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Lifting Induction Theorems

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We show that—in some suitable sense—any induction theorem for the character ring of a finite group can be lifted to an induction theorem for the Green ring. (A precise statement see Theorem A herein.) This provides a uniform proof of important induction theorems, as, for example, the ones of Conlon and Dress. Moreover, we prove an analogue of Brauer's induction theorem for the Green ring (cf. Theorem B herein). © 1998 Academic Press

1. INTRODUCTION

Let us fix a complete discrete valuation ring $\mathscr{O}$ of characteristic 0 with quotient field $\mathbb{K}$ and residue field $\mathbb{F}$ of positive characteristic p. Unless otherwise stated, all our modules will be free of finite rank over $\mathbb{K}$, $\mathscr{O}$, or $\mathbb{F}$, respectively. Moreover, let G be a finite group, and assume $\mathbb{K}$ and $\mathbb{F}$ to be splitting fields for G and all of its subgroups (and hence for all of its subquotients). (For instance, assume that $\mathbb{K}$ contains a primitive $|G|$th root of unity.)

We denote by $\mathrm{R}(G)$ the ring of ($\mathbb{K}$-valued) characters of G, and let $\mathrm{a}(\mathscr{O}G)$ and $\mathrm{a}(\mathbb{F}G)$ be the Green ring of $\mathscr{O}G$ and $\mathbb{F}G$, respectively; i.e., the Grothendieck group of the abelian monoid the elements of which are isomorphism classes $[M]$ of $\mathscr{O}G$-modules ($\mathbb{F}G$-modules) M, with addition defined by $[M] + [L] := [M \oplus L]$ (which is even a ring, where multiplication is given by the tensor product). Then, by the Krull–Schmidt theorem, the isomorphism classes of indecomposable $\mathscr{O}G$-modules ($\mathbb{F}G$-modules) form a $\mathbb{Z}$-basis of $\mathrm{a}(\mathscr{O}G)$ ($\mathrm{a}(\mathbb{F}G)$).

For a subgroup H of G, a character χ of H and an $\mathscr{O}H$-module ($\mathbb{F}H$-module) M, $\mathrm{Ind}_H^G \chi$ and $\mathrm{Ind}_H^G M$ denote the induced character and module, respectively. This yields additive maps $\mathrm{ind}_H^G\colon \mathrm{R}(H) \to \mathrm{R}(G)$, $\chi \mapsto$

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$\mathrm{Ind}_H^G \chi$, and ind_H^G: $\mathrm{a}(\mathscr{O}H) \to \mathrm{a}(\mathscr{O}G)$, $[M] \mapsto [\mathrm{Ind}_H^G M]$, respectively (similarly for the Green ring over $\mathbb{F}$).

Let k be a commutative ring with 1. For simplicity, we write $k\mathrm{R}(G)$, $k\mathrm{a}(\mathscr{O}G)$, and $k\mathrm{a}(\mathbb{F}G)$ instead of $k \otimes_{\mathbb{Z}} \mathrm{R}(G)$, $k \otimes_{\mathbb{Z}} \mathrm{a}(\mathscr{O}G)$, and $k \otimes_{\mathbb{Z}} \mathrm{a}(\mathbb{F}G)$, respectively. Moreover, for $c \in k$, a character χ of G and an $\mathscr{O}G$-module ($\mathbb{F}G$-module) M, $c\chi$ and $c[M]$ are abbreviations of $c \otimes \chi \in k\mathrm{R}(G)$ and $c \otimes [M] \in k\mathrm{a}(\mathscr{O}G)$ ($k\mathrm{a}(\mathbb{F}G)$), respectively. For simplicity, ind_H^G (H a subgroup of G) denotes also the induced map $\mathrm{id}_k \otimes \mathrm{ind}_H^G$: $k\mathrm{R}(H) \to k\mathrm{R}(G)$, and analogously for the Green ring.

The main objective of this paper is to prove the following theorem:

THEOREM A. *Let G be a finite group, let k be a commutative ring with* 1, *and let M be an indecomposable $\mathscr{O}G$-module* ($\mathbb{F}G$*-module*) *with vertex P and P-source N. Moreover, for any subgroup Q of P, let $\mathscr{C}_Q$ be a set of subgroups of* $\mathrm{N}_G(Q)/Q$ *such that*

$$1 \in \sum_{H/Q \in \mathscr{C}_Q} \mathrm{ind}_{H/Q}^{\mathrm{N}_G(Q)/Q} k\mathrm{R}(H/Q).$$

From M, P, and the sets $\mathscr{C}_Q$ construct the set $\mathscr{C}$ of pairs (H, U), where

1. *H is a subgroup of G with normal p-subgroup Q, which is contained in P, such that $H/Q \in \mathscr{C}_Q$.*

2. *U is an indecomposable $\mathscr{O}H$-module* ($\mathbb{F}H$*-module*) *such that, for some $g \in G$, some vertex R of U is contained in $Q \cap {}^gP$, and some R-source of U is a direct summand of the restricted $\mathscr{O}R$-module* ($\mathbb{F}R$*-module*) $\mathrm{Res}_R^{^gP}\, {}^gN$.

Then there are $a_{(H,U)} \in k$ ($(H,U) \in \mathscr{C}$) such that

$$[M] = \sum_{(H,U) \in \mathscr{C}} a_{(H,U)} \,\mathrm{ind}_H^G [U]$$

in $k\mathrm{a}(\mathscr{O}G)$ ($k\mathrm{a}(\mathbb{F}G)$).

Remark. 1. Choose $k := \mathbb{Q}$, and, for each p-subgroup Q of G, let $\mathscr{C}_Q$ be the set of cyclic subgroups of $\mathrm{N}_G(Q)/Q$. By Artin's induction theorem [5, (15.4)], the sets $\mathscr{C}_Q$ satisfy the hypotheses of Theorem A. Hence, using Theorem A, Artin's induction theorem implies Conlon's induction theorem [6, (80.51)].

2. Now take $k := \mathbb{Z}$, and, for a p-subgroup Q of G, denote by $\mathscr{C}_Q$ the set of elementary subgroups of $\mathrm{N}_G(Q)/Q$. Then, using Theorem A, Brauer's induction theorem [5, (15.8)] implies Dress's induction [2, Theorem 5.6.11].

3. We will show that, for the Green ring $\mathrm{a}(\mathbb{F}G)$, Theorem A remains valid if we replace $\mathrm{R}(\mathrm{N}_G(Q)/Q)$ by the Brauer character ring $\mathrm{R}_{p'}(\mathrm{N}_G(Q)/Q)$ of $\mathrm{N}_G(Q)/Q$ for all Q.

4. Theorem A is also valid for a complete discrete valuation ring (of characteristic 0) with arbitrary quotient field $\mathbb{K}$ and arbitrary residue field $\mathbb{F}$ of characteristic p, as long as we replace the character ring $\mathrm{R}(\mathrm{N}_G(Q)/Q)$ by the Grothendieck group $\mathrm{G}_0(\mathbb{K}[\mathrm{N}_G(Q)/Q])$ of the group algebra $\mathbb{K}[\mathrm{N}_G(Q)/Q]$ for all Q. Hence Theorem A can also be applied to the Bermann–Witt induction theorem [2, Theorem 5.6.7].

Of course, the most important applications of Theorem A are the classical induction theorems for the Green ring mentioned in the above remark. For a quite recent account cf. [12, Proposition 3.5], which, as we will do, uses a G-algebra approach. A proof of the well-known fact that the minimal family of subgroups needed to write an indecomposable module as a linear combination of modules induced from these subgroups depends on a vertex of that module is also contained in [12 (cf. Proposition 9.2)]. (It might be new that these induction theorems can be improved if one takes also the source of an indecomposable module into account.) However, we emphasize that Theorem A shows that *any* induction theorem for the character ring of $\mathrm{N}_G(Q)/Q$ for each p-subgroup Q of G "lifts" to an induction theorem for the Green ring of G. Moreover, our proof of Theorem A will demonstrate that this "lifting process" is rather explicit. In fact, the only thing which is not explicit in our proof of Theorem A is that it relies on realizations of all the characters involved; so if we replace throughout character rings by the corresponding Grothendieck groups (the proof of) Theorem A really provides an explicit "lifting method."

Brauer's induction theorem asserts that any character of G can be written as an integral linear combination of monomial characters, i.e., characters which are induced from linear characters. (This is a corollary of what we mean by Brauer's induction theorem in the above remark.) Assume, in addition, that $\mathbb{F}$ is algebraically closed. We offer the following modular analogue of this famous result for the Green ring:

THEOREM B. *Let G be a finite group, and let M be an indecomposable $\mathcal{O}G$-module ($\mathbb{F}G$-module) with vertex P and P-source L. Moreover, denote by $\mathcal{C}$ the set of pairs (H, U) where*

1. *H is a subgroup of G with normal Sylow p-subgroup Q, which is contained in P, such that H/Q is elementary;*

2. *U is an indecomposable $\mathcal{O}H$-module ($\mathbb{F}H$-module) with vertex Q and the restricted $\mathcal{O}Q$-module ($\mathbb{F}Q$-module) $\mathrm{Res}_Q^H U$ is an indecomposable direct summand of $\mathrm{Res}_Q^P L$. In particular, $\mathrm{Res}_Q^H U$ is a Q-source of U.*

Then there are integers $a_{(H,U)}$ $((H,U) \in \mathscr{C})$ *such that*

$$[M] = \sum_{(H,U)\in\mathscr{C}} a_{(H,U)} \operatorname{ind}_H^G[U]$$

in $k\mathrm{a}(\mathscr{O}G)$ $(k\mathrm{a}(\mathbb{F}G))$.

Remark. Note that Theorem B can be regarded as a generalization of Brauer's induction theorem. To see this, choose $\mathscr{O}$ in such a way that p does not divide $|G|$. Then all the $\mathscr{O}H$-modules U above have rank 1. Maybe a better way to see that Theorem B is an analogue of Brauer's induction theorem is to say that the defect multiplicity modules of all the $\mathscr{O}H$-modules U of 2 are one-dimensional. (See [13] for details.)

Moreover, Theorem B generalizes the well-known fact that trivial source modules are virtually monomial, i.e., any trivial source module, regarded as an element of $\mathrm{a}(\mathscr{O}G)$, can be written as an integral linear combination of elements which correspond to monomial modules (i.e., modules which are induced from modules of rank 1). This is a theorem of Dress [7]. Boltje [4] shows that this can be done explicitly and, in his sense, in a canonical way.

Note that these specific modules U of Theorem B(2) also play an important role in Puig's generalization of Brauer's second main theorem for the Green ring (cf. [12, Theorem 8.4 and Section 9]).

In case of an algebraically closed residue field $\mathbb{F}$, Theorem A can be slightly improved.

THEOREM C. *We adopt the notation of Theorem* A. *Assume, in addition, that* $\mathbb{F}$ *is algebraically closed, and that, for any subgroup* Q *of* P, *all the elements of* $\mathscr{C}_Q$ *are p-solvable subgroups of* $\mathrm{N}_G(Q)/Q$. *Denote by* $\mathscr{D}$ *the set of pairs* (H,U) *where*

1. *H is a subgroup of G such that* $\mathrm{O}_p(H)$ *is contained in* P, *and such that* $H/\mathrm{O}_p(H)$ *is a Hall* p'*-subgroup of an element of* $\mathscr{C}_{\mathrm{O}_p(H)}$;
2. *U is an indecomposable $\mathscr{O}H$-module* ($\mathbb{F}H$*-module*) *such that, for some* $g \in G$, *some vertex R of U is contained in* $\mathrm{O}_p(H) \cap {}^gP$, *and some R-source of U is a direct summand of* $\mathrm{Res}_R^{{}^gP}\,{}^gN$.

Then there are $a_{(H,U)} \in k$ $((H,U) \in \mathscr{D})$ *such that*

$$[M] = \sum_{(H,U)\in\mathscr{D}} a_{(H,U)} \operatorname{ind}_H^G[U]$$

in $k\mathrm{a}(\mathscr{O}G)$ $(k\mathrm{a}(\mathbb{F}G))$.

As a by-product, our methods provide the following G-algebra implication. Before we state the result, let us recall that, for a G-algebra A, A_1^G denotes the image of the trace map $\mathrm{Tr}_1^G\colon A \to A^G$, $a \mapsto \sum_{g\in G} {}^ga$.

