Interval Maximum-Entropy Method for Min-Max-Min Problem

Meirong Chen\textsuperscript{a*}, Dexin Cao\textsuperscript{a}, Lei Zhang\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, China University of Mining and Technology, Xuzhou 221008, China
\textsuperscript{b} Agricultural Bank of China, Jiangsu Branch, Nanjing 210002, China

Abstract

An interval maximum-entropy method for a kind of unconstrained min-max-min problem is described, where the constituent objective functions are $C^1$. We transform this problem into a differentiable optimization problem using the maximum-entropy function, discuss the interval extension of the maximum-entropy function, prove relevant properties and provide several region deletion rules. An interval maximum-entropy algorithm is proposed, with a proof of convergence provided. Theoretical analysis and numerical results indicate that the new method is reliable and effective.

Key words: Maximum-entropy function; Interval algorithm; Min-Max-Min problem.

1. Introduction

In this paper, we consider the unconstrained min-max-min problems defined by

$$\min_{x \in X^{(0)}} \{\max_{i \in \{1, \ldots, m\}} \min_{j \in \{1, \ldots, l_i\}} f_{ij}(x)\},$$

(1.1)

where $X^{(0)} = \{x = (x_1, x_2, \ldots, x_n)^T \in R^n \mid x_\in [a_i, b_i] \subseteq R^1, i = 1, \ldots, n\}$, $f_{ij} : X^{(0)} \rightarrow R^l$ is in $C^l$, $m$ and $l_i$ are elements of natural number set of all $i$.

Min-max-min problems are considerably important in engineering design, electronic circuit design, control system design and economics. Due to complexity in theory and computation, the literature dealing with them remains relatively small. Methods for min-max-min problems have been discussed by several authors, e.g. \cite{1}-\cite{3}. Polak, and Royset (\cite{1}) described a set of weaker first-order optimality conditions for finite and semi-infinite min-max-min problems, where a sequence of min-max consistent approximations is constructed. \cite{2} extended an adaptive technique for constructing a sequence of smooth approximations to finite min-max problems and combined it with the approximation results from \cite{1}. \cite{3}
designed an algorithm for tracing the smooth pathway for min-max-min problem by the homotopy idea, but it didn’t present the numerical examples.

Recently, interval analysis method and maximum-entropy method have been discussed by several authors in [4-8] for solving nonlinear programming problems, such as minimax problems, nonlinear ill-posed problems and nonlinear equations. In this paper, An interval maximum-entropy method is described for solving a kind of box-constrained min-max-min problem. With the interval extension of maximum-entropy function and region deletion rules, we suggest interval maximum-entropy algorithm for solving problem (1.1). Also numerical results are given.

The notation that is needed about the interval mathematics is as follows. A real interval $x = [\underline{x}, \overline{x}]$ has infimum $\underline{x} \in \mathbb{R}^l$ and supremum $\overline{x} \in \mathbb{R}^l$ with $\underline{x} \leq \overline{x}$. The interval vector is denoted by $X = (X_1, \ldots, X_n)^T$, with each interval $X_j$, the set of all interval vectors is denoted by $I(R^n)$. If $x^{(0)} \subseteq R^n$, then $I(x^{(0)}) = \{x \in I(R^n) | x \subseteq x^{(0)}\}$. The midpoint $M(X)$, the width $W(X)$, the norm $\|X\|$ of $X \in I(R^n)$ are denoted by $M(X) = \{M(X_1), M(X_2), \ldots, M(X_n)\}$, $W(X) = \max\{W(X_1), W(X_2), \ldots, W(X_n)\}$, $\|X\| = \max\{\|X_1\|, \|X_2\|, \ldots, \|X_n\|\}$ respectively. If the function $f(x)$ and the interval function $F(X)$ satisfy $F(x) = f(x)$, $f(x) \in F(X)$ for each $x \in X \in I(x^{(0)})$, then $F(X)$ is called the interval extension of $f(x)$. And for the interval function $F(X)$, we define $\underline{F}(X)$ and $\overline{F}(X)$ as the left endpoint and right endpoint, i.e. $F(X) = [\underline{F}(X), \overline{F}(X)]$. For background knowledge on interval mathematics, see for example, [4],[10-13].

