-1})$ which contain $x_1$ in their $A^+(g)$ simple expansion. Third, if there are two simple roots in a component of $A^+(m_{j-1})$ which contain $x_1$, then each of these roots has coefficient one in the $A^+(m_{j-1})$-component highest root, for otherwise, the highest root in $A^+(m_{j-1})$ contains $x_1$ with multiplicity greater than two. Fourth, we note that if a Dynkin diagram contains two simple roots with coefficient one in the highest root, then that diagram is a single line diagram. Finally, with the exception of the $A_n$-type diagrams, no Dynkin diagram has adjacent simple roots contained with coefficient one in its highest root.

According to the recursive procedure, of the two $\lambda_{\gamma_{j-1}}$-nondominant, simple roots, we let $x_j$ be the root for which $\langle x_{\gamma_{j-1}}, x_j \rangle$ is greatest. Even though the other simple root may not turn out to be $x_{j+1}$ in the process, for ease of notation in the next few arguments, we call this root $x_{j+1}$.

Theorem 6.8. Assuming ($*$), (ia) and (ii) of the Main Hypotheses, if $\gamma \in C_j$ then $\langle x_{\gamma_{j-1}}, x_j \rangle > 0$ for all $j$.

Proof. By the above paragraph, we only need to consider the case in which $\lambda_{\gamma_{j-1}}$ is nondominant against two simple roots of $A^+(m_{j-1})$ which lie in the same single-lined Dynkin component, and which are contained in the $A^+(m_{j-1})$ highest root with coefficient one.

We know that in order to have $\langle \gamma, x_j \rangle > 0$, when expanded into $A^+(m_{j-1})$ simple roots, $\gamma$ must contain $x_j$. In fact, since $x_j$ is contained in the $A^+(m_{j-1})$ highest root with coefficient one, $\gamma$ contains $x_j$ with coefficient one. Moreover, since we are in a single-line diagram, we have $\langle \gamma, \lambda_{\gamma_{j-1}} \rangle = 1$.

Expand $\gamma$ into its $A^+(m_{j-1})$ simple expansion as

$$\gamma = \sum k_i \lambda_{\gamma_{i-1}} + x_{j+1} + x_j$$
so that
\[ \langle s_j (\lambda_{\gamma^{-1}}), \gamma \rangle = \langle \lambda_{\gamma^{-1}}, s_j (\gamma) \rangle = \langle \lambda_{\gamma^{-1}}, \gamma - \alpha_j \rangle \]
\[
= \langle \lambda_{\gamma^{-1}}, \sum k_i \kappa_i + x \alpha_{j+1} \rangle 
\geq \langle \lambda_{\gamma^{-1}}, x \alpha_{j+1} \rangle \quad \text{since only } \alpha_j \text{ and } \alpha_{j+1} \text{ contain } \alpha_1.
\]

Therefore, we have the result when \( x = 0 \).

So suppose \( x \neq 0 \). By our assumptions, \( x \) must be one. So, as a root of \( \mathcal{A}^+(\gamma) \), \( \gamma \) contains \( \alpha_1 \) with coefficient two and is therefore compact. Hence, by Proposition 6.5, \( \gamma \) is \( \mathcal{A}^+(\mathfrak{m}_{\gamma^{-1}}) \) compact. Writing \( \lambda_{\gamma^{-1}} = \lambda_{j-1} + \delta(\mathfrak{m}_{\gamma^{-1}}) \), and using the fact [7, Prop. 10] that \( A_{j-1} \) is \( \mathcal{A}^+(\mathfrak{m}_{j-1} \cap \mathfrak{t}) \) dominant, we imitate the proof of Proposition 6.3 to write
\[
2 \langle s_j (\lambda_{\gamma^{-1}}), \gamma \rangle \geq \langle \lambda_{j-1}, \gamma - 2 \alpha_j \rangle = \langle \lambda_{j-1}, \sum k_i \kappa_i + 2 \alpha_{j+1} - \alpha_j \rangle
\]

Now, \( \langle \lambda_{j-1}, \sum k_i \kappa_i \rangle \geq 0 \), and
\[
\langle \lambda_{j-1}, \alpha_{j+1} - \alpha_j \rangle = \langle \lambda_{\gamma^{-1}}, \alpha_{j+1} - \alpha_j \rangle \quad \text{since } \alpha_{j+1} \text{ and } \alpha_j \text{ are}
\mathcal{A}^+(\mathfrak{m}_{\gamma^{-1}}) \text{ simple}
\[
= - |\alpha_{j+1}|^2 c_{j+1} + |\alpha_j|^2 c_j
\geq |\alpha_j|^2 (c_j - c_{j+1}) \quad \text{since } |\alpha_j|^2 = |\alpha_{j+1}|^2 \quad \text{by our choice of } c_j.
\]

