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Hermite-based Appell polynomials: Properties and applications ☆

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ABSTRACT

By employing certain operational methods, the authors introduce Hermite-based Appell polynomials. Some properties of Hermite–Appell polynomials are considered, which proved to be useful for the derivation of identities involving these polynomials. The possibility of extending this technique to introduce Hermite-based Sheffer polynomials (for example, Hermite–Laguerre and Hermite–Sister Celine's polynomials) is also investigated.

1. Introduction

Recently an increasing interest has grown around operational techniques (involving differential operators) and special functions. The use of operational techniques combined with the principle of monomiality is a fairly useful tool for treating various families of special polynomials as well as their new and known generalizations. The idea of monomiality came from the concept of poweroid suggested by Steffensen [20]. The monomiality principle is reformulated and developed by Dattoli [6].

According to the principle of monomiality, a family of polynomials $p_n(x)$ $(n \in \mathbb{N},\ x \in \mathbb{C})$ is said to be "quasi-monomial," if two operators $\hat{M}$ and $\hat{P}$, hereafter called "multiplicative" and "derivative" operators respectively can be defined in such a way that

$$\begin{aligned} &\hat{M}\{p_n(x)\} = p_{n+1}(x), \\ &\hat{P}\{p_n(x)\} = np_{n-1}(x), \\ &p_0(x) = 1. \end{aligned} \tag{1.1}$$

The operators $\hat{M}$ and $\hat{P}$ can be recognized as raising and lowering operators acting on the polynomials $p_n(x)$. These operators satisfy the following commutation relation

$$[\hat{P}, \hat{M}] = \hat{1} \tag{1.2}$$

and thus display a Weyl group structure.

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The properties of the polynomials $p_n(x)$ can be deduced from those of the $\hat{M}$ and $\hat{P}$ operators. If the operators $\hat{M}$ and $\hat{P}$ possess a differential realization, then the polynomials $p_n(x)$ satisfy the differential equation

$$\hat{M}\hat{P}\{p_n(x)\} = np_n(x). \tag{1.3}$$

The $p_n(x)$ family can be explicitly constructed through the action of $\hat{M}^n$ on $p_0(x)$ (in the following, we shall always set $p_0(x) = 1$):

$$p_n(x) = \hat{M}^n\{1\} \tag{1.4}$$

and consequently the generating function of $p_n(x)$ can be cast in the form

$$G(x,t) = \exp(t\hat{M})\{1\} = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}. \tag{1.5}$$

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ [3], defined by

$$H_n(x,y) = n!\sum_{r=0}^{[\frac{n}{2}]} \frac{x^{n-2r}y^r}{r!(n-2r)!}, \tag{1.6}$$

have shown to be quasi-monomials under the action of the operators [6, p. 148 (1.9)],

$$\hat{M} = x + 2y\frac{\partial}{\partial x},$$
$$\hat{P} = \frac{\partial}{\partial x}. \tag{1.7}$$

It is easily seen from definition (1.6) that

$$H_n(2x,-1) = H_n(x),$$

and

$$H_n\left(x,-\frac{1}{2}\right) = He_n(x), \tag{1.8}$$

with $H_n(x)$ or $He_n(x)$ being ordinary Hermite polynomials [1]. Also

$$H_n(x,0) = x^n. \tag{1.9}$$

The properties of 2VHKdFP $H_n(x,y)$ are derived by using the monomiality principle, according to which the differential equation and the generating function for $H_n(x,y)$ are given by [6, p. 149, (1.10) and (1.14)]

$$\left(2y\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} - n\right)H_n(x,y) = 0 \tag{1.10}$$

and

$$\exp(xt + yt^2) = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}, \tag{1.11}$$

respectively.

