JOURNAL OF DIFFERENTIAL EQUATIONS **103**, 205–219 (1993)

Eigenvalue Ratios for Sturm–Liouville Operators

MARK S. ASHBAUGH*

*Department of Mathematics, University of Missouri,
Columbia, Missouri 65211*

AND

RAFAEL D. BENGURIA†

*Facultad de Física, P. Universidad Católica de Chile,
Avda. Vicuña Mackenna 4860, Casilla 6177, Santiago 22, Chile*

Received January 17, 1991

In this paper we prove various optimal bounds for eigenvalue ratios for the Sturm–Liouville equation $-[p(x)y']'+q(x)y=\lambda w(x)y$ and certain specializations. Our results primarily concern the regular case with Dirichlet boundary conditions though various extensions and generalizations to other situations are possible. Our results here extend the result $\lambda_m/\lambda_1 \leqslant m^2$ obtained in a previous paper for the one-dimensional Schrödinger equation, $-y''+q(x)y=\lambda y$, on a finite interval with Dirichlet boundary conditions and nonnegative potential ($q \geqslant 0$). In particular, we obtain $\lambda_m/\lambda_1 \leqslant Km^2/k$, where the constants k, K satisfy $0<k\leqslant p(x)\,w(x)\leqslant K$ for all x. If $q\equiv 0$, lower bounds can also be obtained. Our methods involve a slight modification of the Prüfer variable techniques employed in the Schrödinger case. We also examine the consequences of our recent proof of the Payne–Pólya–Weinberger conjecture in the one-dimensional (Sturm–Liouville) setting. Finally, we compare our general bounds to the detailed analyses of Keller and of Mahar and Willner for the special case of the inhomogeneous stretched string.

1. INTRODUCTION AND STATEMENT OF RESULTS

We consider the Sturm–Liouville problem

$$-[p(x)y']'+q(x)y=\lambda w(x)y. \tag{1.1}$$

We are concerned only with the regular case though extensions to other problems with weaker hypotheses would certainly be possible. In par-

* Work partially supported by a Summer Research Fellowship provided by the Research Council of the University of Missouri–Columbia.

† Supported in part by FONDECYT (Chile) under project 1238–90.

0022-0396/93 $5.00

ticular, we consider (1.1) on a finite interval $\Omega = (\alpha, \beta)$ with continuous coefficients p, q, and w on $\bar{\Omega} = [\alpha, \beta]$ satisfying $b \leqslant p(x) \leqslant B$, $c \leqslant w(x) \leqslant C$, $q(x) \geqslant 0$ for all $x \in \bar{\Omega}$, where b, B, c, and C are positive constants. Also, we consider only the case of (homogeneous) Dirichlet boundary conditions here, i.e., $y(\alpha) = 0 = y(\beta)$. Undoubtedly, some of our results could be extended in some form to other boundary conditions or problems with weaker hypotheses on the coefficient functions but since this is not our primary objective here we leave these considerations aside. Since we are concerned only with bounding eigenvalue ratios it is clear that nothing is lost by always considering problems on the standarized interval $\Omega = (0, 1)$ (i.e., a linear change of variable can always accomplish this while leaving all eigenvalue ratios unchanged).

In this paper we prove several generalizations of our optimal bounds [4] (see also the earlier papers [2,3])

$$\lambda_m/\lambda_1 \leqslant m^2 \qquad \text{for} \quad m = 2, 3, 4, \ldots \tag{1.2}$$

and

$$\lambda_m/\lambda_l \leqslant \{m/l\}^2 \qquad \text{for} \quad m > l \geqslant 1 \tag{1.3}$$

for the Dirichlet eigenvalues of the one-dimensional Schrödinger equation $-y'' + q(x)y = \lambda y$ on a finite interval with $q \geqslant 0$, where $\{x\}$ denotes the least integer greater than or equal to x. Specifically, we prove

$$\lambda_m/\lambda_1 \leqslant Km^2/k \qquad \text{for} \quad m = 2, 3, 4, \ldots \tag{1.4}$$

