Liouville-type theorems and bounds of solutions of Hardy–Hénon equations

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We consider the Hardy–Hénon equation $-\Delta u = |x|^a u^p$ with $p > 1$ and $a \in \mathbb{R}$ and we are concerned in particular with the Liouville property, i.e. the nonexistence of positive solutions in the whole space $\mathbb{R}^N$. It has been conjectured that this property is true if (and only if) $p < p_S(a)$, where $p_S(a)$ is the Hardy–Sobolev exponent, given by $(N + 2 + 2a)/(N - 2)$. However, when $N \geq 3$, the conjecture had up to now been proved only for $a \leq 0$. Indeed the case $a > 0$ seems more difficult, due to $p_S(a) > (N + 2)/(N - 2)$.

In this paper, we prove the conjecture for $a > 0$ in dimension $N = 3$, in the case of bounded solutions. Next, for the conjecture in the case $a < 0$, and for related estimates near isolated singularities and at infinity, we give new proofs – based in particular on doubling-rescaling arguments – and we provide some extensions of these estimates. These proofs are significantly simpler than the previously known ones. Finally, we clarify some of the previous results on a priori estimates for the related Dirichlet problem.

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1. Introduction

This article is devoted to the study of positive solutions of the following elliptic equation

$$-\Delta u = |x|^a u^p, \quad x \in \Omega,$$

where $p > 1$, $a \in \mathbb{R}$ and $\Omega$ is a domain of $\mathbb{R}^N$ with $N \geq 2$. (For the case $N = 1$, see Proposition A.1 and Remark A.2 in Appendix A.) Eq. (1) is traditionally called the Hénon (resp., Hardy, or Lane–Emden)
equation for $a > 0$ (resp., $a < 0$, $a = 0$). Throughout this paper, unless otherwise specified, solutions are considered in the class

$$
\begin{cases}
C^2(\Omega), & \text{if } a \geq 0, \\
C^2(\Omega \setminus \{0\}) \cap C(\Omega), & \text{if } a < 0,
\end{cases}
$$

and are assumed to satisfy the equation pointwise, except at $x = 0$ if $a < 0$ and $0 \in \Omega$.

Our primary interest is in the Liouville property – i.e. the nonexistence of positive solution in the entire space $\Omega = \mathbb{R}^N$ – and on singularity and decay estimates of solutions. The case $a = 0$ has been widely studied by many authors. Here, the optimal Liouville-type result has been established by Gidas and Spruck in their celebrated article [15]. Namely, Eq. (1) has no positive solution if and only if

$$p < p_S := \frac{N + 2}{N - 2} \quad (= \infty \text{ if } N \leq 2).$$

The case $a \neq 0$ is less completely understood. Let us first recall that if $a \leq -2$, then (1) has no positive solution in any domain $\Omega$ containing the origin (cf. [15], [1, Lemma 6.2] and [13]). We therefore restrict ourselves to the case $a > -2$ in the rest of this article. Let us introduce the Hardy–Sobolev exponent

$$p_S(a) := \frac{N + 2 + 2a}{N - 2} \quad (= \infty \text{ if } N = 2).$$

In the case of radial solutions, we have the following complete result (stated in [15]; see [2] for a detailed proof).

**Proposition A.** Let $N \geq 2$, $a > -2$ and $p > 1$.

(i) *If* $p < p_S(a)$, *then* Eq. (1) *has no positive radial solution in* $\Omega = \mathbb{R}^N$.

(ii) *If* $p \geq p_S(a)$, *then* Eq. (1) *possesses bounded, positive radial solution in* $\Omega = \mathbb{R}^N$.

The Hardy–Sobolev exponent $p_S(a)$ thus plays a critical role in the radial case and this, in addition to the above mentioned result for $a = 0$, supports the following natural conjecture:

**Conjecture B.** If $N \geq 2$, $a > -2$ and $1 < p < p_S(a)$, then Eq. (1) has no positive solutions in $\Omega = \mathbb{R}^N$.

The condition $p < p_S(a)$ is the best possible due to Proposition A(ii). However, apart from the radial case, the best available nonexistence result up to now is the following.

**Theorem C.** Let $N \geq 2$, $a > -2$ and $p > 1$.

(i) *If*

$$p < \min(p_S, p_S(a)).$$

*then* Eq. (1) *has no positive solution in* $\Omega = \mathbb{R}^N$.

(ii) *The conclusion of part (i) remains true if*

$$p \leq \frac{N + a}{N - 2}.$$


Theorem 4.1. The restriction
namely for
less, for solutions in the regularity class (2), the Liouville property is false in part of the range (5),
$q$
some
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in
$\Omega$

Theorem 1.2.

We now turn to the second topic of this paper, which concerns the closely connected subject of singularity and decay estimates. Namely, we will present a simpler proof, as well as an extension, of results from [15,4] for $p < p_S$. By the same token, we will obtain a new and much simpler proof of Theorem C(i), hence in particular of Conjecture B for $a < 0$. Concerning singularity and decay estimates, we have the following:

Theorem 1.1. Let $N \geq 2$, $a > 0$, $p > 1$ and $N = 3$. If $p < p_S(a)$, then Eq. (1) has no positive bounded solution in $\Omega = \mathbb{R}^N$.

Remark 1.1. (a) The proof of Theorem 1.1 uses the technique introduced by Serrin and Zou in [23] and further developed by the second author in [26], which is based on a combination of Pohozaev identity, Sobolev inequality on $S^{N-1}$ and a measure argument. By using additional interpolation and feedback arguments from [26], one could extend the result to higher dimensions $N \geq 4$, but at the expense of the further restriction $p < (N - 1)/(N - 3) \leq p_S$. Therefore, for $N \geq 4$, these techniques do not seem to lead to an improvement of Theorem C(i).

(b) Theorem 1.1 is still true for polynomially bounded solutions, i.e. if $u(x) \leq C|x|^q$ for $x$ large, with some $q > 0$ (see after the end of the proof). We note that, although it would be desirable to show Theorem 1.1 without any growth restriction on the solutions, Liouville type theorems for bounded solutions are usually sufficient for applications such as a priori estimates and universal bounds, obtained by rescaling arguments (see [16,20]).

On the other hand, the Liouville property is not true in general if the continuity assumption in (2)
(at $x = 0$) is relaxed. For instance, (1) admits a distributional solution of the form $u(x) = C|x|^{-\alpha}$, $\alpha = (2 + a)/(p - 1)$, whenever $N \geq 3$, $p > (N + a)/(N - 2)$ and $a > -2$. However, Theorem C(ii) (for $p \leq (N + a)/(N - 2)$) remains true for distributional supersolutions (see [18] and Remark A.1 below).

(c) Prior to [2], Theorem C(i) had been proved in the special case $a \geq 2$ (with $p < p_S$) in [15, Theorem 4.1]. The restriction $a \geq 2$ comes from the assumption that the $x$-depending coefficient be a $C^2$ function.

(d) It is claimed in [16] that the Liouville property is true for

$$a > -2, \quad 1 < p < p_S, \quad p \neq p_S(a)$$

(cf. Theorem [16, Theorem 4.2], which is not proved there but attributed for $a < 0$ to Ref. [3] in the bibliography of [16], a work which doesn’t seem to have actually appeared). However, solutions in Theorem [16, Theorem 4.2] are assumed to be in $C^2(\mathbb{R}^N)$, which is not relevant for $a < 0$. Nevertheless, for solutions in the regularity class (2), the Liouville property is false in part of the range (5), namely for $-2 < a < 0$, $p_S(a) \leq p < p_S$, as shown by Proposition A(ii).