Therefore, \( \langle s_j (\lambda_{\gamma^{-1}}), \gamma \rangle > 0 \). □

**Remark.** If one wishes to verify (6.2.1) in a particular example, it is often easier to use the following simplification:
\[
\langle s_j (\lambda_{\gamma^{-1}}), \gamma \rangle = \langle s_j ((\lambda + \delta) + \nu^{-1}), \gamma \rangle \quad \text{by (4.2.4)}
\]
\[
= \langle s_j (\lambda + \delta) + \nu^{-1}, \gamma \rangle \quad \text{since } \alpha_j \perp \alpha_i \text{ for } 1 \leq i \leq j - 1
\geq \langle s_j (\lambda + \delta), \gamma \rangle \quad \text{since } \gamma \perp \alpha_i \text{ for } 1 \leq i \leq j - 1.
\]

6.4. **Langlands Parameters of Subquotients of** \( A_k(\lambda) \)

Theorem 6.8. allows us to conclude:
**Theorem 6.9.** Let $G$ be a linear, noncompact simple Lie group with finite center, let $K$ be a maximal compact subgroup, and suppose rank $G = \text{rank } K$. Let $T^0 \subset K$ be a Cartan subgroup, and let $\Delta^+(g, T^0)$ be a positive system of roots such that

(i) there is exactly one noncompact root; call it $\alpha_1$, and

(ii) the coefficient of $\alpha_1$ in the highest root is $\leq 2$.

Let $\lambda$ be an analytically integral form on $T^0$, and set $\Lambda = \lambda + 2\delta(p) = (\lambda + \delta) + (\delta - 2\delta_K)$. Suppose

(iii) $\langle \lambda, \beta \rangle \geq 0$ for all compact simple roots $\beta$, of $\Delta^+(g)$

(iv) $\Lambda$ is $A^+_K$ dominant.

Then there exists a $(g, K)$ map from a standard continuous series module to $\text{Aut}(\lambda)$,

$$\Phi: X_K(\tau^\omega, \nu^\omega) \to A_K(\lambda),$$

whose image contains the (nonzero) $K$ type with highest weight $\Lambda$.

**Corollary 6.10.** In the setting of the above Theorem, let $V$ be the irreducible subquotient of $\text{Aut}(\lambda)$ containing the $K$ type with highest weight $\Lambda$.

1. If $\text{Aut}(\lambda)$ is irreducible then the Conjectural Method produces the Langlands parameters of $\text{Aut}(\lambda)$.

2. If $\text{Aut}(\lambda)$ is infinitesimally unitary, then the Conjectural Method produces the Langlands parameters of $V$.

3. If $\Lambda$ is the minimal $K$ type of $X_K(\tau^\omega, \nu^\omega)$, then the Conjectural Method produces the Langlands parameters of $V$.

4. If $\Lambda$ is the minimal $K$ type of $\text{Aut}(\lambda)$, then the Conjectural Method produces the Langlands parameters of $V$.

5. If $\lambda$ is orthogonal to the compact simple roots of $\Delta^+(g)$, then the Conjectural Method produces the Langlands parameters of $V$.

**Proof.** (1) is clear. For (2), since $\text{Aut}(\lambda)$ is infinitesimally unitary, we can compose the map $\Phi$ of the theorem with a projection onto $V$ to yield the result. (3) is also clear.

For (4), if $\Lambda$ is not also a minimal $K$ type in $X_K(\tau^\omega, \nu^\omega)$, then every minimal $K$ type in $X_K(\tau^\omega, \nu^\omega)$ maps to 0 under $\Phi$. This is a contradiction. Therefore (3) applies.

For (5), setting $\tau = 1 \oplus \nu$ where $l$ is formed from the compact simple roots and $\nu$ from the remaining positive root spaces, we have $\text{Aut}(\lambda) \cong \text{Aut}(\lambda)$. By Corollary 8 of [7], $\Lambda$ is the minimal $K$ type of $\text{Aut}(\lambda)$ and therefore (4) applies. \[\square\]
Final Note. The restriction in (ii) on the multiplicity of the lone noncompact root in the highest root can be removed if $G$ is $E_6$, $F_4$, or $G_2$. If $G = E_7$ (resp. $G = E_8$), then label the simple roots $\{ \beta \}$ in the standard fashion, and let $E_7, \beta$ (resp. $E_8, \beta$) be a real form such that $\beta$ is the lone noncompact simple root of $A^+(g)$. Then we can remove (ii) for $E_7, 3$, $E_7, 4$, $E_8, 3$, $E_8, 4$, and $E_8, 7$. For the remaining cases, $E_7, 5$, $E_8, 2$, $E_8, 5$ and $E_8, 6$, we need to impose a minor nonsingularity condition on $A$ in order to remove (ii).

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