Nearest matrix with two prescribed eigenvalues

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Abstract

Given a square complex matrix $A$ and two complex numbers $z_1$ and $z_2$, we find the distance from $A$ to the set of matrices that have $z_1$, $z_2$ as some of their eigenvalues. We use the distance between two matrices associated with the spectral norm.

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1. Introduction

Denote by $\mathbb{C}^{m \times n}$ the space of complex matrices of dimension $m \times n$. The singular values of a matrix $M \in \mathbb{C}^{m \times n}$ are denoted by $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_p(M)$, ordered in nonincreasing order, where $p := \min(m, n)$. We denote by $A(X)$ the set of distinct eigenvalues of a matrix $X \in \mathbb{C}^{n \times n}$. By $m(\alpha, X)$ and $gm(\alpha, X)$ we denote the algebraic and geometric multiplicities of the complex number $\alpha$ as an eigenvalue of
We agree that \( \alpha \) is not an eigenvalue of \( X \) if and only if \( m(\alpha, X) = 0 \). By \( I \) we denote the identity matrix and when it is precise to indicate its order, let us say \( n \), we will write \( I_n \). We will use the spectral norm for matrices, and the Euclidean norm for vectors of \( \mathbb{C}^{n \times 1} \), associated with the ordinary scalar product \( \langle x, y \rangle := y^*x \) of the vectors \( x, y \in \mathbb{C}^{n \times 1} \). The exponent * denotes the conjugate transpose.

Let \( A \in \mathbb{C}^{n \times n} \) and \( \lambda_0 \) be a given complex number. It is well known that the minimum singular value, \( \sigma_n(\lambda_0 I - A) \), of the matrix \( \lambda_0 I - A \) is equal to the minimum distance from \( A \) to the matrices \( X \) that have \( \lambda_0 \) as one of their eigenvalues. The distance between the matrices \( A \) and \( X \) is measured by \( \| X - A \| \). With mathematical notations,

\[
\min_{\lambda_0 \in \mathcal{A}(X)} \| X - A \| = \sigma_n(\lambda_0 I - A). \tag{1}
\]

Moreover, let

\[
U^*(\lambda_0 I - A)V = \begin{bmatrix} \sigma_1(\lambda_0 I - A) & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & \sigma_{n-1}(\lambda_0 I - A) & \sigma_n(\lambda_0 I - A) \end{bmatrix},
\]

be the singular value decomposition of the matrix \( \lambda_0 I - A \), with \( U, V \in \mathbb{C}^{n \times n} \) unitary matrices. Let

\[
X_0 := \lambda_0 I - U \begin{bmatrix} \sigma_1(\lambda_0 I - A) & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & \sigma_{n-1}(\lambda_0 I - A) \end{bmatrix} V^*. \tag{2}
\]

Then the minimum of (1) is attained at \( X_0 \).

The minimum distance from \( A \) to the matrices \( X \) that have \( \lambda_0 \) as an eigenvalue of geometric multiplicity \( \geq 2 \) is equal to \( \sigma_{n-1}(\lambda_0 I - A) \). In general, if \( k \in \{2, \ldots, n\} \), the minimum distance from \( A \) to the matrices \( X \) that have \( \lambda_0 \) as an eigenvalue of geometric multiplicity \( \geq k \) is equal to \( \sigma_{n-(k-1)}(\lambda_0 I - A) \). Let

\[
X_0 := \lambda_0 I - U \begin{bmatrix} \sigma_1(\lambda_0 I - A) & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & \sigma_{n-k}(\lambda_0 I - A) \end{bmatrix} V^*. \tag{3}
\]

Then

\[
\min_{\text{gm}(\lambda_0, X) \geq k} \| X - A \|
\]

is reached at \( X_0 \) (see [2, Theorem 4.1, Corollary 4.2]).
Suppose the $z_1, z_2$ are two given different complex numbers. In this paper we will try to give a formula for the distance from $A$ to the set of matrices $X$ that have $z_1, z_2$ among their eigenvalues in terms of the singular values of a matrix depending on the data: $A, z_1, z_2$. That is to say, we will find the minimum value

$$\min_{z_1, z_2 \in \text{spec}(X)} \|X - A\|$$

and a matrix $X_0$ where it is attained.

Given a complex number $\lambda_0$ and an $n$-by-$n$ complex matrix $A$, Malyshev proved that the distance from $A$ to the set of matrices $X$ such that $m(\lambda_0, X) \geq 2$ is given by

$$\min_{m(\lambda_0, X) \geq 2} \|X - A\| = \max_{t \geq 0} \sigma_{2n-1} \begin{pmatrix} \lambda_0 I - A & tI \\ 0 & \lambda_0 I - A \end{pmatrix},$$

where $t$ runs over the nonnegative real numbers (see [6]). Moreover, Malyshev also gave a matrix $X_0$ where the minimum in (5) is attained. The matrix $X_0$ was given in terms of the $(2n - 1)$th singular value and its left and right singular vectors of the matrix

$$\begin{pmatrix} \lambda_0 I - A & t_0I \\ 0 & \lambda_0 I - A \end{pmatrix},$$

t_0 being a real number where the maximum in (5) is attained. This gives a solution to the Wilkinson’s problem: What is the distance from a matrix with simple eigenvalues to the nearest matrix having a multiple eigenvalue. Malyshev was inspired by the article [8] of Qiu and others. We, as well, have been inspired by the work of Malyshev to give a partial solution to the problem that we have raised. Lippert and Edelman in [5] gave geometric solutions to the problem of finding the minimum

$$\min_{m(\lambda_0, X) \geq 2} \|X - A\|_F,$$

where $\| \cdot \|_F$ is the Frobenius norm.

