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The Decidability of the D0L–DT0L Equivalence Problem

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It is shown that it is decidable of any endomorphisms $\delta, \delta_1,..., \delta_m$ and any words $P, P_1,..., P_n$ whether or not $\delta^*(P) = \{\delta_1,..., \delta_m\}^*(\{P_1,..., P_n\})$. As a consequence it is decidable of any endomorphisms $\delta_1,..., \delta_m$ and any words $P_1,..., P_n$ whether or not there exist an endomorphism δ and a word P such that the above equality holds true. In the positive case all such endomorphisms δ and words P can be effectively listed.

0. INTRODUCTION

Consider the following decidability problem. Given two finite sets of endomorphisms $\{\delta_1,..., \delta_{m_1}\}$, $\{\sigma_1,..., \sigma_{m_2}\}$ on a word monoid A^* and two words $P, Q \in A^*$, is it decidable whether or not

$$\{\delta_1,..., \delta_{m_1}\}^*(P) = \{\sigma_1,..., \sigma_{m_2}\}^*(Q)?$$

Problems like this first arose in the theory of Lindenmayer systems. In current L systems jargon this is the DT0L *equivalence problem* and it is known to be undecidable in the general case. This result was first obtained by Rozenberg [9] in 1972. It is closely connected with the undecidability of the equivalence problem of sentential forms of context-free grammars, independently obtained by Blattner [2], Rozenberg [8] and Salomaa [17]. Using the standard technique of "cycling tables," see e.g., the proof of Theorem 7.12 in [5], it is straightforward to show that the undecidability is obtained already in the case $m_1 = m_2 = 2$.

The case $m_1 = m_2 = 1$ (the D0L *equivalence problem*) was open for a long time; it was finally solved in 1976 by Čulik and Friš [3] who proved that the problem is decidable in this case, using complicated structural analysis. Later, another solution was obtained by Ehrenfeucht and Rozenberg [4], which uses some elegant algebraic constructs. Further modifications and simplifications of this latter solution can be found in [11] and [6]. All these solutions are based on the pioneering earlier work of Nielsen [7].

The remaining cases $m_1 = 1$, $m_2 \geqslant 2$ (the D0L–DT0L *equivalence problem*) have been open. We show in this paper that they are all decidable. While pursuing another subject (namely the decidability of the inclusion problem for D0L languages) we

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proved in [13, Corollary 3.3] a related result: It is decidable whether or not $\delta_1^*(P)$ satisfies the equation

$$X = \sigma_1(X) + \sigma_2(X) + \cdots + \sigma_{m_2}(X) + \{Q\}. \tag{0}$$

Clearly, $\{\sigma_1,..., \sigma_{m_2}\}^*(Q)$ is the minimal solution of this equation and therefore, if $\delta_1^*(P)$ satisfies it, we have

$$\{\sigma_1,..., \sigma_{m_2}\}^*(Q) \subseteq \delta_1^*(P).$$

The proof is based on the decidability of the D0L equivalence problem and a powerful result of Berstel's and Nielsen's [1] concerning D0L growth sequences.

We take the result of [13]—in more detail than is indicated above—as our starting point in this paper and proceed to show that the D0L–DT0L equivalence problem is decidable. It is interesting to contrast this result and the result of [13] with the result of [12] concerning the structure of the class of solutions of equations like (0) (and even more general equations and systems of equations). Indeed, although the structure seems to be rather complicated in the general case, D0L languages possess strong structural restraints which makes it possible to effectively test their position (solution or not, minimal solution or not) with respect to the class of solutions of (0). Disregarding minimality, this is known to be true for regular languages, too, but not any more for linear languages, 0L languages nor DT0L languages.

A reader unfamiliar with the usual requisites of formal language theory is referred to [16]. The basis for L systems can be found in [11] or [10]. Reference [5] contains the earlier developments of L systems theory up to 1973 as well as a good introduction to the biological origins and applications of L systems written by the "father" of the theory, A. Lindenmayer.

1. Notation

The *length* of a word P is denoted by $|P|$.

