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Note

Completely independent spanning trees in the underlying graph of a line digraph

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Abstract

In this note, we define completely independent spanning trees. We say that $T_1, T_2, \ldots, T_k$ are completely independent spanning trees in a graph H if for any vertex r of H, they are independent spanning trees rooted at r. We present a characterization of completely independent spanning trees. Also, we show that for any k-vertex-connected line digraph $L(G)$, there are k completely independent spanning trees in the underlying graph of $L(G)$. At last, we apply our results to de Bruijn graphs, Kautz graphs, and wrapped butterflies. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Independent spanning trees; Line digraphs; Interconnection networks; Parallel processing; de Bruijn graphs; Kautz graphs; Wrapped butterflies

1. Introduction

In a graph, two paths P_1 and P_2 from a vertex x to another vertex y are called *openly disjoint* if P_1 and P_2 are edge-disjoint and have no common vertex except for x and y. Let $T_1, T_2, \ldots, T_k$ be spanning trees in a graph H. Let r be a vertex of H. If for any vertex $v(\neq r)$ of H, the paths from r to v in $T_1, T_2, \ldots, T_k$, are pairwise openly disjoint, then we say that $T_1, T_2, \ldots, T_k$ are k *independent spanning trees rooted at* r. (When we treat digraphs instead of graphs, a rooted tree is defined as an acyclic digraph in which there is a unique vertex (root) with indegree 0 such that for any other vertex, the indegree is 1. The notion of independent spanning trees in a digraph is similarly defined.) For independent spanning trees, the following conjecture is well-known; 'Let H be a k-vertex-connected graph. Then, for any vertex r of H, there are k independent spanning trees rooted at r'. This conjecture was proved for $k \leqslant 3$ [2,10,14]. Also, it has

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been shown that the conjecture holds for the class of planar graphs [9]. The directed version of the conjecture was proved for $k = 2$ [13] and also for any $k \geqslant 1$ if we restrict ourselves to the class of line digraphs [7]. However, the directed version of the conjecture does not hold for general digraphs when $k \geqslant 3$ [8].

Independent spanning trees have been studied from not only the theoretical point of view but also the practical point of view because of their applications to fault-tolerant broadcasting in parallel computers [10]. Until now, independent spanning trees in several interconnection networks have been studied; product graphs [12], de Bruijn and Kautz digraphs [5,7], and chordal rings [11].

Many papers have presented constructions of independent spanning trees for a given root vertex. However, if one set of spanning trees is always a set of independent spanning trees rooted at any given vertex, then we do not need to reconstruct independent spanning trees when the root is changed with another vertex. Motivated by this point of view, we define the following notion.

Definition 1.1. Let $T_1, T_2, \ldots, T_k$ be spanning trees in a graph H. If for any two vertices u, v of H, the paths from u to v in $T_1, T_2, \ldots, T_k$, are pairwise openly disjoint, then we say that $T_1, T_2, \ldots, T_k$ are completely independent.

Note that completely independent spanning trees must be edge-disjoint (cf. the proof of Theorem 2.1) although independent spanning trees are not always edge-disjoint. It is known that edge-disjoint spanning trees have applications to worm-hole routing in parallel computers [1]. In this note, we present a characterization of completely independent spanning trees.

Unless otherwise stated, a digraph may have loops but not multiarcs. Let G be a digraph. Then, $V(G)$ and $A(G)$ denote the vertex set and the arc set of G, respectively. The *line digraph* $L(G)$ of G is defined as follows. The vertex set of $L(G)$ is the arc set of G, i.e., $V(L(G)) = A(G)$. Then, there is an arc from a vertex (u, v) to a vertex (x, y) in $L(G)$ iff $v = x$, i.e., $A(L(G)) = \{((u, v), (v, w)) \mid (u, v), (v, w) \in A(G)\}$. When we regard '$L$' as an operation on digraphs, the operation is called the *line digraph operation*. The *m-iterated line digraph* $L^m(G)$ of G is the digraph obtained from G by iteratively applying the line digraph operation m times. The *underlying graph* $U(G)$ is a graph obtained from G by replacing each arc with the corresponding edge and deleting loops. Note that $U(G)$ may have a 2-multiedge because G may have a pair of opposite arcs.