We will present a simpler proof, as well as an extension, of results from [15,4] for $p < p_S$. By the same token, we will obtain a new and much simpler proof of Theorem C(i), hence in particular of Conjecture B for $a < 0$. Concerning singularity and decay estimates, we have the following:

Theorem 1.2. Let $N \geq 2$, $a > -2$ and $1 < p < p_S$. There exists a constant $C = C(N, p, a) > 0$ such that the following holds.
(i) Any nonnegative solution of Eq. (1) in \( \Omega = \{ x \in \mathbb{R}^N; 0 < |x| < \rho \} \) (\( \rho > 0 \)) satisfies
\[
    u(x) \leq C|x|^{\frac{2-a}{p-1}} \quad \text{and} \quad |\nabla u(x)| \leq C|x|^{\frac{p+1-a}{p-1}}, \quad 0 < |x| < \rho/2.
\]  
(ii) Any nonnegative solution of Eq. (1) in \( \Omega = \{ x \in \mathbb{R}^N; |x| > \rho \} \) (\( \rho \geq 0 \)) satisfies
\[
    u(x) \leq C|x|^{\frac{2-a}{p-1}} \quad \text{and} \quad |\nabla u(x)| \leq C|x|^{\frac{p+1-a}{p-1}}, \quad |x| > 2\rho.
\]  

The first part of estimate (6) was proved in [15, Theorem 3.1] and [4, Theorem 6.3] (cf. also [3, Corollary 6.4] for the exterior domain case (ii)). In addition, we also estimate the gradient – a feature that will be used for our proof of Theorem C(i).

Our proof of Theorem 1.2 is based on the observation that estimates (6) and (7) for given \( p, a \) can be rather easily reduced to the Liouville property for the same \( p \) but with a replaced by 0.\(^1\) This reduction relies on two ingredients:

(i) a change of variable, that allows to replace the coefficient \( |x|^a \) with a smooth function which is bounded and bounded away from 0 in a suitable spatial domain;
(ii) a generalization of a doubling-rescaling argument from [20] (see Lemma 2.1 below).

We can then obtain an easy derivation of Theorem C(i) from Theorem 1.2, by combining the Pohozaev identity with the decay estimate (7). We note that the gradient part of estimate (7) is crucial for the proof in order to estimate some of the terms appearing in the Pohozaev identity.

As the third topic of this paper, let us finally consider the associated boundary value problem:

\[
\begin{aligned}
    -\Delta u &= |x|^a u^p, & x &\in \Omega, \\
    u(x) &= \varphi(x), & x &\in \partial \Omega.
\end{aligned}
\]  

Here we assume that
\[
\Omega \subset \mathbb{R}^N \text{ is a smoothly bounded domain containing the origin}
\]  

and that \( \varphi \in C(\partial \Omega) \) is a nonnegative function. It is well known that Liouville-type results enable one to derive a priori bounds for positive solutions of elliptic Dirichlet problems, via the blow-up method of [16]. In the case of (8), this was actually done in [16, Theorem 4.1]. Unfortunately, that statement suffers from shortcomings similar to those mentioned in Remark 1.1(d) above. Namely, it is claimed in [16, Theorem 4.1] that, under assumption (5), positive \( C^2 \) solutions of (8) satisfy a uniform a priori bound. However, no such solutions obviously exist when \( a < 0 \), so that one probably has to interpret this as a statement about positive solutions in the natural class \( C^2(\bar{\Omega} \setminus \{0\}) \cap C(\bar{\Omega}) \). But it turns out (see Theorem 1.3(ii) below) that such an a priori bound is not true for \(-2 < a < 0, p_S(a) \leq p < p_S\). We thus provide the following corrected version of [16, Theorem 4.1].

**Theorem 1.3.** Let \( N \geq 2, a > -2, p > 1 \) and assume (9).\(^1\)

(i) Assume (3). Let \( M > 0 \) and \( 0 \leq \varphi \in C(\partial \Omega) \) with \( \|\varphi\|_\infty \leq M \). Then all positive solutions \( u \in C^2(\bar{\Omega} \setminus \{0\}) \cap C(\bar{\Omega}) \) of problem (8) satisfy
\[
    \|u\|_{L^\infty(\Omega)} \leq C,
\]
where the constant \( C > 0 \) depends only on \( \Omega, a, p, M \).

\(^1\) Of course, the Liouville property for \( p < p_S \) and \( a = 0 \) is a deep result, but its proof (see [15,4] and also [21, Chapter 8]) is easier than in the case \( a \neq 0 \) and, furthermore, an alternative proof [12] by the method of moving planes is known.
(ii) Assume \( p \geq p_S(a) \). Then assertion (i) fails. More precisely, there exists a bounded sequence of real numbers \( b_k > 0 \) and a sequence of solutions \( u_k \) of (8) with \( \Omega = B_1 \) and \( \varphi_k \equiv b_k \), such that \( u_k(0) \to \infty \) as \( k \to \infty \).

In particular for \( a \leq 0 \), it follows that the assumption \( p < p_S(a) \) in assertion (i) is optimal.

We close this introduction by mentioning other work related to the boundary value problem (8). The existence and non-existence of positive solutions of (8), especially for the case \( \varphi = 0 \), have been studied (see for instance [14,19,22], and the references therein). More precisely, if \( a < 0 \), one obtains the existence of a positive solution in \( H^1_0(\Omega) \) provided that \( 1 < p < p_S(a) \), by using variational methods and Caffarelli–Kohn–Nirenberg estimates (see [8]); if \( p \geq p_S(a) \), one proves non-existence of nontrivial solutions in starshaped domains as a consequence of a generalized Pohozaev-type identity. If \( a \geq 0 \), one obtains the existence of a solution for \( 1 < p < p_S \) by standard variational argument. On the other hand, if \( \Omega \) is a ball, W.-M. Ni [19] proved the existence of a radial solution in a larger range, namely for \( 1 < p < p_S(a) \), by using the Mountain Pass Lemma in a space of radial functions. Recently, the question of multiplicity and qualitative properties of solutions for the Hénon equation, such as the symmetry-breaking, have been widely studied. If \( \Omega \) is a ball and \( a > 0 \), numerical computation (see [11]) suggested that for some values of the parameter \( a > 0 \), the ground state solutions (i.e. solutions with minimal energy) are nonradial. It was then confirmed by Smets, Su and Willem (see [25]) that, if \( 1 < p < p_S \), there exists \( a^* > 0 \) such that for \( a > a^* \), Ni’s radial solution is not the ground state solution. Further results on the subcritical Hénon equation such as symmetry properties of solutions and blowup profile of ground states as \( a \to \infty \) or \( p \to p_S \) can be found in [5–7,9,10,24].

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.2 by doubling-rescaling arguments. In Section 3 we provide a simple proof of Theorem C(i), based on the Pohozaev-type identity and an interpolation lemma. The proof of Theorem 1.3, along the lines of [16], is also given there.