2. The Maximum Entropy Function

Let

$$f(x) = \max_{1 \leq i \leq m} \min_{1 \leq j \leq l_i} \{f_{ij}(x)\}. \quad (2.1)$$

According to the definition of maximum-entropy function, the function $f(x)$ can be approximated by the following smooth function

$$\varphi(x, p) = \frac{1}{p} \ln \left( \sum_{i=1}^{m} \exp(-p \varphi_i(x, p)) \right), \quad (2.2)$$

where

$$\varphi_i(x, p) = \frac{1}{p} \ln \left( \sum_{j=1}^{l_i} \exp(-p f_{ij}(x)) \right),$$

where $p > 0$ is the precision parameter.

**Lemma 2.1**[9] For each $x \in X^{(0)}$, one has

$$f(x) - \frac{1}{p} \ln(\max_{i \leq m} \{l_i\}) \leq \varphi(x, p) \leq f(x) + \frac{1}{p} \ln m,$$

also given $\varepsilon > 0$, such that

$$|\varphi(x, p) - f(x)| < \varepsilon \quad (\forall p > \bar{p}).$$

**Theorem 2.1** Let the optimal solution $x \in X^{(0)}$ of problem (1.1) exist, and $\{p_k\}$ be a monotonically increasing sequence of integers with $p_k \geq 1(k = 1, 2, \cdots)$, such that $\lim_{k \rightarrow \infty} p_k = +\infty$. Let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of real numbers $(\forall k \geq 1)$, such that $\alpha_k \geq \alpha = \min_{i \leq m} f(x)$, $\lim_{k \rightarrow \infty} \alpha_k = \alpha$. Let

$$L_k = \left\{x \mid \varphi(x, p_k) \leq \alpha_k + \frac{1}{p_k} \ln m\right\}, \quad L_\alpha = \left\{x \mid f(x) = \alpha\right\},$$

$$\alpha_k = \alpha_k^{(0)}.$$
then \( L_\infty = \bigcap_{k=1}^{\infty} L_k \).

**Proof** For each \( x \in L_\infty \), by lemma 2.1

\[
\phi(x, p_k) \leq f(x) + \frac{1}{p_k} \ln m = \alpha + \frac{1}{p_k} \ln m \leq \alpha_k + \frac{1}{p_k} \ln m.
\]

If \( x \in L_k (\forall k \geq 1) \), then \( L_\infty \subseteq \bigcap_{k=1}^{\infty} L_k \). Conversely, suppose that \( x \in \bigcap_{k=1}^{\infty} L_k \), then (\( \forall k \geq 1 \))

\[
\alpha + \frac{1}{p_k} \ln \left( \max_{l \in \mathcal{E}_m} l \right) \leq f(x) + \frac{1}{p_k} \ln \left( \max_{l \in \mathcal{E}_m} l \right) \leq \phi(x, p_k) \leq \alpha_k + \frac{1}{p_k} \ln m.
\]

So letting \( k \to \infty \), one has \( \alpha = f(x) \), if \( x \in L_\infty \), then \( L_\infty \subseteq \bigcap_{k=1}^{\infty} L_k \). Therefore \( L_\infty = \bigcap_{k=1}^{\infty} L_k \).

With the proceeding ideas and Theorem 2.1, an approximate problem of problem (1.1) is written as

\[
\min_{x \in X^{(0)}} \phi(x, p) \quad (p > 0).
\]

Next we define the interval extension of \( \phi(x, p) \) to get the sequence of real number \( \{\alpha_k\}_{k=1}^{\infty} \).

### 3. Interval Extension

The mean value form interval extension of \( f_j(x) (i = 1, \cdots, m, j = 1, \cdots, l_i) \) is defined by

\[
F_j(X) = f_j(m(X)) + (X - m(X))^T F_j'(X),
\]

where

\[
F_j'(X) = \left( F_{ij_1}', X \right) \cdots F_{ij_n}', X \right)^T,
\]

and \( F_{ij_k}'(X) = [L_{ij_k}, T_{ij_k}] \) is the Lipschitz interval extension of \( \frac{\partial f_j(x)}{\partial x_k} \) (\( k = 1, \cdots, n \)) in \( X \). We define

\[
\phi(X, p) = \frac{1}{p} \ln \left\{ \sum_{i=1}^{m} \left( \sum_{j=1}^{l_i} \exp(-pF_j'(X)) \right)^{-1} \right\}
\]

\[
= \frac{1}{p} \left[ \ln \left\{ \sum_{i=1}^{m} \left( \sum_{j=1}^{l_i} \exp(-pF_j'(X)) \right)^{-1} \right\}, \ln \left\{ \sum_{i=1}^{m} \left( \sum_{j=1}^{l_i} \exp(-pF_j'(X)) \right)^{-1} \right\} \right].
\]

**Theorem 3.1** The interval function \( \phi(X, p) \) is the interval extension of the maximum-entropy function \( \phi(x, p) \).