It has been shown in [7] that if a line digraph $L(G)$ is k-vertex-connected, then for any vertex r of $L(G)$, there are k independent spanning trees rooted at r in $L(G)$, thus, in $U(L(G))$ too. In this note, we strengthen such a result in $U(L(G))$, i.e., we show that if a line digraph $L(G)$ is k-vertex-connected, then there are k completely independent spanning trees in $U(L(G))$. Since the class of the underlying graphs of line digraphs contains de Bruijn graphs, Kautz graphs, and wrapped butterflies which are known as interconnection networks of massively parallel computers, we finally apply our results to these interconnection networks.

The set of vertices adjacent from a vertex v in G is denoted by $\Gamma_G^+(v)$, and the outdegree of v in G, i.e., $|\Gamma_G^+(v)|$, is denoted by $\deg_G^+ v$. Analogously, $\Gamma_G^-(v)$ and $\deg_G^- v$ are defined. For a graph H and $v \in V(H)$, $\deg_H v$ denotes the degree of v in H. If for any vertex u of G, $\deg_G^+ u = \deg_G^- u = d$, then we say that G is *d-regular*. Let B be a subset of $A(G)$. Then, the subdigraph of G induced by B is denoted by $\langle B \rangle_G$. Let T be a rooted tree. The *depth* of T is the maximum length of paths from the root in T. (When we do not assume the existence of the root in a tree, the depth of a tree is the maximum length of paths in the tree.) A rooted tree of depth 1 is called a *star*. The trees obtained from T by deleting the root are called the *subtrees* of T.

2. A characterization of completely independent spanning trees

The notion of completely independent spanning trees can be characterized as follows.

Theorem 2.1. *Let $T_1, T_2, \ldots, T_k$ be spanning trees in a graph H. Then, $T_1, T_2, \ldots, T_k$ are completely independent if and only if $T_1, T_2, \ldots, T_k$ are edge-disjoint and for any vertex v of H, there is at most one spanning tree T_i such that* $\deg_{T_i} v > 1$.

Proof. ($\Leftarrow$): Let $T_1, T_2, \ldots, T_k$ be spanning trees such that they satisfy the right-hand side condition in the proposition. Now, assume that $T_1, T_2, \ldots, T_k$ are not completely independent. Then, there exist two vertices u, v and two spanning trees T_i, T_j such that the paths from u to v in T_i and T_j are not openly disjoint. Since T_i and T_j are edge-disjoint, the paths from u to v have a common vertex w except for u and v. This means that $\deg_{T_i} w > 1$ and $\deg_{T_j} w > 1$, which produces a contradiction.

($\Rightarrow$): Suppose that $T_1, T_2, \ldots, T_k$ are completely independent. If the edge $\{u, v\}$ belongs to both T_i and T_j, then the paths from u to v in T_i and T_j are not openly disjoint. Hence, $T_1, T_2, \ldots, T_k$ must be edge-disjoint. Now assume that there exists a vertex w such that $\deg_{T_i} w > 1$ and $\deg_{T_j} w > 1$. Without loss of generality, we can set $i = 1$ and $j = 2$. Let v be a vertex different from w. Let $\{w, t_l\}$ be the first edge on the path from w to v in T_l for $l = 1, 2$. Let x_l be a vertex such that the path from w to x_l in T_l does not contain the edge $\{w, t_l\}$ for $l = 1, 2$. Such vertices exist since $\deg_{T_1} w > 1$ and $\deg_{T_2} w > 1$. Both the path from x_1 to v in T_1 and the path from x_2 to v in T_2 contain w. Thus, $x_1 \neq x_2$. Since the paths from x_1 to v in T_1 and T_2 are openly disjoint, the path from x_1 to v in T_2 does not contain w. Now, we regard T_2 as a tree rooted at w. Then, x_1 and v are in the same subtree of T_2. On the other hand, x_2 and v are in different subtrees of T_2. Thus, x_1 and x_2 are in different subtrees of T_2. Similarly, when we regard T_1 as a tree rooted at w, x_1 and x_2 are in different subtrees of T_1. Therefore, both the paths from x_1 to x_2 in T_1 and in T_2 have w as a common vertex, which contradicts our assumption that T_1 and T_2 are completely independent. Hence, for any vertex v, there is at most one T_i such that $\deg_{T_i} v > 1$. $\square$

