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Connected components of definable groups and o -minimality I

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Abstract

We give examples of definable groups G (in a saturated model, sometimes o -minimal) such that $G^{00} \neq G^{000}$, yielding also new examples of “non G -compact” theories. We also prove that for G definable in a (saturated) o -minimal structure, G has a “bounded orbit” (i.e. there is a type of G whose stabilizer has bounded index) if and only if G is definably amenable, giving a positive answer to a conjecture of Newelski and Petrykowski in this *special case* of groups definable in o -minimal structures.

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1. Introduction and preliminaries

In this paper, groups definable in o -minimal and closely related structures are studied, partly for their own sake and partly as a “testing ground” for general conjectures. Given a \emptyset -definable group G in a saturated structure \overline{M} , G_{\emptyset}^{00} is the smallest subgroup of G of bounded index which is type-definable over \emptyset , and G_{\emptyset}^{000} is the smallest subgroup of G of bounded index which is $\text{Aut}(\overline{M})$ -invariant. In o -minimal structures and more generally theories with NIP , these “connected components” remain unchanged after naming parameters and so are just referred to

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as G^{00} and G^{000} . In any case G_\emptyset^{00} and G_\emptyset^{000} are “definable group” analogues of the groups of KP -strong automorphisms and Lascar strong automorphisms, respectively, of a saturated structure. The relationship between these definable group and automorphism group notions is explored in [10]. Although examples were given in [2] where the strong automorphism groups differ, until now no example was known where $G_\emptyset^{000} \neq G_\emptyset^{00}$. In this paper (Section 3) we give a “natural” example: G is simply a saturated elementary extension of $\widehat{SL_2(\mathbb{R})}$ (the universal cover of $SL_2(\mathbb{R})$) in the language of groups. G is *not* actually definable in an o -minimal structure, but we give another closely related example which is. In any case the two-sorted structure consisting of G and a principal homogeneous space for G is now a (natural) example of a “non G -compact” structure (or theory) i.e. where the group of Lascar strong automorphisms is properly contained in the group of KP -strong automorphisms.

Another fruitful theme in recent years has been the generalization of stable group theory outside the stable context. The o -minimal case has been important and there is now a good understanding of “definably compact” groups from this point of view; for example they are definably amenable, “generically stable for measure”, and G is dominated by G/G^{00} . It should be remarked that for *any* group G definable in a (saturated) o -minimal structure, G/G^{00} , equipped with the logic topology, is a compact Lie group [1]. In the current paper we try to go beyond the definably compact setting, motivated partly by questions of Newelski and Petrykowski. In [11], definable groups G with “finitely satisfiable generics” (which include definably compact groups in o -minimal structures) were shown to be definably amenable by lifting the Haar measure on G/G^{00} to a left invariant Keisler measure on G , making use of a global *generic type* p , whose stabilizer is G^{00} . We guess this encouraged Petrykowski to suggest that if a definable group G (in any structure) has a global type whose stabilizer has “bounded index” then G is definably amenable. Note that a left invariant *type* is a special case of a left invariant Keisler measure, so trivially if there is a global type with stabilizer G then G is definably amenable. In any case, in Section 4 we confirm Petrykowski’s conjecture when G is definable in an o -minimal structure, as well as raise questions about the nature of types with bounded orbit in the o -minimal and more generally NIP environment.

In Section 2 of the paper we give a rather basic decomposition theorem (implicit in the literature) for groups in o -minimal structures, which is useful for understanding the issues around definable amenability and bounded orbits, as well as G^{00} and G^{000} (although Section 3 can be more or less read independently of Section 2). We introduce and discuss the notion of G having a “good decomposition” (Definition 2.7). The o -minimal examples where $G^{00} \neq G^{000}$ will be also examples where good decomposition fails, although good decomposition does hold for algebraic groups.

In a sequel [5] to the current paper we will give a systematic account of G^{00} , G^{000} as well as the quotient G^{00}/G^{000} , for groups G definable in o -minimal structures. The decomposition theorem (2.6), refinements of it, as well as the notion of good decomposition, will play major roles.

In general T will denote a complete theory, M an arbitrary model of T , and G a group definable in M . We sometimes work in a sufficiently saturated and homogeneous model \overline{M} of T , in which case “small” or “bounded” essentially means of cardinality strictly less than the degree of saturation of \overline{M} , but we will make the meaning more precise later in the paper. *Definability* usually means with parameters, and we say A -definable to mean definable with parameters from A for A a subset of M . When we talk about o -minimal theories we will mean *o -minimal expansions of the theory RCF of real closed fields* (and we leave it for later or to others

to consider more general o -minimal contexts). In the o -minimal context, the important notion of definable compactness was introduced by Peterzil and Steinhorn in [22]. For X a definable subset of M^n , definable compactness of X amounts to X being closed and bounded in M^n . In the more general case of X being a *definable manifold*, it means that for any definable function f from $[0, 1)$ to X , $\lim_{x \rightarrow 1^-} f(x)$ exists in X . When G is a definable group, G can be equipped with a definable manifold structure such that multiplication and inversion are continuous [23]. Definable compactness of a definable group G is then meant with respect to this definable manifold structure. But, as we are working in an o -minimal expansion of a real closed field, any definable group manifold G can be assumed to be a definable submanifold of some M^n , and so definable compactness of G reduces to G being closed and bounded. *Definable connectedness* of G is meant with respect to its definable manifold structure mentioned above. But it turns out that G is definably connected in this sense if and only if G has no proper definable subgroup of finite index (i.e. $G = G^0$). Any definable group G is definably connected by finite, and so (in this o -minimal context) we will often assume that our definable groups are definably connected. We will often use the well-known fact that any definably compact, definably connected, solvable normal definable subgroup N of a definably connected group is central. This follows from Corollaries 5.3 and 5.4 of [20]. We will also use the fact that if N is normal and definable in G , then G is definably compact if and only if N and G/N are definably compact. (This can be seen to follow from [11, 13], but has direct proofs too.)

In Section 4 of this paper we will make some references to “stability-type” notions, *NIP* theories, forking, etc. We generally refer the reader to [13] for the definitions, but will make a few explanatory comments here as well as in Section 4. For \overline{M} a saturated model of arbitrary theory T and G a group definable in \overline{M} , recall that $S_G(\overline{M})$ denotes the space of complete types $p(x)$ over \overline{M} such that “ $x \in G$ ” $\in p$. G (namely $G(\overline{M})$) acts on $S_G(\overline{M})$ on the left by $gp = tp(ga/\overline{M})$ where a realizes p in a bigger model. Slightly modifying Definition 5.1 from [13], we will say that $p(x) \in S_G(\overline{M})$ is *left f -generic* if there is a small model M_0 such that for any $g \in G(\overline{M})$, gp does not fork over M_0 .

The second author was partly motivated by some e-mail discussions with Hrushovski and Newelski in the late summer of 2010. Thanks to both of them for the inspiration, and in particular to Hrushovski for allowing us to include (in Section 4) some observations that he made on definable amenability. Thanks also to Gopal Prasad for pointing out some references.

Many of the themes and results of this paper and the sequel appear in one form or another in the first author’s doctoral thesis [4], which is devoted to structural properties of groups definable in o -minimal structures, but does *not* explicitly discuss G^{000} . In particular the o -minimal example where $G^{00} \neq G^{000}$ (Example 2.10/Theorem 3.3) appears in her thesis as an example of a definable group *without* a definable “Levi decomposition”. In any case the first author would like to thank her advisor Alessandro Berarducci, as well as Ya’acov Peterzil for useful conversations.