**Notation.** For \( R > 0 \), we set \( B_R = \{ x \in \mathbb{R}^N ; |x| < R \} \). We shall use spherical coordinates \( r = |x|, \theta = x/|x| \in S^{N-1} \) and write \( u = u(r, \theta) \). The derivative \( \partial / \partial r = \partial / \partial x \cdot \nabla \) will be denoted by \( \cdot \). The surface measures on \( S^{N-1} \) and on the sphere \( \{ x \in \mathbb{R}^N ; |x| = R \} \), \( R > 0 \), will be denoted respectively by \( d\nu \) and \( d\sigma_{\partial B_R} \). For given function \( w = w(\theta) \) on \( S^{N-1} \) and \( 1 \leq k \leq \infty \), we set \( \|w\|_k = \|w\|_{L^k(S^{N-1})} \). When no confusion is likely, we shall denote \( \|u\|_k = \|u(r, \cdot)\|_k \).

**2. Singularity and decay estimates**

In this section, we give a relatively simple proof of Theorem 1.2. We need the following lemma, which is an extension of Theorem 6.1 in [20]. The main difference with that result is that the estimate is uniform with respect to the (Hölder bounded) coefficient \( c(x) \).

**Lemma 2.1.** Let \( N \geq 1, 1 < p < p_S \) and \( \alpha \in (0, 1) \). Let \( c \in C^\alpha(\overline{B}_1) \) satisfy

\[
\|c\|_{C^\alpha(\overline{B}_1)} \leq C_1 \quad \text{and} \quad c(x) \geq C_2, \quad x \in \overline{B}_1, \tag{10}
\]

for some constants \( C_1, C_2 > 0 \). There exists a constant \( C \), depending only on \( \alpha, C_1, C_2, p, N \), such that, for any nonnegative classical solution \( u \) of

\[
-\Delta u = c(x)u^p, \quad x \in B_1, \tag{11}
\]

\( u \) satisfies

\[
|u(x)|^{(p-1)/2} + |\nabla u(x)|^{(p-1)/(p+1)} \leq C \left( 1 + \text{dist}^{-1}(x, \partial B_1) \right), \quad x \in B_1.
\]
Proof. Arguing by contradiction, we suppose that there exist sequences $c_k, u_k$ verifying (10), (11) and points $y_k$, such that the functions

$$M_k = |u_k|^{(p-1)/2} + |\nabla u_k|^{(p-1)/(p+1)}$$

satisfy

$$M_k(y_k) > 2k(1 + \text{dist}^{-1}(y_k, \partial B_1)) \geq 2k \text{dist}^{-1}(y_k, \partial B_1).$$

By the doubling lemma in [20, Lemma 5.1], there exists $x_k$ such that

$$M_k(x_k) \geq M_k(y_k), \quad M_k(y_k) > 2k \text{dist}^{-1}(x_k, \partial B_1),$$

and

$$M_k(z) \leq 2M_k(x_k), \quad \text{for all } z \text{ such that } |z - x_k| \leq kM_k^{-1}(x_k).$$

(12)

We have

$$\lambda_k := M_k^{-1}(x_k) \to 0, \quad k \to \infty,$$

(13)

due to $M_k(x_k) \geq M_k(y_k) > 2k$. Next we let

$$v_k = \lambda_k^{2/(p-1)}u_k(x_k + \lambda_k y), \quad \tilde{c}_k(y) = c_k(x_k + \lambda_k y).$$

We note that $|v_k|^{(p-1)/2}(0) + |\nabla v_k|^{(p-1)/(p+1)}(0) = 1$,

$$\left[|v_k|^{(p-1)/2} + |\nabla v_k|^{(p-1)/(p+1)}\right](y) \leq 2, \quad |y| \leq k,$$

(14)

due to (12), and we see that $v_k$ satisfies

$$-\Delta v_k = \tilde{c}_k(y)v_k^p, \quad |y| \leq k.$$  

(15)

On the other hand, due to (10), we have $C_2 \leq \tilde{c}_k \leq C_1$ and, for each $R > 0$ and $k \geq k_0(R)$ large enough,

$$|\tilde{c}_k(y) - \tilde{c}_k(z)| \leq C_1 |\lambda_k(y - z)|^\alpha \leq C_1 |y - z|^\alpha, \quad |y|, |z| \leq R.$$  

(16)

Therefore, by Ascoli’s theorem, there exists $\tilde{c}$ in $C(\mathbb{R}^N)$, with $\tilde{c} \geq C_2$ such that, after extracting a subsequence, $\tilde{c}_k \to \tilde{c}$ in $C_{\text{loc}}(\mathbb{R}^N)$. Moreover, (16) and (13) imply that $|\tilde{c}_k(y) - \tilde{c}_k(z)| \to 0$ as $k \to \infty$, so that the function $\tilde{c}$ is actually a constant $C > 0$.

Now, for each $R > 0$ and $1 < q < \infty$, by (15), (14) and interior elliptic $L^q$ estimates, the sequence $v_k$ is uniformly bounded in $W^{2,q}(B_R)$. Using standard embeddings and interior elliptic Schauder estimates, after extracting a subsequence, we may assume that $v_k \to v$ in $C^2_{\text{loc}}(\mathbb{R}^N)$. It follows that $v \geq 0$ is a classical solution of

$$-\Delta v = Cv^p, \quad y \in \mathbb{R}^N,$$

and $|v|^{(p-1)/2}(0) + |\nabla v|^{(p-1)/(p+1)}(0) = 1$. Since $p < p_S$, this contradicts the Liouville-type result [15, Theorem 1.1] and concludes the proof. $\square$
Proof of Theorem 1.2. Assume either \( \Omega = \{ x \in \mathbb{R}^N; 0 < |x| < \rho \} \) and \( 0 < |x_0| < \rho/2 \), or \( \Omega = \{ x \in \mathbb{R}^N; |x| > \rho \} \) and \( |x_0| > 2\rho \). We denote
\[
R = \frac{1}{2} |x_0|
\]
and observe that, for all \( y \in B_1, \frac{|x_0|}{2} < |x_0 + R y| < \frac{3|x_0|}{2} \), so that \( x_0 + R y \in \Omega \) in either case. Let us thus define
\[
U(y) = R^{\frac{2+a}{p-1}} u(x_0 + R y).
\]
Then \( U \) is a solution of
\[
-\Delta U = c(y) U^p, \quad y \in B_1, \text{ with } c(y) = \left| y + \frac{x_0}{R} \right|^a.
\]
Notice that \( |y + \frac{x_0}{R}| \in [1, 3] \) for all \( y \in B_1 \). Moreover \( \|c\|_{C^1(B_1)} \leq C(a) \). Then applying Lemma 2.1, we have \( U(0) + |\nabla U(0)| \leq C \), hence
\[
u(x_0) \leq CR^{-\frac{2+a}{p-1}}, \quad |\nabla u(x_0)| \leq CR^{-\frac{p+1+a}{p-1}},
\]
which yields the desired conclusion. \( \square \)

Remark 2.1. Lemma 2.1 does not hold any longer if the Hölder norm in (10) is replaced with the uniform norm, as shown by the following counter-example. Let \( N \geq 3, N/(N - 2) < p < (N + 2)/(N - 2) \) and \( u(x) = (1 + |x|^2)^{-1/(p-1)} \) then
\[
-\Delta u = a(x) u^p, \quad x \in \mathbb{R}^N, \text{ with } a(x) = \left( \frac{2N}{p-1} - \frac{4p}{(p-1)^2} \right) + \frac{4p}{(p-1)^2} \left( 1 + |x|^2 \right)^{-1}.
\]
Since \( p > N/(N - 2) \) then \( \frac{2N}{p-1} - \frac{4p}{(p-1)^2} > 0 \). Thus, \( 0 < C_2 \leq a(x) \leq C_1 \) for all \( x \in \mathbb{R}^N \). Let \( u_\lambda(y) = \lambda^{2/(p-1)} u(\lambda y) \). Then
\[
-\Delta u_\lambda = a_\lambda(y) u_\lambda^p, \quad y \in B(0, 1), \text{ with } a_\lambda(y) = a(\lambda y),
\]
whereas \( u_\lambda(0) = \lambda^{2/(p-1)} \to \infty \) as \( \lambda \to \infty \). Therefore, the conclusion of Lemma 2.1 fails. In fact, we see that \( a_\lambda(y) - a_\lambda(0) = C > 0 \) for \( |y| = \lambda^{-1} \); consequently, the modulus of continuity of \( a_\lambda \) near 0 is not uniform w.r.t. \( \lambda \to \infty \), and in particular assumption (10) of Lemma 2.1 is not satisfied.