for the Dirichlet eigenvalues of (1.1), where k and K are positive constants such that $0 < k \leqslant p(x)\, w(x) \leqslant K$. We also prove the two-sided inequality

$$km^2/Kl^2 \leqslant \lambda_m/\lambda_l \leqslant Km^2/kl^2 \tag{1.5}$$

for the case where $q \equiv 0$ in addition to the hypotheses in effect for (1.4). Finally, we give some improvements of (1.4) that follow from our recent proof of the Payne–Pólya–Weinberger conjecture [5, 6], discuss the relation of our bounds to some results of Keller [11] and of Mahar and Willner [14] for eigenvalue ratios for the stretched string, and give some complementary results based on the transformations available through changes of variables.

2. The Modified Prüfer Transformation

In this section we introduce the modified Prüfer transformation that is the key to our results and present the differential equations for the Prüfer variables. Our variables here represent a slight further modification of the Prüfer variables we used in [4]. Prüfer variables similar to those of [4] were also introduced by Crandall and Reno in [9], though with a different purpose in mind.

The Prüfer variables $r(x)$, $\theta(x)$ that we use here are defined by

$$y = r(x)\sin[a\sqrt{\lambda}\,\theta(x)] \tag{2.1a}$$

$$py' = a\sqrt{\lambda}\,r(x)\cos[a\sqrt{\lambda}\,\theta(x)]. \tag{2.1b}$$

These differ from the standard Prüfer variables by the explicit appearance of factors of $a\sqrt{\lambda}$. In [4] we used this transformation with $a = 1$. Here we take a to be an explicit constant coming from properties of the coefficients functions in (1.1) when we get to the point of making our comparison argument. The explicit factor $\sqrt{\lambda}$ in (2.1) is particularly useful in proving results about eigenvalue ratios. Specifically, it makes the angle variables θ_1 and θ_m for the first and mth eigenfunctions nearly comparable and this can be exploited to get tight comparison results for θ_1 and θ_m that lead to optimal bounds for λ_m/λ_1 via an argument by contradiction. All this is developed in greater detail in the next section.

Using Eq. (1.1) and the definitions (2.1) one finds the following differential equations for $r(x)$ and $\theta(x)$:

$$\frac{r'}{r} = \frac{1}{2}\left[\sqrt{\lambda}\left(\frac{a}{p} - \frac{w}{a}\right) + \frac{q}{a\sqrt{\lambda}}\right]\sin(2a\sqrt{\lambda}\,\theta) \tag{2.2a}$$

and

$$\theta' = \frac{1}{p} - \left(\frac{1}{p} - \frac{w}{a^2}\right)\sin^2(a\sqrt{\lambda}\,\theta) - \frac{q}{a^2\lambda}\sin^2(a\sqrt{\lambda}\,\theta). \tag{2.2b}$$

Equation (2.2b) is really the key equation for our comparison method. In the next section we compare the angle variables θ_1 and θ_m using the respective equations (2.2b) with the values of λ and a specialized appropriately. This is the basis for our optimal bounds on eigenvalue ratios. More generally, by this same device we can compare θ_l and θ_m for arbitrary indices l and m leading to bounds on the eigenvalue ratio λ_m/λ_l in the case where q is identically zero. This case is discussed in Section 4.

3. Optimal Upper Bounds for Eigenvalue Ratios for $-(py')' + qy = \lambda wy$ with $q \geq 0$

We denote the eigenvalues of (1.1) by $\lambda_1, \lambda_2, \lambda_3, \ldots$ and note that it is well known that

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

under the stated hypotheses. We let y_j, $j=1, 2, 3, ...$, denote the corresponding sequence of normalized eigenfunctions. Then to each solution y_j we can associate a Prüfer angle θ_j via the transformation (2.1) if we also specify the initial condition (we take $\Omega=(0, 1)$ throughout this section)

$$\theta_j(0)=0 \qquad \text{for} \quad j=1, 2, 3, ... \tag{3.1}$$

One then has, by standard results for Prüfer variables [7],

$$\theta_j(1)=j\pi/a\sqrt{\lambda_j} \qquad \text{for} \quad j=1, 2, 3, ... \tag{3.2}$$

and also

$$0<\theta_j(x)<j\pi/a\sqrt{\lambda_j} \qquad \text{for} \quad 0<x<1. \tag{3.3}$$