PROPOSITION D. *Let G be a finite group, let A be a G-algebra over $\mathscr{O}$, and let e, f be idempotents in A_1^G. Then e and f are conjugate by a unit in A^G, if and only if, for each cyclic p'-subgroup C of G, e and f are conjugate by a unit in A^C.*

The most important tool for our proof of Theorem A will be the Grothendieck group of projective modules of a certain skew group algebra. The following section will provide the required machinery. It might be convenient to start reading in Section 3 and skip back whenever necessary.

2. SOME REMARKS ON SKEW GROUP ALGEBRAS

Throughout, we fix a finite group G, and let k be a commutative (ground) ring with 1. Unless otherwise stated, all our modules are unitary left modules, and all algebras are assumed to be k-algebras. For an algebra A, A-Mod denotes the category of A-modules.

Let us recall the definition of the skew group algebra. Let A be a G-algebra, i.e., a k-algebra on which G acts via algebra automorphisms. As k-module, the skew group algebra $A * G$ is isomorphic to $A \otimes_k kG$. The image of an element $a \otimes g$ $(a \in A,\ g \in G)$ under this isomorphism is denoted by $a * g$. Multiplication is defined as follows. For $a, b \in A$ and $x, y \in G$, define $(a * x)(b * y) := a({}^x b) * xy$, and extend bilinearly. Then $A * G$ becomes an associative algebra.

For $a \in A$, we identify a with $a * 1 \in A * G$, so we can regard A as a unitary subalgebra of $A * G$. More generally, for a subgroup H of G, $A * H$ may be regarded as a unitary subalgebra of $A * G$ in the canonical way. This yields a restriction functor Res_H^G: $A * G$-Mod $\to A * H$-Mod with left adjoint induction functor Ind_H^G: $A * H$-Mod $\to A * G$-Mod. Moreover, for $g \in G$, we obtain a conjugation functor ${}^g\_$: $A * H$-Mod $\to$ $A * {}^gH$-Mod as restriction along the isomorphism $A * {}^gH \to A * H$, $a * ghg^{-1} \mapsto {}^{g^{-1}}a * h$. (Where ${}^gH$ denotes the conjugate subgroup gHg^{-1}.) It is straightforward to check that we have Mackey decomposition:

2.1. LEMMA. *Let H and K be subgroups of G, and let M be an $A * H$-module. Then there is an isomorphism*

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G M \cong \bigoplus_{HgK \in K\backslash G/H} \mathrm{Ind}_{K \cap {}^gH}^K \mathrm{Res}_{K \cap {}^gH}^{{}^gH} {}^gM$$

of $A * K$-modules.

Also the group algebra kG may be regarded as a unitary subalgebra of $A * G$. However, for $g \in G$, we do not write g instead of $1_A * g \in A * G$, to avoid confusion in case A is interior. (Although this reason is not apparent in this work.)

Let M and N be $A * G$-modules. In particular, M and N may be regarded as kG-modules. Hence the k-module $\mathrm{Hom}_k(M, N)$ of k-homomorphisms $M \to N$ is a kG-module, as well, where

$$({}^g\varphi)(m) := 1_A * g\varphi\big(1_A * g^{-1}\, m\big)$$

for $g \in G$, $\varphi \in \mathrm{Hom}_k(M, N)$, and $m \in M$. Then we have

$$a * g\varphi(m) = ({}^g\varphi)(a * g\, m) \tag{1}$$

for all $a \in A$, $g \in G$, $\varphi \in \mathrm{Hom}_A(M, N)$, and $m \in M$, as one easily verifies. In particular, $\mathrm{Hom}_A(M, N)$ is a kG-submodule of $\mathrm{Hom}_k(M, N)$. Thus the algebra $\mathrm{End}_A(M)$ of A-endomorphisms of M is a G-algebra, and, for a subgroup H of G, we have $\mathrm{End}_A(M)^H = \mathrm{End}_{A*H}(M)$. (Where, for a G-algebra B, $B^H := \{b \in B\colon {}^hb = b \text{ for all } h \in H\}$ denotes the subalgebra of fixed points of H on B.)

The purpose of this section is to define certain functors, related to the category of $A * G$-modules, which will be important for our proof of Theorem A. The starting point of these definitions is the following generalization of the notion of a bimodule.

2.2. DEFINITION. Let G be a finite group, and let A and B be G-algebras. Moreover, let M be both a left $A * G$-module and a right B-module, as well (such that the induced k-module structures coincide). Then M is called a *twisted* (A, G)*-*B*-bimodule* if

$$a * g(mb) = (a * g\, m)^g b$$

for all $a \in A$, $g \in G$, $m \in M$, and $b \in B$.

2.3. EXAMPLE. 1. We may regard A as an $A * G$-module when we define $a * gb := a({}^gb)$ for $a, b \in A$ and $g \in G$. But A is also a right A-module, and we have $a * g\,(bc) = a({}^g(bc)) = a({}^gb)({}^gc) = (a * g b)^g c$ for all $a, b, c \in A$ and $g \in G$; so A is a twisted (A, G)-A-bimodule.

2. Let H be a subgroup of G and let B be a G-algebra. Trivially, any twisted (A, G)-B-bimodule may be regarded as twisted (A, H)-B-bimodule.

3. Let B be an algebra which we regard as G-algebra with trivial G-action. Then a twisted (A, G)-B-bimodule is nothing but an $A * G$-B-bimodule. In particular, any $A * G$-module is a twisted (A, G)-k-bimodule.

4. Let B be a G-algebra. Denote by B° the opposite G-algebra, and regard $A \otimes_k B^{\circ}$ as a G-algebra with diagonal G-action. Then any $(A \otimes_k B^{\circ}) * G$-module M is an $A * G$-module via restriction along the canonical

homomorphism $A * G \to (A \otimes_k B^o) * G$. On the other hand, M becomes a B^o-module (and hence a right B-module) via restriction along the canonical homomorphism $B^o \to (A \otimes_k B^o) * G$. Moreover,

$$\begin{aligned} a * g\,(mb) &= ((a \otimes 1_B) * g)\big((1_A \otimes b) * 1\big)m = \big((a \otimes {}^g b) * g\big)m \\ &= \big((1_A \otimes {}^g b) * 1\big)\big((a \otimes 1_B) * g\big)m = (a * g\, m)^g b, \end{aligned}$$

so M is a twisted (A, G)-B-bimodule.

Conversely, let L be a twisted (A, G)-B-bimodule. Then L can be regarded as an $(A \otimes_k B^o) * G$-module when we define $((a \otimes b) * g)m := (a * g\, m)b$ for $a \in A$, $b \in B$, $g \in G$, and $m \in M$. This is straightforward to check.

2.4. Lemma. *Let G be a finite group, let A and B be G-algebras, and let M be a twisted (A, G)-B-bimodule.*

1. *Let N be a $B * G$-module. Then $M \otimes_B N$ is an $A * G$-module, where*

$$a * g(m \otimes n) := (a * g\, m) \otimes (1_B * g\, n)$$

for $a \in A$, $g \in G$, $m \in M$, and $n \in N$. This gives rise to a k-additive functor $M \otimes_B -\colon B * G\text{-Mod} \to A * G\text{-Mod}$.

2. *Let L be an $A * G$-module. Then $\mathrm{Hom}_A(M, L)$ is a $B * G$-module where, for $b \in B$, $g \in G$, and $\varphi \in \mathrm{Hom}_A(M, L)$, $b * g\, \varphi$ is defined by*

$$(b * g\, \varphi)(m) := ({}^g\varphi)(mb)$$

for $m \in M$. This yields a k-additive functor $\mathrm{Hom}_A(M, -)\colon A * G\text{-Mod} \to B * G\text{-Mod}$.

Proof. ad 1. Let $g \in G$. Set $\mathrm{E}_{g,B}(M) := \{\varphi \in \mathrm{End}_k(M) : \varphi(mb) = \varphi(m)({}^g b)$ for all $m \in M$ and $b \in B\}$, and similarly $\mathrm{E}_{g,B}(N) := \{\psi \in \mathrm{End}_k(N) : \psi(bn) = ({}^g b)\,\psi(n)$ for all $b \in B$, $n \in N\}$. Then $\mathrm{E}_{g,B}(M)$ and $\mathrm{E}_{g,B}(N)$ are k-submodules of $\mathrm{End}_k(M)$ and $\mathrm{End}_k(N)$, respectively.

Let $\varphi \in \mathrm{E}_{g,B}(M)$, and let $\psi \in \mathrm{E}_{g,B}(N)$. Then the universal property of the tensor product yields a k-endomorphism $M \otimes_B N \to M \otimes_B N$, $m \otimes n \mapsto \varphi(m) \otimes \psi(n)$, which we denote by $\varphi \otimes \psi$. It is plain that the corresponding map $\mathrm{E}_{g,B}(M) \times \mathrm{E}_{g,B}(N) \to \mathrm{End}_k(M \otimes_B N)$, $(\varphi, \psi) \mapsto \varphi \otimes \psi$, is k-bilinear inducing a homomorphism of k-modules $\mathrm{E}_{g,B}(M) \otimes_k \mathrm{E}_{g,B}(N) \to \mathrm{End}_k(M \otimes_B N)$.

Set $A * g := \{a * g : a \in A\}$. By definition of a twisted bimodule, we obtain a homomorphism $A * g \to \mathrm{E}_{g,B}(M) \otimes_k \mathrm{E}_{g,B}(N)$, $a * g \mapsto \varphi_{a*g} \otimes \psi_g$, of k-modules, where φ_{a*g} and ψ_g are given by left multiplication with $a * g$ and $1_B * g$, respectively. This yields a k-homomorphism

$$\Gamma\colon A * G = \bigoplus_{g \in G} A * g \to \coprod_{g \in G} E_{g,B}(M) \otimes_k E_{g,B}(N) \to \mathrm{End}_k(M \otimes_B N)$$

such that $(\Gamma(a * g))(m \otimes n) = (a * g\, m) \otimes (1_B * g\, n)$ for all $a \in A$, $g \in G$, $m \in M$, and $n \in N$. Obviously, Γ is a homomorphism of algebras, proving the first assertion. The additional statement is clear.

ad 2. For $b, c \in B$, $g, h \in G$, $\varphi \in \mathrm{Hom}_A(M, L)$, and $m \in M$, we have

$$\begin{aligned}
(b * g(c * h\, \varphi))(m) &= \left({}^g(c * h\, \varphi)\right)(mb) \\
&= 1_A * g(c * h\, \varphi)\left(1_A * g^{-1}(mb)\right) \\
&= 1_A * g\left({}^h\varphi\right)\left(\left(1_A * g^{-1}(mb)\right)c\right) \\
&= 1_A * g\left({}^h\varphi\right)\left(1_A * g^{-1}\left(mb({}^g c)\right)\right) \\
&= \left({}^{gh}\varphi\right)\left(mb({}^g c)\right) = \left(b({}^g c) * gh\, \varphi\right)(m) \\
&= \left(\left((b * g)(c * h)\right)\varphi\right)(m),
\end{aligned}$$

so we are done, the additional assertion being obvious. ∎

Similarly, there are also certain contravariant hom functors, which will not be important for us here. (We refer to [9, Section II.4] for details.)

One might expect the following result:

2.5. LEMMA. *Let G be a finite group, let A and B be G-algebras, and let M be a twisted (A, G)-B-bimodule. Then the functor $M \otimes_B -$: $B * G$-*Mod $\to$ $A * G$-Mod *is left adjoint to* $\mathrm{Hom}_A(M, -)$: $A * G$-Mod $\to B * G$-Mod.

Proof. Let N be a $B * G$-module, and let L be an $A * G$-module. Disregarding the G-action for the moment, there is a well-known isomorphism

$$\alpha\colon \mathrm{Hom}_A(M \otimes_B N, L) \to \mathrm{Hom}_B(N, \mathrm{Hom}_A(M, L))$$

which is natural in N and L, where, for an A-homomorphism φ: $M \otimes_B N \to L$ and $n \in N$, $(\alpha(\varphi))(n)$ is defined by $((\alpha(\varphi))(n))(m) := \varphi(m \otimes n)$ for $m \in M$.