The rest of this paper is organized in this way. In Section 2 we recall a lemma about the extrema of a singular value function $\sigma_t(G(t))$ of an analytic matrix function $G(t)$ where $t$ is a real variable. In Section 3 we present the main result in the paper. In Section 4 we study the function of a real variable $t$, given by,

$$\sigma_{2n-1} \begin{pmatrix} z_1 I - A & tI \\ 0 & z_2 I - A \end{pmatrix}.$$ 

Moreover, at the points $t_0 \neq 0$ where this function has a positive local extremum we obtain special singular vectors of the matrix

$$\begin{pmatrix} z_1 I - A & t_0I \\ 0 & z_2 I - A \end{pmatrix}$$

corresponding to its $(2n - 1)$th singular value. In Section 5 we prove the main result; in the proof we obtain a minimizing matrix $X_0$ for (4). Finally, in Section 6 we
establish a conjecture whose proof would solve the problem in the cases where the function in (6) is identically zero or satisfies that its maximum value is reached only at \( t_0 = 0 \).

2. Singular values of matrix functions

Let \( G : \Omega \to \mathbb{C}^{m \times n} \) be a matrix function analytic and defined on an open set \( \Omega \) of \( \mathbb{R} \). Then the singular values \( \sigma_1(G(t)), \ldots, \sigma_p(G(t)) \), with \( p := \min(m, n) \), are continuous and piecewise analytic functions on \( \Omega \). This were proved in classical results that can be seen in [3] and [6, p. 447]. We need the following lemma that was proved in [6, Lemma 5, p. 448].

**Lemma 1.** Let \( G : \Omega \to \mathbb{C}^{m \times n} \) be a matrix function analytic and defined on an open set \( \Omega \) of \( \mathbb{R} \). Let \( \sigma_1(G(t)) \geq \cdots \geq \sigma_p(G(t)) \geq 0 \), with \( p := \min(m, n) \), be its ordered singular values. If the function \( t \mapsto \sigma_i(G(t)) \) has a local maximum (or minimum) at \( t_0 \in \Omega \), then there exists a pair of left and right singular vectors \( u \in \mathbb{C}^{m \times 1}, v \in \mathbb{C}^{n \times 1} \) of \( G(t_0) \) corresponding to \( \sigma_i(G(t_0)) \) such that

\[
\text{Re} \left( u^* \frac{dG}{dt}(t_0) v \right) = 0.
\]

3. The main result

If a matrix \( X \in \mathbb{C}^{n \times n} \) has \( z_1, z_2 \) as some of its eigenvalues, with \( z_1 \neq z_2 \), then for all real number \( t \neq 0 \) the inequality

\[
\text{rank} \begin{pmatrix} z_1 I - X & tI \\ 0 & z_2 I - X \end{pmatrix} \leq 2n - 2,
\]

holds. In fact, we have that

\[
\begin{pmatrix} t^{-1}I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} z_1 I - X & tI \\ 0 & z_2 I - X \end{pmatrix} \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} z_1 I - X & I \\ 0 & z_2 I - X \end{pmatrix}.
\]

But

\[
\nu \begin{pmatrix} z_1 I - X \\ z_2 I - X \end{pmatrix} = \nu \begin{pmatrix} 0 \\ -(z_1 I - X)(z_2 I - X) \end{pmatrix} = \nu \begin{pmatrix} 0 \\ -(z_1 I - X)(z_2 I - X) \end{pmatrix} = \nu(\nu(z_1 I - X) + \nu(z_2 I - X)),
\]

for all real number \( t \neq 0 \).
where $\nu$ denotes the nullity; therefore
\[
\text{rank} \left( \begin{array}{cc} z_1 I - X & I \\ 0 & z_2 I - X \end{array} \right) = 2n - \text{gm}(z_1, X) - \text{gm}(z_2, X).
\]

Hence,
\[
\text{rank} \left( \begin{array}{cc} z_1 I - X & I \\ 0 & z_2 I - X \end{array} \right) \leq 2n - 2.
\]

By (7) it follows that
\[
\sigma_{2n-1} \left( \begin{array}{cc} z_1 I - X & tI \\ 0 & z_2 I - X \end{array} \right) = 0,
\]

This fact led us to Theorem 2, which is the main result of the paper.

Let us introduce the following notation:
\[
F(t) := \left( \begin{array}{cc} z_1 I - A & tI \\ 0 & z_2 I - A \end{array} \right), \quad f(t) := \sigma_{2n-1} \left( \begin{array}{cc} z_1 I - A & tI \\ 0 & z_2 I - A \end{array} \right).
\]

**Theorem 2.** Let $n \geq 2$, $A \in \mathbb{C}^{n \times n}$ and $z_1, z_2 \in \mathbb{C}$ with $z_1 \neq z_2$. Then
\[
\min_{X \in \mathbb{C}^{n \times n}} \|X - A\| = \max_{t \geq 0} f(t),
\]
whenever $f$ is not identically zero and the maximum of the second member is reached at some point $t_0 > 0$.