We begin by proving

THEOREM 3.1. *Consider the regular Sturm–Liouville problem* (1.1) *with hypotheses as formulated there* (*Dirichlet boundary conditions*, $q \geqslant 0$). *Then*

$$\lambda_m/\lambda_1 \leqslant Km^2/k, \tag{3.4}$$

where k and K are positive constants such that $k \leqslant p(x)\, w(x) \leqslant K$. Moreover, equality occurs if and only if $q=0$ and $pw=k=K$.

Proof. Assume we have a problem for which (3.4) is violated. We compare θ_1 and θ_m, where for θ_1 we take the constant $a=a_1=\sqrt{K}$ and for θ_m we take $a=\tilde{a}_m=\sqrt{k}$. Then from (2.2b) we have

$$\theta_1'=\frac{1}{p}-\left(\frac{1}{p}-\frac{w}{K}\right)\sin^2(\sqrt{K\lambda_1}\,\theta_1)-\frac{q}{K\lambda_1}\sin^2(\sqrt{K\lambda_1}\,\theta_1)\equiv F_1(x, \theta_1) \tag{3.5}$$

and

$$\theta_m'=\frac{1}{p}-\left(\frac{1}{p}-\frac{w}{k}\right)\sin^2(\sqrt{k\lambda_m}\,\theta_m)-\frac{q}{k\lambda_m}\sin^2(\sqrt{k\lambda_m}\,\theta_m)\equiv \tilde{F}_m(x, \theta_m). \tag{3.6}$$

We know from (3.3) that $0<\sqrt{k\lambda_m}\,\theta_m<m\pi$ for $0<x<1$. We intend to show that $\tilde{F}_m(x, \theta) \geqslant F_1(x, \theta)$ for $(x, \theta) \in (0, 1) \times (0, m\pi/\sqrt{k\lambda_m})$ since by the comparison result (see Birkhoff and Rota [7, pp. 26–28]) it will follow that

$$\theta_1(x) \leqslant \theta_m(x) \qquad \text{for} \quad 0 \leqslant x \leqslant 1. \tag{3.7}$$

To see that $F_1(x, \theta) \leqslant \tilde{F}_m(x, \theta)$ it suffices to observe that

$$F_1(x, \theta) \leqslant \frac{1}{p}-\frac{q}{K\lambda_1}\sin^2(\sqrt{K\lambda_1}\,\theta) \leqslant \frac{1}{p}-\frac{q}{k\lambda_m}\sin^2(\sqrt{k\lambda_m}\,\theta) \leqslant \tilde{F}_m(x, \theta) \tag{3.8}$$

where the first and last of these inequalities are simple consequences of $k \leqslant p(x) w(x) \leqslant K$ and the middle inequality follows from Proposition 3.2, which concludes this section. From (3.7) and (3.2) it then follows that

$$\frac{\pi}{\sqrt{K\lambda_1}} = \theta_1(1) \leqslant \theta_m(1) = \frac{m\pi}{\sqrt{k\lambda_m}},$$

implying $\lambda_m/\lambda_1 \leqslant Km^2/k$, which contradicts our assumption that (3.4) was violated.

To see that equality in (3.4) occurs if and only if $q=0$ and $k = pw = K$ it suffices to observe that if one of these conditions fails to hold then there will be some subinterval of [0, 1] on which at least one of the inequalities in (3.8) will become strict. This in turn would force (3.7) to become strict across the remainder of [0, 1], yielding the desired contradiction. ∎

As in [4], the optimal upper bounds

$$\lambda_{jl}/\lambda_l \leqslant Kj^2/k \qquad \text{for} \quad j \geqslant 2 \tag{3.9}$$

and

$$\lambda_m/\lambda_l \leqslant K\{m/l\}^2/k \qquad \text{for} \quad m > l \geqslant 1 \tag{3.10}$$

also follow.