**Proof** First, \( \forall x \in X \in I(X^{(0)}) \), we can obtain \( F_j(x) = f_j(x) = F_j'(x) \), So

\[
\phi(x, p) = \frac{1}{p} \ln \left\{ \sum_{i=1}^{m} \left( \sum_{j=1}^{l_i} \exp(-pF_j(x)) \right)^{-1} \right\} = \phi(x, p). \quad (3.1)
\]

Second, \( \forall x \in X \), \( f_j(x) \in F_j(X) \), then

\[
\frac{1}{p} \ln \left\{ \sum_{i=1}^{m} \left( \sum_{j=1}^{l_i} \exp(-pF_j(x)) \right)^{-1} \right\}
\]

\[
\leq \frac{1}{p} \left[ \ln \left\{ \sum_{i=1}^{m} \left( \sum_{j=1}^{l_i} \exp(-pF_j(X)) \right)^{-1} \right\}, \ln \left\{ \sum_{i=1}^{m} \left( \sum_{j=1}^{l_i} \exp(-pF_j(X)) \right)^{-1} \right\} \right].
\]

Namely,

\[
\phi(x, p) \in \phi(X, p) \quad (3.2)
\]

Therefore, \( \phi(X, p) \) is the interval extension of \( \phi(x, p) \) by (3.1) and (3.2).
Lemma 3.1\cite{11} Let \( f_0(x) \in C^4(X^{(0)}) \) \((i = 1, 2, \cdots, m; j = 1, 2, \cdots, l_j)\), \( F'_0(X) \) is Lipschitz interval extension of \( f'_0(x) \), then interval extensions \( F_0(x) = f_0(M(X)) + (X-M(X))^T F'_0(X) \) of \( f'_0(x) \) is at least of convergent order two.

**Theorem 3.2** Let \( f_0(x) \in C^4(X^{(0)}) \) \((i = 1, 2, \cdots, m; j = 1, 2, \cdots, l_j)\), \( F'_0(X) \) is the Lipschitz interval extension of \( f'_0(x) \), then interval extension \( \phi(X, p) \) is at least of convergent order two.

**Proof** With Lemma 3.1, one has \( F_0(X) \) is at least of convergent order two. Therefore
\[
W(F_0(X)) - W(f_0(x)) \leq c_y \| W(X) \|_{\infty}^2 \quad (i = 1, \cdots, m; j = 1, \cdots, l_j).
\]
One has \( \overline{\phi}(X, p) - \underline{\phi}(X, p) = \frac{1}{p} \ln \left[ \sum_{i=1}^{m} \exp \left( -p \phi(X, p) \right) \right] - \frac{1}{p} \ln \left[ \sum_{i=1}^{m} \exp \left( -p \underline{\phi}(X, p) \right) \right] \)
\[
= \frac{1}{p} \ln \left[ \sum_{i=1}^{m} \exp \left( -p \xi_i \phi(X, p) \right) \right] = \frac{1}{p} \ln \left[ \sum_{i=1}^{m} \exp \left( -p \xi_i \underline{\phi}(X, p) \right) \right] \leq \max_{i \in \mathcal{G}} (\phi(X, p) - \underline{\phi}(X, p)),
\]
in which \( \xi_i \) is between \( \phi(X, p) \) and \( \underline{\phi}(X, p) \), Similarly \( \phi(X, p) - \underline{\phi}(X, p) \leq c_i \| W(X) \|_{\infty}^2 \), where \( \overline{f}_0(X) \leq \eta_j \leq \overline{f}_0(X) \), and \( c_i = \max \{c_y\} \).

Therefore \( W(\phi(X, p)) - W(\phi(X, p)) \leq 2c \| W(X) \|_{\infty}^2 \), hence, interval extension \( \phi(X, p) \) is at least of convergent order two.