The *Parikh vector* of a word P is denoted by $[P]$ (a row vector)

The *empty word* is denoted by Λ.

The set of symbols occurring in a word P is denoted by $\mathrm{alph}(P)$.

The *Parikh matrix* of a word morphism $\delta: A^* \to B^*$ is denoted by $[\delta]$, i.e.,

$$[\delta] = \sum_{a \in A} [a]^T [\delta(a)],$$

where T denotes transpose.

A D0L *system* G with alphabet A, endomorphism δ and axiom ω is denoted by $G = (A, \delta, \omega)$.

A DT0L *system* H with alphabet A, endomorphisms $\delta_1,\ldots,\delta_m$ and axiom P is denoted by $H=(A,\{\delta_1,\ldots,\delta_m\},P)$.

The *language generated by a* D0L *system* G is denoted by $L(G)$, i.e., $L(G)=\{\delta^n(\omega)\mid n\geqslant 0\}$; in the sequel it is always assumed that $L(G)$ is infinite without an explicit mentioning of it; this is to avoid more or less trivial exceptions (finiteness and membership problems are decidable for D0L languages).

The *monoid of endomorphisms* generated by $\delta_1,\ldots,\delta_m$ under composition is denoted by $\{\delta_1,\ldots,\delta_m\}^*$ and also by δ_1^* if $m=1$.

The *language generated by a* DT0L *system* H is denoted by $L(H)$, i.e.,

$$L(H)=\{\sigma(P)\mid\sigma\in\{\delta_1,\ldots,\delta_m\}^*\}=\{\delta_1,\ldots,\delta_m\}^*(P);$$

we will always assume that $L(H)$ is infinite (finiteness and membership problems are decidable for DT0L languages, too).

The *Parikh language generated by a* D0L *system* G is denoted by $P(G)$, i.e.,

$$P(G)=\{[Q]\mid Q\in L(G)\}=\{[\omega][\delta]^n\mid n\geqslant 0\}.$$

The *Parikh language generated by a* DT0L *system* H is denoted by $P(H)$, i.e.,

$$P(H)=\{[Q]\mid Q\in L(H)\}=\{[P][\sigma]\mid\sigma\in\{\delta_1,\ldots,\delta_m\}^*\}.$$

The D0L systems $(A,\delta^p,\delta^j(\omega))$, $j=0,\ldots,p-1$, form a *decomposition* of the D0L system (A,δ,ω); decompositions of D0L systems are used frequently in the sequel for technical convenience.

n is assumed in the sequel to be a free variable ranging through nonnegative integers.

An *A-guarded subword* of a word $P\in(A+B)^*$, where $A\cap B=\Phi$, is a word $Q\in B^*$ such that aQb is a subword of aPb for some $a,b\in A$.

2. The Case of Nonlinear Growth

Let us first recall the result of [13], concerning the D0L–DT0L equivalence problem, in its full generality:

Theorem 2.1. *For any endomorphisms $\delta_1,\ldots,\delta_m$, any word P and any* D0L *language $L=L((A,\delta,\omega))$ it is decidable whether or not*

$$L=\delta_1(L)+\cdots+\delta_m(L)+\{P\}. \tag{1}$$

Furthermore, if (1) *holds true, then there exist computable numbers $p_i\geqslant 1$, $q_{ij}\geqslant 0$, $r_i\geqslant 0$, $s_{ij}\geqslant 0$, where $j=0,\ldots,p_i-1$ and $i=1,\ldots,m$, such that for any $i\in\{1,\ldots,m\}$ and $j\in\{0,\ldots,p_i-1\}$ either $q_{ij}=0$ and $\delta_i(\{\delta^{p_in+j}(\omega)\mid n\geqslant 0\})$ is finite or $q_{ij}>0$ and*

$$\delta_i\delta^{p_in+r_i+j}(\omega)=\delta^{q_{ij}n+s_{ij}}(\omega).\quad\blacksquare$$

For simplicity we will assume in the sequel that $p_1 = p_2 = \cdots = p_m = p$. If this is not the case originally we write $p = \text{l.c.m.}\ \{p_1, ..., p_m\}$ and take suitable decompositions. We may obviously also assume that $r_1 = r_2 = \cdots = r_m = r$. Note that if (1) holds true, then $L((A, \{\delta_1, ..., \delta_m\}, P)) \subseteq L$.