2. Decomposition theorems

In this section T is a complete o -minimal expansion of *RCF*, and we work in a model M of T . G will typically denote a definable, definably connected group, although we usually explicitly state definable connectedness. K will denote the underlying real closed field of M . We first aim towards a useful “basic decomposition theorem”, Proposition 2.6 (which is easily extracted from results in the literature). We begin by pointing out the existence, in every definable group, of a (unique) maximal normal definable torsion-free subgroup. As usual, for a positive integer n , an n -torsion element of G is an element $x \in G$ such that $x^n = 1$, 1 being the identity of the group

(note that we are not assuming G is commutative). We make use of results from [27] connecting the existence of n -torsion elements with the o -minimal Euler characteristic of G . Recall that if \mathcal{P} is a cell decomposition of a definable set X , then the o -minimal Euler characteristic $E(X)$ is the number of even-dimensional cells in \mathcal{P} minus the number of odd-dimensional cells in \mathcal{P} . This does not depend on \mathcal{P} , and when X is finite then $E(X) = |X|$. A definable torsion-free group will be definably connected (Corollary 2.4 of [21] but also follows from the proof of (ii) below). The reader should also bear in mind that any definably compact group G contains nontrivial torsion [8].

- Proposition 2.1.** (i) G is torsion-free if and only if G is “solvable with no definably compact parts” in the sense of [7], namely there are definable subgroups $\{1\} = G_0 < \cdots < G_n = G$ of G such that for each $i < n$, G_i is normal in G_{i+1} and G_{i+1}/G_i is 1-dimensional and torsion-free. (In particular a torsion-free definable group is solvable.)
- (ii) In every definable group G there is a normal definable torsion-free subgroup which contains every normal definable torsion-free subgroup of G . It is the unique normal definable torsion-free subgroup of G of maximal dimension. We will refer to it as the maximal normal definable torsion-free subgroup of G , and note that it is invariant under all automorphisms of (G, \cdot) which are definable in the ambient structure.

Proof. (i) Right to left is obvious. Left to right follows (using induction) from Corollary 2.12 of [21] which states that if G is torsion-free (and nontrivial) then there is a normal definable subgroup H of G such that G/H is 1-dimensional and torsion-free.

(ii) We recall that for definable groups $K < G$,

$$E(K)E(G/K) = E(G),$$

and G is torsion-free if and only if $E(G) = \pm 1$ [27]. It follows that a quotient of torsion-free definable groups is still torsion-free (and hence torsion-free definable groups are definably connected).

Let N be a normal definable torsion-free subgroup of G of maximal dimension, and H any normal definable torsion-free subgroup of G . We want to show that $H \subseteq N$.

We claim that HN is a normal definable torsion-free subgroup of G : the definable group $H/(H \cap N)$ is torsion-free and it is definably isomorphic to HN/N . Thus $E(HN) = E(N)E(HN/N) = \pm 1$ and HN is torsion-free.

But N is of maximal dimension among the normal definable torsion-free subgroups of G , so $\dim(HN) = \dim(N)$. Since definable torsion-free groups are definably connected, it follows that $HN = N$, $H \subseteq N$ and $\dim H < \dim N$, unless $H = N$. \square

Bearing in mind Proposition 2.1, the following proposition is easily deduced from Theorem 5.8 of [7], together with the fact that definably compact, definably connected, solvable definable groups are commutative:

Proposition 2.2. Let G be a definable, solvable, definably connected group, and let W be its maximal normal definable torsion-free subgroup. Then G/W is definably compact and commutative.

Recall that a definable group G is said to be *semisimple* if G has no definable, normal, definably connected, solvable (or commutative), nontrivial subgroups. Then, clearly, for an arbitrary definable group G , we have the exact sequence

$$1 \rightarrow R \rightarrow G \rightarrow G/R \rightarrow 1$$

where R , the *solvable radical* of G , is the maximal definable, normal, solvable, definably connected subgroup of G , and G/R is semisimple. If R is definably compact then it is central in G .

Definition 2.3. We call a definable group G , *definably almost simple*, if G is noncommutative, definably connected, and has no infinite (equivalently nontrivial, definably connected) proper definable normal subgroup.

Note that if G is definably almost simple, then $Z(G)$ is finite and $G/Z(G)$ is definably simple, and moreover G is definably compact if and only if $G/Z(G)$ is definably compact.

Lemma 2.4. *Let the definable group be semisimple and definably connected. Then there are definable, definably almost simple subgroups H_1, \dots, H_t of G such that G is the almost direct product of the H_i , namely there is a definable surjective homomorphism from $H_1 \times \dots \times H_t$ to G with finite kernel.*

Proof. Well known. By Peterzil et al. [17], $G/Z(G)$ is the direct product of definably simple groups B_1, \dots, B_r . Let H_i be the definably connected component of the preimage of B_i under the quotient map $G \rightarrow G/Z(G)$. \square

Definition 2.5. Let G be semisimple and definably connected. We say that G has no definably compact part if in Lemma 2.4, no H_i is definably compact.

We can now observe:

Proposition 2.6. *Let G be a definable (definably connected) group. Then there is a definable, definably connected, normal subgroup W of G , and a definable, definably connected normal subgroup C of G/W , such that*

- (i) W is torsion-free,
- (ii) C is definably compact, and
- (iii) $(G/W)/C$ is semisimple with no definably compact part.

W is the maximal normal definable torsion-free subgroup of G , and C is the maximal normal definable, definably compact, definably connected subgroup of G/W .

Proof. Let R be the solvable radical of G , and let W be the maximal normal definable torsion-free subgroup of R (given by Proposition 2.1). So R/W is definably compact and commutative by 2.2. But let us note for now that since any definable torsion-free group is definably connected and solvable [21, 2.11], then W coincides with the maximal normal definable torsion-free subgroup of G .

Now R/W is the solvable radical of G/W (and is also connected, definably compact, so in fact central in G/W), and G/R is semisimple. Let us denote G/R by H for now, and π the surjective homomorphism from G/W to H . Let H_1, \dots, H_t be given for H by Lemma 2.4, namely the H_i are definable, definably almost simple and H is their (almost direct) product. Let C_1 be the product of those H_i which are definably compact, and D_1 the product of the rest. So $G/R = H$ is the almost direct product of the semisimple definable groups C_1 and D_1 . Let $C = \pi^{-1}(C_1)$. So C is an extension of the definably compact connected group C_1 by the definably compact definably connected group R/W , hence is also definably compact and definably connected. Note that C is normal in G/W , and the quotient $(G/W)/C$ is an image of D_1 (with finite kernel) so is semisimple with no definably compact parts. \square

Let us fix notation for the data obtained in the proof above, so as to be able to refer to them in the future. R denotes the solvable radical of G and W the maximal normal definable torsion-free subgroup of G (equivalently of R).

G/R is the semisimple part of G which can be written uniquely as $C_1 \cdot D_1$ (almost direct product) where C_1 is semisimple and definably compact and D_1 is semisimple with no definably compact parts (and everybody is definably connected).

We have the exact sequence

$$1 \rightarrow R/W \rightarrow G/W \xrightarrow{\pi} G/R = C_1 \cdot D_1 \rightarrow 1$$

and C denotes $\pi^{-1}(C_1)$ which is the maximal normal definable, definably connected, definably compact subgroup of G/W , and we call it the *normal definably compact part* of G .

Finally $(G/W)/C$ is denoted D and called the *semisimple with no definably compact parts* part of G .

Note that R/W is the connected component of the centre of C and

$$1 \rightarrow R/W \rightarrow C \rightarrow C_1 \rightarrow 1$$

definably almost splits by results from [12].

One natural question is whether there is a better decomposition theorem.

Definition 2.7. We will say that G has a *good decomposition*, if, with above notation, the exact sequence $1 \rightarrow C \rightarrow G/W \rightarrow D \rightarrow 1$ definably almost splits, namely G/W can be written as $C \cdot D_2$ for some definable, definably connected, subgroup D_2 of G/W which is semisimple with no definably compact parts (i.e. the map $D_2 \rightarrow D$ is surjective with finite kernel).