From the above relations, we have

$$\frac{\partial}{\partial y}H_n(x,y) = \frac{\partial^2}{\partial x^2}H_n(x,y), \tag{1.12}$$

which in view of Eq. (1.9), gives the following operational definition for $H_n(x,y)$:

$$H_n(x,y) = \exp\left(y\frac{\partial^2}{\partial x^2}\right)\{x^n\}. \tag{1.13}$$

Further, the 3-variable Hermite polynomials (3VHP) $H_n(x,y,z)$ are introduced [7, p. 114 (22)]

$$H_n(x,y,z) = n!\sum_{r=0}^{[\frac{n}{3}]} \frac{z^r H_{n-3r}(x,y)}{r!(n-3r)!}, \tag{1.14}$$

which are quasi-monomials under the action of the operators

$$\hat{M} = x + 2y\frac{\partial}{\partial x} + 3z\frac{\partial^2}{\partial x^2},$$
$$\hat{P} = \frac{\partial}{\partial x}. \tag{1.15}$$

The differential equation and the generating function for $H_n(x, y, z)$ are given by

$$\left(3z\frac{\partial^3}{\partial x^3} + 2y\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} - n\right)H_n(x, y, z) = 0 \tag{1.16}$$

and

$$\exp(xt + yt^2 + zt^3) = \sum_{n=0}^{\infty} H_n(x, y, z)\frac{t^n}{n!}, \tag{1.17}$$

respectively.

Also, the polynomials $H_n(x, y, z)$ satisfy the following relations

$$\frac{\partial}{\partial y} H_n(x, y, z) = \frac{\partial^2}{\partial x^2} H_n(x, y, z)$$

and

$$\frac{\partial}{\partial z} H_n(x, y, z) = \frac{\partial^3}{\partial x^3} H_n(x, y, z), \tag{1.18}$$

which in view of the initial condition

$$H_n(x, 0, 0) = x^n, \tag{1.19}$$

gives the following operational definition for $H_n(x, y, z)$:

$$H_n(x, y, z) = \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\{x^n\}. \tag{1.20}$$

The Appell sets [2] may be defined by either of the following equivalent conditions [19, p. 398]: $\{A_n(x)\}$ $(n = 0, 1, 2, \ldots)$, is an Appell set (A_n being of degree exactly n) if either

(i) $$\frac{d}{dx} A_n(x) = nA_{n-1}(x), \quad n = 0, 1, 2, \ldots, \quad \text{or} \tag{1.21}$$

(ii) there exists a formal power series $A(t) = \sum_{n=0}^{\infty} a_n t^n$, $a_0 \neq 0$ such that (again formally)

$$A(t)\exp(xt) = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}. \tag{1.22}$$

The function $A(t)$ may be called the determining function for the set $\{A_n(x)\}$. The Appell polynomials have shown to be quasi-monomials [10] and characterized by the fact that the relevant derivative operator is just the ordinary derivative. The Appell polynomials are very often found in different applications in pure and applied mathematics. The Appell family includes polynomials ranging from the Hermite to the Euler ones. We recall some of the members of Appell family:

(I) If $A(t) = \frac{t}{(e^t-1)}$, then

$A_n(x) = B_n(x)$: The Bernoulli polynomials [18].

(II) If $A(t) = \frac{2}{(e^t+1)}$, then

$A_n(x) = E_n(x)$: The Euler polynomials [18].

(III) If $A(t) = \frac{t^\alpha}{(e^t-1)^\alpha}$, then

$A_n(x) = B_n^{(\alpha)}(x)$: The generalized Bernoulli polynomials [15].

(IV) If $A(t) = \frac{2^\alpha}{(e^t+1)^\alpha}$, then

$A_n(x) = E_n^{(\alpha)}(x)$: The generalized Euler polynomials [15].

(V) If $A(t) = \alpha_1\alpha_2 \ldots \alpha_m t^m[(e^{\alpha_1 t} - 1)(e^{\alpha_2 t} - 1)\ldots(e^{\alpha_m t} - 1)]^{-1}$, then

$A_n(x) =$ The Bernoulli polynomials of order m [14].

(VI) If $A(t) = \frac{t^m}{e^t - \sum_{h=0}^{m-1}(\frac{t^h}{h!})}$, then

$A_n(x) = B_n^{[m-1]}(x)$, $\quad m \geqslant 1$: The new generalized Bernoulli polynomials [5].

(VII) If $A(t) = 2^m[(e^{\alpha_1 t}+1)(e^{\alpha_2 t}+1)\dots(e^{\alpha_m t}+1)]^{-1}$, then

$A_n(x) =$ The Euler polynomials of order m [14].

(VIII) If $A(t) = \exp(\xi_0 + \xi_1 t + \xi_2 t^2 + \cdots + \xi_{r+1} t^{r+1})$, $\xi_{r+1} \neq 0$, then

$A_n(x) =$ The generalized Gould–Hopper polynomials [13], including the Hermite polynomials when $r = 1$ and classical 2-orthogonal polynomials when $r = 2$.