LEMMA 2.1. *Let $G = (A, \delta, \omega)$ be a* DOL *system and σ an endomorphism on A^*. Let*

$$\sigma\delta^{pn+r}(\omega) = \delta^{qn+s}(\omega) \tag{2}$$

for some $p, q \geqslant 1$ and $r, s \geqslant 0$. If G is nonlinear, then $p = q$. If G is linear, then $q \geqslant p$.

Proof. We may obviously assume that $r = 0$. The case where G is exponential is taken care of in the proof of Lemma 3.3 of [13]. Suppose then that G is polynomial.

We may assume that

$$\text{alph}(\delta^s(\omega)) = \text{alph}(\delta^{1+s}(\omega)) = \text{alph}(\delta^{2+s}(\omega)) = \cdots .$$

We may assume also that

$$|\delta^{n+s}(\omega)| = (\phi_1(n), ..., \phi_l(n)), \tag{3}$$

where $\phi_1, ..., \phi_l$ are nonzero polynomials the set of degrees of which equals $\{0, ..., d\}$, where d is the degree of G, cf., e.g., Lemma 4.1 in [13]. Without restricting the case we may assume that $\phi_1, ..., \phi_t$ are of degree d and $\phi_{t+1}, ..., \phi_l$ are of lower degree. Since (2) holds true, $\phi_1(qn+s), ..., \phi_t(qn+s)$ can be written as linear combinations of $\phi_1(pn), ..., \phi_l(pn)$ with nonnegative integer coefficients. Hence $q \geqslant p$.

In the sequel we assume that G is nonlinear. To prove that $p = q$ "growth arguments," that is, arguments using Parikh sequences only, are not sufficient any more, as is witnessed by the identity $(1, 2n, 4n^2) = (1, 2n, (2n)^2)$. Therefore we need methods which employ the internal order of symbols in words. We recall a characterization result, see, e.g., [13, Sect. 4]:

The degree of the polynomial in (3) corresponding to a symbol is also called the degree of the symbol. Denote by A_i the set of symbols of degree i (for $i = 0, ..., d$). Then, for any $a \in A_i$, $\delta(a)$ is either empty of contains only occurrences of symbols of $A_i + A_{i+1} + \cdots + A_d$ and, since (2) holds true, the same is true for $\sigma(a)$. Therefore we may assume that $d = 2$ simply by ignoring symbols of higher degree.

Let $c'n + d'$ be the total number of $(A_0 + A_1)$-guarded subwords of $\delta^n(\omega)$ and let $N_D(n)$ be the number of $(A_0 + A_1)$-guarded subwords of $\delta^n(\omega)$ of length greater than D. It is easily seen by induction on n that the maximum length $l(n)$ of an $(A_0 + A_1)$-guarded subword of $\delta^n(\omega)$ is linearly bounded, say $l(n) \leqslant en + f$. Then

$$(c'n + d')D + (en + f)N_D(n) \geqslant |\delta^n(\omega)| - c'n - d' + 1,$$

whence $N_D(n) \geqslant cn + d_D$ for some $c > 0$ not depending on D. Assume now, contrary to the claim, that $q > p$. Then, for some k, $(q/p)^k > c'/c$. Equation (2) implies that for each $j = 1, 2, \ldots$

$$(\delta^{sp^{j-1}-s}\sigma)^j \delta^{p^j n}(\omega) = \delta^{q^j n + s_j}(\omega), \qquad \text{where} \quad s_j = \frac{s(p^j - q^j)}{p-q}.$$

Take $j = k$ and let D be the maximum length of an $(A_0 + A_1)$-guarded subword of $(\delta^{sp^{k-1}-s}\sigma)^k(x)$ where $x \in A_0 + A_1$. Then $N_D(q^k n + s_k)$ cannot exceed $c'p^k n + d'$ since no word of $(\delta^{sp^{k-1}-s}\sigma)^k(A_2)$ contains symbols of $A_0 + A_1$. This is a contradiction because, on the other hand,

$$N_D(q^k n + s_k) \geqslant c(q^k n + s_k) + d_D. \quad \blacksquare$$

Remark. In [13] Lemma 3.3 was proved using the decidability of the equivalence problem for polynomial HD0L sequences. The above proof shows that this decidability result can be replaced by weaker arguments.