The second author mistakenly claimed in an early draft of this paper that G always has a good decomposition. The first author pointed out counterexamples from her thesis (see [Example 2.10](#)) which led us to the examples where $G^{00} \neq G^{000}$. Anyway this is partly the reason for giving [Definition 2.7](#). The connection between the general decomposition above and the quotients G/G^{00} , G/G^{000} and G^{00}/G^{000} features prominently in the sequel [5].

Lemma 2.8. *The following are equivalent:*

- (i) G has a good decomposition.
- (ii) G/W has a “definable Levi decomposition”, i.e. is an almost semidirect product of its solvable radical (R/W) and a definable, definably connected semisimple group S .
- (iii) $\pi^{-1}(D_1)$ is an almost direct product of R/W (the connected component of its centre) and a definable semisimple group (again necessarily without definably compact parts).

Proof. First the equivalence of (i) and (ii) is immediate from the fact that

$$1 \rightarrow R/W \rightarrow C \rightarrow C_1 \rightarrow 1$$

definably almost splits, as remarked above.

The rest is clear, because G/W will be the almost direct product of C and some D_2 if and only if $\pi^{-1}(D_1)$ is the almost direct product of R/W and D_2 . \square

Hence the existence of good decompositions depends on the definable almost splitting of central extensions of semisimple groups without definably compact parts by definable compact groups. Note that if G itself has a definable Levi decomposition then G has a good decomposition (see [2.12](#) for a counterexample to the converse).

Lemma 2.9. *G has a definable Levi decomposition (and hence a good decomposition) in either of the cases:*

- (i) *G is linear, namely a definable, in M, subgroup of some $GL(n, K)$, or*
- (ii) *G is algebraic, namely of the form $H(K)^0$ for some algebraic group H defined over K.*

Proof. When *G* is linear this is Theorem 4.5 of [19].

Suppose now that *H* is a connected algebraic group defined over *K*, and $G = H(K)^0$. We have Chevalley’s theorem for *H* yielding the following exact sequence of connected algebraic groups defined over *K*:

$$1 \rightarrow L \rightarrow H \xrightarrow{f} A \rightarrow 1$$

where *L* is linear and *A* is an abelian variety. Then $f(G)$ is a connected semialgebraic subgroup of $A(K)$ so is definably compact and commutative, and the semialgebraic connected component of the group of *K*-points of *L* is a definably connected definable subgroup of $GL(n, K)$ for some *n*. Namely at the level now of definable, definably connected, groups in *M*, we have an exact sequence

$$1 \rightarrow R \rightarrow G \xrightarrow{f} B \rightarrow 1$$

where *R* is linear, and *B* is commutative (and definably compact). Again by Peterzil et al. [18], *R* is an almost semidirect product of a definably connected solvable group R_1 and a definable semisimple group *S*. Let *R* be the solvable radical of *G* (as a definable group). As G/R is semisimple, *R* must map onto *B* under *f*, whereby *G* is the almost direct product of *R* and *S*. □

Finally in this section we give:

Example 2.10. There is a (Nash) group *G* without a good decomposition. The theory *T* will be RCF, *M* the standard model $(\mathbb{R}, +, \times)$, and *G* a certain amalgamated central product of $SO_2(\mathbb{R})$ with the universal cover of $SL_2(\mathbb{R})$.

The model-theoretic setting is the structure $M = (\mathbb{R}, +, \times)$. Let *H* be the definable group $SL_2(\mathbb{R})$ consisting of 2-by-2 matrices over \mathbb{R} of determinant 1. Let $\tilde{H} = \widetilde{SL_2(\mathbb{R})}$ be the universal cover of *H*. \tilde{H} is a connected, simply connected Lie group and we have the exact sequence (of Lie groups)

$$1 \rightarrow \mathbb{Z} \rightarrow \tilde{H} \xrightarrow{\pi} H \rightarrow 1$$

where \mathbb{Z} is the discrete group $(\mathbb{Z}, +)$. \tilde{H} is not definable in *M*, but we will make use of a certain description from Section 8.1 of [12] (see Theorem 8.5 there) of \tilde{H} as a group definable in the 2-sorted structure $((\mathbb{Z}, +), M)$, and this will be used again in the next section:

Fact 2.11. *There is a 2-cocycle $h : H \times H \rightarrow \mathbb{Z}$ with finite image which is moreover definable in M (in the sense that for each $n \in \text{Im}(h)$, $\{(x, y) \in H \times H, h(x, y) = n\}$ is definable in M), and such that the set $\mathbb{Z} \times H$ with group structure $(t_1, x_1) * (t_2, x_2) = (t_1 + t_2 + h(x_1, x_2), x_1x_2)$ and projection to the second coordinate, is isomorphic to the group \tilde{H} with its projection π to *H*.*

Although not needed, let us say a few words of where the cocycle *h* comes from, referring to [12] for more details. The group \tilde{H} is naturally ind-definable in *M*, namely as an increasing union $\bigcup_i X_i$ of definable sets with group operation and projection π to *H* piecewise definable.

For some i , the restriction of π to X_i is surjective and as M has Skolem functions there is a definable section $s : H \rightarrow X_i$ of $\pi|_{X_i}$. Define h on $H \times H$ by $h(x, y) = s(x)s(y)s(xy)^{-1}$. Then h is as required.

Let now consider the circle group $SO_2(\mathbb{R})$ and we use additive notation for it. Let $g \in SO_2(\mathbb{R})$ be an element of infinite order. Define a group operation $*$ on $SO_2(\mathbb{R}) \times H$ by $(t_1, x_1) * (t_2, x_2) = (t_1 + t_2 + h(x_1, x_2)g, x_1x_2)$. Let G be the resulting group, and note that G is now definable (without parameters, taking g algebraic) in M . As $h(g, 1) = h(1, g) = 0$ for all $g \in G$, $SO_2(\mathbb{R})$ is naturally embedded in G by the map taking t to $(t, 1)$. Note that $\{(ng, x) : n \in \mathbb{Z}, x \in H\}$ is a subgroup of $(G, *)$ isomorphic to \tilde{H} (with again projection on second coordinate corresponding to $\pi : \tilde{H} \rightarrow H$). So identifying $\langle g \rangle$ with \mathbb{Z} , we have that

- (i) $SO_2(\mathbb{R})$ is central in $(G, *)$,
- (ii) $G = SO_2(\mathbb{R}) \cdot \tilde{H}$,
- (iii) $SO_2(\mathbb{R}) \cap \tilde{H} = \mathbb{Z}$,

and we have the exact sequence of definable, definably connected, groups in M ,

$$1 \rightarrow SO_2(\mathbb{R}) \rightarrow G \rightarrow H \rightarrow 1$$

(where remember $H = SL_2(\mathbb{R})$).

H is of course definably almost simple and not (definably) compact, whereas $SO_2(\mathbb{R})$ is (definably) compact and central in G . To show that G does not have a good decomposition it suffices to show that the exact sequence above does not definably almost split in M (because of Lemma 2.8, as W is trivial). In fact there is no (even abstract) subgroup H_1 of G such that $SO_2(\mathbb{R}) \cap H_1$ is finite and $SO_2(\mathbb{R}) \cdot H_1 = G$, for otherwise (as $SO_2(\mathbb{R})$ is central in G), the commutator subgroup $[G, G]$ is contained in H_1 so has finite intersection with $SO_2(\mathbb{R})$. But, using (ii) above and the fact that $\widetilde{SL_2(\mathbb{R})}$ is perfect, $[G, G] = \tilde{H}$ and so has infinite intersection with $SO_2(\mathbb{R})$, a contradiction. We have completed the exposition of Example 2.10.

In the next section an elaboration of the above analysis will show that passing to a saturated elementary extension, $G^{00} \neq G^{000}$.

Remark 2.12. A definably connected group G with a good decomposition does not have necessarily a definable Levi decomposition.