**4. Properties of the functions $f$ and $F$**

First, the function $f$ is continuous and even, i.e. for all real $t$, $f(t) = f(-t)$. This is a consequence of the identity
\[
\left( \begin{array}{cc} I & 0 \\ 0 & -I \end{array} \right) \left( \begin{array}{cc} z_1 I - A & tI \\ 0 & z_2 I - A \end{array} \right) \left( \begin{array}{cc} I & 0 \\ 0 & -I \end{array} \right) = \left( \begin{array}{cc} z_1 I - A & -tI \\ 0 & z_2 I - A \end{array} \right)
\]
and of the fact that the matrix
\[
\left( \begin{array}{cc} I & 0 \\ 0 & -I \end{array} \right)
\]
is unitary.

**Lemma 3.** Let $n \geq 2$ and $M_1, M_2 \in \mathbb{C}^{n \times n}$. Then
\[
\lim_{|t| \to \infty} \sigma_{2n-1} \left( \begin{array}{cc} M_1 & tI \\ 0 & M_2 \end{array} \right) = 0.
\]
Proof. Case 1. Suppose first that $M_1$ and $M_2$ are invertible. For all real $t$,

$$\sigma_{2n-1} \begin{pmatrix} M_1 & tI \\ 0 & M_2 \end{pmatrix} = \frac{1}{\sigma_2 \begin{pmatrix} M_1 & tI \\ 0 & M_2 \end{pmatrix}^{-1}};$$

but

$$\begin{pmatrix} M_1 & tI \\ 0 & M_2 \end{pmatrix}^{-1} = \begin{pmatrix} M_1^{-1} & -tM_1^{-1}M_2^{-1} \\ 0 & M_2^{-1} \end{pmatrix}.$$

In virtue of the inequality

$$\left| \sigma_2 \begin{pmatrix} M_1^{-1} & -tM_1^{-1}M_2^{-1} \\ 0 & M_2^{-1} \end{pmatrix} - \sigma_2 \begin{pmatrix} 0 & -tM_1^{-1}M_2^{-1} \\ 0 & 0 \end{pmatrix} \right| \leq \left\| \begin{pmatrix} M_1^{-1} & 0 \\ 0 & M_2^{-1} \end{pmatrix} \right\|,$$

it follows that

$$-\left\| \begin{pmatrix} M_1^{-1} & 0 \\ 0 & M_2^{-1} \end{pmatrix} \right\| \leq \sigma_2 \begin{pmatrix} M_1^{-1} & -tM_1^{-1}M_2^{-1} \\ 0 & M_2^{-1} \end{pmatrix} - \sigma_2 \begin{pmatrix} 0 & -tM_1^{-1}M_2^{-1} \\ 0 & 0 \end{pmatrix};$$

whence

$$\sigma_2 \begin{pmatrix} 0 & -tM_1^{-1}M_2^{-1} \\ 0 & 0 \end{pmatrix} - \left\| \begin{pmatrix} M_1^{-1} & 0 \\ 0 & M_2^{-1} \end{pmatrix} \right\| \leq \sigma_2 \begin{pmatrix} M_1^{-1} & 0 \\ 0 & M_2^{-1} \end{pmatrix} - \sigma_2 \begin{pmatrix} 0 & -tM_1^{-1}M_2^{-1} \\ 0 & 0 \end{pmatrix};$$

which is equivalent to

$$|t| \sigma_2 \begin{pmatrix} M_1^{-1} & M_2^{-1} \\ 0 & 0 \end{pmatrix} - \left\| \begin{pmatrix} M_1^{-1} & 0 \\ 0 & M_2^{-1} \end{pmatrix} \right\| \leq \sigma_2 \begin{pmatrix} M_1^{-1} & 0 \\ 0 & M_2^{-1} \end{pmatrix} - \sigma_2 \begin{pmatrix} 0 & -tM_1^{-1}M_2^{-1} \\ 0 & 0 \end{pmatrix}.$$
As
\[
\sigma_{2n-1} \begin{pmatrix} M_1 & tI \\ 0 & M_2 \end{pmatrix} \leq \sigma_{2n-1} \begin{pmatrix} M'_1 & tI \\ 0 & M'_2 \end{pmatrix} + \max \{ \|M_1 - M'_1\|, \|M_2 - M'_2\| \},
\]
there exists a \( T > 0 \) such that if \(|t| \geq T\), then
\[
\sigma_{2n-1} \begin{pmatrix} M_1 & tI \\ 0 & M_2 \end{pmatrix} < 2\varepsilon.
\]
\[\square\]

From this lemma we have just proved, one deduces the following result immediately.

**Lemma 4**
\[
\lim_{|t| \to \infty} f(t) = 0.
\]

**Remark 1.** Either \( f(t) = 0 \) for all real \( t \) or \( f(t) \neq 0 \) for all real \( t \). In fact, given that
\[
u \begin{pmatrix} z_1I - A & tI \\ 0 & z_2I - A \end{pmatrix} = v(z_1I - A) + v(z_2I - A)
\]
for all \( t \in \mathbb{R} \), then
\[
r := \text{rank} \begin{pmatrix} z_1I - A & tI \\ 0 & z_2I - A \end{pmatrix}
\]
is constant. If \( r \geq 2n - 1 \), then \( f(t) \neq 0 \), for all \( t \in \mathbb{R} \); if \( r < 2n - 1 \), then \( f(t) \equiv 0 \) over \( \mathbb{R} \).

