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# The Banach fixed point theorem in fuzzy quasi-metric spaces with application to the domain of words ☆

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**Abstract**

We present a fuzzy quasi-metric version of the Banach contraction principle, which constitutes an extension of the famous Grabiec fixed point theorem. By using this result we show the existence of fixed point for contraction mappings on the domain of words when it is endowed with certain fuzzy quasi-metrics of Baire type. We apply this approach to deduce the existence of solution for some recurrence equations associated to the analysis of Quicksort algorithms and Divide & Conquer algorithms, respectively.


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*Keywords:* Non-Archimedean fuzzy quasi-metric; G-bicomplete; B-contraction; Domain of words; Algorithm

## 1. Introduction

By using the notion of a fuzzy metric space in the sense of Kramosil and Michalek [12], Grabiec proved in [8] a celebrated fuzzy version of the Banach contraction principle. Although Grabiec's fixed point theorem has the disadvantage that it cannot be applied to the fuzzy metric induced by the Euclidean metric on $\mathbb{R}$ (see [6,21]), we here show that, nevertheless, it provides an efficient tool to obtain fixed points for (fuzzy) contraction mappings on complete non-Archimedean fuzzy metric spaces, and thus it can be applied, for instance, to the domain of words endowed with the fuzzy metric induced by the Baire metric. (Let us recall that the Baire metric provides a suitable mathematical model in denotational semantics of programming languages [1–3,11,15], etc.)

Actually, we will establish our results in the more general framework of fuzzy quasi-metric spaces because, in this context, the measurement of the distance from a word $x$ to another word $y$, automatically indicates if $x$ is a prefix of $y$ or not, while the Baire metric does not provide this information. Finally, we will apply our methods to prove the existence (and uniqueness) of solution for some recurrence equations associated to the asymptotic complexity analysis of Quicksort algorithms and Divide & Conquer algorithms, respectively.

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Following the modern terminology, by a quasi-metric on a nonempty set $X$ we mean a nonnegative real valued function $d$ on $X \times X$ such that for all $x, y, z \in X$:

(i) $x = y$ if and only if $d(x, y) = d(y, x) = 0$;
(ii) $d(x, z) \leqslant d(x, y) + d(y, z)$.

If $d$ satisfies condition (i) above and

(ii′) $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in X$,

then, $d$ is called a non-Archimedean quasi-metric on $X$.

A (non-Archimedean) quasi-metric space is a pair $(X, d)$ such that $X$ is a nonempty set and $d$ is a (non-Archimedean) quasi-metric on $X$.

Each quasi-metric $d$ on $X$ generates a $T_0$ topology $\tau_d$ on $X$ which has as a base the family of open balls $\{B_d(x, r)\colon x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X\colon d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

A topological space $(X, \tau)$ is said to be (non-Archimedean) quasi-metrizable if there is a (non-Archimedean) quasi-metric $d$ on $X$ such that $\tau = \tau_d$.

Given a (non-Archimedean) quasi-metric $d$ on $X$, then the function $d^{-1}$ defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$, is also a (non-Archimedean) quasi-metric on $X$, called the conjugate of $d$, and the function $d^s$ defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a (non-Archimedean) metric on $X$.

A quasi-metric space $(X, d)$ is said to be bicomplete if $(X, d^s)$ is a complete metric space. In this case, we say that $d$ is a bicomplete quasi-metric on $X$.

Next we recall some pertinent fuzzy concepts and facts.

According to [19], a binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is a continuous $t$-norm if $*$ satisfies the following conditions:

(i) $*$ is associative and commutative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for every $a \in [0, 1]$;
(iv) $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$, with $a, b, c, d \in [0, 1]$.

**Definition 1.** (See [9].) A KM-fuzzy quasi-metric on a nonempty set $X$ is a pair $(M, *)$ such that $*$ is a continuous $t$-norm and $M$ is a fuzzy set in $X \times X \times [0, \infty[$ such that for all $x, y, z \in X$:

(i) $M(x, y, 0) = 0$;
(ii) $x = y$ if and only if $M(x, y, t) = M(y, x, t) = 1$ for all $t > 0$;
(iii) $M(x, z, t + s) \geqslant M(x, y, t) * M(y, z, s)$ for all $t, s \geqslant 0$;
(iv) $M(x, y, \_): [0, \infty[ \to [0, 1]$ is left continuous.