(IX) If $A(t) = \frac{1}{(1-t)^{m+1}}$, then

$A_n(x) = n!G_n^{(m)}(x)$: The Miller–Lee polynomials [1,8], including the truncated exponential polynomials $e_n(x)$, when $m = 0$ and modified Laguerre polynomials $f_n^{(\alpha)}(x)$ [17], when $m = \alpha - 1$.

(X) If $A(t) = \frac{2t}{(e^t+1)}$, then

$A_n(x) = G_n(x)$: The Genocchi polynomials [9].

Operational methods can be exploited to simplify the derivation of the properties associated with ordinary and generalized special functions and to define new families of functions. In the case of multi-variable generalized special functions, the use of operational techniques combined with the principle of monomiality provides new means of analysis for the solutions of a wide class of partial differential equations often encountered in physical problems. The importance of the use of operational techniques in the study of special functions and their applications has been recognized by Dattoli and his co-workers, see for example [6–12].

Motivated by these contributions, in this paper, we introduce Hermite-based Appell polynomials and discuss their properties and applications.

2. Hermite-based Appell polynomials

To generate Hermite-based Appell polynomials associated with 3VHP $H_n(x, y, z)$, we introduce the generating function

$$\mathcal{G}(x, y, z; t) = A(t)\exp(\hat{M}t),$$

or, equivalently

$$\mathcal{G}(x, y, z; t) = A(t)\exp\left(\left(x + 2y\frac{\partial}{\partial x} + 3z\frac{\partial^2}{\partial x^2}\right)t\right), \tag{2.1}$$

which is the result of replacement of x by the multiplicative operator $\hat{M}$ of $H_n(x, y, z)$ given in Eq. (1.15).

Now, decoupling the exponential operator appearing in Eq. (2.1), by using the Berry decoupling identity [11]

$$e^{\hat{A}+\hat{B}} = e^{m^2/12} e^{((-\frac{m}{2})\hat{A}^{1/2}+\hat{A})} e^{\hat{B}}, \qquad [\hat{A}, \hat{B}] = m\hat{A}^{1/2}, \tag{2.2}$$

we get the generating function for Hermite-based Appell polynomials ${}_HA_n(x, y, z)$ in the form

$$\mathcal{G}(x, y, z; t) = A(t)\exp\bigl(xt + yt^2 + zt^3\bigr) = \sum_{n=0}^{\infty} {}_HA_n(x, y, z)\frac{t^n}{n!}. \tag{2.3}$$

Differentiating Eq. (2.3) partially with respect to x, y and z, we get the following differential recurrence relations satisfied by the Hermite–Appell polynomials ${}_HA_n(x, y, z)$:

$$\frac{\partial}{\partial x}\,{}_HA_n(x, y, z) = n\,{}_HA_{n-1}(x, y, z),$$
$$\frac{\partial}{\partial y}\,{}_HA_n(x, y, z) = n(n-1)\,{}_HA_{n-2}(x, y, z),$$
$$\frac{\partial}{\partial z}\,{}_HA_n(x, y, z) = n(n-1)(n-2)\,{}_HA_{n-3}(x, y, z). \tag{2.4}$$

From relations (2.4), we observe that ${}_HA_n(x, y, z)$ are solutions of the equations

$$\frac{\partial}{\partial y}\,{}_HA_n(x, y, z) = \frac{\partial^2}{\partial x^2}\,{}_HA_n(x, y, z),$$
$$\frac{\partial}{\partial z}\,{}_HA_n(x, y, z) = \frac{\partial^3}{\partial x^3}\,{}_HA_n(x, y, z), \tag{2.5}$$

under the following initial condition

$$ {}_HA_n(x,0,0) = A_n(x). \tag{2.6}$$

Thus from Eqs. (2.5) and (2.6), it follows that:

$$ {}_HA_n(x,y,z) = \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\{A_n(x)\}. \tag{2.7}$$

According to the aforementioned point of view, the Hermite–Appell polynomials ${}_HA_n(x,y,z)$ can by generated from the corresponding Appell polynomials $A_n(x)$ by merely employing the operational rule (2.7). For example, the Hermite–Bernoulli polynomials ${}_HB_n(x,y,z)$ and Hermite–Euler polynomials ${}_HE_n(x,y,z)$ are defined by means of the operational definitions

$$ {}_HB_n(x,y,z) = \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\{B_n(x)\}, \tag{2.8}$$

and

$$ {}_HE_n(x,y,z) = \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\{E_n(x)\}, \tag{2.9}$$

respectively.