We claim that, here, α is even a homomorphism of kG-modules. In fact, for $g \in G$, $\varphi \in \mathrm{Hom}_A(M \otimes_B N, L)$, $m \in M$, and $n \in N$, we have

$$\begin{aligned}
\big(\big(\alpha({}^g\varphi)\big)(n)\big)(m) &= ({}^g\varphi)(m \otimes n) = 1_A * g\,\varphi\big(1_A * g^{-1}(m \otimes n)\big) \\
&= 1_A * g\varphi\big(\big(1_A * g^{-1}\, m\big) \otimes \big(1_B * g^{-1}\, n\big)\big) \\
&= 1_A * g\,\big(\big(\alpha(\varphi)\big)\big(1_A * g^{-1}\, n\big)\big)\big(1_A * g^{-1}\, m\big) \\
&= \big({}^g\big(\big(\alpha(\varphi)\big)\big(1_B * g^{-1}\, n\big)\big)\big)(m) \\
&= \big(1_B * g\big(\alpha(\varphi)\big)\big(1_B * g^{-1}\, n\big)\big)(m) \\
&= \big(\big({}^g\big(\alpha(\varphi)\big)\big)(n)\big)(m),
\end{aligned}$$

so $\alpha({}^g\varphi) = {}^g(\alpha(\varphi))$, proving our claim.

Therefore, α yields a natural equivalence

$$\mathrm{Hom}_A(M \otimes_B -, -) \to \mathrm{Hom}_B\big(-, \mathrm{Hom}_A(M, -)\big)$$

of functors $B * G\text{-Mod} \times (A * G\text{-Mod})^{\circ} \to kG\text{-Mod}$. Applying the fixed point functor $-^G\colon kG\text{-Mod} \to k\text{-Mod}$ to this equivalence, we obtain a natural equivalence

$$\mathrm{Hom}_{A*G}(M \otimes_B -, -) \to \mathrm{Hom}_{B*G}\big(-, \mathrm{Hom}_A(M, -)\big)$$

of functors $B * G\text{-Mod} \times (A * G\text{-Mod})^{\circ} \to k\text{-Mod}$. In particular, $M \otimes_B -$ is left adjoint to $\mathrm{Hom}_A(M, -)$. ∎

This immediately implies that the functor $M \otimes_B -$ above commutes with induction.

2.6. Corollary. *Let H be a subgroup of a finite group G, let A and B be G-algebras, and let M be a twisted (A, G)-B-bimodule. Then the diagram*

$$\begin{array}{ccc}
B * G\text{-Mod} & \xrightarrow{M \otimes_B -} & A * G\text{-Mod} \\
{\scriptstyle \mathrm{Ind}_H^G}\uparrow & & \uparrow{\scriptstyle \mathrm{Ind}_H^G} \\
B * H\text{-Mod} & \xrightarrow{M \otimes_B -} & A * H\text{-Mod}
\end{array}$$

of functors commutes up to natural equivalence.

Proof. Since Ind_H^G is left adjoint to Res_H^G,

$$\mathrm{Ind}_H^G \circ (M \otimes_B -)\colon B * H\text{-Mod} \to A * G\text{-Mod}$$

is left adjoint to $\operatorname{Hom}_A(M, \_) \circ \operatorname{Res}_H^G$: $A * G$-Mod $\to B * H$-Mod, by Lemma 2.5, and

$$(M \otimes_B -) \circ \operatorname{Ind}_H^G \colon B * H\text{-Mod} \to A * G\text{-Mod}$$

is left adjoint to $\operatorname{Res}_H^G \circ \operatorname{Hom}_A(M, \_)$. Now since $\operatorname{Hom}_A(M, \_)$ clearly "commutes with restriction," the result follows by uniqueness of adjoints. ∎

2.7. *Remark*. In the situation of Corollary 2.6, it is plain that the functor $M \otimes_B -$ "commutes with restriction," as well.

The next result deals with conjugation.

2.8. LEMMA. *Let H be a subgroup of a finite group G, let $g \in G$, let A and B be G-algebras, and let M be a twisted (A, G)-B-bimodule. Then the following diagram of functors commutes up to natural equivalence*:

$$\begin{array}{ccc} B * H\text{-Mod} & \xrightarrow{M \otimes_B -} & A * H\text{-Mod} \\ {}^{g}\_ \downarrow & & \downarrow {}^{g}\_ \\ B * {}^{g}H\text{-Mod} & \xrightarrow{M \otimes_B -} & A * {}^{g}H\text{-Mod}. \end{array}$$

Proof. Let N be a $B * H$-module. Define β: ${}^{g}(M \otimes_B N) \to M \otimes_B {}^{g}N$, $m \otimes n \mapsto (1_A * g\, m) \otimes n$. Note that β is well defined since $(1_A * g\,(mb)) \otimes n = ((1_A * g\, m)^{g}b) \otimes n = (1_A * g\, m) \otimes bn$ for all $m \in M$, $b \in B$, and $n \in N$. (Recall that $B * {}^{g}H$ acts on ${}^{g}N$ via restriction along $B * {}^{g}H \to B * H$, $b * ghg^{-1} \to {}^{g^{-1}}b * h$.)

Obviously, β is an isomorphism of k-modules (an inverse being defined analogously) which is natural in N. Moreover, for $a \in A$, $h \in H$, $m \in M$ and $n \in N$, we have

$$\begin{aligned} \beta\big({}^{g}a * ghg^{-1}\,(m \otimes n)\big) &= \beta(a * h\,(m \otimes n)) \\ &= \beta\big((a * h\, m) \otimes (1_B * h\, n)\big) \\ &= ({}^{g}a * gh\, m) \otimes (1_B * h\, n) \\ &= \big(({}^{g}a * ghg^{-1})(1_A * g\, m)\big) \otimes (1_B * h\, n) \\ &= {}^{g}a * ghg^{-1}\ \beta(m \otimes n). \end{aligned}$$

This completes the proof. ∎

The following lemma introduces the second class of functors which will be important for our proof of Theorem A.

2.9. LEMMA. *Let G be a finite group, let A be a G-algebra, let N be a kG-module, and let M be an $A * G$-module.*

1. *Then $M \otimes_k N$ is an $A * G$-module when we define $a * g(m \otimes n) := (a * g\ m) \otimes (gn)$ for $a \in A$, $g \in G$, $m \in M$, and $n \in N$. This yields a k-additive endofunctor $\_ \otimes_k N\colon A * G\text{-Mod} \to A * G\text{-Mod}$.*

2. *Also $\operatorname{Hom}_k(N, M)$ is an $A * G$-module where, for $a \in A$, $g \in G$, and $\varphi \in \operatorname{Hom}_k(N, M)$, $a * g\varphi$ is defined by*

$$(a * g\ \varphi)(n) := a({}^g\varphi)(n)$$

for $n \in N$. Thus N gives rise to a k-additive endofunctor $\operatorname{Hom}_k(N, \_)\colon A * G\text{-Mod} \to A * G\text{-Mod}$.

Proof. By Example 2.3(3), N is a twisted (A, G)-k-bimodule. Thus the first assertion of item 1 is a special case of Lemma 2.4; the second assertion is then immediate.

Let $a, b \in A$, $g, h \in G$, and $\varphi \in \operatorname{Hom}_k(N, M)$. First, $a * g\ \varphi$, as defined in item 2, is clearly a well-defined k-homomorphism $N \to M$. Second, we have

$$\begin{aligned}((a * g)(b * h\ \varphi))(n) &= a({}^g(b * h\ \varphi))(n) = a * g(b * h\ \varphi)(g^{-1}n)\\ &= a({}^g b)({}^{gh}\varphi)(n) = (a({}^g b) * gh\ \varphi)(n)\\ &= (((a * g)(b * h))\ \varphi)\end{aligned}$$

for all $n \in N$, and we are done. ∎

Again, we have an adjointness relation.

2.10. LEMMA. *Let G be a finite group, let A be a G-algebra, and let N be a kG-module. Then the functor $\_ \otimes_k N\colon A * G\text{-Mod} \to A * G\text{-Mod}$ is left adjoint to $\operatorname{Hom}_k(N, \_)\colon A * G\text{-Mod} \to A * G\text{-Mod}$.*

Proof. Let M and L be $A * G$-modules. As in the proof of Lemma 2.5, there is an isomorphism of k-modules $\beta\colon \operatorname{Hom}_A(M \otimes_k N, L) \to \operatorname{Hom}_A(M, \operatorname{Hom}_k(N, L))$, where, for an A-homomorphism $\varphi\colon M \otimes_k N \to L$, $\beta(\varphi)$ is defined by $((\beta(\varphi))(n))(m) := \varphi(m \otimes n)$ for $m \in M$ and $n \in N$. Moreover, this isomorphism is natural in M and L.

We show that β behaves well with respect to the G-action. Let $g \in G$ and $\varphi \in \operatorname{Hom}_A(M \otimes_k N, L)$. Then

$$\begin{aligned}((\beta({}^g\varphi))(n))(m) &= ({}^g\varphi)(m \otimes n) = 1_A * g\varphi\big((1_A * g^{-1} m) \otimes (g^{-1}n)\big) \\ &= 1_A * g\big((\beta(\varphi))(1_A * g^{-1} m)\big)(g^{-1}n) \\ &= \big({}^g((\beta(\varphi))(1_A * g^{-1} m))\big)(n) \\ &= \big(1_A * g(\beta(\varphi))(1_A * g^{-1} m)\big)(n) \\ &= (({}^g(\beta(\varphi)))(m))(n)\end{aligned}$$

for all $m \in M$ and $n \in N$, so $\beta({}^g\varphi) = {}^g(\beta(\varphi))$. As in the proof of Lemma 2.5, this immediately implies the assertion of our lemma. ∎

In particular, this implies that, with notation of Lemma 2.10, the functor $\_ \otimes_k N$ "commutes with induction"; this fact will be important for us.

2.11. COROLLARY. *Let H be a subgroup of a finite group G, let A be a G-algebra, and let N be a kG-module. Then the following diagram of functors commutes up to natural equivalence*:

$$\begin{array}{ccc} A * G\text{-Mod} & \xrightarrow{\_\otimes_k N} & A * G\text{-Mod} \\ {\scriptstyle \operatorname{Ind}_H^G}\uparrow & & \uparrow{\scriptstyle \operatorname{Ind}_H^G} \\ A * H\text{-Mod} & \xrightarrow{\_\otimes_k N} & A * H\text{-Mod}. \end{array}$$

Proof. Proceed as in the proof of Corollary 2.6 using Lemma 2.10. ∎

2.12. *Remark.* In the situation of Corollary 2.11 the functor $\_ \otimes_k N$ trivially "commutes with restriction" also; and it is not at all surprising that it behaves well with respect to conjugation, as well:

2.13. LEMMA. *Let H be a subgroup of a finite group G, let $g \in G$, let A be a G-algebra, let M be an $A * G$-module, and let N be a kG-module. Then the diagrams*

$$\begin{array}{ccc} A * H\text{-Mod} & \xrightarrow{\_\otimes_k N} & A * H\text{-Mod} \\ {\scriptstyle {}^g\_}\downarrow & & \downarrow{\scriptstyle {}^g\_} \\ A * {}^gH\text{-Mod} & \xrightarrow{\_\otimes_k {}^gN} & A * {}^gH\text{-Mod}, \end{array} \qquad \begin{array}{ccc} kH\text{-Mod} & \xrightarrow{M\otimes_k \_} & A * H\text{-Mod} \\ {\scriptstyle {}^g\_}\downarrow & & \downarrow{\scriptstyle {}^g\_} \\ k[{}^gH]\text{-Mod} & \xrightarrow{{}^gM\otimes_k \_} & A * {}^gH\text{-Mod} \end{array}$$

of functors commute up to natural equivalence.

Proof. Obviously, φ: ${}^{g}M \otimes_k {}^{g}N \to {}^{g}(M \otimes_k N)$, $m \otimes n \mapsto m \otimes n$, is a k-linear bijection which is natural in M and N. Moreover, for $a \in A$, $h \in H$, $m \in M$, and $n \in N$, we have

$$\begin{aligned}\varphi\big({}^{g}a * ghg^{-1}(m \otimes n)\big) &= \varphi\big(\big({}^{g}a * ghg^{-1}\, m\big) \otimes \big(ghg^{-1}n\big)\big)\\ &= \varphi((a * h\, m) \otimes (hn))\\ &= (a * h\, m) \otimes (hn) = a * h(m \otimes n)\\ &= {}^{g}a * ghg^{-1}\ \varphi(m \otimes n).\end{aligned}$$

Thus φ is an isomorphism of $A * {}^{g}H$-modules, and we are done. ∎

We also need a Higman criterion for modules of skew group algebras. Let us recall the definition of the relative trace map. Let H and K be subgroups of G such that K is contained in H. Then the relative trace map Tr_K^H: $A^K \to A^H$ is defined by $a \mapsto \sum_{gK \in H/K} {}^{g}a$. The image of Tr_K^H is denoted by A_K^H. We refer to [11] or [13] for basic properties of relative trace maps.