Note that a KM-fuzzy quasi-metric $(M, *)$ satisfying for all $x, y \in X$ and $t > 0$ the symmetry axiom $M(x, y, t) = M(y, x, t)$, is a fuzzy metric in the sense of Kramosil and Michalek [12] (see also [6,7], where a slight but interesting modification of this notion of fuzzy metric space is discussed).

In the following, KM-fuzzy quasi-metrics will we simply called fuzzy quasi-metrics. In particular, by a fuzzy metric we will mean a fuzzy quasi-metric satisfying the symmetry axiom; thus, fuzzy metrics are taken in the sense [12].

A triple $(X, M, *)$ where $X$ is a nonempty set and $(M, *)$ is a fuzzy (quasi-)metric on $X$, is said to be a fuzzy (quasi-)metric space.

It was shown in Proposition 1 of [9] that if $(M, *)$ is a fuzzy quasi-metric on $X$, then for each $x, y \in X$, $M(x, y, \_)$ is nondecreasing, i.e., $M(x, y, t) \leqslant M(x, y, s)$ whenever $t \leqslant s$.

If $(M, *)$ is a fuzzy quasi-metric on $X$, then $(M^{-1}, *)$ is also a fuzzy quasi-metric on $X$, where $M^{-1}$ is the fuzzy set in $X \times X \times [0, \infty)$ defined by $M^{-1}(x, y, t) = M(y, x, t)$. Moreover, if we denote by $M^i$ the fuzzy set in $X \times X \times [0, \infty)$ given by $M^i(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}$, then $(M^i, *)$ is a fuzzy metric on $X$ [9].

Similarly to the fuzzy metric case (compare [9]), each fuzzy quasi-metric $(M,*)$ on $X$ generates a $T_0$ topology $\tau_M$ on $X$ which has as a base the family of open balls $\{B_M(x,r,t)\colon x\in X,\ 0<r<1,\ t>0\}$, where $B_M(x,r,t)=\{y\in X\colon M(x,y,t)>1-r\}$.

Therefore, a sequence $(x_n)_n$ in $(X,M,*)$ converges to $x\in X$ with respect to $\tau_M$ if and only if $\lim_n M(x,x_n,t)=1$ for all $t>0$.

Now let $(X,d)$ be a quasi-metric space. For $a,b\in[0,1]$ let $a\cdot b$ be the usual multiplication, and let $M_d$ be the function defined on $X\times X\times[0,\infty[$ by $M_d(x,y,0)=0$ and

$$M_d(x,y,t)=\frac{t}{t+d(x,y)}\quad \text{whenever } t>0.$$

It is easily seen [9] that $(M_d,\cdot)$ is a fuzzy quasi-metric on $X$ which will be called the fuzzy quasi-metric induced by $d$. Moreover $\tau_d=\tau_{M_d}$ and $\tau_{d^{-1}}=\tau_{(M_d)^{-1}}$, and hence $\tau_{d^s}=\tau_{(M_d)^i}$ on X. If $d$ is a metric, then $(M_d,\cdot)$ is obviously a fuzzy metric on $X$ (compare [6]).

## 2. The Banach fixed point theorem in fuzzy quasi-metric spaces

In this section we extend Grabiec's fuzzy version of the Banach fixed point theorem to fuzzy quasi-metric spaces.

According to [18], a B-contraction on a fuzzy metric space $(X,M,*)$ is a self-mapping $f$ on $X$ such that there is a constant $k\in\,]0,1[$ satisfying

$$M\big(f(x),f(y),kt\big)\geqslant M(x,y,t)$$

for all $x,y\in X$, $t>0$.

In [8], M. Grabiec introduced the following notions in order to obtain a fuzzy version of the classical Banach fixed point theorem: A sequence $(x_n)_n$ in a fuzzy metric space $(X,M,*)$ is Cauchy provided that $\lim_{n\to\infty}M(x_n,x_{n+p},t)=1$ for each $t>0$ and $p\in\mathbb{N}$. A fuzzy metric space $(X,M,*)$ is complete provided that every Cauchy sequence in $X$ is convergent. In this case, $(M,*)$ is called a complete fuzzy metric on $X$.