Also, in view of Eq. (2.3), we get the generating function for Hermite–Appell polynomials by taking $A(t)$ of the corresponding Appell polynomials. For example, by choosing $A(t)$ given in (I), (II) and (IX) of previous section, we get the generating functions for some polynomials belonging to Hermite–Appell family.

For $A(t) = \frac{t}{(e^t-1)}$, i.e. corresponding to the generating function for Bernoulli polynomials $B_n(x)$ [18]

$$\frac{t}{(e^t-1)}\exp(xt) = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi, \tag{2.10}$$

we get the following generating function for Hermite–Bernoulli polynomials ${}_HB_n(x,y,z)$:

$$\frac{t}{(e^t-1)}\exp\left(xt + yt^2 + zt^3\right) = \sum_{n=0}^{\infty} {}_HB_n(x,y,z)\frac{t^n}{n!}. \tag{2.11}$$

Next, for $A(t) = \frac{2}{(e^t+1)}$, i.e. corresponding to the generating function for Euler polynomials $E_n(x)$ [18]

$$\frac{2}{(e^t+1)}\exp(xt) = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad |t| < \pi, \tag{2.12}$$

we get the following generating function for Hermite–Euler polynomials ${}_HE_n(x,y,z)$:

$$\frac{2}{(e^t+1)}\exp\left(xt + yt^2 + zt^3\right) = \sum_{n=0}^{\infty} {}_HE_n(x,y,z)\frac{t^n}{n!}. \tag{2.13}$$

Again, for $A(t) = \frac{1}{(1-t)^{m+1}}$, i.e. corresponding to the generating function for Miller–Lee polynomials $G_n^{(m)}(x)$ [8, p. 21, (1.11)]

$$\frac{1}{(1-t)^{m+1}}\exp(xt) = \sum_{n=0}^{\infty} G_n^{(m)}(x)t^n, \quad |t| < 1, \tag{2.14}$$

we get the following generating function for Hermite–Miller–Lee polynomials ${}_HG_n^{(m)}(x,y,z)$:

$$\frac{1}{(1-t)^{m+1}}\exp\left(xt + yt^2 + zt^3\right) = \sum_{n=0}^{\infty} {}_HG_n^{(m)}(x,y,z)t^n, \tag{2.15}$$

which for $m = 0$, gives the generating function for Hermite-truncated exponential polynomials ${}_He_n(x,y,z)$:

$$\frac{1}{(1-t)}\exp\left(xt + yt^2 + zt^3\right) = \sum_{n=0}^{\infty} {}_He_n(x,y,z)t^n \tag{2.16}$$

and for $m = \alpha - 1$, gives the generating function for Hermite-modified Laguerre polynomials ${}_Hf_n^{(\alpha)}(x,y,z)$:

$$\frac{1}{(1-t)^{\alpha}}\exp\left(xt + yt^2 + zt^3\right) = \sum_{n=0}^{\infty} {}_Hf_n^{(\alpha)}(x,y,z)t^n. \tag{2.17}$$

Further, we recall that the Bernoulli polynomials $B_n(x)$ are defined by means of the following series:

$$B_n(x)=\sum_{r=0}^{n}\binom{n}{r}B_r x^{n-r},\quad n\geqslant 0,\tag{2.18}$$

where $B_n:=B_n(0)$ are the Bernoulli numbers defined by the generating function

$$\frac{t}{(e^t-1)}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}.\tag{2.19}$$