2.14. PROPOSITION (Higman's criterion). *Let H be a subgroup of a finite group G, let A be a G-algebra, and let M be an $A * G$-module. Then the following assertions are equivalent*:

1. *The module M is isomorphic to a direct summand of* $\mathrm{Ind}_H^G \mathrm{Res}_H^G M$.

2. *There is an $A * H$-module N such that M is isomorphic to a direct summand of* $\mathrm{Ind}_H^G N$.

3. *There is* $\varphi \in \mathrm{End}_{A*H}(M)$ $(= \mathrm{End}_A(M)^H)$ *such that* $\mathrm{Tr}_H^G(\varphi) = \mathrm{id}_M$.

Sketch of the proof. It is straightforward to check that the well-known proof of this fact for ordinary group algebras (cf. [13, Proposition (17.7)], for instance) carries over to this more general setup. ∎

3. A MODULE FOR THE CHARACTER RING

In this section we will show that the Grothendieck groups of projective modules of certain skew group algebras can be regarded as (Green functor) modules of the character ring (functor).

Throughout, we fix a finite group G; let $\mathscr{O}$ be a complete discrete valuation ring of characteristic 0 with residue field $\mathbb{F}$ of characteristic $p \neq 0$. Moreover, we denote by $\mathbb{K}$ a quotient field of $\mathscr{O}$. From now on, all our algebras and modules will be free of finite rank over either $\mathbb{K}$, $\mathscr{O}$, or $\mathbb{F}$. For an algebra A, A-mod denotes the category of A-modules in this sense.

We refer to [4] for the definition of Mackey functors, Green functors, pairings, and so forth. Partly, these definitions can also be found in [13].

Let A be an algebra. For an A-module M, we denote by $[M]$ its isomorphism class. The isomorphism classes of A-modules form an abelian monoid where, for A-modules M and N, addition is defined by $[M] + [N] = [M \oplus N]$. We denote by $\mathrm{a}(A)$ the corresponding Grothendieck group. Then the set of isomorphism classes of indecomposable A-modules is a $\mathbb{Z}$-basis of $\mathrm{a}(A)$ since the Krull–Schmidt theorem holds for A-modules, by our specific choice of ground rings.

Denote by $\mathrm{K}_0(A)$ the subgroup of $\mathrm{a}(A)$ generated by isomorphism classes of projective A-modules. Then the isomorphism classes of indecomposable projective A-modules form a (finite) $\mathbb{Z}$-basis of $\mathrm{K}_0(A)$.

Apart from that, let $\mathrm{ses}(A)$ be the subgroup of $\mathrm{a}(A)$ generated by elements of the form $[E] - [M] - [L]$, where E, M, and L are A-modules such that there is a short exact sequence $0 \to L \to E \to M \to 0$ of A-modules. The corresponding quotient group $\mathrm{G}_0(A) := \mathrm{a}(A)/\mathrm{ses}(A)$ is called the Grothendieck group of A.

For an A-module M, denote by $[\![M]\!]$ the coset $[M] + \mathrm{ses}(A)$. If A is an algebra over $\mathbb{K}$ or $\mathbb{F}$, then the set $\{[\![S]\!]\}$, where S runs through a complete system of representatives of simple A-modules, is a $\mathbb{Z}$-basis of $\mathrm{G}_0(A)$, by the Jordan–Hölder theorem. In the case $A = \mathscr{O}G$, the group algebra over $\mathscr{O}$, the structure of $\mathrm{G}_0(A)$ is also known by a theorem of Swan:

3.1. THEOREM [6, Theorem (39.10)]. *Let G be a finite group. Then the map* $\mathrm{G}_0(\mathscr{O}G) \to \mathrm{G}_0(\mathbb{K}G)$, $[\![M]\!] \mapsto [\![K \otimes_{\mathscr{O}} M]\!]$, *is an isomorphism of groups* (*or rings*).

In fact this is just a special case of Swan's result adapted to our needs. (Note that what we understand by $\mathrm{G}_0(\mathscr{O}G)$ is denoted by $\mathrm{G}_0^{\mathscr{O}}(\mathscr{O}G)$ in [6]. Whereas, with notation of [6], $\mathrm{G}_0(\mathscr{O}G)$ is a different but at least isomorphic group (cf. [6, Theorem (38.42)].)

We now bring the group G into play. Let A be a G-algebra. Moreover, let $g \in G$, let H be a subgroup of G, and let K be a subgroup of H. Then the additive and exact functors ${}^g\_\colon A * H\text{-mod} \to A * {}^gH\text{-mod}$, $\mathrm{Res}_K^H\colon A * H\text{-mod} \to A * K\text{-mod}$ and $\mathrm{Ind}_K^H\colon A * K\text{-mod} \to A * H\text{-mod}$ induce homomorphisms $\mathrm{c}_{g,H}\colon \mathrm{a}(A * H) \to \mathrm{a}(A * {}^gH)$, $[M] \mapsto [{}^gM]$, $\mathrm{res}_K^H\colon \mathrm{a}(A * H) \to \mathrm{a}(A * K)$, $[M] \mapsto [\mathrm{Res}_K^H M]$, and $\mathrm{ind}_K^H\colon \mathrm{a}(A * K) \to \mathrm{a}(A * H)$, $[M] \mapsto [\mathrm{Ind}_K^H M]$, respectively. (Note that $A * H$ is free, and hence flat, as a right $A * K$-module.)

Plainly, the restrictions $\mathrm{c}_{g,H}\colon \mathrm{K}_0(A * H) \to \mathrm{K}_0(A * {}^gH)$, $\mathrm{res}_K^H\colon \mathrm{K}_0(A * H) \to \mathrm{K}_0(A * K)$, and $\mathrm{ind}_K^H\colon \mathrm{K}_0(A * K) \to \mathrm{K}_0(A * H)$ of the above homomorphisms are well defined. (In case of restriction, one needs that $A * H$ is also a free left $A * K$-module, which is straightforward to check.)

Moreover, we have $\mathrm{c}_{g,H}(\mathrm{ses}(A * H)) \subseteq \mathrm{ses}(A * {}^{g}H)$, $\mathrm{res}_K^H(\mathrm{ses}(A * H)) \subseteq \mathrm{ses}(A * K)$, and $\mathrm{ind}_K^H(\mathrm{ses}(A * K)) \subseteq \mathrm{ses}(A * H)$ since the underlying functors are exact. Therefore, these homomorphisms induce homomorphisms $\mathrm{c}_{g,H}\colon \mathrm{G}_0(A * H) \to \mathrm{G}_0(A * {}^{g}H)$, $[\![M]\!] \mapsto [\![{}^{g}M]\!]$, $\mathrm{res}_K^H\colon \mathrm{G}_0(A * H) \to \mathrm{G}_0(A * K)$, $[\![M]\!] \mapsto [\![\mathrm{Res}_K^H M]\!]$, and $\mathrm{ind}_K^H\colon \mathrm{G}_0(A * K) \to \mathrm{G}_0(A * H)$, $[\![M]\!] \mapsto [\![\mathrm{Ind}_K^H M]\!]$.

It is straightforward to check that the families $(\mathrm{a}(A * H))_{H \le G}$, $(\mathrm{K}_0(A * H))_{H \le G}$, and $(\mathrm{G}_0(A * H))_{H \le G}$ of abelian groups, together with the corresponding homomorphisms $\mathrm{c}_{g,H}$, res_K^H, and ind_K^H $(g \in G,\ K \le H \le G)$, form Mackey functors for G (over $\mathbb{Z}$). We denote these Mackey functors by $\mathbf{a}(A, G)$, $\mathbf{K}_0(A, G)$, and $\mathbf{G}_0(A, G)$, respectively.

Also let B be a G-algebra, and let M be a twisted (A, G)-B-bimodule (cf. Definition 2.2) which is finitely generated and projective as a right B-module. Then, for any subgroup H of G, there is an additive functor $M \otimes_B -\colon B * H\text{-mod} \to A * H\text{-mod}$, by Lemma 2.4(1), which induces a homomorphism $[M \otimes_B -]_H\colon \mathrm{a}(B * H) \to \mathrm{a}(A * H)$, $[L] \mapsto [M \otimes_B L]$. (Note that, for a $B * H$-module L, $M \otimes_B L$ is finitely generated and free over the ground ring since this is the case for L, and M is projective as a right B-module.) By Corollary 2.6, Remark 2.7, and Lemma 2.8, the family $[M \otimes_B -] := ([M \otimes_B -]_H)_{H \le G}$ is a homomorphism of Mackey functors $\mathbf{a}(B, G) \to \mathbf{a}(A, G)$.

3.2. LEMMA. *Let G be a finite group, let A and B be G-algebras, and let M be a twisted (A, G)-B-bimodule which is finitely generated and projective, both as a left A-module and as a right B-module, as well. Then the restriction*

$$[M \otimes_B -]\colon \mathbf{K}_0(B, G) \to \mathbf{K}_0(A, G)$$

of $[M \otimes_B -]\colon \mathbf{a}(B, G) \to \mathbf{a}(A, G)$ is a well-defined homomorphism of Mackey functors.

Proof. Let H be a subgroup of G. It suffices to show that the functor

$$M \otimes_B -\colon B * H\text{-mod} \to A * H\text{-mod}$$

preserves projectives. But this follows from the fact that $M \otimes_B -$ is left adjoint to $\mathrm{Hom}_A(M, \_)\colon A * H\text{-mod} \to B * H\text{-mod}$, by Lemma 2.5, which preserves epimorphisms since M is projective as an A-module, by assumption. ∎

We also have a similar homomorphism between the Grothendieck groups.

3.3. LEMMA. *Let G be a finite group, let A and B be G-algebras, and let M be a twisted (A, G)-B-bimodule which is finitely generated and projective as* *a right B-module. Then, for any subgroup H of G,* $[\![M \otimes_{\underline{B}}]\!]_H\colon \mathrm{G}_0(B * H) \to \mathrm{G}_0(A * H)$, $[\![L]\!] \mapsto [\![M \otimes_B L]\!]$, *is a homomorphism of groups. Thus*

$$[\![M \otimes_B -]\!] := ([\![M \otimes_B -]\!]_H)_{H \le G}\colon \mathbf{G}_0(B, G) \to \mathbf{G}_0(A, G)$$

is a homomorphism of Mackey functors.

Proof. In view of the observations preceding Lemma 3.2, the only thing we need to show is that, for any subgroup H of G, $[M \otimes_B -]_H(\mathrm{ses}(B * H)) \subseteq \mathrm{ses}(A * H)$, i.e., that the functor $M \otimes_B -\colon B * H\text{-mod} \to A * H\text{-mod}$ is exact. But this is certainly the case since M is a flat right B-module, by assumption. ∎

Let A be a G-algebra over $\mathscr{O}$, and let N be an $\mathscr{O}G$-module. Then, for any subgroup H of G, the additive endofunctor $- \otimes_{\mathscr{O}} N\colon A * H\text{-mod} \to A * H\text{-mod}$ of Lemma 2.9 induces an endomorphism of groups $[- \otimes_{\mathscr{O}} N]_H\colon \mathrm{a}(A * H) \to \mathrm{a}(A * H)$, $[M] \mapsto [M \otimes_{\mathscr{O}} N]$. By Corollary 2.11, Remark 2.12, and Lemma 2.13, the family $[- \otimes_{\mathscr{O}} N] := ([- \otimes_{\mathscr{O}} N]_H)_{H \le G}$ is an endomorphism of Mackey functors of $\mathbf{a}(A, G)$.