Proof. If one replaces $SO_2(\mathbb{R})$ with $(\mathbb{R}, +)$ in Example 2.10, then one obtains a group with a good decomposition ($G/W = SL_2(\mathbb{R})$), but without a definable Levi decomposition (for the same reason as in Example 2.10). \square

3. G^{00} , G^{000} and the examples

We will first repeat the definitions and geneses of the various notions of “connected components” of a definable group. To begin with let T be an arbitrary complete theory. We can identify a definable set with the formula $\phi(x)$ which defines it, or rather the functor taking M to $\phi(M)$ from the category $\text{Mod}(T)$ (of models of T with elementary embeddings) to Set given by that formula. If the formula has parameters from a set A in a given model of T , then the functor is from $\text{Mod}(Th(M, a)_{a \in A})$ to Set . Likewise for type-definable sets, and also hyperdefinable sets (a type-definable set quotiented by a type-definable equivalence relation). If X is a type-definable set over $A \subseteq M$, then we sometimes identify X with its interpretation in an $|A|^+$ -saturated model \bar{M} containing M . If X is a type-definable (over A) set, defined by partial type $\Phi(x)$ and E a type-definable (over A) equivalence relation on X given by partial type $\Psi(x, y)$ then we say that X/E

is “bounded” if $|\Phi(N)/\Psi(N)|$ is bounded as the model N (containing A) varies. If X/E is bounded it is not hard to see that $|\Phi(N)/\Psi(N)| \leq 2^{|T|+|A|}$ for all N , and if $N_1 < N_2$ are $|A|$ -saturated models containing A then the natural embedding of $\Phi(N_1)/\Psi(N_1)$ in $\Phi(N_2)/\Psi(N_2)$ is a bijection. In fact, assuming X/E bounded, for a fixed model M containing A , and N a saturated model containing M , the E -class of some $b \in X$ depends only on $tp(b/M)$, hence the map $X \rightarrow X/E$ factors through the space $S_\Phi(M)$ of complete types over M extending $\Phi(x)$. We give X/E the quotient topology (considering it as a quotient, not of X which has no topology, but of the type-space $S_\Phi(M)$) which we call the logic topology. It does not depend on the choice of M . In any case equipped with this logic topology X/E is a compact Hausdorff space.

Now suppose that the equivalence relation E on X is given instead by a possibly infinite disjunction $\bigvee_i \Psi_i(x, y)$ of partial types over A (i.e. working in a saturated model \bar{M} , is $\text{Aut}(\bar{M}/A)$ -invariant, or as we often just say A -invariant). The whole discussion above regarding boundedness of E goes through in this more general case, including the fact that the map $X \rightarrow X/E$ factors through the type space $S_\Phi(M)$ (for M any model containing A). However the “logic topology” on X/E is no longer Hausdorff, and it is not really clear how to view X/E as a mathematical object. In [2] it was suggested that the descriptive set theoretic point of view might be useful.

Let us first consider the case where X is a sort of T . Work again in a saturated model \bar{M} . Given any (small) set A of parameters, there is a finest bounded type-definable over A equivalence relation on X which we call $E_{X,A,KP}$. Likewise there is a finest bounded A -invariant equivalence relation on X which we call $E_{X,A,L}$. For $a \in X$, the KP -strong type of a over A is precisely the $E_{X,A,KP}$ -class of a , and the Lascar strong type of a over A is precisely the $E_{X,A,L}$ -class of a . There is also of course the usual strong type of a over A , which is the $E_{X,A,Sh}$ -class of a where $E_{X,A,Sh}$ is the intersection of all A -definable equivalence relations on X with finitely many classes. In stable theories all these strong types coincide. In [2] an example was given where KP -strong types differ from Lascar strong types. More (natural) examples will be given later.

We now consider the case where $X = G$ is a definable group, and E comes from an appropriate subgroup of G . So we assume G to be a group definable in a saturated model \bar{M} , and we fix a small set A of parameters over which G is defined. G_A^0 denotes the intersection of all A -definable subgroups of G of finite index. It is clearly a type-definable (normal) subgroup of G of bounded index, and equipped with the logic topology the quotient G/G_A^0 is a profinite group. We let G_A^{00} denote the smallest type-definable over A subgroup of G of bounded index. It is also normal, the quotient G/G_A^{00} , equipped with the logic topology is a compact (Hausdorff) topological group, and G/G_A^0 is its maximal profinite quotient. Finally G_A^{000} is the smallest A -invariant subgroup of G , of bounded index, which is again normal. We have that $G_A^{000} \leq G_A^{00} \leq G_A^0$.

A well-known construction links these different “connected components” of definable groups with the various strong types. We refer the reader to [10] for the details, although it would not be hard to work it out for oneself. Let T be a complete theory and let G be a \emptyset -definable group in T . Adjoin a new sort S together with a regular action of G on S . Call the new theory T' . An argument by automorphisms for example shows that no “new structure” is imposed on T . Work in a saturated model of T' . Then

Fact 3.1. (i) $E_{S,\emptyset,Sh}$ is the orbit equivalence relation on S induced by G_\emptyset^0 , namely for $a, b \in S$, $E_{S,\emptyset,Sh}(a, b)$ iff there is $g \in G_\emptyset^0$ such that $g \cdot a = b$.

- (ii) Likewise $E_{S,\emptyset,KP}$ is the orbit equivalence relation on S induced by G_\emptyset^{00} , and
- (iii) $E_{S,\emptyset,L}$ is the orbit equivalence relation on S induced by G_\emptyset^{000} .

Hence, if for example $G^{00} \neq G^{000}$, then we obtain in this way examples where KP -strong type differs from Lascar strong type.

There are plenty of examples where $G_\emptyset^0 \neq G_\emptyset^{00}$ (such as definably compact groups definable in o -minimal structures). However, until now no examples had been worked out where $G_\emptyset^{00} \neq G_\emptyset^{000}$. We give a brief description of the current state of knowledge regarding this objects.

We say, for example, that “ G^0 exists” if for some set A of parameters, for all $B \supseteq A$, $G_A^0 = G_B^0$. If G^0 exists, then, assuming G is \emptyset -definable, we can take A to be \emptyset and we define G^0 to be G_\emptyset^0 . Likewise for G^{00} and G^{000} . If G^{000} exists then so do G^{00} and G^0 . Shelah [25] was the first to prove that G^{00} exists when T has NIP . Moreover, following this and related work of Shelah, Gismatullin [9] proves that G^{000} exists when T has NIP . When T is stable, $G^0 = G^{00} = G^{000}$. For T simple, G^0 may not exist, but it is known that for any A , $G_A^{00} = G_A^{000}$. It is conjectured (for T simple) that $G_A^0 = G_A^{00}$ and this is known in the supersimple case [28]. Lemmas 5.6 and 5.9 in [13] yield that if G is definable in an NIP theory and G is *definably amenable* then $G^{00} = G^{000}$. (See the beginning of Section 4 for the precise definition of definable amenability.) In [11] it is shown that definably compact groups in o -minimal structures are definably amenable. Hence for definably compact groups G , $G^{00} = G^{000}$, and this will be used below.

When we are working with either o -minimal theories, or closely related NIP theories, we just say G^0, G^{00}, G^{000} .

We now give examples of G (including o -minimal examples) where $G^{00} \neq G^{000}$. In the sequel to this paper we will make a systematic analysis of G^{00} and G^{000} in the o -minimal case, showing that the behaviour in Theorem 3.3 for example is typical.

Theorem 3.2. *Let $T = Th(\widetilde{SL_2(\mathbb{R})}, \cdot)$. Then T has NIP , and if (G, \cdot) denotes a saturated model, then $G^{00} \neq G^{000}$. In fact $G = G^{00}$ and G/G^{000} is isomorphic to $\widehat{\mathbb{Z}}/\mathbb{Z}$ where $\widehat{\mathbb{Z}}$ is the profinite completion of $(\mathbb{Z}, +)$.*

Proof. From Fact 2.11 and the discussion following it (taken from [12]) the group $(\widetilde{SL_2(\mathbb{R})}, \cdot)$ is interpretable (with parameters) in the 2-sorted structure

$$((\mathbb{Z}, +), (\mathbb{R}, +, \times))$$

(where there are no additional basic relations between the sorts). As $Th(\mathbb{Z}, +)$ is stable (in fact superstable of U -rank 1) and RCF has NIP clearly the 2-sorted structure has NIP too, and hence the interpretable group $(\widetilde{SL_2(\mathbb{R})}, \cdot)$ has NIP .