3. A simple proof of Theorem C(i)

A basic ingredient to the proof of both Theorems C(i) and 1.1 is the following Pohozaev-type identity. It is more or less known, but we give a proof in Appendix A for completeness, especially since there is a slight technical difficulty when \( a < 0 \).

Lemma 3.1 (Rellich–Pohozaev identity). Let \( p > 1, N \geq 2, a > -2 \) and let \( u \) be a positive solution of (1) in \( \mathbb{R}^N \). For all \( R > 0 \), there holds
\[
\left(\frac{N+a}{p+1} - N + 2\right) \int_{B_R} |x|^a u^{p+1} \, dx \\
= \int_{|x|=R} \left(2R^{1+a} \frac{u^{p+1}}{p+1} + 2R^{-1} |x| \nabla u|^2 - R |\nabla u|^2 + (N-2) uu'\right) \, d\sigma_R.
\]  

(17)

Proof of Theorem C(i). Let \( u \) be a positive solution of (1) and define

\[ F(R) = \int_{B_R} |x|^a u^{p+1} \, dx. \]  

(18)

By Rellich–Pohozaev identity, we have

\[ F(R) \leq C \left( G_1(R) + G_2(R) \right), \]  

(19)

where

\[ G_1(R) = R^{N+a} \int_{S^{N-1}} u^{p+1}(R, \theta) \, d\theta \]  

(20)

and

\[ G_2(R) = R^N \int_{S^{N-1}} \left( |D_x u(R, \theta)|^2 + R^{-2} u^2(R, \theta) \right) \, d\theta. \]  

(21)

Now, by (7) in Theorem 1.2, we have

\[ u(x) \leq C |x|^{\frac{2a}{p-1}} \quad \text{and} \quad |\nabla u(x)| \leq C |x|^{-\frac{p+1+a}{p-1}}, \quad x \neq 0. \]

Due to \( p < p_S(a) \), it follows that

\[ G_1(R) + G_2(R) \leq CR^{N-\frac{2(p+1+a)}{p-1}} \to 0, \quad \text{as} \ R \to \infty. \]

Therefore, \( u \equiv 0 \) by (19). \( \square \)

4. Proof of Theorem 1.1

4.1. Functional inequalities and basic estimates

Lemma 4.1 (Sobolev inequalities on \( S^{N-1} \)). Let \( N \geq 2 \), let \( j \geq 1 \) be integer and \( 1 < k < \lambda \leq \infty \) satisfy \( k \neq (N-1)/j \). For \( w = w(\theta) \in W^{j,k}(S^{N-1}) \), we have

\[ \|w\|_{\lambda} \leq C \left( \|D_\theta^j w\|_k + \|w\|_1 \right), \]

where \( C = C(j, k, N) > 0 \) and

\[
\left\{
\begin{array}{ll}
\frac{1}{k} - \frac{1}{\lambda} = \frac{j}{N-1}, & \text{if } k < (N-1)/j, \\
\lambda = \infty, & \text{if } k > (N-1)/j.
\end{array}
\right.
\]

See e.g. [23].
Lemma 4.2 (Elliptic $L^k$-estimates on an annulus). Let $N \geq 2$ and $1 < k < \infty$. For $z = z(x) \in W^{2,k}(B_{2R} \setminus B_{R/4})$ and $R > 0$, we have

$$\int_{B_R \setminus B_{R/2}} |D^2 z|^k \, dx \leq C \left( \int_{B_{2R} \setminus B_{R/4}} |\Delta z|^k \, dx + R^{-2k} \int_{B_{2R} \setminus B_{R/4}} |z|^k \, dx \right),$$

(22)

with $C = C(N, k) > 0$.

Lemma 4.3 (An interpolation inequality on an annulus). Let $N \geq 2$. For $z = z(x) \in W^{2,1}(B_{2R})$ and $R > 0$, we have

$$\int_{B_R \setminus B_{R/2}} |D_x z| \, dx \leq CR \int_{B_{2R} \setminus B_{R/4}} |\Delta z| \, dx + CR^{-1} \int_{B_{2R} \setminus B_{R/4}} |z| \, dx,$$

(23)

with $C = C(N) > 0$.

Lemmas 4.2 and 4.3 follow from the case $R = 1$ and an obvious dilation argument. For $R = 1$, (22) is just the standard interior elliptic estimate. As for (23) with $R = 1$, this is a variant of an estimate stated without proof in [26, Lemma 2.3]. If the $L^1$ norms in (23) are replaced with $L^k$ norms with $1 < k < \infty$, then it follows from standard elliptic and interpolation inequalities. However for $k = 1$, we could not find a reference in the literature and we therefore provide a proof in Appendix A. Note that [26, Lemma 2.3] can be proved by a very similar argument.

The following basic integral estimates for solutions of (1) follows from the rescaled test-function method (see [18, Section I.3]). We give a proof in Appendix A for completeness.

Lemma 4.4. Let $N \geq 2$, $a > -2$ and $u$ be a positive solution of (1) with $\Omega = \mathbb{R}^N$. Then there holds

$$\int_{B_R} |x|^a u^p \, dx \leq CR^{N-2-\frac{2+a}{p-1}}, \quad R > 0,$$

(24)

with $C = C(N, p, a) > 0$.

We now deduce the following lemma.

Lemma 4.5. Let $N \geq 2$, $a > -2$ and $u$ be a positive solution of (1) with $\Omega = \mathbb{R}^N$. Then, for all $R > 0$, there hold

$$\int_{B_R \setminus B_{R/2}} u \, dx \leq CR^{N-\frac{2+a}{p-1}},$$

(25)

$$\int_{B_R \setminus B_{R/2}} |D_x u| \, dx \leq CR^{N-1-\frac{2+a}{p-1}},$$

(26)

$$\int_{B_R} |\Delta u| \, dx \leq CR^{N-2-\frac{2+a}{p-1}}.$$

(27)
Proof. Estimate (27) is just (24). Next, by Hölder’s inequality and (24), we obtain
\[
\int_{B_R \setminus B_{R/2}} u \, dx \leq C R^{N - \frac{p-1}{p}} \left( \int_{B_R \setminus B_{R/2}} u^p \, dx \right)^{1/p} \leq C R^{-\frac{N}{p} + \frac{N(p-1)}{p}} \left( \int_{B_R} |x|^a u^p \, dx \right)^{1/p} \leq C R^{N - \frac{2 \alpha q}{p-1}},
\]
hence (25). Finally, adding up estimates (25) for \( R/2 \), \( R \) and \( 2R \), we obtain (25) on \( B_{2R} \setminus B_{R/4} \) and this, along with (27) and Lemma 4.3 yields (26). \( \square \)

4.2. Proof of Theorem 1.1

The proof consists of 4 steps. Starting from the Pohozaev inequality, which yields formulas (18)–(21), we shall control the terms \( G_1(R), G_2(R) \) suitably for appropriate values of \( R \). For sake of clarity, although here \( N = 3 \), we shall keep the letter \( N \) in the proof. We fix a number \( \varepsilon > 0 \), which will be ultimately chosen small. In what follows, \( C \) denotes any positive constant independent of \( R \) (but possibly depending on \( \varepsilon \)).