Suppose that \( X^{(0)} \) is subdivided into \( k \) segments \( X^{(l)}(l = 1, 2, \cdots, k) \) of equal width, \( X^{(0)} = \bigcup_{l=1}^{k} X^{(l)} \), so that \( c_k = \max_{1 \leq j \leq k} \| X^{(l)} \|_{\infty} \to 0(k \to +\infty) \). Let \( \{p_k\} \) be monotonically sequences of real numbers, and \( p_k \to \infty(k \to \infty) \). Let
\[
\alpha_k = \min_{1 \leq j \leq k} \phi(X^{(l)}, p_k).
\]

**Theorem 3.3** Let \( f^* \) is the optimal value of problem (1.1). Given \( \varepsilon > 0 \), \( \forall K > 0 \), such that for \( \forall k > K \), one has
\[
| f^* - \alpha_k | < \varepsilon.
\]

**Proof** By Lemma 2.1, if \( k > K_1 \), where \( K_1 \) and \( p_k \) are sufficiently large, then
\[
| f^* - \min_{x \in X^{(0)}} \phi(x, p_k) | = | \min_{x \in X^{(0)}} f(x) - \min_{x \in X^{(0)}} \phi(x, p_k) | \leq \varepsilon / 2.
\]

On the other hand, there exists \( X^{(r)} \) such that
\[
\phi(X^{(r)}, p_k) = \min_{1 \leq j \leq k} \phi(X^{(l)}, p_k) = \alpha_k.
\]

By theorem 3.2, we notice that
\[
| \min_{x \in X^{(r)}} \phi(x, p_k) - \alpha_k | = | \min_{x \in X^{(r)}} \phi(x, p_k) - \phi(X^{(r)}, p_k) | \leq | \min_{x \in X^{(r)}} \phi(x, p_k) - \phi(X^{(r)}, p_k) | \leq c \| W(X^{(r)}) \|_{\infty}^2 \leq c \alpha_k^2.
\]
So for \( k > K_2(K_2 > 0) \), we have
\[
| \min_{x \in X^{(r)}} \phi(x, p_k) - \alpha_k | \leq \varepsilon / 2.
\]
Furthermore \( K = \max \{K_1, K_2\} \), it follows from (3.4), (3.5) that for \( \forall k > K \), there is
\[ |f^* - \alpha_k| \leq |f^* - \min_{x \in X^0} \varphi(x, p_k)| + |\min_{x \in X^0} \varphi(x, p_k) - \alpha_k| < \varepsilon. \]

4. The Algorithm

To find the min-max-min value and the set of min-max-min points of the problem (1.1), one generates a list of subintervals which must contain the min-max-min points. Subintervals that don't contain min-max-min points may be deleted from \(X^{(0)}\) by using the following region deletion rules. Region deletion rules can reduce the computational cost and accelerate the convergent speed of algorithm.

**Deletion rule 1** (evaluate test)

Let \(\bar{\varphi}\) be an upper bound of \(\min_{x \in X^0} \varphi(x, p)\). If \(\bar{\varphi}(X, p) > \bar{\varphi}\) for \(X \in I(X^{(0)})\), then there is no optimal point in \(X\). \(X\) can be discarded from \(X^{(0)}\).

**Deletion Rule 2** (monotone test)

Let \(\phi'_{ik}(X, p)\) be interval extension of \(\frac{\partial \bar{\varphi}(X, p)}{\partial x_k}\). If \(\phi'_{ik}(X, p) > 0\), then there is no optimal point in \(X\) except \((X_1, \ldots, X_{k-1}, X_k, X_{k+1}, \ldots, X_n)^T\). If \(\phi'_{ik}(X, p) < 0\), then there is no optimal point in \(X\) except \((X_1, \ldots, X_{k-1}, X_k, X_{k+1}, \ldots, X_n)^T\).

For \(X = (X_1, \ldots, X_n)^T \in I(X^{(0)})\), let \(X_k = [x_k, \bar{x}_k]\), \(K = \{k \in \{1, \ldots, n\} | W(X_k) \neq 0\}\), \(L(X) = \sum_{k \in K} |\phi'_{ik}(X, p)|\). For \(\forall x, y \in X\), one has \(|\varphi(x, p) - \varphi(y, p)| \leq \sum_{k \in K} |\phi'_{ik}(X, p)||x - y||_{\alpha} = L(X)||x - y||_{\alpha}.