3.4. LEMMA. *Let G be a finite group, let A be a G-algebra over* $\mathscr{O}$, *and let N be an* $\mathscr{O}G$*-module. Then the restriction*

$$[- \otimes_{\mathscr{O}} N]\colon \mathbf{K}_0(A, G) \to \mathbf{K}_0(A, G)$$

of $[- \otimes_{\mathscr{O}} N]\colon \mathbf{a}(A, G) \to \mathbf{a}(A, G)$ *is a well-defined endomorphism of Mackey functors.*

Proof. Proceed as in Lemma 3.2. ∎

3.5. LEMMA. *Let G be a finite group, let A be a G-algebra, and let N be an* $\mathscr{O}G$*-module. Then, for any subgroup H of G,* $[\![- \otimes_{\mathscr{O}} N]\!]_H\colon \mathrm{G}_0(A * H) \to \mathrm{G}_0(A * H)$, $[\![M]\!] \mapsto [\![M \otimes_{\mathscr{O}} N]\!]$, *is a well-defined homomorphism of groups. Thus*

$$[\![- \otimes_{\mathscr{O}} N]\!] := ([\![- \otimes_{\mathscr{O}} N]\!]_H)_{H \le G}\colon \mathbf{G}_0(A, G) \to \mathbf{G}_0(A, G)$$

is an endomorphism of Mackey functors.

Proof. As in the proof of Lemma 3.3, the only thing one needs is that N is flat as an $\mathscr{O}$-module. But this is certainly the case, by our general assumption on modules in this section. ∎

Let H be a subgroup of G. Then

$$[M \otimes_{\mathscr{O}} -]_H([N]) = [M \otimes_{\mathscr{O}} N] = [- \otimes_{\mathscr{O}} N]_H([M])$$

for any $A * H$-module M and any $\mathscr{O}G$-module N. Thus

$$[\_\otimes_{\mathscr{O}}\_]_H \colon \mathrm{a}(A * H) \times \mathrm{a}(\mathscr{O}H) \to \mathrm{a}(A * H),$$
$$([M],[N]) \mapsto [M \otimes_{\mathscr{O}} N]$$

is a $\mathbb{Z}$-bilinear map. Therefore, by Corollaries 2.6 and 2.11, Remarks 2.7 and 2.12, and Lemma 2.13, the family $[\_\otimes_{\mathscr{O}}\_] := ([\_\otimes_{\mathscr{O}}\_])_{H \leq G}$ is a pairing of Mackey functors

$$\mathbf{a}(A,G) \times \mathbf{a}(\mathscr{O},G) \to \mathbf{a}(A,G).$$

Moreover,

$$[M \otimes_{\mathscr{O}} \mathscr{O}] = [M] \quad \text{and} \quad [(M \otimes_{\mathscr{O}} N) \otimes_{\mathscr{O}} L] = [M \otimes_{\mathscr{O}} (N \otimes_{\mathscr{O}} L)]$$

for all subgroups H of G, all $A * H$-modules M, and all $\mathscr{O}G$-modules N and L, as one easily verifies. Choosing $A = \mathscr{O}$, the G-algebra with trivial G-action, this shows on the one hand that, for any subgroup H of G, $\mathrm{a}(\mathscr{O}H)$ is a ring with multiplicative identity $[\mathscr{O}]$, called the Green ring of H (which is clearly commutative). Thus $\mathbf{a}(\mathscr{O},G)$ is a Green functor for G.

On the other hand, it demonstrates that, for any subgroup H of G, $\mathrm{a}(A * H)$ is an $\mathrm{a}(\mathscr{O}H)$-module. Therefore, the above pairing turns $\mathbf{a}(A,G)$ into a right module for the Green functor $\mathbf{a}(\mathscr{O},G)$.

By Lemma 3.4, the restriction

$$[\_\otimes_{\mathscr{O}}\_] \colon \mathbf{K}_0(A,G) \times \mathbf{a}(\mathscr{O},G) \to \mathbf{K}_0(A,G)$$

of the above pairing is well defined. Thus $\mathbf{K}_0(A,G)$ is a right $\mathbf{a}(\mathscr{O},G)$-module, as well.

Moreover, in view of Lemmata 3.3 and 3.5, it is clear that the above pairing induces a pairing

$$\mathbf{G}_0(A,G) \times \mathbf{G}_0(\mathscr{O},G) \to \mathbf{G}_0(\mathscr{O},G).$$

We can take $A = \mathscr{O}$, and obtain that $\mathbf{G}_0(\mathscr{O},G)$ is a Green functor for G. Besides, $\mathbf{G}_0(A,G)$ is a right $\mathbf{G}_0(\mathscr{O},G)$-module.

Let H be a subgroup of G, and let $0 \to N \overset{\mu}{\to} E \overset{\varepsilon}{\to} L \to 0$ be a short exact sequence of $\mathscr{O}H$-modules and $\mathscr{O}H$-homomorphisms. Moreover, let P be a projective $A * H$-module. Then, by Lemma 3.4, $P \otimes_{\mathscr{O}} L$ is a projective $A * H$-module, as well. Besides,

$$\xi \colon 0 \to P \otimes_{\mathscr{O}} N \overset{\mathrm{id}_P \otimes \mu}{\to} P \otimes_{\mathscr{O}} E \overset{\mathrm{id}_P \otimes \varepsilon}{\to} P \otimes_{\mathscr{O}} L \to 0$$

is a short exact sequence of $A * H$-modules since P is flat as an $\mathscr{O}$-module (or because the original sequence splits as a sequence of $\mathscr{O}$-modules). Thus ξ splits. Therefore,

$$[P]\cdot[E] = [P \otimes_{\mathscr{O}} E] = \big[(P \otimes_{\mathscr{O}} N) \oplus (P \otimes_{\mathscr{O}} L)\big]$$
$$= [P \otimes_{\mathscr{O}} N] + [P \otimes_{\mathscr{O}} L] = [P]([N] + [L]).$$

This shows that $\mathrm{K}_0(A * H)\mathrm{ses}(A * H) = 0$. Hence

$$[-\otimes_{\mathscr{O}}-]_H\colon \mathrm{K}_0(A * H) \times \mathrm{G}_0(\mathscr{O}H) \to \mathrm{K}_0(A * H),$$
$$([P], [\![N]\!]) \mapsto [P \otimes_{\mathscr{O}} N]$$

is a well-defined $\mathbb{Z}$-bilinear map such that $[-\otimes_{\mathscr{O}}-] := ([-\otimes_{\mathscr{O}}-]_H)_{H \le G}$ is a pairing of Mackey functors $\mathbf{K}_0(A, G) \times \mathbf{G}_0(\mathscr{O}, G) \to \mathbf{K}_0(A, G)$.

We summarize:

3.6. PROPOSITION. *Let G be a finite group, and let A be a G-algebra over $\mathscr{O}$. Then the Mackey functor $\mathbf{K}_0(A, G)$ is a right module of the Green functor $\mathbf{G}_0(\mathscr{O}, G)$, where $[P]\cdot[\![M]\!] := [P \otimes_{\mathscr{O}} M]$, for a subgroup H of G, a projective $A * H$-module P, and an $\mathscr{O}H$-module M.*

3.7. *Remark*. Of course, all the results above are equally valid if we replace the ground ring $\mathscr{O}$ by $\mathbb{K}$ or $\mathbb{F}$, respectively. We will use this fact without further comment.

Let H be a subgroup of G. By Swan's theorem (cf. Theorem 3.1), we have an isomorphism $\mathrm{G}_0(\mathscr{O}H) \to \mathrm{G}_0(\mathbb{K}H)$ of rings. Obviously, this yields an isomorphism of Green functors $\mathbf{G}_0(\mathscr{O}, G) \to \mathbf{G}_0(\mathbb{K}, G)$.

Assume that $\mathbb{K}$ is a splitting field for G and all of its subgroups. Then there is a well-known isomorphism $\mathrm{G}_0(\mathbb{K}H) \to \mathrm{R}(H)$ which, for a $\mathbb{K}H$-module M, maps $[\![M]\!]$ to the character afforded by M. This gives rise to an isomorphism of Green functors $\mathbf{G}_0(\mathbb{K}, G) \to \mathbf{R}(G)$, where $\mathbf{R}(G)$ denotes the character ring functor of G. Therefore, in the situation of Proposition 3.6, we can regard $\mathbf{K}_0(A, G)$ also as a right $\mathbf{R}(G)$-module via restriction along these isomorphisms.

Assume now that $\mathbb{F}$ is a splitting field for G and all of its subgroups. Similarly, there is an isomorphism $\mathrm{G}_0(\mathbb{F}H) \to \mathrm{R}_{p'}(H)$ which, for an $\mathbb{F}H$-module M, maps $[\![M]\!]$ to the Brauer character afforded by M. Again, this yields an isomorphism of Green functors $\mathbf{G}_0(\mathbb{F}, G) \to \mathbf{R}_{p'}(G)$, where $\mathbf{R}_{p'}(G)$ denotes the Brauer character ring functor of G. Moreover, the decomposition map $\mathrm{R}(H) \to \mathrm{R}_{p'}(H)$ (which is given by restricting characters to p-regular elements) gives rise to an epimorphism of Green functors $\mathbf{R}(G) \to \mathbf{R}_{p'}(G)$. So, in case A is a G-algebra over $\mathbb{F}$, we may regard $\mathbf{K}_0(A, G)$ as a right $\mathbf{R}_{p'}(G)$-module and as a right $\mathbf{R}(G)$-module, as well.

4. LIFTING INDUCTION THEOREMS

In this section we will prove Theorem A.

Let us briefly recall the definition of vertices and sources of modules of skew group algebras. (Details can be found in [3] in the more general setup of group-graded algebras.) Let A be a G-algebra over $\mathcal{O}$ or $\mathbb{F}$, and let M be an $A * G$-module. Then, for a subgroup H of G, M is called relatively H-projective if M is isomorphic to a direct summand of $\operatorname{Ind}_H^G N$ for some $A * H$-module N.

Assume that M is indecomposable. Then a subgroup of G which is minimal among the subgroups H of G such that M is relatively H-projective is called a vertex of M. In view of Higman's criterion (Proposition 2.14), it is clear that vertices of M are p-subgroups of G. Moreover, it is straightforward to check that the vertices of M form a conjugacy class of p-subgroups of G since the Krull–Schmidt theorem as well as Mackey decomposition hold also in this more general setup.

Let P be a vertex of M. Then an indecomposable $A * P$-module N is called a P-source of M if N is isomorphic to a direct summand of $\operatorname{Res}_P^G M$, and M is isomorphic to a direct summand of $\operatorname{Ind}_P^G N$. In this case, P is necessarily a vertex of N. Again, using the Krull–Schmidt theorem and Mackey's theorem, it is easy to see that P-sources of M are uniquely determined up to isomorphism and $\mathrm{N}_G(P)$-conjugacy. Moreover, any indecomposable $A * P$-module L such that M is isomorphic to a direct summand of $\operatorname{Ind}_P^G L$ is a P-source of M.

It is not surprising that we also have Green correspondence in this situation.

4.1. THEOREM (Green correspondence). *Let P be a subgroup of a finite group G, let H be a subgroup of G containing $\mathrm{N}_G(P)$, and let A be a G-algebra over $\mathcal{O}$ or $\mathbb{F}$. Then the following assertions hold*:

1. *Let M be an indecomposable $A * G$-module with vertex P. Then, up to isomorphism, there is a uniquely determined indecomposable direct summand L of $\operatorname{Res}_H^G M$ with vertex P. Moreover, L has multiplicity 1 in $\operatorname{Res}_H^G M$.*

2. *Let L be an indecomposable $A * H$-module with vertex P. Then, up to isomorphism, there is a uniquely determined direct summand M of $\operatorname{Ind}_H^G L$ with vertex P. Moreover, M has multiplicity 1 in $\operatorname{Ind}_H^G L$, and any indecomposable direct summand of $\operatorname{Ind}_H^G L$ not isomorphic to M has a vertex strictly contained in P.*

3. *Assertions 1 and 2 set up mutually inverse bijections between the set of isomorphism classes of indecomposable $A * G$-modules with vertex P and the set of isomorphism classes of indecomposable $A * H$-modules with vertex P.*

Sketch of the proof. See [1, Theorem 11.1], for example. It is straightforward to check that the proof stated in [1] carries over to the skew group algebra situation since the only things the proof requires are Mackey decomposition and the Krull–Schmidt theorem. ∎

4.2. *Remark*. With notation of Theorem 4.1, L is called the Green correspondent of M, and M is called the Green correspondent of L. It is clear that any P-source of L is a P-source of M, as well. Since P-sources of M are uniquely determined up to $\mathrm{N}_G(P)$-conjugacy (and isomorphism), this shows that, conversely, any P-source of M is also a P-source of L.

The following elementary construction for Mackey functors is an analogue (or a generalization) of the fixed point submodule M^N of a normal subgroup N of G on an $\mathscr{O}G$-module M.