Step 1: Estimation of \( G_1(R) \) and \( G_2(R) \) in terms of suitable norms. Recall that \( \|u\|_k \) denotes \( \|u(\cdot, \cdot)\|_{L^k(S^{N-1})} \). By Lemma 4.1, since \( N = 3 \), we have
\[
\|u\|_{p+1} \leq \|u\|_{\infty} \leq C \left( \|D_\theta^2 u\|_{1+\varepsilon} + \|u\|_1 \right) \leq C(R^2 \|D_\theta^2 u\|_{1+\varepsilon} + \|u\|_1)
\]
and
\[
\|D_\chi u\|_2 \leq C \left( \|D_\theta D_\chi u\|_{1+\varepsilon} + \|D_\chi u\|_1 \right) \leq C(R \|D_\chi^2 u\|_{1+\varepsilon} + \|D_\chi u\|_1),
\]
\[
\|u\|_2 \leq \|u\|_{\infty} \leq C \left( R^2 \|D_\chi^2 u\|_{1+\varepsilon} + \|u\|_1 \right).
\]
Therefore,
\[
G_1(R) \leq CR^{N + \alpha + 2(p+1)\left( \|D_\chi^2 u\|_{1+\varepsilon} + R^{-2} \|u\|_1 \right)^{p+1}} \tag{28}
\]
and
\[
G_2(R) \leq CR^{N + 2(p+1)\left( \|D_\chi^2 u\|_{1+\varepsilon} + R^{-1} \|D_\chi u\|_1 + R^{-2} \|u\|_1 \right)^2}. \tag{29}
\]

Step 2: Control of the averages. For any \( R > 1 \), we claim that
\[
\int_{R/2}^R \|u(r)\|_1 r^{N-1} \, dr \leq CR^{N - \frac{2 + \alpha q}{p-1}}, \tag{30}
\]
\[
\int_{R/2}^R \|D_\chi u(r)\|_1 r^{N-1} \, dr \leq CR^{N - 1 - \frac{2 + \alpha q}{p-1}} \tag{31}
\]
and
\[
\int_{R/2}^R \|D_\chi^2 u(r)\|_{1+\varepsilon} r^{N-1} \, dr \leq CR^{N - 2 - \frac{2 + \alpha q}{p-1} + \alpha \varepsilon}. \tag{32}
\]
Estimates (30)–(31) follow from (25)–(26) in Lemma 4.5. Next, by using Lemma 4.2, Eq. (1), the boundedness of \( u \) and (27), we obtain

\[
\int_{R/2}^{R} \left\| D_{x}^{2} u(r) \right\|_{1+\varepsilon}^{1+\varepsilon} r^{N-1} dr = \int_{B_{R} \setminus B_{R/2}} |D_{x}^{2} u|^{1+\varepsilon} dx
\]

\[
\leq C \int_{B_{2R} \setminus B_{R/4}} |\Delta u|^{1+\varepsilon} dx + C R^{-2(1+\varepsilon)} \int_{B_{2R} \setminus B_{R/4}} u^{1+\varepsilon} dx
\]

\[
\leq C \int_{B_{2R} \setminus B_{R/4}} |x|^{a_{\varepsilon} u^{p_{\varepsilon}}} |\Delta u| dx + C R^{-2(1+\varepsilon)} \int_{B_{2R} \setminus B_{R/4}} u^{1+\varepsilon} dx
\]

\[
\leq C R^{a_{\varepsilon}} \int_{B_{2R} \setminus B_{R/4}} |\Delta u| + C R^{-2(1+\varepsilon)} \int_{B_{2R} \setminus B_{R/4}} u dx
\]

\[
\leq C R^{-2-\frac{2a}{p-1}+a_{\varepsilon}} + R^{-2-\frac{2a}{p-1}+a_{\varepsilon}} \leq C R^{-2-\frac{2a}{p-1}+a_{\varepsilon}}.
\]

Hence (32) holds.

**Step 3: Measure argument.** For a given \( K > 0 \), let us define the sets

\[
\Gamma_{1}(R) = \{ r \in (R, 2R) : \| u(r) \|_{1} > KR^{-\frac{2a}{p-1}} \},
\]

\[
\Gamma_{2}(R) = \{ r \in (R, 2R) : \| D_{x} u(r) \|_{1} > KR^{-1-\frac{2a}{p-1}} \},
\]

\[
\Gamma_{3}(R) = \{ r \in (R, 2R) : \| D_{x}^{2} u(r) \|_{1+\varepsilon} > KR^{-2-\frac{2a}{p-1}+a_{\varepsilon}} \}.
\]

By estimate (30), for \( R > 1 \), we have

\[
C \geq R^{-N+\frac{2a}{p-1}} \int_{R}^{2R} \| u(r) \|_{1} r^{N-1} dr \geq R^{-N+\frac{2a}{p-1}} \left| \Gamma_{1}(R) \right| R^{N-1} K R^{-\frac{2a}{p-1}} = K \left| \Gamma_{1}(R) \right| R^{-1}.
\]

Consequently, \( |\Gamma_{1}| \leq R/4 \) for \( K \geq 4C \). Similarly, from estimates (31) and (32), we obtain \( |\Gamma_{2}|, |\Gamma_{3}| \leq R/4 \). Therefore, for each \( R \geq 1 \), we can assert the existence of

\[
\tilde{R} \in (R, 2R) \setminus \bigcup_{i=1}^{3} \Gamma_{i}(R) \neq \emptyset.
\]

**Step 4: Conclusion.** If follows from (28)–(29) in Step 1 and (33) in Step 3 that

\[
G_{1}(\tilde{R}) \leq C R^{N+a+2(p+1)} \left( R^{-2-\frac{2a}{p-1}+(a_{\varepsilon})/(1+\varepsilon)} + R^{-2-\frac{2a}{p-1}} \right)^{p+1} \leq C (R^{-a_{1}(\varepsilon)} + R^{-a_{1}(0)}),
\]

where

\[
a_{1}(\varepsilon) = (p+1) \left[ 2 + \frac{2+a}{p-1} - a_{\varepsilon} \right] \frac{1}{1+\varepsilon} - 2 - \frac{N+a}{p+1},
\]

and
\[ G_2(\tilde{R}) \leq CR^{N+2} \left( R^{-(\frac{2ap}{p-1}+a\varepsilon)/(1+\varepsilon)} + R^{-\frac{2a\varepsilon}{p-1}} \right)^2 \leq C \left( R^{-a_2(\varepsilon)} + R^{-a_2(0)} \right), \]  