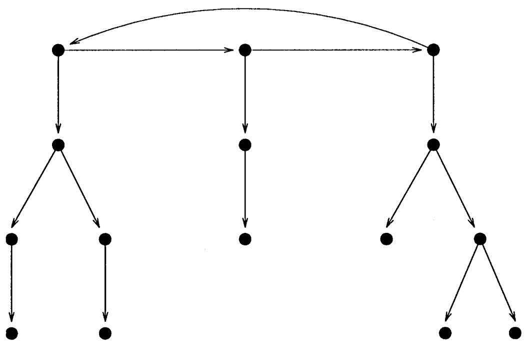

Fig. 1. A cycle-rooted tree.

3. Completely independent spanning trees in the underlying graph of a line digraph

First, we define a cycle-rooted tree (Fig. 1).

Definition 3.1. A cycle-rooted tree is a digraph such that it contains a single cycle and all its vertices have indegree 1. The cycle of a cycle-rooted tree F is denoted by $C(F)$.

A cycle-rooted tree is structurally invariant with respect to the line digraph operation.

Lemma 3.2. *Let F be a cycle-rooted tree. Then, $L(F) \cong F$.*

Proof. Define a bijection φ from $V(L(F))$ to $V(F)$ as $\varphi((u,v)) = v$. Then, for an arc $((u,v),(v,w)) \in A(L(F))$, $(\varphi((u,v)), \varphi((v,w))) = (v,w) \in A(F)$. Suppose that $((u,v), (x,y)) \notin A(L(F))$, i.e., $v \neq x$. Then, $(\varphi((u,v)), \varphi((x,y))) = (v,y)$. Since the indegree of y in F is one, $\Gamma_F^-(y) = \{x\}$. Hence, $(v,y) \notin A(F)$. Therefore, φ is an isomorphism from $L(F)$ to F. □

Lemma 3.3. *Let G be a digraph. Suppose that there are k arc-disjoint spanning cycle-rooted trees $G_1, G_2, \ldots, G_k$ in G. Then, there are k arc-disjoint spanning cycle-rooted trees $F_1, F_2, \ldots, F_k$ in $L(G)$ such that for any F_i and any vertex v of $L(G)$, $\deg^+_{F_i} v = \deg^+_{L(G)} v$, or $\deg^+_{F_i} v = 0$.*

Proof. Let $G_1, G_2, \ldots, G_k$ be arc-disjoint spanning cycle-rooted trees in G. For each G_i, we consider the following set of arcs of $L(G)$:

$$A_i = \{((u,v),(v,w)) \mid (u,v) \in A(G_i), (v,w) \in A(G)\}.$$

Clearly, $A_i \cap A_j = \emptyset$ for $1 \leqslant i < j \leqslant k$ since $A(G_i) \cap A(G_j) = \emptyset$ for $1 \leqslant i < j \leqslant k$. Now we divide A_i into two subsets A_i' and A_i'' as follows:

$$A_i' = \{((u,v),(v,w)) \mid (u,v),(v,w) \in A(G_i)\},$$

$$A_i'' = \{((u,v),(v,w)) \mid (u,v) \in A(G_i),(v,w) \notin A(G_i)\}.$$

From Lemma 3.2, $\langle A_i' \rangle_{L(G)} \cong G_i$. Clearly, $\langle A_i'' \rangle_{L(G)}$ is a union of stars such that each root is a vertex of $\langle A_i' \rangle_{L(G)}$ and each leaf is not a vertex of $\langle A_i' \rangle_{L(G)}$. Hence, $\langle A_i \rangle_{L(G)} = \langle A_i' \cup A_i'' \rangle_{L(G)}$ is also a cycle-rooted tree. Since G_i is spanning, it is easily checked that $\langle A_i \rangle_{L(G)}$ is also spanning. Here, let $F_i = \langle A_i \rangle_{L(G)}$ for $i = 1,2,\dots,k$.