In the sequel, and according to [10,21], a Cauchy sequence in Grabiec's sense will be called G-Cauchy and a complete fuzzy metric space in Grabiec's sense will be called G-complete.

**Theorem 1.** *(See [8].) Let $(X,M,*)$ be a G-complete fuzzy metric space such that $\lim_{t\to\infty}M(x,y,t)=1$ for all $x,y\in X$. Then every B-contraction on $X$ has a unique fixed point.*

Generalizing in a natural way the notions of B-contraction and G-completeness to fuzzy quasi-metric spaces we introduce the following concepts.

**Definition 2.** A B-contraction on a fuzzy quasi-metric space $(X,M,*)$ is a self-mapping $f$ on $X$ such that there is a constant $k\in\,]0,1[$ satisfying

$$M\big(f(x),f(y),kt\big)\geqslant M(x,y,t)$$

for all $x,y\in X$, $t>0$. The number $k$ is then called a contraction constant of $f$.

**Definition 3.** A sequence $(x_n)_n$ in a fuzzy quasi-metric space $(X,M,*)$ is called G-Cauchy if it is a G-Cauchy sequence in the fuzzy metric space $(X,M^i,*)$.

**Definition 4.** A fuzzy quasi-metric space $(X,M,*)$ is called G-bicomplete if the fuzzy metric space $(X,M^i,*)$ is G-complete. In this case, we say that $(M,*)$ is a G-bicomplete fuzzy quasi-metric on $X$.

Then, Grabiec's theorem can be extended to fuzzy quasi-metric spaces as follows.

**Theorem 2.** *Let $(X,M,*)$ be a G-bicomplete fuzzy quasi-metric space such that $\lim_{t\to\infty}M(x,y,t)=1$ for all $x,y\in X$. Then every B-contraction on $X$ has a unique fixed point.*

**Proof.** Let $f : X \to X$ be a B-contraction on $X$ with contraction constant $k \in ]0, 1[$. Then

$$M\big(f(x), f(y), kt\big) \geqslant M(x, y, t)$$

for all $x, y \in X$, $t > 0$. It immediately follows that

$$M^i\big(f(x), f(y), kt\big) \geqslant M^i(x, y, t)$$

for all $x, y \in X$, $t > 0$. Hence $f$ is a B-contraction on the G-complete fuzzy metric space $(X, M^i, *)$ and, by Theorem 1, $f$ has a unique fixed point. □

## 3. G-bicompleteness in non-Archimedean fuzzy quasi-metric spaces

As we indicated in Section 1, G-completeness has the disadvantage that the fuzzy metric induced by the Euclidean metric on $\mathbb{R}$ is not G-complete. This fact motivates the following well-known alternative notion of fuzzy (quasi-)metric completeness.

A sequence $(x_n)_n$ in a fuzzy metric space $(X, M, *)$ is a Cauchy sequence provided that for each $\varepsilon \in ]0, 1[$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geqslant n_0$. The fuzzy metric space $(X, M, *)$ is complete provided that every Cauchy sequence in $X$ is convergent [6]. In this case, we say that $(M, *)$ is a complete fuzzy metric on $X$.

A fuzzy quasi-metric space $(X, M, *)$ is called bicomplete [9], if $(X, M^i, *)$ is a complete fuzzy metric space. In this case, we say that $(M, *)$ is a bicomplete fuzzy quasi-metric on $X$.

It is obvious that each G-(bi)complete fuzzy (quasi-)metric space is (bi)complete, but the fuzzy metric induced by the Euclidean metric on $\mathbb{R}$ provides an example of a complete non G-complete fuzzy metric [21].

Next we shall show that each (bi)complete non-Archimedean fuzzy (quasi-)metric space is G-(bi)complete. Thus, we obtain an interesting class of spaces for which Theorem 2 applies.

The notion of a non-Archimedean fuzzy metric space was introduced by Sapena [16]. We give a natural generalization of this concept to the quasi-metric setting.