We conclude this section with the proposition which established the middle inequality in (3.8), i.e., the inequality

$$\frac{\sin^2(\sqrt{K\lambda_1}\,\theta)}{K\lambda_1} > \frac{\sin^2(\sqrt{k\lambda_m}\,\theta)}{k\lambda_m} \tag{3.11}$$

for $0 < \theta < m\pi/\sqrt{k\lambda_m}$. This is equivalent to

$$\sin x > \frac{|\sin \sqrt{k\lambda_m/K\lambda_1}\, x|}{\sqrt{k\lambda_m/K\lambda_1}} \tag{3.12}$$

for $0 < x < m\pi/\sqrt{k\lambda_m/K\lambda_1}$ and for the sake of contradiction we have assumed $\lambda_m/\lambda_1 > Km^2/k$ so that $m\pi/\sqrt{k\lambda_m/K\lambda_1} < \pi$ and thus $m\pi/\sqrt{k\lambda_m/K\lambda_1}$ is a zero of the right-hand side of (3.12) less than π, the first positive zero of the left-hand side (put differently, by $\lambda_m/\lambda_1 > Km^2/k$ we have $m < \sqrt{k\lambda_m/K\lambda_1}$ and thus $m \leqslant [\sqrt{k\lambda_m/K\lambda_1}]$, where $[\cdot]$ denotes the greatest integer function). Thus the following proposition is sufficient to establish the inequality (3.11) and hence (3.8).

PROPOSITION 3.2. *Let $c>1$. Then for $0<\theta<[c]\,\pi/c$, where $[c]\equiv$ the greatest integer less than or equal to c,*

$$\left|\frac{\sin c\theta}{c}\right| < \sin\theta. \tag{3.13}$$

Proof. We first prove the result when c is an integer $m>1$. That is, we prove that

$$\left|\frac{\sin m\theta}{m\sin\theta}\right| < 1 \qquad \text{for} \quad 0<\theta<\pi. \tag{3.14}$$

To this end, we consider the function

$$f(\theta)=\ln\left|\frac{\sin m\theta}{m\sin\theta}\right|, \tag{3.15}$$

which is well-defined on each of the subintervals $((l-1)\,\pi/m, l\pi/m)$ for $l=1, 2, ..., m$. In addition, f goes to 0 at $\theta=0$ and π and goes to $-\infty$ at each of $\theta=l\pi/m$ for $l=1, 2, ..., m-1$. Further,

$$f(\theta)\sim\frac{-1}{6}(m^2-1)\,\theta^2+O(\theta^4) \qquad \text{at} \quad \theta=0$$

with a similar relation holding at $\theta=\pi$.

Since the inequality (3.14) is equivalent to $f<0$ for $0<\theta<\pi$, we will be done if we can show that $f''<0$ on $(0, \pi/m)$ since this will also take care of the symmetrically placed interval $((m-1)\,\pi/m, \pi)$ and on each of the intermediate subintervals it is clear that $m\sin\theta>1\geqslant\sin m\theta$, the first inequality following from $m\sin\pi/2m>\sin m(\pi/2m)=1$ (implying $m\sin\theta>1$ for $\theta\in[\pi/2m, \pi-\pi/2m]$), which will follow from $f''<0$ on $(0, \pi/m)$.

By using the product representation for $\sin x$, we find

$$f'(\theta)=-\sum_{\substack{l=1\\ l\notin m\mathbb{Z}}}^{\infty}\frac{2m^2\theta}{l^2\pi^2-m^2\theta^2}<0 \qquad \text{for} \quad 0<\theta<\pi/m \tag{3.16}$$

and

$$f''(\theta)=-\sum_{\substack{l=1\\ l\notin m\mathbb{Z}}}^{\infty}\frac{2m^2(l^2\pi^2+m^2\theta^2)}{(l^2\pi^2-m^2\theta^2)^2}<0 \tag{3.17}$$

for all $\theta\neq l\pi/m$ with l an integer which is not a multiple of m. This finishes the proof of (3.14).