In fact we will work with the theory $T = Th((\mathbb{Z}, +), (\mathbb{R}, +, \times))$ and will point out how the results are also valid for the “reduct” $Th(\widetilde{SL_2(\mathbb{R})}, \cdot)$, namely the statement of the theorem holds.

Let M denote $(\mathbb{R}, +, \times)$, and N denote the 2-sorted structure $((\mathbb{Z}, +), (\mathbb{R}, +, \times))$. Then a saturated model \overline{N} of T will be of the form $((\Gamma, +), \overline{M})$ where \overline{M} is a saturated real closed field $(K, +, \times)$ say, and $(\Gamma, +)$ is a saturated elementary extension of $(\mathbb{Z}, +)$. (We hope this notation is not confusing.) Let now G denote the interpretation in the big model \overline{N} of the formula(s) defining the group $\widetilde{SL_2(\mathbb{R})}$ in N , as described in 2.11. So clearly G has universe the definable set $\Gamma \times SL_2(K)$ and group operation given by $(t_1, x_1) * (t_2, x_2) = (t_1 + t_2 + h(x_1, x_2), x_1 x_2)$. Here $h(x_1, x_2) \in \mathbb{Z} < \Gamma$ so everything makes sense. We write the group G as (G, \cdot) hopefully without ambiguity. We identify the group Γ with the subgroup $\{(t, 1) : t \in \Gamma\}$, $*$) of G via the

(definable) isomorphism ι which takes $t \in \Gamma$ to $(t, 1) \in G$. As such Γ is central in G and we have the exact sequence

$$1 \rightarrow \Gamma \rightarrow G \rightarrow SL_2(K) \rightarrow 1. \tag{1}$$

We again identify $\mathbb{Z} < \Gamma$ with the subgroup $(\{(t, 1) : t \in \mathbb{Z}\}, *)$ of G via ι . Note that $(\{(t, x) : t \in \mathbb{Z}, x \in SL_2(K)\}, *)$ is a (non definable) subgroup of G , which we will take the liberty to call $\widetilde{SL_2(K)}$. (In fact the latter will identify with the so-called o -minimal universal cover of $SL_2(K)$, an ind-definable group in \overline{M} , but this fact will not be needed.) From (1) we obtain:

$$1 \rightarrow \mathbb{Z} \rightarrow \widetilde{SL_2(K)} \rightarrow SL_2(K) \rightarrow 1 \tag{2}$$

(where only $SL_2(K)$ is definable).

So with the above identifications we write

$$G = \Gamma \cdot \widetilde{SL_2(K)} \tag{3}$$

where the subgroup Γ of G is definable and central, the subgroup $\widetilde{SL_2(K)}$ of G is not definable and $\mathbb{Z} = \Gamma \cap \widetilde{SL_2(K)}$.

We now aim to understand G^{000} in terms of this decomposition (even though $\widetilde{SL_2(K)}$ is not definable).

Claim 1. $\Gamma^{000} = \Gamma^0 = \Gamma^0 = \bigcap_n n\Gamma$, and is contained in G^{000} .

Proof of Claim 1. Γ (as a group definable in \overline{N}) is simply a model of $Th(\mathbb{Z}, +)$ which is stable, so we have equality of the various connected components and Γ^0 is the intersection of all definable subgroups of finite index which is as described. Also $G^{000} \cap \Gamma$ clearly contains Γ^{000} . \square

Claim 2. $\widetilde{SL_2(K)}$ is perfect, namely equals its own commutator subgroup.

Proof of Claim 2. Because of the exact sequence (2) and the well-known fact that $SL_2(K)$ is perfect, it is enough to show that the subgroup \mathbb{Z} of $\widetilde{SL_2(K)}$ is contained in $[\widetilde{SL_2(K)}, \widetilde{SL_2(K)}]$. But this follows immediately because \mathbb{Z} is contained in the (naturally embedded) subgroup $SL_2(\mathbb{R})$ of $\widetilde{SL_2(K)}$, and again $SL_2(\mathbb{R})$ is known to be perfect. \square

Claim 3. $\widetilde{SL_2(K)} \subseteq G^{000}$.

Proof of Claim 3. Let $H = \widetilde{SL_2(K)} \cap G^{000}$. H is then a normal subgroup of $\widetilde{SL_2(K)}$ of index at most the continuum. Hence $\pi(H)$ the image of H under $\pi : \widetilde{SL_2(K)} \rightarrow SL_2(K)$ is an infinite normal subgroup of $SL_2(K)$. As $SL_2(K)$ is simple as an abstract group modulo its finite centre, it follows that $\pi(H) = SL_2(K)$. Hence $\widetilde{SL_2(K)} = \mathbb{Z} \cdot H$, and as \mathbb{Z} is central, the commutator subgroup of $\widetilde{SL_2(K)}$ is contained in H . By Claim 2, $H = \widetilde{SL_2(K)}$, as required. \square

(Note that we have shown that every proper normal subgroup of $\widetilde{SL_2(K)}$ is central.)

Claim 4. $[G, G] = \widetilde{SL_2(K)}$.

Proof of Claim 4. By the description of G in (3), $[G, G]$ is a subgroup of $\widetilde{SL_2(K)}$. By Claim 2, we get equality. \square

Claim 5. $G^{000} = \Gamma^0 \cdot \widetilde{SL_2(K)}$.

Proof of Claim 5. By [Claims 1](#) and [3](#), G^{000} contains $\Gamma^0 \cdot \widetilde{SL_2(K)}$. On the other hand $\Gamma^0 \cdot \widetilde{SL_2(K)}$ is clearly of bounded index in G , and using [Claim 4](#) is also clearly invariant under automorphisms of N which fix the parameters defining G . So we get equality. In fact note at this point that $\Gamma^0 \cdot \widetilde{SL_2(K)}$ is also invariant under automorphisms of the structure (G, \cdot) , so coincides with G^{000} in the reduct (G, \cdot) of \overline{N} . \square

Claim 6. $G = G^{00}$.

Proof of Claim 6. By [Claim 5](#) and [\(3\)](#), $G^{000} \cap \Gamma = \Gamma^0 \cdot \mathbb{Z}$. So as $G^{000} \subseteq G^{00}$,

(*) $G^{00} \cap \Gamma$ contains $\Gamma^0 \cdot \mathbb{Z}$ and must be type-definable.

We will argue that this implies that $G^{00} \cap \Gamma = \Gamma$, namely $\Gamma \leq G^{00}$. Consider the surjective homomorphism f say taking Γ to $\Gamma/\Gamma^0 = \widehat{\mathbb{Z}}$ (profinite completion of \mathbb{Z}). As $\Gamma^0 \cap \mathbb{Z} = 0$, the subgroup \mathbb{Z} of Γ is sent isomorphically under f to the dense subgroup \mathbb{Z} of $\widehat{\mathbb{Z}}$. Now (bearing in mind the logic topology on Γ/Γ^0), and as $G^{00} \cap \Gamma$ is type-definable, by (*) we see that $f(G^{00} \cap \Gamma)$ is a closed subgroup of $\widehat{\mathbb{Z}}$ containing the dense subgroup \mathbb{Z} , hence equals $\widehat{\mathbb{Z}}$. So $G^{00} \cap \Gamma = \Gamma$ as required. As G^{00} maps onto $SL_2(K)$ we see that $G^{00} = G$. \square

Bearing in mind that G^{000} maps onto $SL_2(K)$, we have by [Claim 6](#) and its proof, that $G/G^{000} = G^{00}/G^{000}$ is isomorphic to $\Gamma/\Gamma \cap G^{000} = \Gamma/\Gamma^0 \cdot \mathbb{Z}$ which is isomorphic to $\widehat{\mathbb{Z}}/\mathbb{Z}$. We have been working in the structure \overline{N} . However G^{000} in the sense of \overline{N} coincides with G^{000} in the sense of the structure (G, \cdot) as pointed out at the end of the proof of [Claim 5](#). And clearly $G = G^{00}$ in the structure (G, \cdot) too. So we have proved [Theorem 3.2](#). \square

We now give a similar o -minimal example. We will use the fact, pointed out above, that for a definably compact group H (such as SO_2) in a saturated o -minimal structure, $H^{00} = H^{000}$.