Suppose that at \( t_0 \in \mathbb{R}, t_0 \neq 0 \), the function \( f \) has a local extremum and that
\[
\sigma_{2n-1} \begin{pmatrix} z_1I - A & t_0I \\ 0 & z_2I - A \end{pmatrix} =: \sigma_0 > 0.
\]

If
\[
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{C}^{2n \times 1}, \quad \text{with} \ u_1, u_2, v_1, v_2 \in \mathbb{C}^{n \times 1},
\]
are, respectively, right and left singular vectors of \( F(t_0) \), then
\[
\begin{pmatrix} z_1I - A & t_0I \\ 0 & z_2I - A \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sigma_0 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{10}
\]
\[
\begin{pmatrix} \bar{z}_1I - A^* & 0 \\ t_0I & \bar{z}_2I - A^* \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \sigma_0 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{11}
\]
\[
u_1^*u_1 + u_2^*u_2 = 1, \quad v_1^*v_1 + v_2^*v_2 = 1. \tag{12}
\]
By Lemma 1 we can choose \( u_1, u_2, v_1, v_2 \) in such a way that, besides (10)–(12), they satisfy the condition

\[
0 = \Re \left( (u_1^*, u_2^*) \frac{dF}{dt}(t_0) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right);
\]

and as

\[
\frac{dF}{dt}(t_0) = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix},
\]

we deduce

\[
0 = \Re(u_1^*v_2).
\]

(13)

Multiplying (10) by \((u_1^*, -u_2^*)\) on the left we have

\[
(u_1^*, -u_2^*) \begin{pmatrix} z_1I - A & t_0I \\ 0 & z_2I - A \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sigma_0(u_1^*u_1 - u_2^*u_2);
\]

whence the next identities hold

\[
(u_1^*(z_1I - A), t_0u_1^* - u_2^*(z_2I - A)) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sigma_0(u_1^*u_1 - u_2^*u_2);
\]

\[
u_1^*(z_1I - A)v_1 + t_0u_1^*v_2 - u_2^*(z_2I - A)v_2 = \sigma_0(u_1^*u_1 - u_2^*u_2).
\]

(14)

Multiplying (11) by \((v_1^*, -v_2^*)\) on the left, we have the identities

\[
(v_1^*, -v_2^*) \begin{pmatrix} \bar{z}_1I - A^* & 0 \\ t_0I & \bar{z}_2I - A^* \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \sigma_0(v_1^*v_1 - v_2^*v_2);
\]

\[
(v_1^*(\bar{z}_1I - A^*) - t_0v_1^*u_1 - v_2^*(\bar{z}_2I - A^*)u_2 = \sigma_0(v_1^*v_1 - v_2^*v_2);
\]

\[
v_1^*(\bar{z}_1I - A^*)u_1 + t_0v_2^*u_1 - v_2^*(\bar{z}_2I - A^*)u_2 = \sigma_0(v_1^*v_1 - v_2^*v_2).
\]

(15)

Taking the conjugate transpose of identity (14), we obtain

\[
v_1^*(\bar{z}_1I - A^*)v_1 + t_0v_2^*u_1 - v_2^*(\bar{z}_2I - A^*)u_2 = \sigma_0(v_1^*v_1 - v_2^*v_2).
\]

(16)

By (15) and (16), we deduce

\[
\sigma_0(v_1^*v_1 - v_2^*v_2) + t_0v_2^*u_1 = \sigma_0(u_1^*u_1 - u_2^*u_2) - t_0v_2^*u_1;
\]

reason why

\[
\sigma_0(v_1^*v_1 - v_2^*v_2) + 2t_0v_2^*u_1 = \sigma_0(u_1^*u_1 - u_2^*u_2);
\]

(17)

consequently, as \( t_0 \) is distinct from zero, the number \( v_2^*u_1 \) is real; therefore \( u_1^*v_2 \) is real too. Now (13) implies

\[
u_1^*v_2 = 0.
\]

(18)

We have proved the following lemma:

Lemma 5. If \( t_0 > 0 \) is a local extremum of \( f \) and \( f(t_0) = \sigma_0 > 0 \), then there exist right and left singular vectors associated with the singular value \( f(t_0) \) of the matrix \( F(t_0) \).
\[
\begin{pmatrix}
  v_1 \\
  v_2
\end{pmatrix},
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix},
\]
with \(u_1, u_2, v_1, v_2 \in \mathbb{C}^{n \times 1}\), such that
\[u_1^* v_2 = 0.\]

**Lemma 6.** Let \(u_1, u_2, v_1, v_2\) be the vectors given in Lemma 5, let
\[U := \begin{bmatrix} u_1 & u_2 \end{bmatrix}, \quad V := \begin{bmatrix} v_1 & v_2 \end{bmatrix}\]
of \(\mathbb{C}^{n \times 2}\). Then
\[U^* U = V^* V. \quad (19)\]