**Definition 5.** A fuzzy quasi-metric space $(X, M, *)$ such that $M(x, y, t) \geqslant \min\{M(x, z, t), M(z, y, t)\}$ for all $x, y, z, \in X$, $t > 0$, is called a non-Archimedean fuzzy quasi-metric space, and $(M, *)$ is called a non-Archimedean fuzzy quasi-metric.

**Example 1.** Let $(X, d)$ be a quasi-metric space. It is immediate to show that $(X, d)$ is a non-Archimedean quasi-metric space if and only if $(X, M_d, \cdot)$ is a non-Archimedean fuzzy quasi-metric space.

**Lemma 1.** *Each G-Cauchy sequence in a non-Archimedean fuzzy quasi-metric space is a Cauchy sequence.*

**Proof.** Let $(x_n)_n$ be a G-Cauchy sequence in the non-Archimedean fuzzy quasi-metric space $(X, M, *)$. Fix $\varepsilon \in ]0, 1[$ and $t > 0$. Since $\lim_{n\to\infty} M^i(x_n, x_{n+1}, t) = 1$, there is $n_0 \in \mathbb{N}$ such that $M^i(x_n, x_{n+1}, t) > 1 - \varepsilon$ for all $n \geqslant n_0$.

Now let $n, m \geqslant n_0$ with $m > n$. Then $m = n + j$, for some $j \in \mathbb{N}$. So

$$M^i(x_n, x_m, t) \geqslant \min\big\{M^i(x_n, x_{n+1}, t), \ldots, M^i(x_{n+j-1}, x_{n+j}, t)\big\} > 1 - \varepsilon.$$

We conclude that $(x_n)_n$ is a Cauchy sequence in $(X, M, *)$. □

**Theorem 3.** *Each bicomplete non-Archimedean fuzzy quasi-metric space is G-bicomplete.*

**Proof.** Let $(x_n)_n$ be a G-Cauchy sequence in the bicomplete non-Archimedean fuzzy quasi-metric space $(X, M, *)$. By Lemma 1, $(x_n)_n$ is a Cauchy sequence in $(X, M, *)$. Hence, there is $x \in X$ such that $\lim_{n\to\infty} M^i(x, x_n, t) = 1$ for all $t > 0$. We conclude that $(X, M^i, *)$ is G-complete, i.e., $(X, M, *)$ is G-bicomplete. □

**Corollary.** *Each complete non-Archimedean fuzzy metric space is G-complete.*

The following result, whose easy proof is omitted, permits us to construct in an easy way a non-Archimedean fuzzy quasi-metric from a bounded non-Archimedean quasi-metric $d$, which is different from the fuzzy quasi-metric induced by $d$ as defined in Section 1. It will be consider in the next section.

**Proposition 1.** *Let $d$ be a non-Archimedean quasi-metric on a set $X$ such that $d(x,y) \leqslant 1$ for all $x, y \in X$. Let*

$$M_{d1}(x,y,0) = 0 \quad \text{for all } x, y \in X,$$
$$M_{d1}(x,y,t) = 1 - d(x,y) \quad \text{for all } x, y \in X \text{ and } t \in ]0,1],$$
$$M_{d1}(x,y,t) = 1 \quad \text{for all } x, y \in X \text{ and } t > 1.$$

*Then the following statements hold*:

(1) *$(M_{d1}, \wedge)$ is a non-Archimedean fuzzy quasi-metric on $X$, where by $\wedge$ we denote the continuous t-norm given by $a \wedge b = \min\{a,b\}$, for all $a, b \in [0,1]$.*
(2) *For each $x, y \in X$, $t \in ]0,1]$ and $\varepsilon \in ]0,1[$:*

$$M_{d1}(x,y,t) > 1 - \varepsilon \quad \Leftrightarrow \quad d(x,y) < \varepsilon.$$

(3) *$\tau_{M_{d1}} = \tau_d$ and $\tau_{(M_{d1})^{-1}} = \tau_{d^{-1}}$.*
(4) *A sequence in $X$ is Cauchy in $(X, (M_{d1})^i, \wedge)$ if and only if it is Cauchy in $(X, d^s)$.*
(5) *$(X, M_{d1}, \wedge)$ is G-bicomplete if and only if $(X, d)$ is bicomplete.*

## 4. Applications to the domain of words

Let $\Sigma$ be a nonempty alphabet. Let $\Sigma^\infty$ be the set of all finite and infinite sequences ("words") over $\Sigma$, where we adopt the convention that the empty sequence $\phi$ is an element of $\Sigma^\infty$. Denote by $\sqsubseteq$ the prefix order on $\Sigma^\infty$, i.e., $x \sqsubseteq y \Leftrightarrow x$ is a prefix of $y$.