Theorem 3.3. *Let T be RCF, and G the group from [Example 2.10](#). Let G_1 be $G(\overline{M})$ for $\overline{M} = (K, +, \times)$ a saturated model. Then $G_1 = G_1^{00}$, but $G_1 \neq G_1^{000}$ and in fact G_1/G_1^{000} is naturally isomorphic to the quotient of the circle group $SO_2(\mathbb{R})$ by a dense cyclic subgroup.*

Proof. The proof is more or less identical to that of [Theorem 3.2](#), so we just give a sketch. In analogy with [\(3\)](#) from the proof of [3.2](#) and with the same notation we have:

(*) G_1 is a central product of its subgroups $SO_2(K)$ (which is definable) and $\widetilde{SL_2(K)}$ which is not definable, and with intersection “ \mathbb{Z} ” (an infinite cyclic subgroup $\langle g \rangle$ of $SO_2(\mathbb{R}) < SO_2(K)$).

As in [Claims 3](#) and [4](#) in the proof of [3.2](#), G_1^{000} contains $\widetilde{SL_2(K)}$, and (using (*)) $[G_1, G_1] = \widetilde{SL_2(K)}$. Also $G^{000} \cap SO_2(K)$ contains $SO_2(K)^{000}$ which we know to be equal to $SO_2(K)^{00}$. Hence we conclude that

(**) $G_1^{000} = SO_2(K)^{00} \cdot \widetilde{SL_2(K)}$.

Now the quotient map $SO_2(K) \rightarrow SO_2(K)/SO_2(K)^{00}$ identifies with the standard part map $SO_2(K) \rightarrow SO_2(\mathbb{R})$ which is the identity on $SO_2(\mathbb{R})$ and in particular on $\langle g \rangle$ (so $\langle g \rangle \cap SO_2(K)^{00}$ is trivial).

By (**) $G_1^{00} \cap SO_2(K)$ is type-definable and contains $SO_2(K)^{00} \cdot \langle g \rangle$, so its image under the standard part map $SO_2(K) \rightarrow SO_2(\mathbb{R})$ is a closed subgroup which contains the dense subgroup $\langle g \rangle$, hence has to be $SO_2(\mathbb{R})$. So G_1^{00} contains $SO_2(K)$ hence by (*) $G_1^{00} = G_1$. \square

We conclude this section with a few comments on the examples. Note that in the context of 3.2, G^{000} is the group product of a type-definable subgroup with the commutator subgroup $[G, G]$. As G^{000} is not type-definable, $[G, G]$ is not definable. Likewise in 3.3. The isomorphisms in 3.2 and 3.3 (e.g. in 3.2 between G/G^{000} and $\widehat{\mathbb{Z}}/\mathbb{Z}$) are on the face of it just isomorphisms of abstract groups. In the sequel to this paper we will show for arbitrary definable groups G in a saturated o -minimal expansion of a real closed field, G^{00}/G^{000} is either trivial or isomorphic to the quotient of a connected commutative compact Lie group by a finitely generated dense subgroup. One can ask whether there is a finer notion of isomorphism which holds in all these cases, and this will be treated in future work. As remarked earlier the above theorems provide new examples of non G -compact theories, i.e. where Lascar strong types differ from KP -strong types. A natural problem at this point is to find G such that $G_\emptyset^{00}/G_\emptyset^{000}$ is noncommutative. Also we see, via the examples above, some relationships between universal covers and fundamental groups on the one hand, and Lascar groups on the other, and maybe the connection is more than just accidental.

4. Definable amenability and bounded orbits

We begin with an arbitrary theory T . We recall that if M is a model, and X a definable set in M , then a Keisler measure μ on X (over M) is a finitely additive probability measure on the family of subsets of X which are definable (with parameters) in M . As explained in the introduction to Section 4 of [13], a Keisler measure μ on X over M induces and is induced by a (unique) regular Borel probability measure on the space $S_X(M)$ of complete types over M containing the formula defining X . Sometimes we identify this measure on the type space with μ . Of course a special case of a Keisler measure is a complete type (a Dirac measure on the type space).

When $X = G$ is a definable group, namely is equipped with a definable group structure, then $G(M)$ acts (on both the left and right) on the set (in fact space) of Keisler measures μ on G over M : if Y is an M -definable subset of G then, $(g \cdot \mu)(Y) = \mu(g^{-1} \cdot Y)$. In particular it makes sense for a Keisler measure μ on G over M to be left (or right) $G(M)$ -invariant. If G has such a left $G(M)$ -invariant Keisler measure over M then we say that $G(M)$ is *definably amenable*. Let us note for the record (assuming G is definable without parameters), that is a property of $Th(M)$, in the sense that if N is another model of T and $G(N)$ is the interpretation in N of the formula defining G , then $G(M)$ is definably amenable iff $G(N)$ is. This follows from Proposition 5.4 of [11].

In the above context we also have the (left and right) actions of $G(M)$ on the space $S_G(M)$ (of complete types over M concentrating on G). When M is a “big” model, and $p(x) \in S_G(M)$, we have the notion “ p has bounded orbit” from [15] for example. We will take our working definition as the following rather crude one, which on the face of it depends on set theory.

Definition 4.1. Suppose $\bar{\kappa}$ is an inaccessible cardinal, and \bar{M} a saturated model of cardinality $\bar{\kappa}$.

- (i) We will say that $p(x) \in S_G(\bar{M})$ has *bounded orbit* if the orbit of p under the (left) action of $G(\bar{M})$ is of cardinality $< \bar{\kappa}$, equivalently if $Stab(p) = \{g \in G(\bar{M}) : gp = p\}$ is a subgroup of $G(\bar{M})$ of index $< \bar{\kappa}$.
- (ii) We say that G has a *bounded orbit* if some $p(x) \in S_G(\bar{M})$ has bounded orbit.

In [15] some more careful definitions (see Definition 1.1 there) are given of “bounded orbit” avoiding the dependence on set theory (and some problems are mentioned concerning the possible sizes of bounded orbits), and our results in this section hold with these more refined definitions. The same paper [15] states a conjecture attributed to Petrykowski:

Conjecture 4.2. *If G has a bounded orbit then G is definably amenable.*

As discussed in the introduction the motivation for this conjecture seems to be also closely connected to G^{00} and G^{000} , in the sense that one may hope, given a global type p with bounded orbit, to be able to show that $G^{00} = G^{000} = \text{Stab}(p)$ and then to lift the Haar measure on G/G^{00} to a translation invariant Keisler measure on G . Note that a special case of a type $p(x) \in S_G(\overline{M})$ with bounded orbit, is a type $p(x)$ which is $G(\overline{M})$ -invariant. And in this case p itself witnesses definable amenability of G . The aim of this section is to prove **Conjecture 4.2** in the o -minimal context (although we have not yet “identified” those types with bounded orbit). We do this by characterizing each of the properties “definable amenability” and “having a bounded orbit” in terms of the decomposition given in **Proposition 2.6** and concluding that they coincide. So in a sense it is a proof by inspection.

We first describe when a definable group in an o -minimal structure is definably amenable. The proof is basically due to Hrushovski.

We begin with some preparatory lemmas, the first two of which are in a general context.