Now, operating $\exp(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3})$ on both sides of Eq. (2.18), we find

$$\exp\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}\right)\{B_n(x)\}=\sum_{r=0}^{n}\binom{n}{r}B_r\exp\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}\right)\{x^{n-r}\},\tag{2.20}$$

which on using the operational definitions (2.8) and (1.20) in the L.H.S. and R.H.S. respectively, yields the series defining the Hermite–Bernoulli polynomials ${}_HB_n(x,y,z)$ in terms of 3VHP $H_n(x,y,z)$ as

$${}_HB_n(x,y,z)=\sum_{r=0}^{n}\binom{n}{r}B_r\ H_{n-r}(x,y,z).\tag{2.21}$$

Similarly, from the series defining the Euler polynomials $E_n(x)$:

$$E_n(x)=\sum_{k=0}^{n}2^{-k}\binom{n}{k}E_k\left(x-\frac{1}{2}\right)^{n-k},\tag{2.22}$$

where $E_n:=2^nE_n(\frac{1}{2})$ are the Euler numbers defined by the generating function

$$\frac{2e^t}{(e^{2t}+1)}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},\tag{2.23}$$

we get the series definition for Hermite–Euler polynomials ${}_HE_n(x,y,z)$ in terms of 3VHP $H_n(x,y,z)$ as

$${}_HE_n(x,y,z)=\sum_{k=0}^{n}2^{-k}\binom{n}{k}E_k\,H_{n-k}\left(x-\frac{1}{2},y,z\right).\tag{2.24}$$

Thus, we conclude that the series definition for Hermite–Appell polynomials ${}_HA_n(x,y,z)$ can be obtained from the series defining the corresponding Appell polynomials on replacing the monomial x^n by the 3VHP $H_n(x,y,z)$.

3. Applications

Several identities involving Appell polynomials are known. The formalism developed in the previous section can be used to obtain the corresponding identities involving Hermite–Appell polynomials.

To achieve this, we perform the following operation:

($\mathcal{O}$) Operating $\exp(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3})$ on both sides of a given relation.

First, we recall the following functional equations involving Bernoulli polynomials $B_n(x)$ [16, p. 26]:

$$B_n(x+1)-B_n(x)=nx^{n-1},\quad n=0,1,2,\dots,$$

$$\sum_{m=0}^{n-1}\binom{n}{m}B_m(x)=nx^{n-1},\quad n=2,3,4,\dots,$$

$$B_n(mx)=m^{n-1}\sum_{l=0}^{m-1}B_n\left(x+\frac{l}{m}\right),\quad n=0,1,2,\dots;\ m=1,2,3,\dots.$$

Now, performing the operation ($\mathcal{O}$) on the above equations and using the operational definitions (1.20) and (2.8) on the resultant equations we get the following identities involving Hermite–Bernoulli polynomials ${}_HB_n(x,y,z)$:

$$ {}_HB_n(x+1,y,z) - {}_HB_n(x,y,z) = nH_{n-1}(x,y,z), \quad n=0,1,2,\ldots, \tag{3.1} $$

$$ \sum_{m=0}^{n-1}\binom{n}{m}{}_HB_m(x,y,z) = nH_{n-1}(x,y,z), \quad n=2,3,4,\ldots, \tag{3.2} $$

$$ {}_HB_n\big(mx,m^2y,m^3z\big) = m^{n-1}\sum_{l=0}^{m-1}{}_HB_n\left(x+\frac{l}{m},y,z\right), \quad n=0,1,2,\ldots;\ m=1,2,3,\ldots. \tag{3.3} $$

Similarly, corresponding to the functional equations involving Euler polynomials $E_n(x)$ [16, p. 30]:

$$ E_n(x+1)+E_n(x) = 2x^n, $$
$$ E_n(mx) = m^n\sum_{l=0}^{m-1}(-1)^lE_n\left(x+\frac{l}{m}\right), \quad n=0,1,2,\ldots;\ m \text{ odd}, $$

we find the following identities involving Hermite–Euler polynomials ${}_HE_n(x,y,z)$:

$$ {}_HE_n(x+1,y,z) + {}_HE_n(x,y,z) = 2H_n(x,y,z), \tag{3.4} $$

$$ {}_HE_n\big(mx,m^2y,m^3z\big) = m^n\sum_{l=0}^{m-1}(-1)^l{}_HE_n\left(x+\frac{l}{m},y,z\right), \quad n=0,1,2,\ldots;\ m \text{ odd}. \tag{3.5} $$