Now let $\delta_1, \ldots, \delta_m, P$ and $G = (A, \delta, \omega)$ be such that (1) in Theorem 2.1 holds true. Denote $H = (A, \{\delta_1, \ldots, \delta_m\}, P)$. Let G be nonlinear. By Lemma 2.1 it follows that $q_{ij} = 0$ or $q_{ij} = p$ for each $i = 1, \ldots, m$ and $j = 0, \ldots, p-1$. We may assume that $\{\omega, \delta(\omega), \ldots, \delta^u(\omega)\} \subseteq L(H)$, after separate testing of membership, for any fixed u.

Let us construct a finite directed graph with labelled vertices and labelled (directed) edges as follows:

(A) the vertices are labelled by $0, \ldots, p-1$,

(B) the edges are labelled by $1, \ldots, m$,

(C) an edge labelled by i joins two vertices labelled by j_1 and j_2 (in that order) iff $s_{ij_1} \equiv j_2 + r \pmod p$ and $q_{ij_1} \neq 0$.

This digraph has the following property which follows from (1):

(D) for any vertex j_1 there exists a vertex j_2 which is joined to j_1 (in that order).

Note that there need not be an edge labelled by i joining j_1 to any of the vertices for each label i (or for any i, for that matter), owing to the fact that q_{ij_1} may be zero.

Now (D) implies that the digraph contains cycles. For any cycle with vertices $j_1, \ldots, j_k = j_1$ and edges $i_1, \ldots, i_k = i_1$ (denoted by $(j_1, \ldots, j_k; i_1, \ldots, i_k)$) we have

$$\delta_{i_{k-1}} \cdots \delta_{i_1} \delta^{pn+r+j_1}(\omega) = \delta^{p(n+q)+r+j_1}(\omega)$$

for some q. The cycles are divided into two types: (I) those cycles for which $q > 0$ and (II) those cycles for which $q \leqslant 0$. For a cycle $(j_1, \ldots, j_k; i_1, \ldots, i_k)$ of type (I) we have

$$(\delta_{i_{k-1}} \cdots \delta_{i_1})^n \delta^{r+j_1+pl}(\omega) = \delta^{p(qn+l)+r+j_1}(\omega),$$

for $l = 0, \ldots, q-1$, whence we know that $\{\delta^{pn+r+j_1}(\omega) \mid n \geqslant 0\} \subseteq L(H)$ once we assume that $\delta^{r+j_1}(\omega), \delta^{r+j_1+p}(\omega), \ldots, \delta^{r+j_1+p(q-1)}(\omega)$ are in $L(H)$.

Let then, more generally, j be a vertex such that a directed path $(j'_1,\dots, j'_h = j$: $i'_1,\dots, i'_h)$ joins a vertex j'_1 to j and j'_1 belongs to a cycle $(j_1 = j'_1,\dots, j_k; i_1,\dots, i_k)$ of type (I). Then $\{\delta^{pn+r+j_1}(\omega) \mid n \geqslant 0\} \subseteq L(H)$ and hence also

$$\delta_{i'_{h-1}} \cdots \delta_{i'_1}(\{\delta^{pn+r+j_1}(\omega) \mid n \geqslant 0\}) = \{\delta^{p(n+q')+r+j}(\omega) \mid n \geqslant 0\} \subseteq L(H)$$

for some q'. We may thus assume that for any $j \in \{0,\dots, p-1\}$ such that the vertex labelled by j can be reached from a cycle of type (I), $\{\delta^{pn+r+j}(\omega) \mid n \geqslant 0\} \subseteq L(H)$. Our aim, then, is to show that if $L(G) = L(H)$ then all vertices satisfy this condition, whence we can decide whether or not $L(G) \subseteq L(H)$ since the membership problem is decidable for DT0L languages and so is reachability from cycles of type (I).