Now, consider a vertex (u,v) of $L(G)$. Suppose that (u,v) is contained in G_j. Then, for any $(v,w) \in A(G)$, $((u,v),(v,w))$ is contained in F_j, i.e., $\deg^+_{F_j}(u,v) = \deg^+_{L(G)}(u,v)$. Thus, in this case, for any F_i, $i \neq j$, $\deg^+_{F_i}(u,v) = 0$. Suppose that (u,v) is not contained in any G_i. In this case, $\deg^+_{F_i}(u,v) = 0$ for any F_i. □

Lemma 3.4. *Let G be a digraph. Suppose that there are k arc-disjoint spanning cycle-rooted trees in G. Then, there are k completely independent spanning trees in $U(L(G))$.*

Proof. Let $G_1, G_2, \dots, G_k$ be k arc-disjoint spanning cycle-rooted trees in G. Then, let F_i be the digraph defined as $\langle A_i \rangle_{L(G)}$ in the proof of Lemma 3.3 for $i = 1,2,\dots,k$. Let T_i be the spanning tree in $U(L(G))$ obtained from $U(F_i)$ by deleting one edge in $U(C(F_i))$ for $i = 1,2,\dots,k$. Then, clearly $T_1, T_2, \dots, T_k$ are edge-disjoint. Also, for any vertex v of $U(L(G))$,

$$\deg_{T_i} v \leqslant \deg^+_{F_i} v + \deg^-_{F_i} v = \deg^+_{F_i} v + 1.$$

From Lemma 3.3, there is at most one F_j such that $\deg^+_{F_j} v \geqslant 1$. Therefore, from Theorem 2.1, $T_1, T_2, \dots, T_k$ are completely independent spanning trees in $U(L(G))$. □

The following theorem was shown by Edmonds [4].

Theorem 3.5 (Edmonds [3]). *Let G be a k-arc-connected digraph. Then, for any vertex r of G, there are k arc-disjoint spanning trees rooted at r in G.*

Theorem 3.5 corresponds to the arc-version of the conjecture mentioned in the introduction.

Theorem 3.6. *Let $L(G)$ be a k-vertex-connected line digraph. Then, there are k completely independent spanning trees in $U(L(G))$.*

Proof. It is easily checked that if $L(G)$ is k-vertex-connected, then G is k-arc-connected. From Theorem 3.5, there are k arc-disjoint spanning trees rooted at any vertex r. Since G is k-arc-connected, $\deg^-_G r \geqslant k$. Adding an arc adjacent to the root r to each spanning tree disjointly, we can obtain k arc-disjoint spanning cycle-rooted trees in G. Hence, by Lemma 3.4, there are k completely independent spanning trees in $U(L(G))$. □

4. Applications to de Bruijn graphs, Kautz graphs, and wrapped butterflies

Applying Lemma 3.3 iteratively and a discussion similar to the proof of Lemma 3.4 shows that the following proposition holds.

Proposition 4.1. *Let G be a digraph. Suppose that there are k arc-disjoint spanning cycle-rooted trees in G. Then, there are k completely independent spanning trees in $U(L^m(G))$.*

In the above proposition, if we add some conditions, then we can obtain a more interesting result. The depth of a cycle-rooted tree T is the maximum depth of the trees obtained from T by deleting all the arcs in the cycle.