4.3. LEMMA. *Let k be a commutative ring with* 1, *let N be a normal subgroup of a finite group G, and let $\mathbf{M}$ be a Mackey functor for G over k. Then $\mathbf{M}^N = (\mathbf{M}^N, {}_N\mathrm{res}, {}_N\mathrm{ind}, {}_N\mathrm{c})$ is a Mackey functor for G/N over k, where*

1. $\mathbf{M}^N(H/N) := \mathbf{M}(H)$ *for a subgroup H/N of G/N*;
2. ${}_N\mathrm{res}^{H/N}_{K/N} := \mathrm{res}^H_K$ *and* ${}_N\mathrm{ind}^{H/N}_{K/N} := \mathrm{ind}^H_K$ *for subgroups H/N, K/N of G/N such that $K/N \subseteq H/N$*;
3. ${}_N\mathrm{c}_{gN, H/N} := \mathrm{c}_{g,H}$ *for $gN \in G/N$ and a subgroup H/N of G/N.*

Proof. This is straightforward to check. Note that definition 2 is independent of the choice of representatives since $\mathrm{c}_{n,H} = \mathrm{id}_{M(H)}$ and $\mathrm{c}_{gn,H} = \mathrm{c}_{g,H} \circ \mathrm{c}_{n,H}$ for all $g \in G$, $n \in N$, and all subgroups H of G, by definition of a Mackey functor. Moreover, for subgroups H/N and K/N of G/N, a transversal T of $H \backslash G/K$ yields a transversal $TN := \{tN\colon t \in T\}$ of $(H/N)\backslash(G/N)/(K/N)$. ∎

4.4. DEFINITION. Let G be a finite group, let A be a G-algebra, and let $\mathbf{M}$ be a Mackey subfunctor of $\mathbf{a}(A, G)$. We say that $\mathbf{M}$ is *closed under taking direct summands* if, for any subgroup H of G and any $A * H$-module N with $[N] \in \mathbf{M}(H)$, any direct summand L of N satisfies $[L] \in \mathbf{M}(H)$.

4.5. EXAMPLE. 1. Obviously, $\mathbf{K}_0(A, G)$ is a Mackey subfunctor of $\mathbf{a}(A, G)$ which is closed under taking direct summands.

2. Denote by $\mathbf{a}(\mathbb{F}G, \mathrm{triv})$ the trivial source ring functor of G. Clearly, $\mathbf{a}(\mathbb{F}G, \mathrm{triv})$ is a Mackey subfunctor of $\mathbf{a}(\mathbb{F}, G)$ which is closed under taking direct summands.

3. Let A be a G-algebra over $\mathscr{O}$ or $\mathbb{F}$, and let M be an indecomposable $A * G$-module with vertex P and P-source N. For a subgroup H of G, denote by $\mathbf{M}(H)$ the subgroup of $\mathrm{a}(A * H)$ generated by elements $[L]$,

where L is an indecomposable $A * H$-module such that some vertex Q of L satisfies $Q \subseteq {}^gP$ for some $g \in G$, and such that some Q-source of L is a direct summand of $\mathrm{Res}_Q^{{}^gP}\,{}^gN$. It is straightforward to check that $\mathbf{M} := (\mathbf{M}(H))_{H \leq G}$ is a Mackey subfunctor of $\mathbf{a}(A, G)$. By construction, $\mathbf{M}$ is closed under taking direct summands.

Apart from Proposition 3.6, our proof of Theorem A will essentially be based on the following result.

4.6. PROPOSITION. *Let N be a normal subgroup of a finite group G, let A be a G-algebra, and let L be a relatively N-projective $A * G$-module. Moreover, let $\mathbf{M}$ be a Mackey subfunctor of $\mathbf{a}(A, G)$, which is closed under taking direct summands, such that $[L] \in \mathbf{M}(G)$. Then there are a G/N-algebra B and a homomorphism of Mackey functors for G/N*

$$\varphi = (\varphi_{H/N})_{H/N \leq G/N} : \mathbf{K}_0(B, G/N) \to \mathbf{M}^N$$

such that $[B] \in \mathrm{K}_0(B * G/N)$ and $\varphi_{G/N}([B]) = [L]$. Moreover, if, for a subgroup H/N of G/N and an $A * H$-module U, $[U]$ is contained in the image of φ_H, then U is relatively N-projective.

Proof. Set $\overline{G} := G/N$, for $g \in G$, set $\bar{g} := gN$, and, for a subgroup H of G containing N, set $\overline{H} := H/N$. Consider the G-algebra

$$B := \mathrm{End}_{A * N}(L)^{\circ} = \left(\mathrm{End}_A(L)^{\circ}\right)^N,$$

which can also be regarded as a $\overline{G}$-algebra. By Example 2.3(1), B is a $B * \overline{G}$-module. We have

$$1_{B^{\circ}} = \mathrm{id}_L = \mathrm{Tr}_N^G(\alpha) = \mathrm{Tr}_1^{\overline{G}}(\alpha)$$

for some $\alpha \in \mathrm{End}_{A * N}(L) = B^{\circ}$, by Higman's criterion (Proposition 2.14), since M is relatively N-projective, by assumption. It is straightforward to check that $B^{\circ} \to \mathrm{End}_B(B)$, $\psi \mapsto \psi(1_B)$ is an isomorphism of $\overline{G}$-algebras. Thus B is a relatively 1-projective $B * \overline{G}$-module, again by Higman's criterion. Therefore, B is a projective $B * \overline{G}$-module since B is certainly projective as a B-module. Hence $[B] \in \mathrm{K}_0(B * G/N)$.

It is well known that L is a right B-module, where $l\beta := \beta(l)$ for $l \in L$ and $\beta \in B$. Moreover,

$$a * g\,(l\beta) = a * g\ \beta(l) = ({}^g\beta)(a * gl) = (a * gl)^g\beta$$

for all $a \in A$, $g \in G$, $l \in L$, and $\beta \in B$, by (1). Thus L is a twisted (A, G)-B-bimodule. This yields an additive functor $L \otimes_B -$: $B * G$-Mod $\to$ $A * G$-Mod, by Lemma 2.4(1).

For a G-algebra C, denote by $C * G\text{-pro}_C$ the category of $C * G$-modules which are finitely generated and projective as C-modules. Then the restriction $L \otimes_B -\colon B * G\text{-pro}_B \to A * G\text{-mod}$ of the above functor is well defined. (Note that, for a finitely generated projective B-module N, $L \otimes_B N$ is finitely generated and free over the ground ring since L is.)

Let H be a subgroup of G containing N. There is an epimorphism of algebras $\sigma_H\colon B * H \to B * \overline{H}$, $b * h \mapsto b * (hN)$. Denote by $\operatorname{Res}_{\sigma_H}\colon B * \overline{H}\text{-pro}_B \to B * H\text{-pro}_B$ restriction along σ_H, which is an additive functor. (Note that σ_H is the identity on B.) In particular, this yields a homomorphism of groups

$$\varphi_{H/N}\colon \mathrm{K}_0(B * \overline{H}) \to \mathrm{a}(A * H), \qquad [P] \mapsto \left[L \otimes_B \operatorname{Res}_{\sigma_H} P\right].$$

Moreover, $\varphi_{G/N}([B]) = [L \otimes_B \operatorname{Res}_{\sigma_H} B] = [L]$, as one easily verifies.

We claim that $\varphi := (\varphi_U)_{U \leq \overline{G}}$ is a homomorphism of Mackey functors

$$\mathbf{K}_0(B, \overline{G}) \to (\mathbf{a}(A, G))^N.$$

Let K/N be a subgroup of H/N, and let M be a $B * \overline{K}$-module. Define

$$\begin{aligned} \beta\colon\ & B * H \otimes_{B * K} \operatorname{Res}_{\sigma_K} M \\ & \to \operatorname{Res}_{\sigma_H}\left(B * \overline{H} \otimes_{B * \overline{K}} M\right), \qquad b * h \otimes m \mapsto b * (hN) \otimes m. \end{aligned}$$

It is straightforward to check that β is a well-defined isomorphism of $B * H$-modules which is natural in M. Thus the following diagram of functors commutes up to natural equivalence:

$$\begin{array}{ccc} B * \overline{H}\text{-mod} & \xrightarrow{\operatorname{Res}_{\sigma_H}} & B * H\text{-mod} \\ {\scriptstyle \operatorname{Ind}_{\overline{K}}^{\overline{H}}}\big\uparrow & & \big\uparrow{\scriptstyle \operatorname{Ind}_K^H} \\ B * \overline{K}\text{-mod} & \xrightarrow{\operatorname{Res}_{\sigma_K}} & B * K\text{-mod}. \end{array}$$

Moreover, for $g \in G$, it is obvious that the diagrams of functors

$$\begin{array}{ccc} B * \overline{H}\text{-mod} & \xrightarrow{\operatorname{Res}_{\sigma_H}} & B * H\text{-mod} \\ {\scriptstyle {}^{\bar g}-}\big\downarrow & & \big\downarrow{\scriptstyle {}^{g}-} \\ B * {}^{\bar g}\overline{H}\text{-mod} & \xrightarrow{\operatorname{Res}_{\sigma_K}} & B * {}^{g}H\text{-mod}, \end{array} \qquad \begin{array}{ccc} B * \overline{H}\text{-mod} & \xrightarrow{\operatorname{Res}_{\sigma_H}} & B * H\text{-mod} \\ {\scriptstyle \operatorname{Res}_{\overline{K}}^{\overline{H}}}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Res}_K^H} \\ B * \overline{K}\text{-mod} & \xrightarrow{\operatorname{Res}_{\sigma_K}} & B * K\text{-mod} \end{array}$$

commute up to natural equivalence. In view of Corollary 2.6, Remark 2.7, and Lemma 2.8, this proves our claim.

Next we show that the image of $\varphi_{\overline{H}}$ is contained in $\mathbf{M}(H)$. Let P be a projective $B * \overline{H}$-module. Then there is a natural number n such that P is isomorphic to a direct summand of $(B * \overline{H})^n \cong (\operatorname{Ind}_1^{\overline{H}} B)^n$. Therefore, $L \otimes_B \operatorname{Res}_{\sigma_H} P$ is isomorphic to a direct summand of

$$L \otimes_B \left(\operatorname{Res}_{\sigma_H}\left(\operatorname{Ind}_1^{\overline{H}} B\right)^n\right) \cong \operatorname{Ind}_N^H\left(L \otimes_B \operatorname{Res}_{\sigma_H} B\right)^n \cong \operatorname{Ind}_N^H \operatorname{Res}_N^G L^n. \quad (2)$$

Since $[L] \in \mathbf{M}(G)$, we have $[\operatorname{Ind}_N^H \operatorname{Res}_N^G L^n] = n \operatorname{ind}_N^H \operatorname{res}_N^G [L] \in \mathbf{M}(H)$, so

$$\varphi_{\overline{H}}([P]) = \left[L \otimes_B \operatorname{Res}_{\sigma_H} P\right] \in \mathbf{M}(H)$$

as well, by our assumption on $\mathbf{M}$.

Finally, Eq. (2) also implies the additional assertion. ∎

Let k be a commutative ground ring with 1, and let $\mathbf{M}$ be a Mackey functor for G over $\mathbb{Z}$. We denote by $k\mathbf{M} := k \otimes_{\mathbb{Z}} \mathbf{M}$ the Mackey functor for G over k which is given by extending scalars. In consistency with this notation, we write $k\mathbf{M}(H)$ instead of $k \otimes_{\mathbb{Z}} \mathbf{M}(H)$ for any subgroup H of G. Moreover, for $c \in k$, a subgroup H of G, and $m \in \mathbf{M}(H)$, cm is an abbreviation of $c \otimes m$, and m stands for $1m \in k\mathbf{M}(H)$. Note that one has to be careful with this notation, for if k is of positive characteristic, then an equation $[M] = [L]$ in $k\mathrm{a}(A * G)$ does not necessarily imply that M and L are isomorphic unless M and L are indecomposable.

Now we are in a position to prove Theorem A. Our proof will be based on the following skew group algebra version of Theorem A.