We note first that $q = 0$ for type (II) cycles. Assume the contrary, i.e., $q < 0$, and denote

$$\varepsilon_1 = \delta_{i_{k-1}} \cdots \delta_{i_1}, \qquad \varepsilon_2 = \delta^{p|q|}, \qquad \omega' = \delta^{r+j_1}(\omega).$$

Then $\varepsilon_1^n \varepsilon_2^n(\omega') = \omega'$. We may assume that $|\omega'| < |\varepsilon_2(\omega')| < |\varepsilon_2^2(\omega')| < \cdots$, by König's Lemma, and that

$$\mathrm{alph}(\omega') = \mathrm{alph}(\varepsilon_2(\omega')) = \mathrm{alph}(\varepsilon_2^2(\omega')) = \cdots .$$

Now, for any DOL system (B, ε, θ) there exists a constant $c > 0$ depending on the cardinality of B only such that $|\varepsilon^{x+c}(\theta)| > |\varepsilon^x(\theta)|$ for all large enough x, cf., e.g., the proof of Lemma 3.1 in [14]. Thus, for some $c > 0$,

$$|\varepsilon_1^{x+c}\varepsilon_2^c(\omega')| > |\varepsilon_1^x\varepsilon_2^c(\omega')| = |\varepsilon_1^{x-c}(\omega')| = |\varepsilon_1^{x+c}\varepsilon_2^{2c}(\omega')| \geqslant |\varepsilon_1^{x+c}\varepsilon_2^c(\omega')|,$$

which is absurd, unless $K = \{\varepsilon_1^n(\theta) \mid n \geqslant 0\}$ is finite for all words θ in $\{\varepsilon_2^n(\omega') \mid n \geqslant 0\}$. This is, however, impossible since the size of K is then bounded by a constant depending on the cardinality of $\mathrm{alph}(\omega')$ only, see, e.g., Corollary 4 of [18], whence $\varepsilon_1^{d+e}\varepsilon_2^n(\omega') = \varepsilon_1^e\varepsilon_2^n(\omega')$ for some $d \geqslant 1$ and e.

Consider an arbitrary $j \in \{0,\dots, p-1\}$. For any word $\delta^{pn+j+r}(\omega) \neq P$, say $\delta^{pn_1+j+r}(\omega)$, we must have a sequence of indices $i_1,\dots, i_v \in \{1,\dots, m\}$ such that

$$\delta_{i_v} \cdots \delta_{i_1}(P) = \delta^{pn_1+j+r}(\omega),$$

if $L(H) = L(G)$. We may assume, of course, that the words P, $\delta_{i_1}(P)$, $\delta_{i_2}\delta_{i_1}(P),\dots, \delta_{i_v} \cdots \delta_{i_1}(P)$ are distinct, i.e., v is minimal. We may, of course, also assume that v and n_1 are arbitrarily large. Let

$$\delta_{i_w} \cdots \delta_{i_1}(P) = \delta^{pn_w+j_w+r}(\omega), \qquad 0 \leqslant j_w \leqslant p-1, \tag{4}$$

for all $w = u,\dots, v$, where u is chosen to be so large that it is possible to write (4). Clearly, we may assume $v - u$ to be arbitrarily large, too. Now, for a large enough u, there is always an edge labelled by i_w joining j_{w-1} to j_w in our digraph for each $w \in \{u,\dots, v\}$. Otherwise $\delta_{i_w}(\{\delta^{pn+j_{w-1}+r}(\omega) \mid n \geqslant 0\})$ is finite, i.e., $q_{i_w j_{w-1}} = 0$, and we may assume that u is so large that these finite languages do not contain $\delta_{i_u} \cdots \delta_{i_1}(P),..., \delta_{i_v} \cdots \delta_{i_1}(P)$. If $v-u$ is large enough, the path $(j_u,..., j_{v-1}; i_{u+1},..., i_v)$ contains a cycle. This cycle cannot be of type (II) by the minimality of v. Hence j is reached from a cycle of type (I).