Proposition 4.2. *Let G be a regular digraph. Suppose that there are k isomorphic arc-disjoint spanning cycle-rooted trees of cycle-length l and depth c in G. Then, there are k isomorphic completely independent spanning trees of depth at most $2(m+c)+l-1$ in $U(L^m(G))$.*

Proof. Let G be d-regular. We use the same notations introduced in the proof of Lemma 3.3. By the assumption, $\langle A_i' \rangle_{L(G)} \cong \langle A_j' \rangle_{L(G)}$ for $1 \leqslant i < j \leqslant k$. By adding arcs in A_i'' to $\langle A_i' \rangle_{L(G)}$, for any vertex of $\langle A_i' \rangle_{L(G)}$, if the outdegree is not equal to d, then it becomes d in $\langle A_i \rangle_G (= F_i)$. Thus, we can see that $F_i \cong F_j$ for $1 \leqslant i < j \leqslant k$. From this observation, the isomorphic property in the proposition is induced.

By adding arcs in A_i'' to $\langle A_i' \rangle_{L(G)}$, the depth of the cycle-rooted tree increases by one. On the other hand, the cycle-length is invariant with respect to the line digraph operation. Since we consider the underlying graph of a spanning cycle-rooted tree and delete one edge in the cycle, the upper bound on the depth shown in the proposition is obtained. □

Let K_d^* denote the complete symmetric digraph with d vertices. Also, let K_d° denote the complete digraph with d vertices, i.e., the digraph obtained from K_d^* by adding a loop to each vertex (Fig. 2). Then, the *de Bruijn digraph* $B(d,D)$ and the *Kautz digraph* $K(d,D)$ are defined as follows [4]:

$$B(d,D) = L^{D-1}(K_d^\circ),$$

$$K(d,D) = L^{D-1}(K_{d+1}^*).$$

We abbreviate $U(B(d,D))$ and $U(K(d,D))$ to $UB(d,D)$ and $UK(d,D)$, respectively.

It is easily checked that K_d° and K_{d+1}^* have d arc-disjoint spanning cycle-rooted trees isomorphic to the digraphs shown in Fig. 3(a) and (b), respectively. Hence, from

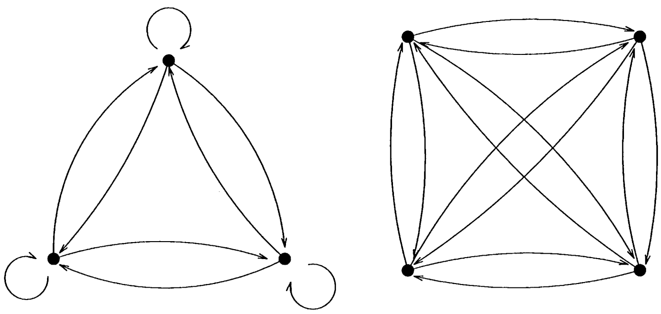

Fig. 2. K_3° and K_4^*.

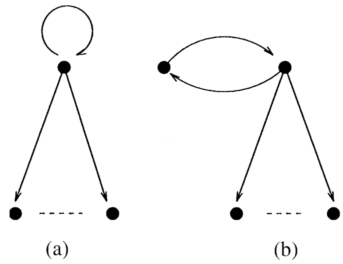

Fig. 3. Spanning cycle-rooted trees in K_d° and K_{d+1}^*.

Proposition 4.2, the following corollaries are obtained. The fact of Corollary 4.3 has been shown in [5].

Corollary 4.3. *There are d isomorphic completely independent spanning trees of depth* $2D$ *in* $UB(d,D)$.

Corollary 4.4. *There are d isomorphic completely independent spanning trees of depth* $2D$ *in* $UK(d,D)$.

For example, $UB(3,3)$ and $UK(3,3)$ have three completely independent spanning trees isomorphic to the graphs shown in Fig. 4(a) and (b), respectively.

The *wrapped butterfly* $wb(k,l)$ can be defined by the underlying graph of $L^{l-1}(K_k^\circ \otimes C_l)$ [6], where C_l is the cycle of length l, and $\otimes$ is the Kronecker product, i.e., for two digraphs G_1 and G_2 (Fig. 5),

$$V(G_1 \otimes G_2) = V(G_1) \times V(G_2),$$

$$A(G_1 \otimes G_2) = \{((u_1,u_2),(v_1,v_2)) \mid (u_1,v_1) \in A(G_1) \text{ and } (u_2,v_2) \in A(G_2)\}.$$

Since $K_k^\circ \otimes C_l$ has k arc-disjoint spanning cycle-rooted trees isomorphic to the digraph shown in Fig. 6, the next corollary follows from Proposition 4.2.