4.7. THEOREM. *Let G be a finite group, let A be a G-algebra over $\mathscr{O}$ or $\mathbb{F}$, and let M be an indecomposable $A * G$-module with vertex P and P-source N. Moreover, let k be a commutative ring with* 1, *and, for any subgroup Q of P, let $\mathscr{C}_Q$ be a set of subgroups of $\mathrm{N}_G(Q)/Q$ such that*

$$1 \in \sum_{H/Q \in \mathscr{C}_Q} \operatorname{ind}_{H/Q}^{\mathrm{N}_G(Q)/Q} k\mathrm{G}_0(\mathscr{O}[H/Q])$$

in $k\mathrm{G}_0(\mathscr{O}[\mathrm{N}_G(Q)/Q])$. (In case A is a G-algebra over $\mathbb{F}$, it suffices to have an analogous equation in $k\mathrm{G}_0(\mathbb{F}[\mathrm{N}_G(Q)/Q])$.) From M, P, and the sets $\mathscr{C}_Q$ construct the set $\mathscr{C}$ of pairs (H, U) where

1. *H is a subgroup of G with normal p-subgroup Q, which is contained in P, such that $H/Q \in \mathscr{C}_Q$;*

2. *U is an indecomposable $A * H$-module such that, for some $g \in G$, some vertex R of U is contained in $Q \cap {}^gP$, and some R-source of U is a direct summand of $\operatorname{Res}_R^{^gP} {}^gN$.*

Then there are $a_{(H,U)} \in k$ $((H,U) \in \mathscr{C})$ *such that*

$$[M] = \sum_{(H,U)\in\mathscr{C}} a_{(H,U)} \operatorname{ind}_H^G[U]$$

in $\mathrm{a}(A * G)$.

Proof. Let $\mathbf{M}$ be the Mackey subfunctor of $\mathbf{a}(A, G)$ defined in Example 4.5(3) (with respect to M), so $[M] \in \mathbf{M}(G)$, and $\mathbf{M}$ is closed under taking direct summands. Note that, for a subgroup H of G as in item 1 with normal p-subgroup Q contained in P, an indecomposable $A * H$-module U satisfies the assertions of item 2, if and only if U is relatively Q-projective and $[U] \in \mathbf{M}(H)$, by definition of $\mathbf{M}$.

Let L be an indecomposable $A * \mathrm{N}_G(P)$-module which is the Green correspondent of M. By Theorem 4.1,

$$\operatorname{Ind}_{\mathrm{N}_G(P)}^G L \cong M \oplus M_0 \oplus \cdots \oplus M_n,$$

where $M_0, \dots, M_n$ are indecomposable $A * G$-modules having a vertex properly contained in P. Since $\mathbf{M}$ is closed under taking direct summands, we have $[L] \in \mathbf{M}(\mathrm{N}_G(P))$, and thus $[M_0], \dots, [M_n] \in \mathbf{M}(G)$. This yields an equation

$$[M] = \operatorname{ind}_{\mathrm{N}_G(P)}^G[L] - [M_0] - \cdots - [M_n]$$

in $\mathbf{M}(G)$. Arguing inductively on the order of P, this demonstrates that there are integers $c_0, \dots, c_r$, subgroups $P_0, \dots, P_r$ of P, and, for $i = 0, \dots, r$, there is an indecomposable $A * \mathrm{N}_G(P_i)$-module L_i with $[L_i] \in \mathbf{M}(\mathrm{N}_G(P_i))$ such that

$$[M] = \sum_{i=0}^{r} c_i \operatorname{ind}_{\mathrm{N}_G(P_i)}^G[L_i] \tag{3}$$

in $\mathbf{M}(G)$.

Fix $i \in \{0, \dots, r\}$. By Proposition 4.6, there are an $\mathrm{N}_G(P_i)/P_i$-algebra B and a homomorphism of Mackey functors for $\mathrm{N}_G(P_i)/P_i$

$$\varphi = (\varphi_{H/P_i})_{H/P_i \le \mathrm{N}_G(P_i)/P_i} : k\mathbf{K}_0(B, \mathrm{N}_G(P_i)/P_i) \to k\mathbf{M}^{P_i}(\mathrm{N}_G(P_i))$$

(depending on i, of course) such that $\varphi_{\mathrm{N}_G(P_i)/P_i}([B]) = [L_i]$. (Where in the equation above $\mathbf{M}$ is regarded as Mackey functor for $\mathrm{N}_G(P_i)$ by restriction, which is a Mackey subfunctor of $\mathbf{a}(A, \mathrm{N}_G(P_i))$, and which is closed under taking direct summands, as well.) Moreover, $k\mathbf{K}_0(B, \mathrm{N}_G(P_i)/P_i)$ is a right $k\mathbf{G}_0(\mathscr{O}, \mathrm{N}_G(P_i)/P_i)$-module ($k\mathbf{G}_0(\mathbb{F}, \mathrm{N}_G(P_i)/P_i)$-module), by Proposition 3.6. (See also Remark 3.7.) By assumption, there are $b_{H/P_i} \in k$ and

$\mathscr{O}[H/P_i]$-modules ($\mathbb{F}[H/P_i]$-modules) N_{H/P_i} ($H/P_i \in \mathscr{C}_{P_i}$) such that

$$1 = \sum_{H/P_i \in \mathscr{C}_{P_i}} b_{H/P_i} \operatorname{ind}_{H/P_i}^{\mathrm{N}_G(P_i)/P_i} [\![N_{H/P_i}]\!]$$

in $k\mathrm{G}_0(\mathscr{O}[\mathrm{N}_G(P_i)/P_i])$ (resp. $k\mathrm{G}_0(\mathbb{F}[\mathrm{N}_G(P_i)/P_i])$). Thus, in $k\mathrm{K}_0(B * \mathrm{N}_G(P_i)/P_i)$, we have an equation

$$\begin{aligned} [B] = [B] \cdot 1 &= \sum_{H/P_i \in \mathscr{C}_{P_i}} b_{H/P_i} [B] \cdot \left(\operatorname{ind}_{H/P_i}^{\mathrm{N}_G(P_i)/P_i} [\![N_{H/P_i}]\!] \right) \\ &= \sum_{H/P_i \in \mathscr{C}_{P_i}} b_{H/P_i} \operatorname{ind}_{H/P_i}^{\mathrm{N}_G(P_i)/P_i} \left(\left(\operatorname{res}_{H/P_i}^{\mathrm{N}_G(P_i)/P_i} [B] \right) \cdot [\![N_{H/P_i}]\!] \right) \\ &= \sum_{H/P_i \in \mathscr{C}_{P_i}} b_{H/P_i} \operatorname{ind}_{H/P_i}^{\mathrm{N}_G(P_i)/P_i} \left(\left[B \otimes_{\mathscr{O}} N_{H/P_i} \right] \right). \end{aligned}$$

Applying $\varphi_{\mathrm{N}_G(P_i)/P_i}$ to this equation and using that φ is a homomorphism of Mackey functors, we obtain an equation

$$[L_i] = \sum_{H/P_i \in \mathscr{C}_{P_i}} c_H \operatorname{ind}_H^{\mathrm{N}_G(P_i)} \left[L_H^{(i)} \right]$$

in $\mathbf{M}(\mathrm{N}_G(P_i))$, where, for $H/P_i \in \mathscr{C}_{P_i}$, $c_H \in k$ and $L_H^{(i)}$ is a relatively P_i-projective $A * H$-module such that $[L_H^{(i)}] \in \mathbf{M}(H)$, by Proposition 4.6. Since i was arbitrary, this equation together with Eq. (3) and the observation of the first paragraph of this proof yields an equation of the desired form. ∎

We adopt the notation of Theorem 4.7 and its proof. Note that the above proof demonstrates that it would suffice to assume that $1 \in k\mathrm{G}_0(\mathscr{O}[\mathrm{N}_G(Q)/Q])$ can be written as a k-linear combination of modules induced from modules of groups contained in $\mathscr{C}_Q$ for all $Q \in \{P_0, \ldots, P_r\}$, rather than for all subgroups Q of P. Apart from that, the above proof shows that the assertion of Theorem 4.7 can be improved if one replaces throughout the Green correspondents with respect to the normalizer of a vertex by the Green correspondents with respect to the inertia group of a source in this normalizer (cf. [13, Proposition (20.8) and Exercise (20.4)(c)], for instance). In this case, one does not necessarily need induction theorems for $k\mathrm{G}_0(\mathscr{O}[\mathrm{N}_G(Q)/Q])$ (where Q runs through the subgroups of P), but only for the Grothendieck group of the group algebra of certain (possibly strict) subgroups of $\mathrm{N}_G(Q)/Q$.

We can now easily finish our proof of Theorem A.

Proof of Theorem A. Assume that $\mathbb{K}$ and $\mathbb{F}$ are splitting fields for G and all of its subgroups. For each subgroup Q of P, there is an isomorphism of Mackey functors

$$\mathbf{R}(\mathrm{N}_G(Q)/Q) \to \mathbf{G}_0(\mathscr{O}, \mathrm{N}_G(Q)/Q),$$

and there is an epimorphism of Mackey functors $\mathbf{R}(\mathrm{N}_G(Q)/Q) \to \mathbf{G}_0(\mathbb{F}, \mathrm{N}_G(Q)/Q)$, by Remark 3.7; thus the assumption gives rise to an equation in $\mathbf{G}_0(\mathscr{O}[\mathrm{N}_G(Q)/Q])$ ($\mathbf{G}_0(\mathbb{F}[\mathrm{N}_G(Q)/Q])$) as in the hypotheses of Theorem 4.7. Now, apply Theorem 4.7 to the G-algebra $A := \mathscr{O}$ ($A := \mathbb{F}$) with trivial G-action, and the result follows.

To prove the assertion of item 3 of the remark succeeding Theorem A use the canonical isomorphism of Mackey functors $\mathbf{R}(\mathrm{N}_G(Q)/Q)_{p'} \to \mathbf{G}_0(\mathbb{F}, \mathrm{N}_G(Q)/Q)$ mentioned in Remark 3.7 for all Q. ∎

5. AN ANALOGUE OF BRAUER'S INDUCTION THEOREM

From now on, let the residue field $\mathbb{F}$ be algebraically closed, and assume that $\mathbb{K}$ is a splitting field for G and all of its subgroups.

In this section we will prove the following result:

5.1. THEOREM. *Let G be a finite group, let A be a G-algebra, and let M be an indecomposable $A * G$-module with vertex P and P-source L. Moreover, denote by $\mathscr{C}$ the set of pairs (H, U) where*

1. *H is a subgroup of G with normal Sylow p-subgroup Q, which is contained in P, such that H/Q is elementary;*
2. *U is an indecomposable $A * H$-module with vertex Q, and $\mathrm{Res}^H_Q U$ is an indecomposable direct summand of $\mathrm{Res}^P_Q L$. In particular, $\mathrm{Res}^H_Q U$ is a Q-source of U.*

Then there are integers $a_{(H,U)}$ ($(H, U) \in \mathscr{C}$) such that

$$[M] = \sum_{(H,U)\in\mathscr{C}} a_{(H,U)} \operatorname{ind}^G_H [U]$$

in $\mathrm{a}(A * G)$.

Apart from Theorem 4.7, our proof of Theorem 5.1 will be based on the following modular analogue of the well-known fact that supersolvable groups are M-groups.

Before we state the result, let us briefly recall the notion of defect groups of (points of) G-algebras. Let A be a G-algebra, and let e be a primitive idempotent in A^G. Then a subgroup D of G is called a defect group of e if $e \in A_D^G$ but $e \notin A_Q^G$ for any proper subgroup Q of D. It is well known that the defect groups of e form a conjugacy class of p-subgroups of G. Moreover, any idempotent which is conjugate to e by a unit of A^G has the same defect groups as e. A $\mathrm{U}(A^G)$-conjugacy class of primitive idempotents is called a point of G on A (where $\mathrm{U}(A^G)$ denotes the group of units of A^G). Hence we may speak of defect groups of points also. We refer to [11] or [13] for details.

For the purpose of this section a module theoretic viewpoint will be convenient. By Example 2.3(1), A may be regarded as an $A * G$-module, and it is clear that Ae is a direct summand of this module. Moreover, it is straightforward to check that $\mathrm{End}_A(Ae) \to eA^{\circ}e$, $\psi \mapsto \psi(e)$, is an isomorphism of G-algebras. Since e is primitive in A^G (and hence in eA^Ge), this shows that Ae is an indecomposable $A * G$-module. Besides, for all subgroups H of G, $e(A_H^G)e = (eAe)_H^G$, so $e \in A_H^G$, if and only if $e \in (eAe)_H^G$. Therefore, by Higman's criterion (Proposition 2.14), the defect groups of e are precisely the vertices of Ae.