We have proved

THEOREM 2.2. *For any* DT0L *system* H *and any nonlinear* D0L *system* G *it is decidable whether or not* $L(H)=L(G)$. *Moreover, if* $L(H)=L(G)$, *there exist (effectively) a decomposition* $G_1,..., G_k$ *of* G, *endomorphisms* $\sigma_1,..., \sigma_k, \tau_1,..., \tau_k$ *generated by the endomorphisms of* H, *words* $P_1,..., P_k \in L(H)$ *and finite languages* $F_1,..., F_k$ *such that*

$$L(G_i)=\sigma_i(L((A, \tau_i, P_i)))+F_i \qquad \textit{for} \quad i=1,..., k,$$

where A *is the alphabet of* H. ∎

3. The Case of Linear Growth and Another Proof for the Case of Nonlinear Growth

We give here a proof which shows that the D0L–DT0L equivalence problem is decidable in the case of linear growth. In fact, the proof is general and does not use the linearity of growth. Thus we have another proof for the nonlinear case. The second sentence of Theorem 2.2 does not hold true, in general, in the case of linear growth as is witnessed by the fact that the D0L system $(\{a, b\}, \delta, b)$ and the DT0L system $(\{a, b\}, \{\delta_1, \delta_2\}, b)$, where

$$\delta(a)=a, \qquad \delta(b)=ba, \qquad \delta_1(a)=\delta_2(a)=a^2, \qquad \delta_1(b)=b, \qquad \delta_2(b)=ba,$$

both generate the language ba^*. Therefore our proof in this section, being quite general, does not readily produce the second sentence of Theorem 2.2, which is why we chose to present the more elaborate proof of the previous section.

Let us denote, for brevity,

$$\pi_j(n)=\delta^{pn+r+j}(\omega) \qquad \text{for} \quad j=0,..., p-1$$

in Theorem 2.1. Then

$$\delta_i(\pi_j(n))=\delta^{q_{ij}n+s_{ij}}(\omega).$$

After separate testing of membership we may assume that

$$F=L(G)\backslash\{\pi_j(n) \mid j=0,..., p-1 \text{ and } n \geqslant 0\} \subseteq L(H).$$

If we wish we may always omit some initial terms of a sequence $(\pi_j(n))$, i.e., replace it by $(\pi_j(n+q))$, where $q>0$ and add $\pi_j(0), \pi_j(1),..., \pi_j(q-1)$ to F. In fact, in order to carry out our argumentation this may have to be done. For this purpose we define a finite directed graph with labelled vertices and labelled (directed) edges as follows:

(A) the vertices are labelled by $0,..., p-1$;

(B) an edge joins two vertices labelled by j_1 and j_2 (in that order) iff $s_{ij_1} \equiv j_2 + r \pmod p$ and $q_{ij_1} = p$ for some $i \in \{1,..., m\}$; the edge is then labelled by the smallest number q, such that

$$\delta_i(\pi_{j_1}(n)) = \pi_{j_2}(n+q) \qquad \text{if} \quad q \geqslant 0$$

or

$$\delta_i(\pi_{j_1}(n-q)) = \pi_{j_2}(n) \qquad \text{if} \quad q < 0,$$

for some such i; any two vertices are joined by at most one edge.

An edge with a negative (resp. zero, positive) label is called a negative edge (resp. a 0-edge, a positive edge). Let E be the sum of the labels of all negative edges, if any. If $E = 0$, i.e., there are no negative edges, then nothing further needs to be done.

Suppose then that $E < 0$. Then there exists a vertex j_0 which is joined to one of the vertices by a negative edge. Denote by $j_1,..., j_u$ those vertices from which j_0 is reached through paths consisting solely of 0-edges. We may assume that no vertex is joined to one of $j_0, j_1,..., j_u$ by a negative edge. Indeed, if j' is joined to j_v, say, by a negative edge, then we replace j_0 by j'. Such replacement can be carried out only a certain finite number of times because otherwise we end up with a cycle consisting of nonpositive edges only and having a negative sum of labels of edges. Such cycles cannot exist, cf. the proof in the previous section of the fact that $q = 0$ for type (II) cycles. For the same reason no vertex is joined to itself by a negative edge.