(35)

where

\[ a_2(\varepsilon) = -N - 2 + \frac{2}{1+\varepsilon} \left( 2 + \frac{2 + a}{p-1} - a\varepsilon \right). \]

Let \( \tilde{a} = \min(a_1(\varepsilon), a_1(0), a_2(\varepsilon), a_2(0)) \). Combining (34) and (35), we obtain

\[ F(R) \leq F(\tilde{R}) \leq CR^{-\tilde{a}}, \quad R \geq 1. \]

By straightforward computation, we see that

\[ a_1(0) = a_2(0) = \frac{N + 2 + 2a - (N - 2)p}{p - 1} > 0, \]

due to \( p < p_S(a) \). Therefore, for \( \varepsilon > 0 \) small enough, we have \( \tilde{a} > 0 \), so that \( \int_{\mathbb{R}^N} |x|^{\tilde{a}p+1} = 0 \), hence \( u \equiv 0 \): a contradiction. The proof is complete. \( \Box \)

Finally we note that the above proof still works if, instead of assuming \( u \) bounded, one assumes that \( u(x) \leq C|x|^q \) for \( x \) large, with some \( q > 0 \). Indeed, estimate (32) above can be replaced with

\[ \int_{R/2}^R \left\| D_x^2 u(r) \right\|^{1+\varepsilon}_{1+\varepsilon} r^{N-1} dr \leq CR^{N-2 - \frac{2\varepsilon}{p+1} + (qp+a)\varepsilon} \]

and the rest of proof is similar.

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**Appendix A. Proof of Lemmas 3.1, 4.3, 4.4, Theorem 1.3, and the case \( N = 1 \)**

We start with the following simple Lemma.

**Lemma A.1.** Let \( N \geq 2, a > -2, p > 1, 0 \in \Omega \) and \( u \) be a positive solution of (1). For any \( R > 0 \) such that \( B_R \subset \Omega \), we have

\[ \int_{B_R} |\nabla u|^2 \, dx = \int_{B_R} |x|^a u^{p+1} \, dx + \int_{|x|=R} uu' \, d\sigma_R < \infty. \]  

(A.1)

In particular,

\[ \text{there exists a sequence } \varepsilon_i \to 0^+ \text{ such that } \varepsilon_i \int_{|x|=\varepsilon_i} |\nabla u|^2 \, d\sigma_{\varepsilon_i} \to 0. \]  

(A.2)

Moreover, \( u \) is a distributional solution of (1).
Proof. If \( a \geq 0 \), the result is immediate, so we may assume \( a < 0 \). Recall that solutions are assumed to belong to the class (2). For \( 0 < \rho < R \) such that \( \mathcal{B}_R \subset \Omega \), we have

\[
\int_{\mathcal{B}_R \setminus \mathcal{B}_\rho} |\nabla u|^2 \, dx = - \int_{\mathcal{B}_R \setminus \mathcal{B}_\rho} u \Delta u \, dx + \int_{|x|=R} uu' \, d\sigma_R - \int_{|x|=\rho} uu' \, d\sigma_\rho.
\]

\[
= \int_{\mathcal{B}_R \setminus \mathcal{B}_\rho} |x|^a u^{p+1} \, dx + \int_{|x|=R} uu' \, d\sigma_R - \int_{|x|=\rho} uu' \, d\sigma_\rho. \quad \text{(A.3)}
\]

On the other hand, we have

\[
\int_{|x|=\rho} uu' \, d\sigma_\rho = \rho^{N-1} f'(\rho), \quad \text{where } f(\rho) := \frac{1}{2} \int_{S^{N-1}} u^2(\rho, \theta) \, d\theta.
\]

Since \( f \in C^1((0, R]) \cap C([0, R]) \) due to (2), we infer the existence of a sequence \( \rho_i \to 0^+ \) such that

\[
\lim_{i \to \infty} \rho_i f'(\rho_i) = 0.
\]

Since \( N \geq 2 \), passing to the limit in (A.3) with \( \rho = \rho_i \), we obtain (A.1), where the RHS is finite due to \( a > -2 > -N \) and (2). Since

\[
R \int_{|x| = \varepsilon} |\nabla u| \, d\varepsilon = \int_{\mathcal{B}_R} |\nabla u|^2 \, dx,
\]

assertion (A.2) follows.

Let now \( \varphi \in C^\infty_{0}(\Omega) \) and denote \( \Omega_\varepsilon = \Omega \cap \{|x| > \varepsilon\} \) for \( \varepsilon > 0 \) small. From (1), using Green’s formula, we obtain

\[
\left| \int_{\Omega_\varepsilon} |x|^a u^{p} \varphi \, dx + \int_{\Omega_\varepsilon} u \Delta \varphi \, dx \right| = \left| - \int_{\Omega_\varepsilon} \varphi \Delta u \, dx + \int_{\Omega_\varepsilon} u \Delta \varphi \, dx \right| = \left| \int_{|x| = \varepsilon} \varphi' u \, d\sigma_\varepsilon - \int_{|x| = \varepsilon} u' \varphi \, d\sigma_\varepsilon \right|.
\]

\[
\text{(A.4)}
\]

We note that, by (A.2),

\[
\int_{|x| = \varepsilon_i} |\nabla u| \, d\sigma_{\varepsilon_i} \leq \left( \varepsilon_i^{N-1} \right)^{1/2} \left( \int_{|x| = \varepsilon_i} |\nabla u|^2 \, d\sigma_{\varepsilon_i} \right)^{1/2} \to 0, \quad \text{as } i \to \infty. \quad \text{(A.5)}
\]

Passing to the limit in (A.4) with \( \varepsilon = \varepsilon_i \) and using (A.5) and the continuity of \( u \) at 0, we obtain

\[
\int_{\Omega} |x|^a u^{p} \varphi \, dx + \int_{\Omega} u \Delta \varphi \, dx = 0,
\]

so that \( u \) is a distributional solution of (1). \( \square \)

Proof of Lemma 3.1. Since \( u \) is a solution of (1) then

\[
(x, \nabla u) \Delta u = -(x, \nabla u)|x|^a u^{p} = -\text{div} \left( x|x|^a u^{p+1} \right) + \frac{N + a}{p + 1} |x|^a u^{p+1}.
\]
Thus, for $0 < \varepsilon < R$, we have

$$
\int_{B_R \setminus B_\varepsilon} (x \nabla u) \Delta u \, dx = \frac{N + a}{p + 1} \int_{B_R \setminus B_\varepsilon} |x|^a u^{p+1} \, dx - R^{1+a} \int_{|x|=R} u^{p+1} \frac{d\sigma_R}{p + 1} + \varepsilon^{1+a} \int_{|x|=\varepsilon} u^{p+1} \frac{d\sigma_\varepsilon}{p + 1}.
$$

Letting $\varepsilon \to 0$, using the continuity of $u$, we obtain

$$
\int_{B_R} (x \nabla u) \Delta u \, dx = \frac{N + a}{p + 1} \int_{B_R} |x|^a u^{p+1} \, dx - R^{1+a} \int_{|x|=R} u^{p+1} \frac{d\sigma_R}{p + 1}.
$$

(A.6)

Next, by direct computation, we have the following identity

$$
div(2(x \nabla u)^2) = 2(x \nabla u) \Delta u - (N - 2) |\nabla u|^2.
$$

(A.7)