By Theorem 2.3 from Ref. [14], one has

**Deletion Rule 3** (Lipschitz test)

Let \(L(X) = \sum_{k \in K} |\phi'_{ik}(X, p)|\), \(\bar{\varphi}\) be an upper bound of \(\min_{x \in X^0} \varphi(x, p)\), \(R(X)\) is the radius of \(X\), \(\alpha = (\varphi(M(X), p) - \bar{\varphi})/L(X)\). If \(\alpha\) is greater than the second largest component of \(R(X)\), then discard \(X_{\alpha}\) from \(X\), where \(X_{\alpha} = [(M(X_1) \pm \alpha], \ldots, [M(X_n) \pm \alpha]^T\).

We suggest the following interval algorithm of problem (1.1), with the region bisection rule of Moore in [10], the interval extension and region deletion rules. The algorithm starts from \(X^{(0)}\) and generates a list \(L\) by the bisection rule and computing accordingly. The elements in the list \(L\) are of the form \((X, F, \bar{f}, L(X))\). Each \(X\) can be obtained from the region of the first element in the list \(L\) in two ways, one is that the region is deleted by Lipschitz test, the other is that the region is bisected normal to the coordinate direction which \(X\) has maximum width.

The detailed steps of the algorithm are the following.

(Notice: Given \(\varepsilon\) and \(p\), let \(F_{ij}(X) = f_{ij}(M(X)) + (X - M(X))^T F_{ij}(M(X))\))

**STEP 1** Set \(X = X^{(0)}, I_x = \{1, 2, \ldots, m\}, J_x = \{1, 2, \ldots, l\}\).

**STEP 2** 1 Compute \(\phi(X, p), L_k = \phi'_{ik}(X, p), \bar{L}_k = \bar{\varphi}(X, p)

2 If there exists \(k\) such that \(L_k > 0\) or \(\bar{L}_k < 0\), \(X\) can be obtained by monotone test, then go to 1. Otherwise, go to STEP 3.

**STEP 3** Compute \(L(X) = \sum_{k \in K} |\phi'_{ik}(X, p)|, F_1 = \frac{1}{p} \ln \left[ \sum_{i=1}^{m} \left( \sum_{j=1}^{l} \exp(-pF_{ij}(X)) \right)^{-1} \right] \)
\[ f = \frac{1}{p} \ln \left( \sum_{i=1}^{m} \left( \sum_{j=1}^{l} \exp(-pf_{ij}(M(X))) \right) \right), \] enter the item \((X,F,f,L(X))\) in the list \(L\).

Let \( \bar{f}_0 = \bar{f}, B_0 = +\infty \).

**STEP 4** Extract the first element \((X,F,f,L(X))\) from the list \(L\) and remove it from the list \(L\). Set \(B = \min\{B_0,F\}\).

**STEP 5** If \( \bar{f} - B < \varepsilon \), go to **STEP 12**; otherwise, go to **STEP 6**.

**STEP 6** Set \( \alpha = (\bar{f} - \bar{f}_0) / L(X) \). If \( \alpha \) is greater than the second largest component of \( R(X) \), then discard \( (X_a \cap X) \) from \( X \) and obtain two parts \( X^{(1)}, X^{(2)} \); otherwise bisect \( X \) normal to the coordinate direction in which \( X \) has maximum width to obtain \( X^{(1)}, X^{(2)} \).

**STEP 7** Set \( h = 1,2 \):

1. Compute \( \phi'(X^{(h)})(i \in I_X) \); \( \bar{L}^{(h)}(k \in K_X) \);
2. If \( \bar{L}^{(h)}(h) > 0 \) or \( \bar{L}^{(h)}(h) < 0 \), \( X^{(h)} \) can be obtained by monotone test, go to **STEP 8**. Otherwise, go to **STEP 8**.

**STEP 8** For each \( h = 1,2 \), compute \( L(X^{(h)}) = \sum_{k \in K_X} |\phi'_{ik}(X^{(k)}(p))| \),

\[ F^{(1)} = \frac{1}{p} \ln \left( \sum_{i=1}^{m} \left( \sum_{j=1}^{l} \exp(-pf_{ij}(X^{(h)})) \right) \right), \quad \bar{f}^{(h)} = \frac{1}{p} \ln \left( \sum_{i=1}^{m} \left( \sum_{j=1}^{l} \exp(-pf_{ij}(M(X^{(h)}))) \right) \right), \]

set \( \bar{f}_0 = \min(\bar{f}^{(1)}, \bar{f}^{(2)}) \).

**STEP 9** enter the item \((X^{(h)},F^{(h)},\bar{f}^{(h)},L(X^{(h)}))(h = 1,2)\) in the list \(L\) in an increasing order of \( F^{(h)} \).