We now replace $\pi_{j_v}(n)$ by $\pi_{j_v}(n+1)$ for $v = 0,..., u$. Then

(i) the label of an edge joining one of $j_0,..., j_u$ to one of the other vertices increases by 1;

(ii) the label of an edge joining a vertex of $\{0,..., p-1\} \setminus \{j_0,..., j_u\}$ to one of $j_0,..., j_u$ decreases by 1;

(iii) the label of an edge joining two of the vertices $j_0,..., j_u$ remains the same.

Since there are no negative edges joining one of the vertices to one of $j_0,..., j_u$, and there is at least one negative edge joining one of $j_0,..., j_u$ to one of the other vertices, the net result is that E is increased. The procedure is repeated till $E = 0$. Finally we may assume that there are no negative edges.

Suppose then that $L(H) \subsetneq L(G)$. Let n_1 be the smallest number such that $\pi_k(n_1) \notin L(H)$ for some k. We will show that there exists a computable upper bound for n_1. Therefore we may assume, given a computable bound w, that $n_1 \geqslant w$. Now $\pi_k(n_1 - d) \in L(H)$ if $n_1 \geqslant d$, where

$$d = \text{l.c.m.}\{q_{ij} \neq 0 \mid i = 1,..., m \text{ and } j = 0,..., p-1\}.$$

We may assume, provided that n_1 is large enough, that

$$\pi_k(n_1 - d) = \delta_i(\pi_l(n_2))$$

for some i and l. Since, for an n_1 so large that $q_{il} \neq 0$,

$$\delta_i(\pi_l(n)) = \delta^{q_{il}(n+q_2)+s_{il}}(\omega)$$

for some $q_2 \geqslant 0$, we have

$$q_{il}n_2 + q_{il}q_2 + s_{il} = p(n_1 - d + q_1) + r + k$$

for some $q_1 \geqslant 0$. Thus

$$\delta_i\left(\pi_l\left(n_2 + \frac{dp}{q_{il}}\right)\right) = \pi_k(n_1) \notin L(H)$$

and, by the minimality of n_1, we have

$$n_1 \leqslant n_2 + \frac{dp}{q_{il}} = \frac{p}{q_{il}}n_1 + \frac{pq_1 + r + k - s_{il}}{q_{il}} - q_2.$$

If n_1 is large enough, this means that $p = q_{il}$ (recall that, by Lemma 2.1, $q_{il} \geqslant p$). But then $n_1 = n_2 + d$. Now $\pi_l(n_2 + d) \notin L(H)$ and we may repeat the above process starting from it. The number of steps like this can be made finite, because there exists a sequence $i_1, \ldots, i_v$ such that

$$\pi_k(n_1 - d) = \delta_{i_v} \cdots \delta_{i_1}(P).$$

We have proved

THEOREM 3.1. *For any* DOL *system* G *and any* DTOL *system* H *it is decidable whether or not* $L(G) = L(H)$. ∎

4. CONCLUSIONS

We may in fact present our results for FDTOL systems, i.e., DTOL systems with finite sets of axioms, instead of DTOL systems only. This follows from the simple observation that if G is a DOL system and H is an FDTOL system and c is a symbol not in the alphabets of G and H, then $L(G) + \{c\}$ is a DOL language and $L(H) + \{c\}$ is a DTOL language. Therefore we have

THEOREM 4.1. *For any* DOL *system* G *and any* FDTOL *system* H *it is decidable whether or not* $L(G) = L(H)$. *Moreover, if* $L(G) = L(H)$ *and* G *is nonlinear, then there exist (effectively) a decomposition* $G_1, \ldots, G_k$ *of* G, *endomorphisms*

$\sigma_1,\ldots,\sigma_k,\tau_1,\ldots,\tau_k$ generated by the endomorphisms of H, words $P_1,\ldots,P_k \in L(H)$ and finite languages $F_1,\ldots,F_k$ such that $L(G_i)=\sigma_i(L((A,\tau_i,P_i)))+F_i$ for $i=1,\ldots,k$, where A is the alphabet of H. ∎

To put it in another way, it is decidable of any endomorphisms $\delta,\delta_1,\ldots,\delta_m$ on a word monoid A^* and any words $\omega,P_1,\ldots,P_l \in A^*$ whether or not $\delta^*(\omega)=\{\delta_1,\ldots,\delta_m\}^*(\{P_1,\ldots,P_l\})$.