**Proof.** By (17) and (18),
\[\sigma_0 (v_1^* v_1 - v_2^* v_2) = \sigma_0 (u_1^* u_1 - u_2^* u_2);\]
as \(\sigma_0 > 0\), this implies
\[v_1^* v_1 - v_2^* v_2 = u_1^* u_1 - u_2^* u_2.\]
Let us call \(\alpha := v_1^* v_1 - v_2^* v_2\); then \(\alpha = u_1^* u_1 - u_2^* u_2\). From this and (12) we deduce
\[2v_1^* v_1 = 1 + \alpha; \quad 2u_1^* u_1 = 1 + \alpha; \quad 2v_2^* v_2 = 1 - \alpha; \quad 2u_2^* u_2 = 1 - \alpha;\]
and, thus,
\[v_1^* v_1 = \frac{1 + \alpha}{2} = u_1^* u_1, \quad (20)\]
\[v_2^* v_2 = \frac{1 - \alpha}{2} = u_2^* u_2. \quad (21)\]
Multiplying (10) on the left by \((0, u_1^*)\) and (11) on the left by \((v_2^*, 0)\) the following identities are obtained:
\[\begin{pmatrix} 0, u_1^* \\ 0 \end{pmatrix} (z_2 I - A) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sigma_0 u_1^* u_2;\]
and
\[\begin{pmatrix} v_2^* (\bar{z}_1 I - A^*) \end{pmatrix} \begin{pmatrix} u_1 \\ 0 \end{pmatrix} = \sigma_0 v_2^* v_1;\]
hence
\[u_1^* (z_2 I - A) v_2 = \sigma_0 u_1^* u_2, \quad (22)\]
\[v_2^* (\bar{z}_1 I - A^*) u_1 = \sigma_0 v_2^* v_1, \quad (23)\]
Taking the conjugate transpose in (23),
\[z_1 u_1^* v_2 - u_1^* A v_2 = \sigma_0 v_1^* v_2. \quad (24)\]
From (22) and (24),
\[ z_2 u_1^* v_2 - \sigma_0 u_1^* u_2 = z_1 u_1^* v_2 - \sigma_0 v_1^* v_2; \]
and since, by Lemma 5, \( u_1^* v_2 = 0 \), then
\[-\sigma_0 u_1^* u_2 = -\sigma_0 v_1^* v_2. \]
Thus,
\[ u_1^* u_2 = v_1^* v_2. \tag{25} \]
Because
\[ U^* U = \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix} [u_1, u_2] = \begin{bmatrix} u_1^* u_1 & u_1^* u_2 \\ u_2^* u_1 & u_2^* u_2 \end{bmatrix} \]
and
\[ V^* V = \begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix} [v_1, v_2] = \begin{bmatrix} v_1^* v_1 & v_1^* v_2 \\ v_2^* v_1 & v_2^* v_2 \end{bmatrix}, \]
due to (20), (21) and (25) we deduce that \( U^* U = V^* V. \)

**Proposition 7.** Let \( u_1, u_2, v_1, v_2 \) be the vectors given in Lemma 5. The vectors \( u_1, u_2, v_1, v_2 \) are different from zero.

**Proof.** (1) If \( u_1 = 0 \), then \( v_1^* v_1 = u_1^* u_1, v_1 = 0 \) would also hold. From (10) it would follow then that
\[ \begin{pmatrix} z_1 I - A & t_0 I \\ 0 & z_2 I - A \end{pmatrix} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = \sigma_0 \begin{pmatrix} 0 \\ u_2 \end{pmatrix}; \]
which would imply \( t_0 v_2 = 0 \) and, as \( t_0 \neq 0 \), it would be true that \( v_2 = 0 \); and, in view of \( u_2^* u_2 = v_2^* v_2 \), we would have that \( u_2 = 0 \). But this is impossible, because
\[ u_1^* u_1 + u_2^* u_2 = 1 = v_1^* v_1 + v_2^* v_2. \]
Therefore, \( u_1 \neq 0 \) (and \( v_1 \neq 0 \)).

(2) If \( u_2 = 0 \), then \( v_2 = 0 \) and from (11) we would deduce
\[ \begin{pmatrix} \tilde{z}_1 I - A^* & 0 \\ t_0 I & \tilde{z}_2 I - A^* \end{pmatrix} \begin{pmatrix} u_1 \\ 0 \end{pmatrix} = \sigma_0 \begin{pmatrix} v_1 \\ 0 \end{pmatrix}; \]
which would imply \( t_0 u_1 = 0 \); hence \( u_1 = 0 \); which is absurd.

5. **Proof of Theorem 2**

For each matrix \( X \in \mathbb{C}^{n \times n} \) that has \( z_1 \) and \( z_2 \) as some of its eigenvalues and for all real \( t \neq 0 \),
\[
\sigma_{2n-1} \left( \begin{array}{cc}
    z_1 I - A & t I \\
    0 & z_2 I - A
\end{array} \right) - \sigma_{2n-1} \left( \begin{array}{cc}
    z_1 I - X & t I \\
    0 & z_2 I - X
\end{array} \right) \\
\leq \left\| \begin{array}{cc}
    -A + X & 0 \\
    0 & -A + X
\end{array} \right\| = \|X - A\|. \tag{26}
\]

By (8),
\[
\sigma_{2n-1} \left( \begin{array}{cc}
    z_1 I - A & t I \\
    0 & z_2 I - A
\end{array} \right) \leq \|X - A\|. \tag{27}
\]

As a consequence,
\[
\sup_{t \in \mathbb{R}} f(t) \leq \|X - A\|. \tag{28}
\]

Observe that we have included the value \(t = 0\) taking into account that the singular values are continuous functions of the entries of the matrix. Therefore,
\[
\inf_{X \in \mathbb{C}^{n \times n}, \det(z_1 I - X) = 0, \det(z_2 I - X) = 0} \|X - A\| \geq \sup_{t \in \mathbb{R}} f(t). \tag{29}
\]

In this inequality we can substitute \(\inf\) by \(\min\), given that the subset of \(\mathbb{C}^{n \times n}\) formed by the matrices \(X\) that satisfy \(\det(z_1 I - X) = 0\) and \(\det(z_2 I - X) = 0\) is closed; this follows from the fact that for each \(i = 1, 2\), \(\det(z_i I - X)\) is a polynomial in the entries of the matrix \(X\). Due to Lemma 4 and the continuity of the function \(f\), we can change \(\sup\) by \(\max\) in (28). Thus, we have
\[
\min_{X \in \mathbb{C}^{n \times n}, \det(z_1 I - X) = 0, \det(z_2 I - X) = 0} \|X - A\| \geq \max_{t \in \mathbb{R}} f(t). \tag{30}
\]

In order to prove the opposite inequality in (29) we will construct explicitly a matrix \(X_0\), with \(z_1\) and \(z_2\) as some of its eigenvalues such that
\[
\|X_0 - A\| = \max_{t \in \mathbb{R}} f(t). \tag{31}
\]

Note that \(\max_{t \in \mathbb{R}} f(t) = \max_{t \geq 0} f(t)\), because \(f\) is even.