**STEP 10** Discard all elements \((X,F,f,L(X))\) from the list \(L\) that satisfy \( F_i > \bar{f}_0 \).

**STEP 11** If the list \(L\) is empty, go to **STEP 14**; otherwise, return to **STEP 4**.

**STEP 12** Print interval min-max-min value \([F_1, \bar{f}]\) and min-max-min point \(X\).

**STEP 13** If the list \(L\) is empty, go to **STEP 14**; otherwise, set \( B_0 = B \) and return to **STEP 4**.

**STEP 14** End.

5. Numerical Results

The algorithm has been implemented using Matlab6.5 on P4 computer. Numerical results are reported in this section. The symbols \( x^*, f^*, \varepsilon, K \) denote the exact solution, the entropy factor, the precision of optimal value, the numbers of processing bisections on termination of the algorithm respectively.

**Example 1.** \( f_{11}(x) = -(x_1^2 + x_2^2), \quad f_{12}(x) = x_1 - x_2, \quad f_{21}(x) = 5x_1 + x_2, \quad f_{22}(x) = x_2 - x_1, \)

\( X^{(0)} = \{[-1,1],[-4,1]\} \), the exact solution \( f^* = -9, \quad x^* = (-1,-4) \)

The first result (\( \varepsilon = 10^{-9}, K = 2, p = 10^6 \)):

min-max-min value \( f = [-9.00000000000000,-9.00000000000000] \)

min-max-min point \( x = [(-1.00000000000000,-1.00000000000000)] \)

**Example 2.** \( f_{11}(x) = \sin(10x), \quad f_{12}(x) = \cos(10x), \quad f_{21}(x) = x, \quad f_{22}(x) = x - 1, \quad X^{(0)} = [-1,2] \)

The exact solution \( f^* = -1, \quad x^* = -3\pi/10, -\pi/4, -\pi/10, -\pi/20 \)

The first result of interval algorithm (\( \varepsilon = 10^{-12}, K = 271, p = 10^6 \)):

min-max-min value \( f = [-1.00000000000000,-1.00000000000000] \)

min-max-min point \( x = [-0.94247779250145,-0.94247778132558] \),
\[
x_2 = [-0.78539817780256, -0.78539816662669],
x_3 = [-0.31415926665068, -0.31415925547481],
x_4 = [-0.15707964077592, -0.15707962960005].
\]

**Example 3.** \( f_{11}(x) = x/4, \quad f_{12}(x) = x(1-x), \quad f_{21}(x) = x, \quad f_{22}(x) = 1-x, \quad X^{(0)} = [0,1.5] \)

The exact solution \( f^* = -0.5, \quad x^* = 1.5, \)

The first result of interval algorithm (\( \varepsilon = 10^{-14}, K = 2, p = 10^6 \)):

\begin{align*}
\text{min-max-min} & \quad f = [-0.50000000000000, -0.50000000000000] \\
\text{min-max-min} & \quad x = [1.50000000000000, 1.50000000000000]
\end{align*}

**Example 4.** \( f_{11}(x) = x^2 + x, \quad f_{12}(x) = 3 + x, \quad f_{21}(x) = x - 1, \quad f_{22}(x) = x + 9, \quad X^{(0)} = [-1,4] \)

The exact solution \( f^* = -0.25, \quad x^* = -0.5, \)

The first result of interval algorithm (\( \varepsilon = 10^{-14}, K = 72, p = 10^6 \)):

\begin{align*}
\text{min-max-min} & \quad f = [-0.25000000000000, -0.25000000000000] \\
\text{min-max-min} & \quad x = [-0.500000010430813, -0.50000002980232]
\end{align*}

**Example 5.** \( f_{11}(x) = x(x-1)(x-2)(x-3), \quad f_{12}(x) = (0.5-x)(x-1.5)(x-2.5), \quad f_{21}(x) = (x-1)(x-4), \quad f_{22}(x) = (x-2)(x-5), \quad X^{(0)} = [0,6] \)

The exact solution \( f^* = -2.25, \quad x^* = 3.5 \)

The first result of interval algorithm (\( \varepsilon = 10^{-14}, K = 87, p = 10^6 \)):

\begin{align*}
\text{min-max-min} & \quad f = [-2.25000000000001, -2.25000000000000] \\
\text{min-max-min} & \quad x = [3.49999997019768, 3.50000005960464]
\end{align*}

**References**