It follows that, for $0 < \varepsilon < R$,

$$
\int_{B_R \setminus B_\varepsilon} (2(x \nabla u) \Delta u - (N - 2) |\nabla u|^2) \, dx = \int_{|x|=R} (2(x \nabla u) \nabla u - x |\nabla u|^2) \frac{x}{|x|} \, d\sigma_R - \int_{|x|=\varepsilon} (2(x \nabla u) \nabla u - x |\nabla u|^2) \frac{x}{|x|} \, d\sigma_\varepsilon.
$$

Letting $\varepsilon = \varepsilon_i \to 0$, where $\varepsilon_i$ is given by Lemma A.1, we obtain

$$
\int_{B_R} (2(x \nabla u) \Delta u - (N - 2) |\nabla u|^2) \, dx = \int_{|x|=R} (2(x \nabla u) \nabla u - x |\nabla u|^2) \frac{x}{|x|} \, d\sigma_R.
$$

(A.8)

From (A.6), (A.1) and (A.8) we deduce (17). $\Box$

**Proof of Lemma 4.3.** As mentioned before, it suffices to consider the case $R = 1$, and we can also assume that $u$ is smooth. For $r > 0$, set $A_r := [r/4 < |x| < 3r/2]$ and let $v_r$ and $dS_r$ respectively denote the outer unit normal and surface measure on $\partial A_r$. Next we denote by $G_r(x; y)$ the Green kernel of the $-\Delta$ in with Dirichlet boundary conditions. By a simple rescaling argument, we see that $G_r(x; y) = r^{2-N} G_1(r^{-1}x; r^{-1}y)$. Also, we shall denote by $\tilde{x}, \tilde{y}$ the variables for $G_1 = G_1(\tilde{x}, \tilde{y})$.

Let $1/2 < |x| < 1$ and $1 < r < 4/3$. It follows from the Green representation formula that

$$
u u(x) = -\int_{A_r} \Delta u(y) G_r(x; y) \, dy - \int_{\partial A_r} u(y) \partial_i G_r(x; y) \, dS_r(y),$$

$$= -r^{2-N} \int_{A_r} \Delta u(y) G_1(r^{-1}x; r^{-1}y) \, dy - r^{1-N} \int_{\partial A_r} u(y) v_r \cdot \nabla \tilde{x} G_1(r^{-1}x; r^{-1}y) \, dS_r(y),$$

hence

$$\nabla u(x) = -r^{1-N} \int_{A_r} \Delta u(y) \nabla \tilde{x} G_1(r^{-1}x; r^{-1}y) \, dy - r^{1-N} \int_{\partial A_r} u(y) v_r \cdot \nabla \tilde{x} \nabla \tilde{y} G_1(r^{-1}x; r^{-1}y) \, dS_r(y).$$
We now use the estimates $|\nabla_x G_1(\tilde{x}, \tilde{y})| \leq C|\tilde{x} - \tilde{y}|^{1-N}$ and $|\nabla_{\tilde{x}} \nabla_{\tilde{y}} G_1(\tilde{x}, \tilde{y})| \leq C|\tilde{x} - \tilde{y}|^{-N}$ (see e.g. [17]). It follows that

$$|\nabla u(x)| \leq C \int_{A_r} |\Delta u(y)| |x - y|^{1-N} dy + \int_{\partial A_r} |u(y)| |x - y|^{-N} dS_r(y). \quad (A.9)$$

Now, for $|y| \leq 2$, we note that $\int_{1/2 < |x| < 1} |x - y|^{1-N} dx \leq \int_{B_1} |z|^{1-N} dz < \infty$. Moreover we have $|x - y| > 1/6$ for any $y \in \partial A_r$ (recalling that $1/2 < |x| < 1$ and $1 < r < 4/3$). Combining this with (A.9) and using Fubini's Theorem, we thus obtain, for $1 < r < 4/3,$

$$\int_{1/2 < |x| < 1} |\nabla u(x)| dx \leq C \int_{A_r} |\Delta u(y)| \left( \int_{1/2 < |x| < 1} |x - y|^{1-N} dx \right) dy + C \int_{\partial A_r} |u(y)| \left( \int_{1/2 < |x| < 1} |x - y|^{-N} dS_r(y) \right) dy$$

Integrating over $r \in (1, 4/3)$, we obtain

$$\frac{1}{3} \int_{1/2 < |x| < 1} |\nabla u(x)| dx \leq C \int_{r=1}^{4/3} \int_{A_r} |\Delta u(y)| dS_r(y) dy$$

and the lemma is proved. $\Box$

**Proof of Lemma 4.4.** We use the rescaled test-function method (see e.g. [18]). Fix $\phi \in \mathcal{D}(\mathbb{R}^N)$, $0 \leq \phi \leq 1$, such that $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| > 2$. For each $R > 0$, put $\phi_R(x) = \phi(x/R)$. Let $m = 2p/(p-1) > 2$. We have

$$|\Delta \phi_R^m(x)| = |m \phi_R^{m-1} \Delta \phi_R + (m - 1) \phi_R^{m-2} \nabla \phi_R| \leq C R^{-2} \phi_R^{m-2}.$$

By Lemma A.1 in Appendix A, we know that $u$ is a distributional solution. We thus have

$$\int_{\mathbb{R}^N} |x|^p u^p \phi_R^m dx = - \int_{\mathbb{R}^N} \Delta u \phi_R^m dx = - \int_{\mathbb{R}^N} u \Delta (\phi_R^m) dx \leq C R^{-2} \int_{R < |x| < 2R} u \phi_R^{m-2} dx.$$

Now applying the Hölder's inequality, it follows that

$$\int_{\mathbb{R}^N} |x|^p u^p \phi_R^m dx \leq C R^{-2} \left( \int_{R < |x| < 2R} u^p \phi_R^{p(m-2)} dx \right)^{1/p} \leq C R^{-2} \left( \int_{R < |x| < 2R} u^p \phi_R^m dx \right)^{1/p}.$$

Therefore,
\[
\int_{\mathbb{R}^N} |x|^\alpha u^p \phi_R^m \, dx \leq CR^\theta \left( \int_{R < |x| < 2R} |x|^\alpha u^p \phi_R^m \, dx \right)^{1/p},
\]  
(A.10)

with \( \theta = \frac{(N-2)(p-1)-(2+a)}{p} \), hence \( \int_{\mathbb{R}^N} |x|^\alpha u^p \phi_R^m \, dx \leq CR^{\theta/(p-1)} \) and (24) follows. \( \square \)

**Remark A.1.** Recall from [18] that Theorem C(ii) for \( N - 2 - \frac{2+a}{p-1} < 0 \) is a direct consequence of estimate (24); whereas, in case \( N - 2 - \frac{2+a}{p-1} = 0 \), (24) implies \( \int_{\mathbb{R}^N} |x|^\alpha u^p \, dx = \infty \) and, letting \( R \to \infty \) in (A.10), we then obtain \( \int_{\mathbb{R}^N} |x|^\alpha u^p \, dx = 0 \), hence \( u \equiv 0 \). Note that, by the same token, Theorem C(ii) remains in fact true for distributional supersolutions.

We finally prove Theorem 1.3. The proof of assertion (i) is similar to that of [16, Theorem 4.1]. However, due to the problem with that result, mentioned in the paragraph preceding Theorem 1.3, we prefer to sketch the proof. Moreover, the proof is facilitated by the availability of the universal bounds in Theorem 1.2 (cf. the case \( P = 0 \) below).