Now, for each $x \in \Sigma^\infty$ denote by $\ell(x)$ the length of $x$. Then $\ell(x) \in [1, \infty]$ whenever $x \neq \phi$ and $\ell(\phi) = 0$. For each $x, y \in \Sigma^\infty$ let $x \sqcap y$ be the common prefix of $x$ and $y$.

Thus, the function $d_\sqsubseteq$ defined on $\Sigma^\infty \times \Sigma^\infty$ by

$$d_\sqsubseteq(x,y) = 0 \quad \text{if } x \sqsubseteq y,$$
$$d_\sqsubseteq(x,y) = 2^{-\ell(x \sqcap y)} \quad \text{otherwise},$$

is a quasi-metric on $\Sigma^\infty$. (We adopt the convention that $2^{-\infty} = 0$.)

Actually $d_\sqsubseteq$ is a non-Archimedean quasi-metric on $\Sigma^\infty$ (see, for instance, [14, Example 8(b)]).

We also observe that the non-Archimedean metric $(d_\sqsubseteq)^s$ is the Baire metric on $\Sigma^\infty$, i.e.,

$$(d_\sqsubseteq)^s(x,x) = 0$$

and

$$(d_\sqsubseteq)^s(x,y) = 2^{-\ell(x \sqcap y)}$$

for all $x, y \in \Sigma^\infty$ such that $x \neq y$.

It is well known that $(d_\sqsubseteq)^s$ is complete. From this fact it clearly follows that $d_\sqsubseteq$ is bicomplete.

The quasi-metric $d_\sqsubseteq$, which was introduced by Smyth [20], will be called the Baire quasi-metric. Observe that condition $d_\sqsubseteq(x,y) = 0$ can be used to distinguish between the case that $x$ is a prefix of $y$ and the remaining cases.

From Example 1 and Theorem 3, we obtain the following.

**Proposition 2.** *$(\Sigma^\infty, M_{d_\sqsubseteq}, \cdot)$ is a G-bicomplete non-Archimedean fuzzy quasi-metric space.*

Consequently, Theorem 2 can be applied to this useful space. Next we construct other examples of bicomplete non-Archimedean fuzzy quasi-metrics on $\Sigma^\infty$ that are related to the Baire quasi-metric defined above.

**Example 2.** Let $d$ be a (non-Archimedean) quasi-metric on a set $X$ and let $M_{d\exp}$ be the fuzzy set in $X \times X \times [0, \infty[$ given by $M_{d\exp}(x, y, 0) = 0$ and

$$M_{d\exp}(x, y, t) = \exp\left(-\frac{d(x, y)}{t}\right)$$

for all $x, y \in X$ and $t > 0$. Then $(M_{d\exp}, \wedge)$ is a (non-Archimedean) fuzzy quasi-metric on $X$ (compare [6] for the fuzzy metric case). Furthermore, it is easy to see that

$$M_{d\exp}(x, y, t) \leqslant M_d(x, y, t),$$

for all $x, y \in X$ and $t \geqslant 0$. It is also immediate to show that $\tau_{M_{d\exp}} = \tau_d$ and that $(X, M_{d\exp}, \wedge)$ is bicomplete if and only if $(X, d)$ is bicomplete.

From Example 2 and Theorem 3 we obtain the following.

**Proposition 3.** *$(\Sigma^\infty, M_{d_\sqsubseteq \exp}, \wedge)$ is a G-bicomplete non-Archimedean fuzzy quasi-metric space.*

On the other hand, since $d_\sqsubseteq(x, y) \leqslant 1$ for all $x, y \in \Sigma^\infty$, we deduce from Proposition 1 the following.