Now suppose that \(f\) attains its maximum value at \(t_0 > 0\) and that \(f(t_0) = \sigma_0 > 0\). Let \(U, V \in \mathbb{C}^{n \times 2}\) be as in Lemma 6. Let \(D := \sigma_0 U V^\dagger\), where \(V^\dagger\) is the Moore–Penrose inverse of \(V\). We are going to prove that \(\|D\| = \sigma_0\) and also
\[
Dv_2 = \sigma_0 u_2, \tag{32}
\]
\[
u_2^\dagger D = \sigma_0 v_2^\dagger. \tag{33}
\]

Defining
\[
X_0 := A + D, \tag{34}
\]
from these equalities we shall deduce
\[
\|X_0 - A\| = \sigma_0, \tag{35}
\]
\[
z_1, z_2 \in A(X_0). \tag{36}
\]
We will prove
\[ U^* D = \sigma_0 V^*. \]  
(36)
This equality is equivalent to \( D^* U = \sigma_0 V \). From the definition of \( D \) and Lemma 6 we have
\[ D^* U = (\sigma_0 U V^\dagger)^* U = \sigma_0 (V^\dagger)^* U^* U = \sigma_0 (V^\dagger)^* V^* V, \]
but \( (V^\dagger)^* V^* = (VV^\dagger)^* = V V^\dagger \). Hence, \( D^* U = \sigma_0 V V^\dagger V = \sigma_0 V \).

Next we are going to prove
\[ D V = \sigma_0 U. \]  
(37)

**Case 1.** Let us suppose that \( \text{rank } U = 1 \). Then \( \text{rank } V = 1 \), and in virtue of [1, Exercise 5, p. 25] we have
\[ V^\dagger = \frac{1}{a} V^*, \]
where
\[ a := \| V \|^2 := \sum_{i=1}^{n} \sum_{j=1}^{2} |v_{ij}|^2 \]
and \( V = (v_{ij}) \). By Exercise 8, p. 359 of [4] \( a = \text{tr}(V^* V) \), where \( \text{tr} \) denotes the trace of a matrix. By Lemma 6, \( V^* V = U^* U \); hence, \( a = \text{tr}(U^* U) \); therefore,
\[ U^\dagger = \frac{1}{a} U^*. \]
Then,
\[ D V = \sigma_0 U V^\dagger V = \frac{\sigma_0}{a} U V^* V = \frac{\sigma_0}{a} U U^* U = \sigma_0 U U^\dagger U = \sigma_0 U. \]

**Case 2.** If \( \text{rank } U = 2 \), then \( \text{rank } V = 2 \). In this case by Exercise 1, p. 433 of [4] we have
\[ V^\dagger = (V^* V)^{-1} V^*; \]
therefore \( V^\dagger V = I_2 \); hence,
\[ D V = \sigma_0 U V^\dagger V = \sigma_0 U. \]

From (37) and (36) we deduce that (31) and (32), respectively, are true. Let us see that \( z_1, z_2 \in \lambda(A + D) \). Indeed, from (10) we infer that \( (z_2 I - A) v_2 = \sigma_0 v_2 \). From here and (31), \( z_2 v_2 = (A + D) v_2 \). As \( v_2 \neq 0 \) by Proposition 7, the number \( z_2 \) is an eigenvalue of \( A + D \). By (11), \( (z_1 I - A^*) u_1 = \sigma_0 v_1 \); taking the conjugate transpose in the two members of this equality, \( u_1^* (z_1 I - A) = \sigma_0 u_1^* \). Due to (32), \( z_1 u_1^* = u_1^* (A + D) \). Since \( u_1 \neq 0 \) by Proposition 7, one has that \( u_1 \) is an eigenvector on the left of \( A + D \) and \( z_1 \) is an eigenvalue of \( A + D \).

Since \( U^* U = V^* V \), there exists a unitary matrix \( W \in \mathbb{C}^{n \times n} \) such that \( U = WV \); therefore,
\[ \|D\| = \sigma_0\|UV^\dagger\| = \sigma_0\|VV^\dagger\| = \sigma_0, \]
because \(VV^\dagger\) is a nonzero orthogonal projector. This concludes the proof. \(\Box\)

Remark 2. Let us note from (29) that
\[ \max_{t \in \mathbb{R}} \sigma_{2n-1} \begin{pmatrix} z_1I - A & tI \\ 0 & z_2I - A \end{pmatrix} \]
is in any case a lower bound of the distance from \(A\) to the set of matrices \(X\) such that \(z_1, z_2 \in A(X)\), whether the function \(f\) satisfies the hypotheses in Theorem 2 or not.