**Proof of Theorem 1.3.** (i) Suppose that assertion (i) is false. Then there exists a sequence of solutions \( u_k \) and a sequence of points \( P_k \in \Omega \) such that
\[
M_k = \sup_{x \in \Omega} u_k(x) = u_k(P_k) \to \infty \text{ as } k \to \infty.
\]
We may assume that \( P_k \to P \in \overline{\Omega} \) as \( k \to \infty \).

**Case 1:** \( P \in \Omega \setminus \{0\} \) or \( P \in \partial \Omega \). We rescale the solution according to
\[
U_k(y) = \lambda_k^{\frac{2}{p-1}} u_k(P_k + \lambda_k y), \quad \lambda_k = M_k^{\frac{p-1}{2}}.
\]
Then \( U_k \) is a solution of
\[
-\Delta U_k = |P_k + \lambda_k y|^a U_k^p
\]
in a rescaled domain, with \( 0 \leq U_k \leq 1 \) and \( U_k(0) = 1 \). Using elliptic estimates and standard embeddings similarly as in [16], we deduce that some subsequence of \( U_k \) converges to a solution \( v > 0 \) of the equation \( -\Delta v = \ell v^p \), for some \( \ell > 0 \), either in \( \mathbb{R}^N \), or in a half-space with 0 boundary conditions. Since \( p < p_S \), this contradicts one of the Liouville-type results [15, Theorem 1.1] or [16, Theorem 1.3].

**Case 2:** \( P = 0 \). We now rescale the solution according to
\[
U_k(y) = \lambda_k^{\frac{2+a}{p-1}} u_k(P_k + \lambda_k y), \quad \lambda_k = M_k^{\frac{p-1}{2+a}}.
\]
Then \( U_k \) is a solution of
\[
-\Delta U_k = |y + \lambda_k^{-1} P_k|^a U_k^p
\]
in a rescaled domain containing \( B(0, \rho \lambda_k^{-1}) \) for some \( \rho > 0 \). Moreover, it follows from estimate (6) in Theorem 1.2 that the sequence \( \lambda_k^{-1} |P_k| = |P_k| u_k^{\frac{a}{p-1}}(P_k) \) is bounded. We may thus assume that \( \lambda_k^{-1} P_k \to x_0 \in \mathbb{R}^N \) as \( k \to \infty \). A similar limiting procedure as in Case 1 then produces a positive solution \( v \) of
\[
-\Delta v = |y + x_0|^a v^p, \quad y \in \mathbb{R}^N.
\]
(A.11)
More precisely, in the case $-2 < a < 0$, by elliptic regularity (which is applicable since the $u_k$ are distributional solutions in virtue of Lemma A.1), the $u_k$ satisfy a local $W^{2,m}$ bound for $N/2 < m < N/|a|$, hence a local Hölder bound, and this is sufficient to pass to the limit to obtain a solution of (A.11), with $v(-x_0)$ in the class (2).

Since we assumed (3), after a space shift, this gives a contradiction with Theorem C(i).

(ii) Assume $p > p_S(a)$. Then we know that (1) has a bounded, positive radial solution $U$ in $\mathbb{R}^N$ (see [15, Appendix A] and [2]). Moreover, as $r \to \infty$, we have

$$U(r) \sim \frac{C_0}{r^{(2+a)/(p-1)}}, \quad \text{if } p > p_S(a),$$

$$U(r) \sim C_0 r^{-N+2}, \quad \text{if } p = p_S(a).$$

For $\lambda > 0$, let

$$U_{\lambda}(y) = \lambda^{(2+a)/(p-1)} U(\lambda|y|),$$

then $-\Delta U_{\lambda} = |y|^a U_{\lambda}^p$ on $B_1$ and $U_{\lambda,|B_1} = \lambda^{(2+a)/(p-1)} U(\lambda) \to C_0$ (resp. 0), as $\lambda \to \infty$, if $p > p_S(a)$ (resp., $p = p_S(a)$). The assertion follows by observing that

$$U_{\lambda}(0) = \lambda^{(2+a)/(p-1)} \to \infty \quad \text{as } \lambda \to \infty. \quad \Box$$

In the following Proposition, we briefly comment on the very particular case $N = 1$, where somewhat stronger conclusions can be obtained. Namely, we have a nonexistence result which is stronger than Theorem C and Theorem 1.2(ii) becomes void for $N = 1$. Also we get universal bounds which are stronger than in Theorem 1.2(i) and this in turn imposes an a priori restriction on the size of the boundary data $\varphi(\rho)$ in Theorem 1.3.

**Proposition A.1.** Assume $N = 1$, $a > -2$ and $p > 1$.

(i) For any $b > 0$, there exist no nontrivial nonnegative $C^2$ solution of $-u'' \geq |x|^a u^p$ in $(b, \infty)$.

(ii) For any $\rho > 0$, there exists a constant $C = C(\rho, p, a) > 0$ such that any nonnegative $C^2$ solution of $-u'' = |x|^a u^p$ in $(0, \rho)$ satisfies $u \leq C$ in $(0, \rho)$.

**Proof.** (i) Assume the contrary. Since $u'' \leq 0$, there exists $\ell = \lim_{x \to \infty} u'(x) \in [-\infty, \infty)$, and necessarily $\ell \geq 0$ due to $u \geq 0$. Therefore, $u' \geq 0$ in $(0, \infty)$, hence $u \geq c > 0$ for $x > x_0 > 0$ large enough. For each $R > 1$, define $\varphi_R(x) := u(x_0 + 2R + x)$ for $x \in [-R, R]$. The function $\varphi_R$ satisfies $-\varphi''_R \geq \bar{c} R^a \varphi_R$ in $[-R, R]$, with $\bar{c} > 0$ independent of $R$. Multiplying this inequality with $\varphi_R(\pm R) = \cos(\pi x/(2R))$, which satisfies $-\varphi''_R = 2^{-2} \pi^2 R^{-2} \varphi_R$ in $[-R, R]$, $\varphi_R(\pm R) = 0$, $\varphi'_R(R) \leq 0$ and $\varphi'_R(-R) \geq 0$, we obtain

$$\bar{c} R^a \int_{-R}^{R} \varphi_R \varphi' \ dx \leq 2^{-2} \pi^2 R^{-2} \int_{-R}^{R} \varphi_R \varphi' \ dx + \int_{-R}^{R} \varphi_R \varphi'(R) \ dx - \int_{-R}^{R} \varphi_R \varphi'(R) \ dx \leq 2^{-2} \pi^2 R^{-2},$$

hence $4\bar{c} R^{a+2} \leq \pi^2$, which is a contradiction with $a > -2$ for $R$ large.

(ii) First note that, by the proof of Theorem 1.2(i), we have $u(\rho/2) + |u'(\rho/2)| \leq K(\rho, p, a)$. Since $u$ is concave, we deduce that
since there exist local near 0 – and even global – solutions $u$ not distributional solutions near 0.

Remark A.2. When $a < -2$, there is an important difference between the cases $N = 1$ and $N \geq 2$, since there exist local near 0 – and even global – solutions $u \in C(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$, of the form $u(x) = c|x|^\gamma$, where $\gamma = -(a + 2)/(p - 1) \in (0, 1)$ for $1 < p < -a - 1$. We note that such solutions are not distributional solutions near 0.

References