**Proposition 4.** *$(\Sigma^\infty, M_{d_\sqsubseteq 1}, \wedge)$ is a G-bicomplete non-Archimedean fuzzy quasi-metric space.*

Note (see Proposition 1) that the fuzzy non-Archimedean quasi-metric $(M_{d_\sqsubseteq 1}, \wedge)$ is given by

$$\begin{aligned}
&M_{d_\sqsubseteq 1}(x, y, 0) = 0 \quad \text{for all } x, y \in \Sigma^\infty,\\
&M_{d_\sqsubseteq 1}(x, y, t) = 1 \quad \text{if } x \text{ is a prefix of } y, \text{ and } t > 0,\\
&M_{d_\sqsubseteq 1}(x, y, t) = 1 - 2^{-\ell(x \sqcap y)} \quad \text{if } x \text{ is not a prefix of } y, \text{ and } t \in ]0, 1],\\
&M_{d_\sqsubseteq 1}(x, y, t) = 1 \quad \text{if } x \text{ is not a prefix of } y, \text{ and } t > 1.
\end{aligned}$$

Next we apply any of Proposition 2 and Theorem 2 to the complexity analysis of Quicksort algorithms. The average case analysis of Quicksort is discussed in [13] (see also [5]), where the following recurrence equation is obtained:

$$T(1) = 0, \quad \text{and} \quad T(n) = \frac{2(n-1)}{n} + \frac{n+1}{n} T(n-1), \quad n \geqslant 2.$$

Consider as an alphabet $\Sigma$ the set of nonnegative real numbers, i.e., $\Sigma = [0, \infty)$. We associate to $T$ the functional $\Phi : \Sigma^\infty \to \Sigma^\infty$ given by $(\Phi(x))_1 = T(1)$ and

$$\big(\Phi(x)\big)_n = \frac{2(n-1)}{n} + \frac{n+1}{n} x_{n-1}$$

for all $n \geqslant 2$ (if $x \in \Sigma^\infty$ has length $n < \infty$, we write $x := x_1 x_2 \ldots x_n$, and if $x$ is an infinite word we write $x := x_1 x_2 \ldots$).

Next we show that $\Phi$ is a B-contraction on the G-bicomplete non-Archimedean fuzzy quasi-metric space $(\Sigma^\infty, M_{d_\sqsubseteq}, \cdot)$, with contraction constant $1/2$.

To this end, we first note that, by construction, we have $\ell(\Phi(x)) = \ell(x) + 1$ for all $x \in \Sigma^\infty$ (in particular, $\ell(\Phi(x)) = \infty$ whenever $\ell(x) = \infty$).

Furthermore, it is clear that

$$x \sqsubseteq y \iff \Phi(x) \sqsubseteq \Phi(y),$$

and consequently

$$\Phi(x \sqcap y) \sqsubseteq \Phi(x) \sqcap \Phi(y)$$

for all $x, y \in \Sigma^\infty$. Hence

$$\ell\big(\Phi(x \sqcap y)\big) \leqslant \ell\big(\Phi(x) \sqcap \Phi(y)\big)$$

for all $x, y \in \Sigma^\infty$.

From the preceding observations we deduce that if $x$ is a prefix of $y$, then

$$M_{d_{\sqsubseteq}}\big(\Phi(x), \Phi(y), t/2\big) = M_{d_{\sqsubseteq}}(x, y, t) = 1,$$

and if $x$ is not a prefix of $y$, then

$$\begin{aligned} M_{d_{\sqsubseteq}}\big(\Phi(x), \Phi(y), t/2\big) &= \frac{t/2}{(t/2) + 2^{-\ell(\Phi(x)\sqcap\Phi(y))}} \\ &\geqslant \frac{t/2}{(t/2) + 2^{-\ell(\Phi(x\sqcap y))}} = \frac{t/2}{(t/2) + 2^{-(\ell(x\sqcap y)+1)}} \\ &= \frac{t}{t + 2^{-\ell(x\sqcap y)}} = M_{d_{\sqsubseteq}}(x, y, t) \end{aligned}$$

for all $t > 0$.

Therefore $\Phi$ is a B-contraction on $(\Sigma^\infty, M_{d_{\sqsubseteq}}, \cdot)$ with contraction constant $1/2$. So, by Theorem 2, $\Phi$ has a unique fixed point $z = z_1 z_2 \ldots$, which is obviously the unique solution to the recurrence equation $T$, i.e., $z_1 = 0$ and

$$z_n = \frac{2(n-1)}{n} + \frac{n+1}{n} z_{n-1}$$

for all $n \geqslant 2$.