Remark 3. The Theorem of Schur says that for any matrix \(A \in \mathbb{C}^{n \times n}\) there exists an unitary matrix \(U \in \mathbb{C}^{n \times n}\) such that
\[ U^*AU = T, \]
where \(T\) is an upper triangular matrix. Whence, the eigenvalues of \(A\) are the diagonal elements of \(T\). Since the spectral norm of matrices \(\|\cdot\|\) is unitarily invariant, there is no loss of generality if we substitute \(A\) for \(T\) when we try to find
\[ \min_{X \in \mathbb{C}^{n \times n}} \min_{z_1, z_2 \in A(X)} \|X - A\|. \]

6. Nongeneric case

If the maximum of the right hand side of the formula of Theorem 2 is not attained at a point \(t_0 > 0\), but at \(t_0 = 0\), then the equality can be false; to see this it suffices to consider the next counterexample.

Example 1. Let \(A\) be the matrix
\[ \begin{pmatrix} 1 & 1 & 2 - i \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix} \]
and the numbers \(z_1 = 80 + i, z_2 = 0\). Then
\[ \max_{t \geq 0} \sigma_5 \begin{pmatrix} z_1I - A & tI \\ 0 & z_2I - A \end{pmatrix} = \sigma_5 \begin{pmatrix} z_1I - A & 0 \\ 0 & z_2I - A \end{pmatrix} \approx 2.4029. \]
The graphic of the function
\[ f(t) := \sigma_5 \begin{pmatrix} z_1I - A & tI \\ 0 & z_2I - A \end{pmatrix} \]
can be seen in Fig. 1.

But \(\sigma_5(z_1I - A) \approx 77.5724\); therefore, by (1), the distance from \(A\) to the set of matrices that have \(z_1\) as an eigenvalue is equal to 77.5724. Hence, there cannot exist
a matrix $X$ such that $z_1, z_2 \in \sigma(X)$ and $\|X - A\| = 2.4029$. We think that in this case,
\[
\min_{z_1, z_2 \in \sigma(X)} \|X - A\| = \max \{\sigma_3(z_1I - A), \sigma_3(z_2I - A)\},
\]
because with MATLAB we have found the matrix
\[
\tilde{X}_2 = \begin{bmatrix}
16.8082 - 0.619645i & -20.7263 + 10.6021i & 20.6408 - 8.41734i \\
-21.569 - 11.0929i & 36.4091 + 0.6548i & -31.1807 + 0.401264i \\
18.6397 + 9.84594i & -31.6862 - 1.09343i & 27.3765 - 0.110069i
\end{bmatrix}
\]
whose approximated eigenvalues are
\[
80.0000 + 1.0000i \\
0.5938 - 1.0749i \\
-0.0000 - 0.0000i
\]
That is to say, $\tilde{X}_2$ satisfies with approximation $z_1, z_2 \in \sigma(X)$ and $\|X - A\| \approx 77.5724$; moreover, $\sigma_3(z_2I - A) \approx 0.7679$. The method we have followed to find $\tilde{X}_2$ is based on the deflation of Wielandt, as we are going to explain next. Given any matrix $B \in \mathbb{C}^{n \times n}$, let $\alpha \in \mathbb{C}$ and $c \in \mathbb{C}^{n \times 1}$, with $c \neq 0$, be such that $Bc = \alpha c$. Then for any vector $d \in \mathbb{C}^{n \times 1}$, the eigenvalues of $B$ and $B + cd^*$ are the same except for $\alpha$, which is changed into $\alpha + d^*c$ (see Exercise 7.1.17.(b) of [7]). Let $s_n, u_n, v_n$ be the $n$th singular value, left and right singular vectors of the matrix $z_1I - A$. Then it is known that the matrix $X_1 := A + s_n u_n v_n^*$ has the eigenvalues $\lambda_1 = z_1, \lambda_2, \ldots, \lambda_n$, not necessarily different. If we want to substitute the eigenvalue $\lambda_k$, where $k \in \{2, \ldots, n\}$, by $z_2$ we proceed in the following way: let $c_k \in \mathbb{C}^{n \times 1}$ be an eigenvector of $X_1$ associated with $\lambda_k$, i.e.,
we look for \( d \in \mathbb{C}^{n \times 1} \) such that \( \lambda_k + d^* c_k = z_2 \); then the matrix \( X_2 = X_1 + c_k d^* \) has the spectrum
\[
z_1, \lambda_2, \ldots, \lambda_{k-1}, z_2, \lambda_{k+1}, \ldots, \lambda_n.
\]
If we wish to obtain a matrix \( X_2 \) such that \( z_1, z_2 \in \sigma(X_2) \) and the value \( \|X_2 - A\| \) is minimum, we must find the constrained minimum
\[
\min_{k=2,\ldots,n} \min_{d \in \mathbb{C}^{n \times 1}} \|X_1 + c_k d^* - A\|.
\]
Let us suppose that this minimum is reached for \( k = k_0 \) and \( d_0 \), with \( d_0^* c_{k_0} = z_2 - \lambda_{k_0} \). Thus, we are sure that \( z_1 \) and \( z_2 \) are eigenvalues of \( \tilde{X}_2 := X_1 + c_{k_0} d_0^* \) and that \( \tilde{X}_2 \) is one of the matrices obtained by deflation that are closest to \( X_1 \). The matrix \( X_1 \) is one of the nearest matrices to \( A \) among the matrices that have \( z_1 \) as an eigenvalue. Therefore, we may hope that \( \tilde{X}_2 \) is “close” to \( A \).