Further, we recall the following relations between Bernoulli and Euler polynomials [16, pp. 29–30]

$$ B_n(x) = 2^{-n}\sum_{m=0}^{n}\binom{n}{m}B_{n-m}E_m(2x), \quad n=0,1,2,\ldots, $$
$$ E_n(x) = \frac{2^{n+1}}{(n+1)}\left[B_{n+1}\left(\frac{x+1}{2}\right) - B_{n+1}\left(\frac{x}{2}\right)\right], \quad n=0,1,2,\ldots, $$
$$ E_n(mx) = -\frac{2m^n}{(n+1)}\sum_{l=0}^{m-1}(-1)^l\ B_{n+1}\left(x+\frac{l}{m}\right), \quad n=0,1,2,\ldots;\ m \text{ even}, $$

which on using the operational definitions (2.8) and (2.9), after performing the operation ($\mathcal{O}$), yield the following relations between Hermite–Bernoulli and Hermite–Euler polynomials:

$$ {}_HB_n(x,y,z) = 2^{-n}\sum_{m=0}^{n}\binom{n}{m}B_{n-m}\,{}_HE_m(2x,4y,8z), \quad n=0,1,2,\ldots, \tag{3.6} $$

$$ {}_HE_n(x,y,z) = \frac{2^{n+1}}{(n+1)}\left[{}_HB_{n+1}\left(\frac{x+1}{2},\frac{y}{4},\frac{z}{8}\right) - {}_HB_{n+1}\left(\frac{x}{2},\frac{y}{4},\frac{z}{8}\right)\right], \quad n=0,1,2,\ldots, \tag{3.7} $$

$$ {}_HE_n\big(mx,m^2y,m^3z\big) = -\frac{2m^n}{(n+1)}\sum_{l=0}^{m-1}(-1)^l{}_HB_{n+1}\left(x+\frac{l}{m},y,z\right), \quad n=0,1,2,\ldots;\ m \text{ even}. \tag{3.8} $$

The above examples show that by using the operation ($\mathcal{O}$) on an identity involving Appell polynomials and then using the operational definition of Hermite–Appell polynomials, we get the corresponding identity involving Hermite–Appell polynomials. To provide further examples, we consider the following recently derived recurrence relation involving Genocchi polynomials $G_n(x)$ [9, p. 1038, (42)]

$$ 2nx^{n-1} = G_{n+1}(x) + G_n(x), $$

which yields the following recurrence relation involving 3VHP $H_n(x,y,z)$ and Hermite–Genocchi polynomials ${}_HG_n(x,y,z)$:

$$ 2nH_{n-1}(x,y,z) = {}_HG_{n+1}(x,y,z) + {}_HG_n(x,y,z). \tag{3.9} $$

Also, corresponding to the summation formula involving Genocchi polynomials $G_n(x)$ [9, p. 1038, (43)]

$$ \sum_{k=1}^{m}(-1)^k(x+k)^n = \frac{1}{2(n+1)}\big[(-1)^mG_{n+1}(x+m+1) - G_{n+1}(x)\big], $$

we find the following summation formula involving 3VHP $H_n(x,y,z)$ and Hermite–Genocchi polynomials ${}_HG_n(x,y,z)$:

$$ \sum_{k=1}^{m}(-1)^kH_n(x+k,y,z) = \frac{1}{2(n+1)}\big[(-1)^m\,{}_HG_{n+1}(x+m+1,y,z) - {}_HG_{n+1}(x,y,z)\big]. \tag{3.10} $$

4. Concluding remarks

The Appell family $\{A_n(x)\}_{n=0}^{\infty}$ generated by (1.22) is obviously rather restrictive; it does not allow the treatment of some other polynomial sets on the Laguerre or the Bessel polynomials within the context of the operational formalism. Recently, Dattoli et al. [9] have shown that the extension of Appell family to Sheffer family [18] allows such a possibility.

A polynomial sequence $\{S_n(x)\}$ $(n = 0, 1, 2, \ldots)$ ($S_n(x)$ being a polynomial of degree n), is called of a Sheffer A-type zero [18] (which we shall call here Sheffer-type) if $S_n(x)$ possesses the following exponential generating function:

$$S(t)\exp\bigl(xH(t)\bigr) = \sum_{n=0}^{\infty} S_n(x)\frac{t^n}{n!}, \tag{4.1}$$

in which $H(t)$ and $S(t)$ have (at least the formal) expansions

$$H(t) = \sum_{n=0}^{\infty} h_n t^{n+1}, \quad h_0 \neq 0,$$

and

$$S(t) = \sum_{n=0}^{\infty} a_n t^n, \quad a_0 \neq 0,$$

respectively.