The following result was proved in [10]:

5.2. PROPOSITION [10, Proposition 3.3]. *Let P be a normal Sylow p-subgroup of a finite group G such that G/P is supersolvable, let A be a G-algebra, and let e be a primitive idempotent in A^G with defect group D. Then there are a subgroup H of G with Sylow p-subgroup D and an idempotent $f \in A^H$ such that f is primitive in A^D, $\mathrm{Tr}_H^G(f) = e$, and $f({}^g f) = 0$ for all $g \in G \setminus H$.*

Proof of Theorem 5.1. For any subgroup Q of P, let $\mathscr{C}_Q$ be the set of elementary subgroups of $\mathrm{N}_G(Q)/Q$. Denote by $\mathscr{E}$ the set of pairs (H, U) where

1. H is a subgroup of G with normal p-subgroup Q, which is contained in P, such that $H/Q \in \mathscr{C}_Q$ (i.e., H/Q is elementary);
2. U is an indecomposable $A * H$-module such that, for some $g \in G$, some vertex R of U is contained in $Q \cap {}^gP$, and some R-source of U is a direct summand of $\mathrm{Res}_R^{{}^gP}\, {}^gL$.

By Theorem 4.7, applied to Brauer's induction theorem, there are integers $b_{(H,U)}$ ($(H, U) \in \mathscr{E}$) such that

$$[M] = \sum_{(H,U)\in\mathscr{E}} b_{(H,U)} \,\mathrm{ind}_H^G[U] \tag{4}$$

in $\mathrm{a}(A * G)$.

Let $(H, U) \in \mathscr{E}$. Then $H/\mathrm{O}_p(H)$ is an elementary p'-group. In particular, $\mathrm{O}_p(H)$ is a normal Sylow p-subgroup of H such that $H/\mathrm{O}_p(H)$ is supersolvable. Moreover, for $x \in G$, we have

$$\mathrm{Ind}_H^G U \cong {}^x\big(\mathrm{Ind}_H^G U\big) \cong \mathrm{Ind}_{{}^xH}^G\, {}^xU,$$

i.e., $\mathrm{ind}_H^G[U] = \mathrm{ind}_{{}^xH}^G[{}^xU]$. Replacing H and U in (4) by suitable conjugates, we obtain

$$[M] = \sum_{(H,U)\in\tilde{\mathscr{E}}} b_{(H,U)}\, \mathrm{ind}_H^G[U]$$

in $\mathrm{a}(A * G)$, where $\tilde{\mathscr{E}}$ denotes the set of pairs (H,U) consisting of:

1. a subgroup H of G with normal Sylow p-subgroup $\mathrm{O}_p(H)$ such that $H/\mathrm{O}_p(H)$ is elementary,

2. an indecomposable $A * H$-module U such that some vertex R of U is contained in P, and some R-source of U is a direct summand of $\mathrm{Res}_R^P L$.

Now let $(H,U) \in \tilde{\mathscr{E}}$. Consider the H-algebra $B := \mathrm{End}_A(U)$. Since U is an indecomposable $A * H$-module with vertex R, 1_B is a primitive idempotent in $B^H = \mathrm{End}_{A*H}(U)$ with defect group R, by Higman's criterion. By Proposition 5.2, there is a subgroup K of H with (normal) Sylow p-subgroup R, and there is an idempotent $f \in B^K$ such that f is primitive in B^R, $f({}^hf) = 0$ for all $h \in H \setminus K$, and $\mathrm{Tr}_K^H(f) = 1_B$.

Then $V := f(U)$ is a direct summand of $\mathrm{Res}_K^H U$, and we have

$$U = \bigoplus_{hK\in H/K} 1_A * hV \cong \mathrm{Ind}_K^H V,$$

as one easily verifies. Moreover, $\mathrm{Res}_R^K V$ is indecomposable. (Note that $\mathrm{End}_{A*R}(V)$ is isomorphic to fB^Rf.) Thus R is a vertex of V, and $\mathrm{Res}_R^K V$ is an R-source of V. Hence $\mathrm{Res}_R^K V$ is an R-source of U, as well. Therefore, $\mathrm{Res}_R^K V$ is (isomorphic to) an indecomposable direct summand of $\mathrm{Res}_R^{{}^xP}{}^xL$ for some $x \in \mathrm{N}_H(R)$. Replacing K and V by ${}^{x^{-1}}K$ and ${}^{x^{-1}}V$, respectively, we may assume that $x = 1$. Then

$$\mathrm{ind}_H^G[U] = \mathrm{ind}_H^G\left[\mathrm{Ind}_K^H V\right] = \mathrm{ind}_K^G[V],$$

and K and V have the desired properties, so the proof is complete. ∎

Proof of Theorem B. Apply Theorem 5.1 to the G-algebra $A := \mathscr{O}$ with trivial G-action. ∎

We prove a skew group algebra version of Theorem C.

5.3. THEOREM. *We adopt the notation of Theorem* 4.7. *Assume, in addition, that* $\mathbb{F}$ *is algebraically closed, and that, for any subgroup* Q *of* P, *all the elements of* $\mathscr{C}_Q$ *are p-solvable subgroups of* $\mathrm{N}_G(Q)/Q$. *Denote by* $\mathscr{D}$ *the set of pairs* (H,U) *where*

1. *H is a subgroup of P such that $\mathrm{O}_p(H)$ is contained in P, and such that $H/\mathrm{O}_p(H)$ is a Hall p'-subgroup of an element of $\mathscr{C}_{\mathrm{O}_p(H)}$;*

2. *U is an indecomposable $A * H$-module such that, for some $g \in G$, some vertex R of U is contained in $O_p(H) \cap {}^gP$, and some R-source of U is a direct summand of* $\mathrm{Res}_R^{{}^gP}\,{}^gN$.

Then there are $a_{(H,U)} \in k$ $((H,U) \in \mathscr{D})$ such that

$$[M] = \sum_{(H,U)\in\mathscr{D}} a_{(H,U)} \operatorname{ind}_H^G [U]$$

in $k\mathrm{a}(A * G)$.

Theorem 5.3 is a corollary of Theorem 4.7 and the following result of [10]:

5.4. THEOREM [10, Theorem 3.1]. *Let H be a Hall p'-subgroup of a finite p-solvable group G, let A be a G-algebra, and let e be an idempotent in A_H^G. Then there is an idempotent $f \in A^H$ such that $f({}^gf) = 0$ for all $g \in G \setminus H$ and* $\mathrm{Tr}_H^G(f) = e$.

Theorem 5.4 serves as a basis for the proof of Proposition 5.2 in [10]. We need a module version of this result:

5.5. COROLLARY. *Let N be a normal subgroup of a finite group G such G/N is p-solvable, let H/N be a Hall p'-subgroup of G, and let A be a G-algebra. Then any relatively N-projective $A * G$-module M is induced from H, i.e., $M \cong \mathrm{Ind}_H^G L$ for some $A * H$-module L.*

In particular, any projective $\mathscr{O}G$-module, for some finite p-solvable group G, is induced from a Hall p'-subgroup of G. This is a classical result of Fong [8]. On the other hand, Theorem 5.4 can also be regarded as a generalization of Green's indecomposability theorem.

Proof of Corollary 5.5. Set $\overline{G} := G/N$ and set $\overline{H} := H/N$. Consider the $\overline{G}$-algebra $B := \mathrm{End}_{A*N}(M)$. Since P is relatively N-projective, we have

$$1_B = \mathrm{id}_M = \mathrm{Tr}_N^G(\beta) = \mathrm{Tr}_1^{\overline{G}}(\beta)$$

for some $\beta \in \mathrm{End}_{A*N}(M) = B$, by Higman's criterion, i.e., $1_B \in B_1^{\overline{G}}$. Thus, by Theorem 5.4, there is an idempotent in $B^{\overline{H}}$ such that $f({}^gf) = 0$ for all $g \in \overline{G} \setminus \overline{H}$ and $\mathrm{id}_M = 1_B = \mathrm{Tr}_{\overline{H}}^{\overline{G}}(f) = \mathrm{Tr}_H^G(f)$. Thus $f(M)$ is a direct summand of $\mathrm{Res}_H^G M$, and we have

$$M \cong \bigoplus_{gH\in G/H} 1_A * gf(M) \simeq \mathrm{Ind}_H^G f(M),$$

so we are done. ∎

Proof of Theorem 5.3. In view of Theorem 4.7, it suffices to consider the case that P is contained in some normal p-subgroup Q of G, and that

$G/Q \in \mathscr{C}_Q$. In particular, G is p-solvable. Moreover, M is relatively Q-projective. By Corollary 5.5, $M \cong \mathrm{Ind}_H^G L$ for some $A * H$-module L, where H/Q is a Hall p'-subgroup of G/Q. Thus $[M] = \mathrm{ind}_H^G[L]$ in $k\mathrm{a}(A * G)$, and Q is a normal Sylow p-subgroup of H. Conjugating if necessary, we may assume that P is also a vertex of L, and N is a P-source of L. This completes the proof. ∎

Theorem C now follows immediately by choosing $A := \mathscr{O}$ in Theorem 5.3.

Our proof of Proposition D will essentially be based on Proposition 3.6. Before we embark on the proof, let us recall a simple fact about idempotents. Let A be a G-algebra, and let e and f be idempotents in A^G. As mentioned in the paragraph preceding Proposition 5.2, Ae and Af are $A * G$-direct summands of A, where $A(1_A - e)$ and $A(1_A - f)$ are complements of Ae and Af in A, respectively. It is easy to see that Ae and Af are isomorphic $A * G$-modules, if and only if there are elements $a \in eA^G f$ and $b \in fA^G e$ such that $e = ab$ and $f = ba$. The latter assertion in turn is equivalent to saying that e and f are conjugate by a unit in A^G, as one easily shows using the Krull–Schmidt theorem.

Proof of Proposition D. ($\Rightarrow$) The implication "$\Rightarrow$" is trivial.

($\Leftarrow$) First of all, assume that A is a G-algebra over $\mathbb{F}$. Denote by $\mathscr{C}$ the set of cyclic p'-subgroups of G. By assumption and the above remarks, $\mathrm{Res}_C^G Ae$ and $\mathrm{Res}_C^G Af$ are isomorphic $A * C$-modules for each $C \in \mathscr{C}$. Moreover, Ae and Af are relatively 1-projective $A * G$-modules since $e, f \in A_1^G$. Hence Ae and Af are projective $A * G$-modules since these modules are plainly projective as A-modules. Thus $\mathrm{res}_C^G[Ae] = \mathrm{res}_C^G[Af]$ in $\mathrm{K}_0(A * C)$ for all $C \in \mathscr{C}$.

By Artin's induction theorem for the Brauer character ring, there are a rational number a_C and an $\mathbb{F}C$-module L_C for $C \in \mathscr{C}$ such that

$$1 = \sum_{C \in \mathscr{C}} a_C \, \mathrm{ind}_C^G [\![L_H]\!]$$

in $\mathbb{Q}\mathrm{G}_0(\mathbb{F}G)$. Therefore, by Proposition 4.6, we have

$$\begin{aligned}
[Ae] = [Ae] \cdot 1 &= \sum_{C \in \mathscr{C}} a_C [Ae] \cdot \left(\mathrm{ind}_C^G [\![L_H]\!]\right) \\
&= \sum_{C \in \mathscr{C}} a_C \, \mathrm{ind}_C^G\left(\left(\mathrm{res}_C^G [Ae]\right) \cdot [\![L_H]\!]\right) \\
&= \sum_{C \in \mathscr{C}} a_C \, \mathrm{ind}_C^G\left(\left(\mathrm{res}_C^G [Af]\right) \cdot [\![L_H]\!]\right) = \cdots = [Af]
\end{aligned}$$

in $\mathbb{Q}\mathrm{K}_0(A * G)$. Since $\mathrm{K}_0(A * G)$ is a free abelian group, the canonical map

$$\mathrm{K}_0(A * G) \to \mathbb{Q}\mathrm{K}_0(A * G)$$

is injective. Thus $[Ae] = [Af]$ in $\mathrm{K}_0(A * G)$. Hence Ae and Af are isomorphic $A * G$-modules, so e and f are conjugate by a unit in A^G.

Now, if A is a G-algebra over $\mathcal{O}$ the result follows from the above and the well-known fact that the canonical map $\mathrm{K}_0(A * G) \to \mathrm{K}_0((A/\mathrm{J}(\mathcal{O})A) * G)$ is an isomorphism of groups. ∎

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