Lemma 4.3. *Suppose T has definable Skolem functions and elimination of imaginaries. Let G be definable and definably amenable. Then any definable subgroup H of G is also definably amenable.*

Proof. Let μ be a left G -invariant Keisler measure on G . By the assumptions there is a definable subset S of G which meets each coset of H in G in exactly one point. Define λ on definable subsets of H by: for Y a definable subset of H , $\lambda(Y) = \mu(Y \cdot S)$ where $Y \cdot S = \{a \cdot b : a \in Y, b \in S\}$.

It is easy to see that λ is a Keisler measure on H . Left H -invariance, is because, for $Y \subseteq H$ definable and $h \in H$, $\lambda(h \cdot Y) = \mu((h \cdot Y) \cdot S) = \mu(h \cdot (Y \cdot S)) =$ (by left invariance of μ) $\mu(Y \cdot S) = \lambda(Y)$. \square

Before the next lemma we recall the notion of “definability” of a Keisler measure (from [11] for example). So let $\mu(x)$ be a Keisler measure over \overline{M} . Let A be a small set of parameters. μ is said to be definable over A if for each closed $C \subseteq [0, 1]$, and formula $\phi(x, y)$ in the language L of T , $\{b \in \overline{M} : \mu(\phi(x, b)) \in C\}$ is type-definable over A , i.e. is the set of realizations of some partial type $\Sigma(y)$ over A . We will make use below of Lemma 5.8 from [13]. This lemma states that (assuming T has NIP) if G is a definable group, μ is a left G -invariant Keisler measure over \overline{M} , and M_0 is a small model over which G is defined, then there is a left G -invariant Keisler measure μ' over \overline{M} which agrees with μ on formulas over M_0 , and is definable (over some small subset of \overline{M}).

Lemma 4.4. *Suppose G is definable and H is a definable normal subgroup.*

- (i) *If G is definably amenable, so is G/H .*
- (ii) *(Assume T has NIP.) If both H and G/H are definably amenable, so is G .*

Proof. (i) Let $\pi : G \rightarrow G/H$ be the canonical surjective homomorphism. If μ is a left G -invariant Keisler measure on G , then the “pushforward measure” on G/H defined by $\lambda(Y) = \mu(\pi^{-1}(Y))$ is a left invariant Keisler measure on G/H .

(ii) We work in a saturated model \overline{M} . Let μ, λ be translation-invariant Keisler measures on H and G/H respectively over \overline{M} (i.e. “global” Keisler measures). By remarks above we may assume that μ is *definable*. We define a global Keisler measure χ on G by integration: namely, let X be a definable subset of G , and we may assume that both X and μ are definable over a small

model M . For $g/H \in G/H$, let $f(g/H) = \mu((g^{-1}X) \cap H)$, noting by translation invariance of μ , that this is well-defined. By definability of μ over M , $f(g/H)$ depends on $tp((g/H)/M)$ and the corresponding map from the relevant space of complete types $S_{G/H}(M)$ to $[0, 1]$ is continuous. So considering λ as inducing a Borel measure on $S_{G/H}(M)$ we can form $\int f d\lambda$, which we define to be $\chi(X)$. It is easily checked that χ is a global translation invariant Keisler measure on G . \square

Lemma 4.5. *Suppose G is a definably almost simple, non definably compact group, definable in an o -minimal expansion M of a real closed field K say. Then G is not definably amenable.*

Proof. The main point is to observe that,

(*) working up to definable isogeny, G contains a definable subgroup definably isomorphic to $PSL_2(K)$.

Granting this observation, the lemma follows from Lemma 4.4 together with Remark 5.2(iv) of [11] (which states that $PSL_2(K)$ is not definably amenable).

Let us sketch a proof of (*), using some basic language and notions from algebraic groups. First of all by Peterzil et al. [17] we may assume that $G = H(K)^0$ for some K -simple linear algebraic group, defined over K . By Remark 6.2 of [19], H is K -isotropic (meaning that H has a nontrivial K -split torus). This implies that $H(K)$ contains nontrivial unipotent elements and thus its Lie algebra $L(H(K)) = L(G)$ contains nontrivial unipotent elements. From this we conclude by the Jacobson–Morozov lemma [14] which says that $L(G)$ contains an “ sl_2 -triple”, in particular a subalgebra L isomorphic to $sl_2(K)$. L will be the Lie algebra of an algebraic subgroup of $H(K)$, isogeneous to $SL_2(K)$, and we finish. \square

We can now conclude, where notation comes from the paragraph following the proof of Proposition 2.6.

Proposition 4.6. *Let G be a definable, definably connected, group in an o -minimal expansion M of a real closed field. Then G is definably amenable if and only if D (the semisimple with no definably compact parts, part of G) is trivial, i.e. G is (definably) an extension of a definably compact group by a solvable group.*

Proof. First suppose that D is trivial, so we have a short exact sequence

$$1 \rightarrow W \rightarrow G \rightarrow C \rightarrow 1$$

where W is solvable and C is definably compact. Now W is amenable as an abstract group, so in particular definably amenable, and by Hrushovski et al. [11], C is definable amenable. As $Th(M)$ has NIP , by Lemma 4.4(ii) G is definably amenable.

Conversely, if G is definably amenable, then by Lemma 4.4(i), D is too, as it is a quotient of G . If D is nontrivial then it contains a definably almost simple (non definably compact) definable subgroup, which by Lemma 4.3 is definably amenable. This contradicts Lemma 4.5. \square

We give a little more information around definable amenability by noting:

Proposition 4.7 (*T an o -minimal expansion of RCF*). *Suppose G is definable, definably connected, and torsion-free. Then G has a (left) invariant, definable, global complete type.*

Proof. We again argue by induction on $\dim(G)$. By Proposition 2.1(i), G contains a normal definable subgroup H such that G/H is 1-dimensional. From results in [24] we may assume

that G/H is an open interval in 1-space with continuous group operation. The global type at “ $+\infty$ ”, p say, is both definable and translation invariant. On the other hand the induction hypothesis gives a definable translation invariant global complete type q of H . The argument (by integration) in the proof of [Lemma 4.4\(ii\)](#) produces a global complete type of G which is both translation invariant and definable. We give a few details (as requested by the referee). We will give an explicit description of the complete type $r \in S_G(\overline{M})$ obtained by integration. Let $\phi(x)$ be a formula (where x ranges over G), possibly with parameters. As q is H -invariant we see that whether or not $g^{-1}\phi(x) \cap H$ is in q depends only on the coset of $g \bmod H$. Moreover $\{g \in G : g^{-1}\phi(x) \cap H \in p\}$ is definable, by a formula $\chi(x)$, say. Let $\psi(z)$ be the image of χ under the map $G \rightarrow G/H$. Then we put $\phi(x)$ in r just if $\psi(z)$ is in p . Invariance under G and definability of r are routine to check. \square

We now focus on [Conjecture 4.2](#). From now on \overline{M} denotes a saturated model of (arbitrary complete countable) T , of cardinality $\bar{\kappa}$ where $\bar{\kappa}$ is inaccessible, and G an \emptyset -definable group. Let us first remark that the converse to [Conjecture 4.2](#) holds for *NIP* theories.

Remark 4.8 (Assume T has *NIP*). Suppose G is definably amenable. Then G has a bounded orbit.

Proof. By Proposition 5.12 of [13], G has a global f -generic type p . Fix a small model M_0 which witnesses this. There will then be a bounded number of global complete types which do not fork over M_0 , as there are a bounded number of complete types over M_0 , and by *NIP* any complete type over M_0 has a bounded number of global nonforking extensions (Proposition 2.1 of [13]). As every $G(\overline{M})$ -translate of p does not fork over M_0 there are a bounded number of such translates so p has bounded orbit. \square

Lemma 4.9. Suppose $G = G(\overline{M})$ is almost simple as an abstract group, in the sense that G has no infinite proper normal subgroups. Then G has no proper subgroup of index $< \bar{\kappa}$. In particular any bounded orbit of G is a singleton (namely a translation invariant type).