Clearly the Appell polynomials belong to this family too. If $S_n(x)$ are of Sheffer type, then it is always possible to find the explicit representations of the multiplicative and derivative operators $\hat{M}$ and $\hat{P}$. Conversely, if $\hat{M} = \hat{M}(x, D)$ and $\hat{P} = \hat{P}(x, D)$ $(D \equiv \frac{d}{dx})$, then $p_n(x)$ of Eq. (1.1) are necessarily of Sheffer-type, see for details [4] and references cited therein. Among the polynomials encountered in quantum mechanics, Hermite and Laguerre polynomials are of Sheffer type, whereas Legendre, Jacobi and Gegenbauer polynomials are not.

We note that if

$$S(t) = \frac{1}{1-t} \quad \text{and} \quad H(t) = \frac{-t}{(1-t)},$$

then $S_n(x) = n!L_n(x)$, where $L_n(x)$ are the Laguerre polynomials [18].

Again, if

$$S(t) = \frac{1}{1-t} \quad \text{and} \quad H(t) = \frac{-4t}{(1-t)^2},$$

then $S_n(x) = n!f_n(x)$, where $f_n(x)$ are the Sister Celine polynomials [18].

Let us explore the possibility of introducing some polynomials belonging to Hermite–Sheffer family.

Starting from the generating function for Laguerre polynomials

$$\frac{1}{(1-t)}\exp\left(-\frac{xt}{(1-t)}\right) = \sum_{n=0}^{\infty} L_n(x)t^n \tag{4.2}$$

and replacing x by the multiplicative operator $\hat{M}$ of $H_n(x, y, z)$, we find

$$\mathcal{H}(x, y, z; t) = \frac{1}{(1-t)}\exp\left(-\left(x + 2y\frac{\partial}{\partial x} + 3z\frac{\partial^2}{\partial x^2}\right)\left(\frac{t}{1-t}\right)\right). \tag{4.3}$$

Now, decoupling the exponential operator in (4.3) by using the identity (2.2), we get the generating function for Hermite–Laguerre polynomials ${}_HL_n(x, y, z)$ in the form:

$$\mathcal{H}(x, y, z; t) = \frac{1}{(1-t)}\exp\left(-\frac{xt}{1-t} + \frac{yt^2}{(1-t)^2} - \frac{zt^3}{(1-t)^3}\right) = \sum_{n=0}^{\infty} {}_HL_n(x, y, z)t^n. \tag{4.4}$$

Next, we consider the generating function for Sister Celine's polynomials $f_n(x)$ [18, p. 292, (18)]

$$\frac{1}{(1-t)}\exp\left(-\frac{4xt}{(1-t)^2}\right) = \sum_{n=0}^{\infty} f_n(x)t^n, \tag{4.5}$$

which on replacing x by the multiplicative operator $\hat{M}$ of $H_n(x, y, z)$ and decoupling the exponential operator in the resultant equation using identity (2.2) yields the generating function for Hermite–Sister Celine's polynomials ${}_Hf_n(x, y, z)$ in the form:

$$\frac{1}{(1-t)}\exp\left(-\frac{4xt}{(1-t)^2} + \frac{16yt^2}{(1-t)^4} - \frac{64zt^3}{(1-t)^6}\right) = \sum_{n=0}^{\infty} {}_Hf_n(x, y, z)t^n. \tag{4.6}$$

In this paper, it has been shown that the Hermite-based Appell polynomials can be “generated” by replacing x with the multiplicative operator $\hat{M}$ of 3VHP $H_n(x, y, z)$ in the generating function of Appell polynomials. The Sheffer polynomials, which include Appell polynomials as a special case along with the underlying operational formalism, offer a powerful tool for investigation of the properties of a wide class of polynomials. Here, we have introduced Hermite–Laguerre and Hermite–Sister Celine’s polynomials as two members of the family of Hermite-based Sheffer polynomials. A general approach to generate Hermite-based Sheffer polynomials associated with 3VHP $H_n(x, y, z)$ will be discussed in a forthcoming investigation. Also, there are possibilities to generate Hermite-based Appell and Hermite-based Sheffer polynomials associated with m-variable Hermite polynomials $H_n(x_1, x_2, \ldots, x_{m-1}, x_m)$.

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