It may be noted that Theorem 4.1 holds true for Parikh languages, too, i.e., when A^* is a free Abelian monoid, with the restriction that G should be exponential for the second sentence of the theorem to hold true in general.

We have also a corollary concerning "D0Lness" of FDT0L (Parikh) languages.

COROLLARY 4.1. *For any* FDT0L *system H it is decidable whether or not there exist* D0L *systems G such that $L(G)=L(H)$* (resp. *$P(G)=P(H)$*) *and in the positive case all such* D0L *systems can be effectively listed (there will be a finite number of them).*

Proof. Cf. the proof of Corollary 6.2 in [14] and the proof of Corollary 2 in [15]. ∎

REFERENCES

1. J. BERSTEL AND M. NIELSEN, The growth range equivalence problem for D0L systems is decidable, *in* "Automata, Languages, Development" (G. Rozenberg and A. Lindenmayer, Eds.), pp. 161–178, North-Holland, Amsterdam, 1976.
2. M. BLATTNER, The unsolvability of the equality problem for sentential forms of context-free languages, *J. Comput. System Sci.* **7** (1973), 463–468.
3. K. ČULIK II AND I. FRIŠ, The decidability of the equivalence problem for D0L systems, *Inform. Contr.* **35** (1977), 20–39.
4. A. EHRENFEUCHT AND G. ROZENBERG, Elementary homomorphisms and a solution of the D0L sequence equivalence problem, *Theoret. Comput. Sci.* **7** (1978), 169–184.
5. G. T. HERMAN AND G. ROZENBERG, "Developmental Systems and Languages," North-Holland, Amsterdam, 1975.
6. J. KARHUMÄKI AND I. SIMON, A note on elementary homomorphisms and the regularity of equality sets, *Bull. EATCS* No. 9 (1979), 16–24.
7. M. NIELSEN, On the decidability of some equivalence problems for D0L systems, *Inform. Contr.* **25** (1974), 166–193.
8. G. ROZENBERG, Direct proofs of the undecidability of the equivalence problem for sentential forms of linear context-free grammars and the equivalence problem for 0L systems, *Inform. Process. Lett.* **1**(1972), 233–235.
9. G. ROZENBERG, The equivalence problem for deterministic T0L systems is undecidable, *Inform. Process. Lett.* **1** (1972), 201–204.
10. G. ROZENBERG AND A. SALOMAA, The mathematical theory of L systems, *in* "Advances in Information Systems Science 6," (J. Tou, Ed.), pp. 161–206, Plenum, New York, 1976.
11. G. ROZENBERG AND A. SALOMAA, "The Mathematical Theory of L Systems," Academic Press, New York, 1980.
12. K. RUOHONEN, A note on language equations involving morphisms, *Inform. Process. Lett.* **7** (1978), 209–212.

13. K. Ruohonen, The inclusion problem for D0L languages, *Elektron. Informationsverarbeit. Kybernetik* **15** (1979), 535–548.
14. K. Ruohonen, On the decidability of the 0L–D0L equivalence problem, *Inform. Contr.* **40** (1979), 301–318.
15. K. Ruohonen, The decidability of the F0L–D0L equivalence problem, *Inform. Process. Lett.* **8** (1979), 257–260.
16. A. Salomaa, "Formal Languages," Academic Press, New York, 1973.
17. A. Salomaa, On sentential forms of context-free languages, *Acta Inform.* **2** (1973), 40–49.
18. P. M. B. Vitányi, On the size of D0L languages, *in* "L Systems" (G. Rozenberg and A. Salomaa, Eds.), Lecture Notes in Computer Science No. 15, pp. 78–92, Springer-Verlag, Berlin, 1974.