Proof. Suppose H were a proper subgroup of G of bounded index. Then G acts transitively on the homogeneous space $X = G/H$. Let $N = \{g \in G : gx = x \text{ for all } x \in X\}$. Then N is a proper normal subgroup of G . As G/N acts faithfully on X and $|X| < \bar{\kappa}$, also $|G/N| < \bar{\kappa}$, in particular N is an infinite proper normal subgroup of G , contradiction.

For the “in particular” clause: if $p \in S_G(\overline{M})$ has bounded orbit, then $\text{Stab}(p)$ is a subgroup of G of bounded index. By what has just been shown $\text{Stab}(p) = G$ so p is left G -invariant. \square

Lemma 4.10. Let $f : G \rightarrow H$ be a definable surjective homomorphism. If G has a bounded orbit, so does H .

Proof. Let $p \in S_G(\overline{M})$ have bounded orbit. Then $q = f(p) \in S_H(\overline{M})$, and if $g \in \text{Stab}_G(p)$ then $q = f(p) = f(gp) = f(g)q$ hence $f(\text{Stab}_G(p)) \subseteq \text{Stab}_H(q)$. As $\text{Stab}_G(p)$ has bounded index in G , also $\text{Stab}_H(q)$ has bounded index in H . \square

Proposition 4.11. Assume T is an *o-minimal* expansion of *RCF* and G is definably connected. Suppose G has a bounded orbit. Then D (the semisimple with no definably compact parts, part of G) from [Proposition 2.6](#) is trivial.

Proof. Suppose for a contradiction that D is nontrivial. Then D is an almost direct product of definable, definably almost simple non definably compact groups D_i . But then for $i = 0$ say there

is a definable surjective homomorphism f from G to D_0 . By Lemma 4.10, D_0 has a bounded orbit. As remarked earlier (Corollary 6.3 of [19]) D_0 is almost simple as an abstract group, so by Lemma 4.9, D_0 has an invariant (global) type. This contradicts non definable amenability of D_0 (Lemma 4.5). \square

Corollary 4.12 (*T an o-minimal expansion of RCF*). G has a bounded orbit if and only if G is definably amenable.

Proof. If G has a bounded orbit, then by Propositions 4.6 and 4.11, G is definably amenable. The converse is Remark 4.8. \square

Finally we discuss a strengthening of Conjecture 4.2 in which we try to describe bounded orbits themselves. As we are not completely sure which way it will go we state the new conjecture as a question (with notation as above).

Problem 4.13 (*Assume T has NIP*). Is it the case that $p \in S_G(\bar{M})$ has bounded orbit (equivalently stabilizer of bounded index) if and only if p is f -generic?

Again the right to left direction holds with proof contained in the proof of Remark 4.8. In the o -minimal case we hope to give an explicit description of global types with bounded orbit from which a positive answer to Problem 4.13 can be just read off. By Corollary 4.12 and Proposition 4.6 we may restrict ourselves to definable groups G for which D (from the discussion after Proposition 2.6) is trivial, hence G is built up from a definably compact group, and 1-dimensional torsion-free groups. Here we just point out that Problem 4.13 has a positive answer for these constituents, and leave the general (o -minimal case) to later work. For the next lemma we recall that a definable subset of G (or the formula defining it) is said to be left generic if finitely many left translates of X cover G . Likewise for right generic. Definably compact groups G in o -minimal expansions of real closed fields have the so-called “finitely satisfiable generics” property (see [11]) which says that there is a global type of G every left translate of which is finitely satisfied in some given small model. The fsg property implies among other things that left genericity coincides with right genericity for definable subsets of G , so we just say *generic*. A generic type $p \in S_G(\bar{M})$ is one all of whose formulas are generic, and again such global types exist when G is definably compact in o -minimal T .

Lemma 4.14 (*T o-minimal*). Suppose G is definably compact, and $p(x) \in S_G(\bar{M})$. Then the following are equivalent:

- (i) p has bounded G -orbit,
- (ii) p is generic,
- (iii) p is f -generic.

Proof. In fact the implications (ii) \rightarrow (iii) \rightarrow (i) hold for fsg groups in arbitrary NIP theories and the proof will be at this level of generality.

(iii) implies (i) is given by the proof of Remark 4.8.

(ii) implies (iii): By Peterzil et al. [11] (see also Fact 5.2 of [13]), any generic formula $\phi(x)$ over \bar{M} is satisfied in any small model M_0 (over which G is defined). So if $p \in S_G(\bar{M})$ is generic, then every left translate of p is finitely satisfied in M_0 (where M_0 is any small model over which G is defined), so in particular every left translate of p does not fork over M_0 , hence p is left generic.

(i) implies (ii): Here we give the proof assuming o -minimality of T and definable compactness of G . Suppose p is not generic. Let X be a definable set (or formula) in p which is not generic. Note that we may assume G to be a closed bounded definable subset of some \overline{M}^n . The closure of X in G equals $X \cup Y$ where $\dim(Y) < \dim(G)$. So Y is not generic in G . Hence as the set of non generic definable sets is an ideal, the closure of X is also non generic (and of course in p). The upshot is that we may assume X to be closed. Let M_0 be a small model over which G and X are defined. If for every $g \in G$, the left translate $g \cdot X$ meets $G(M_0)$, then by compactness X is right generic, so generic, a contradiction. Hence for some $g \in G$, $(g \cdot X) \cap G(M_0) = \emptyset$. Now $g \cdot X$ is also closed in G . So by results in [6,16] (see also [26]), $g \cdot X$ forks over M_0 . By the main result of [3] (which is maybe implicit in other papers in the o -minimal case), $g \cdot X$ divides over M_0 . As X is defined over M_0 this means that for some M_0 -indiscernible sequence $(g_i : i < \omega)$ and some $k < \omega$, $\{g_i \cdot X : i < \omega\}$ is k -inconsistent, in the sense that for every (some) $i_1 < \dots < i_k$, $(g_{i_1} \cdot X) \cap \dots \cap (g_{i_k} \cdot X) = \emptyset$. We can stretch the M_0 -indiscernible sequence $(g_i : i < \omega)$ to $(g_i : i < \bar{\kappa})$. So $\{(g_i \cdot X) : i < \bar{\kappa}\}$ is also k -inconsistent. It follows easily that among the set $\{g_i p : i < \bar{\kappa}\}$ of complete global types there are $\bar{\kappa}$ many distinct types. So p does not have bounded orbit. \square

Let us note that various ingredients of the proof of (i) implies (ii) above also appear in earlier papers such as [13]. In fact there is a proof of (i) implies (ii) (so of the whole lemma) in the more general context of fsg groups in NIP theories, but depending on some additional machinery. It will appear in a subsequent paper.

Lemma 4.15. *Suppose G is 1-dimensional and torsion-free (divisible), and $p \in S_G(\overline{M})$. Then the following are equivalent:*

- (i) p has bounded G -orbit,
- (ii) p is G -invariant,
- (iii) p is the type at $+\infty$ or the type at $-\infty$ (so definable and G -invariant, hence f -generic).

Proof. As remarked earlier we can and will identify G with an open interval on which the group operation is continuous, and write G additively (it is commutative). We know (or it is clear) that the types at $+\infty$ and $-\infty$ are G invariant hence have bounded orbit. So it suffices to prove that any other type $q(x) \in S_G(\overline{M})$ has unbounded G -orbit. This is really obvious but we go through details. So q defines a cut in G with nonempty left hand side L and right hand side R . Let $a \in L, b \in R$ and $c = b - a > 0$. By compactness and saturation we can clearly find an increasing sequence $(d_i : i < \bar{\kappa})$ in G , such that $i < j$ implies $(d_j - d_i) \geq c$. Hence $\{d_i + q : i < \bar{\kappa}\}$ witnesses that q has unbounded orbit. \square

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