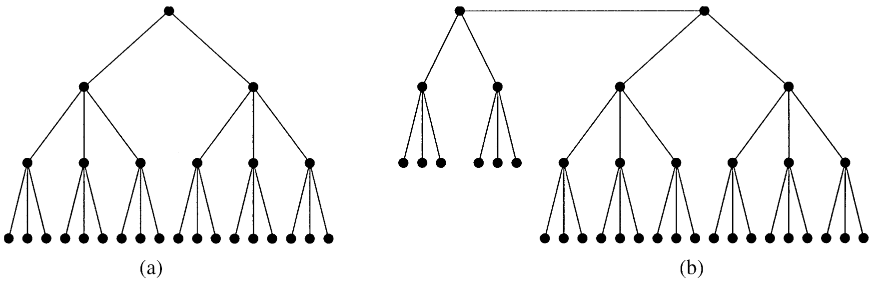

Fig. 4. Spanning trees in $UB(3,3)$ and $UK(3,3)$.

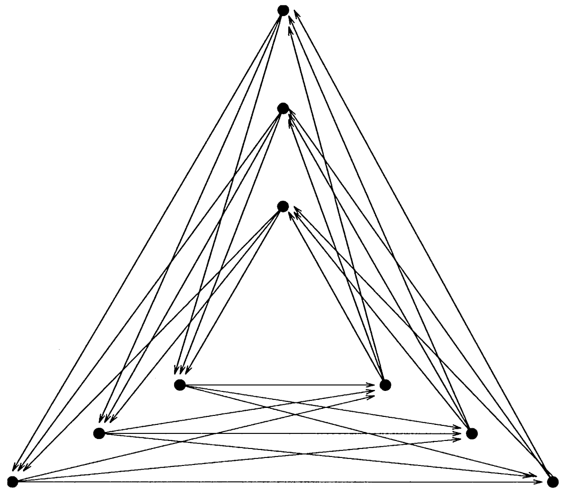

Fig. 5. $K_3^{\circ} \otimes C_3$.

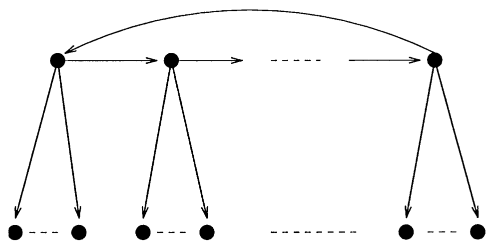

Fig. 6. A spanning cycle-rooted tree in $K_k^{\circ} \otimes C_l$.

Corollary 4.5. *There are k isomorphic completely independent spanning trees of depth* $3l-1$ *in* $wb(k,l)$.

For example, $wb(3,3)$ has three completely independent spanning trees isomorphic to the graph shown in Fig. 7.

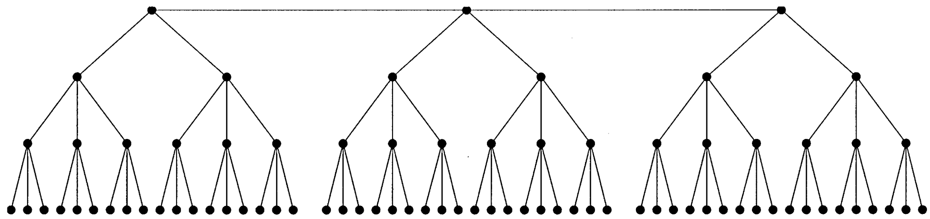

Fig. 7. A spanning tree in $wb(3,3)$.

Note that the numbers of completely independent spanning trees in $UB(d,D)$, $UK(d,D)$ and $wb(k,l)$ shown in the corollaries are best possible. In fact, there is no remaining edge in $UB(d,D)$. Also, there are only d (resp, k) remaining edges in $UK(d,D)$ (resp, $wb(k,l)$).

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