In the current example we have used the function \texttt{fmincon} of MATLAB to compute the constrained minimum
\[
\min_{d \in \mathbb{C}^{n \times 1}, \lambda = \lambda_{k_0}, \sigma = \sigma_{k_0}} \|X_1 + c_{k_0} d^* - A\| = \|X_1 + c_{k_0} d_0^* - A\|,
\]
corresponding to the value \( k = 2 \); afterwards we have defined \( \tilde{X}_2 := X_1 + c_{k_0} d_0^* \), which has been displayed in (38).

However, as the following example shows there are cases in which the maximum of the right hand side of the formula of Theorem 2 is attained only at \( t_0 = 0 \) and the formula is true.

**Example 2.** Let \( A \) be the matrix
\[
\begin{pmatrix}
1 & 2 & -13 & 4 & 5 \\
6 & 7 & 8 & 9 & 10i \\
i & 2i & 3i & 4i & 5i \\
0 & 9 & 8 & 7 & 6 \\
5 & 4 & i & -23 & 0
\end{pmatrix}
\]
and the numbers \( z_1 = 12.2232i \), \( z_2 = 0 \). If we define the function
\[
g(t) := \sigma_9 \begin{pmatrix} z_1 I - A & tI \\
0 & z_2 I - A \end{pmatrix},
\]
the computation of the constrained minimum
\[
\min_{z_1, z_2 \in \sigma(X)} \|X - A\|
\]
with \texttt{fmincon} of MATLAB suggests that the equality
\[
\min_{z_1, z_2 \in \sigma(X)} \|X - A\| = \max_{t \geq 0} g(t)
\]
is true.
is true. In this case $\max_{t \geq 0} g(t) = g(0) \approx 4.2100$. The minimum
\[
\min_{z_1, z_2 \in A(X)} \|X - A\|
\]
is attained approximately at the matrix
\[
X_0 = \begin{bmatrix}
0.35 + 0.43i & 1.62 + 0.73i & -10.52 - 0.62i \\
6.33 - 0.19i & 5.45 + 0.16i & 8.00 - 0.10i \\
1.79 + 0.67i & 0.27 + 2.15i & -0.70 + 3.86i \\
0.76 - 0.29i & 7.31 + 1.52i & 7.55 - 0.40i \\
4.94 - 0.06i & 2.98 + 0.45i & 0.13 + 1.27i \\
5.07 + 0.03i & 5.77 - 1.04i \\
8.79 + 0.21i & -0.19 + 8.59i \\
0.73 + 3.56i & -1.53 + 2.77i \\
5.83 - 0.87i & 6.40 + 0.99i \\
-21.28 - 1.17i & -0.61 + 0.04i
\end{bmatrix},
\]
because the eigenvalues of $X_0$ are approximately equal to
\[
14.1422 - 2.1948i \\
-5.7458 + 10.4274i \\
-0.0000 + 12.2232i \\
1.9114 + 4.0309i \\
0.0000 - 0.0000i
\]
and $\|X_0 - A\| \approx 4.2100$. The graphic of the function $g(t)$ can be seen in Fig. 2.

The constrained minimum
\[
\min_{z_1, z_2 \in A(X)} \|X - A\| \approx 4.2100
\]
has been computed directly with fmincon too. Furthermore, $\sigma_5(z_1I - A) \approx 4.2100$, $\sigma_5(z_2I - A) \approx 2.1796$. Thus, in this example it also seems to be true that
\[
\min_{z_1, z_2 \in A(X)} \|X - A\| = \max \{\sigma_5(z_1I - A), \sigma_5(z_2I - A)\}.
\]

**Remark 4.** If one of the numbers $z_1$ or $z_2$ is an eigenvalue of $A$ with geometric multiplicity $\geq 2$, then for all real $t$, $f(t) = 0$, since
\[
\nu \begin{bmatrix}
z_1I & tI \\
0 & z_2I - A
\end{bmatrix} \geq 2.
\]

After an exhaustive computation with MATLAB we think that the next conjecture will give the answer that lacks to solve the problem.

**Conjecture 8.** If $f(t) \equiv 0$ or the maximum
\[
\max_{t \geq 0} f(t)
\]
is attained only at \( t = 0 \), then
\[
\min_{X \in \mathbb{C}^{n \times n}} \|X - A\| = \max \{\sigma_n(z_1 I - A), \sigma_n(z_2 I - A)\}. \tag{39}
\]

Let us suppose that \( \sigma_n(z_1 I - A) \geq \sigma_n(z_2 I - A) \) and let \( s_n := \sigma_n(z_1 I - A) \). Let \( u_n, v_n \) be the left and right singular vectors, respectively, associated with \( s_n \). Let \( X_1 := A + s_n u_n v_n^* \) and \( \lambda_1 = z_1, \lambda_2, \ldots, \lambda_n \) be the eigenvalues of \( X_1 \). Let \( c_2, \ldots, c_n \in \mathbb{C}^{n \times 1} \) be unitary eigenvectors of the matrix \( X_1 \) corresponding to \( \lambda_2, \ldots, \lambda_n \), respectively. The Weyl inequalities for the singular values of a sum of two matrices imply for all \( k \) and \( d \),
\[
s_n \leq \|s_n u_n v_n^* + c_k d^*\|.
\]
see [9, p. 45]. We guess
\[
s_n = \min_{k \in \{2, \ldots, n\}} \min_{d \in \mathbb{C}^{n \times 1}} \|s_n u_n v_n^* + c_k d^*\|.
\]
This would prove (39). If \( \sigma_n(z_1 I - A) < \sigma_n(z_2 I - A) \), exchanging the roles of \( z_1 \) and \( z_2 \) a similar conclusion is guessed.

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