**Remark.** Note that in the above procedure we can use $(\Sigma^\infty, M_{d_{\sqsubseteq \exp}}, \wedge)$ instead of $(\Sigma^\infty, M_{d_{\sqsubseteq}}, \cdot)$, to deduce the existence of a unique solution for the recurrence $T$.

We conclude the paper by applying our results to the complexity analysis of Divide & Conquer algorithms. Recall [4,17] that Divide & Conquer algorithms solve a problem by recursively splitting it into subproblems each of which is solved separately by the same algorithm, after which the results are combined into a solution of the original problem. Thus, the complexity of a Divide & Conquer algorithm typically is the solution to the recurrence equation given by

$$T(1) = c, \quad \text{and} \quad T(n) = aT\left(\frac{n}{b}\right) + h(n),$$

where $a, b, c \in \mathbb{N}$ with $a, b \geqslant 2$, $n$ range over the set $\{b^p\colon\ p = 0, 1, 2, \ldots\}$, and $h(n) \geqslant 0$ for all $n \in \mathbb{N}$.

As in the case of Quicksort algorithm, take $\Sigma = [0, \infty)$ and put

$$\Sigma^{\mathbb{N}} := \big\{x \in \Sigma^\infty\colon\ \ell(x) = \infty\big\}.$$

Clearly $\Sigma^{\mathbb{N}}$ is a closed subset of $(\Sigma^\infty, (M_{d_{\sqsubseteq}})^i, \cdot)$, so $(\Sigma^{\mathbb{N}}, M_{d_{\sqsubseteq}}, \cdot)$ is a non-Archimedean G-bicomplete fuzzy quasi-metric space by Proposition 2.

Now we associate to $T$ the functional $\Phi : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ given by $(\Phi(x))_1 = T(1)$, and

$$\begin{aligned} &\big(\Phi(x)\big)_n = a x_{n/b} + h(n) \quad &&\text{if } n \in \{b^p\colon\ p = 1, 2, \ldots\}, \quad \text{and} \\ &\big(\Phi(x)\big)_n = 0 &&\text{otherwise} \end{aligned}$$

for all $x \in \Sigma^{\mathbb{N}}$.

For our purposes here it suffices to observe that for each $x, y \in \Sigma^{\mathbb{N}}$, the following inequality holds

$$\ell\big(\Phi(x) \sqcap \Phi(y)\big) \geqslant 1 + \ell(x \sqcap y).$$

In fact, if $\ell(x \sqcap y) = 0$, then $\ell(\Phi(x) \sqcap \Phi(y)) \geqslant 1$; and if $b^p > \ell(x \sqcap y) \geqslant b^{p-1}$, $p \geqslant 1$, then $b^{p+1} > \ell(\Phi(x) \sqcap \Phi(y)) \geqslant b^p$.

Hence, for each $x, y \in \Sigma^{\mathbb{N}}$ and $t > 0$, we obtain

$$\begin{aligned} M_{d_{\sqsubseteq}}\big(\Phi(x), \Phi(y), t/2\big) &= \frac{t/2}{(t/2) + 2^{-\ell(\Phi(x)\sqcap\Phi(y))}} \\ &\geqslant \frac{t/2}{(t/2) + 2^{-(\ell(x\sqcap y)+1)}} = \frac{t}{t + 2^{-\ell(x\sqcap y)}} \\ &= M_{d_{\sqsubseteq}}(x, y, t). \end{aligned}$$

Therefore $\Phi$ is a B-contraction on $(\Sigma^{\mathbb{N}}, M_{d_{\sqsubseteq}}, \cdot)$ with contraction constant $1/2$. So, by Theorem 2, $\Phi$ has a unique fixed point $z = z_1 z_2 \ldots$.

Consequently, the function $F$ defined on $\{b^p\colon\ p = 0, 1, 2 \ldots\}$ by $F(b^p) = z_{b^p}$ for all $p \geqslant 0$, is the unique solution to the recurrence equation of the given Divide & Conquer algorithm.

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