To complete the proof of Proposition 3.2 we fix c and take $m=[c]$ in (3.14). Next observe that the graph of

$$y=\frac{|\sin cx|}{c}$$

is obtained from that of $y=|\sin mx|/m$ by a contraction of the xy-plane by the factor $[c]/c=m/c$ (i.e., $y=|\sin mx|/m$ goes into $y'=|\sin cx'|/c$ under $(x',y')=m(x,y)/c$). Thus, the graph of $y=|\sin cx|/c$ is just that of $y=|\sin mx|/m$ shrunk toward the origin by the factor m/c. Since $y=|\sin mx|/m$ lies below $y=\sin x$ and $\sin x$ is concave down on $[0,\pi]$ (so that any ray from the origin intersects $y=\sin x$ for $0<x\leqslant\pi$ at most once) it is clear that $y=|\sin cx|/c$ also lies below $y=\sin x$ for $0<x<[c]\,\pi/c$, thus completing the proof. ∎

Remarks. Another approach to this proposition is sketched on page 407 of [4]. We give this alternative proof here since it illuminates some of the ideas that go into the similar analyses involving Bessel functions that are needed in our proof of the Payne–Pólya–Weinberger conjecture. In fact, the proof given here can be viewed as an alternative proof of the $p=1/2$ case of our Lemma 3.5 and its consequence, Eq. (3.30), in [6].

4. Eigenvalue Ratios for Sturm-Liouville Equations with $q\equiv 0$

In this section we derive the more general bounds on eigenvalue ratios that can be obtained in the absence of the potential function q. It turns out that the argument is now much simpler in that the detailed comparison of $\sin x$ and $|\sin mx/m|$ is no longer required and this allows us to find both upper and lower bounds on eigenvalue ratios.

For arbitrary indices m and l, with $\tilde{a}_m=\sqrt{k}$ and $a_l=\sqrt{K}$, we have the equations

$$\theta_l'=\frac{1}{p}-\left(\frac{1}{p}-\frac{w}{K}\right)\sin^2(\sqrt{K\lambda_l}\,\theta_l)\equiv G_l(x,\theta_l) \tag{4.1}$$

and

$$\theta_m'=\frac{1}{p}-\left(\frac{1}{p}-\frac{w}{k}\right)\sin^2(\sqrt{k\lambda_m}\,\theta_m)\equiv \tilde{G}_m(x,\theta_m). \tag{4.2}$$

Then we can use the same differential comparison argument that we used in the previous section to obtain

THEOREM 4.1. *Consider the regular Sturm–Liouville problem* $-(py')' = \lambda wy$ *on a finite interval with Dirichlet boundary conditions and suppose* k *and* K *are positive constants such that* $k \leqslant p(x)\, w(x) \leqslant K$ *for all* x *in the interval. Then*

$$\lambda_m/\lambda_l \leqslant Km^2/kl^2 \tag{4.3}$$

with equality if and only if $pw = k = K$.

Proof. By (4.1) and (4.2) it is clear that

$$G_l(x, \theta) \leqslant 1/p \leqslant \tilde{G}_m(x, \theta). \tag{4.4}$$

It then follows by the differential comparison argument [7, pp. 26–28] that (taking $\Omega = (0, 1)$ again)

$$\theta_l(x) \leqslant \theta_m(x) \qquad \text{for} \quad 0 \leqslant x \leqslant 1 \tag{4.5}$$

and thus

$$l\pi/\sqrt{K\lambda_l} = \theta_l(1) \leqslant \theta_m(1) = m\pi/\sqrt{k\lambda_m},$$

which is equivalent to (4.3). The characterizations of the cases of equality follow as in our proof of Theorem 3.1 in Section 3 above. ∎

By interchanging the roles of m and l in (4.3) we arrive at the two-sided bounds

$$km^2/Kl^2 \leqslant \lambda_m/\lambda_l \leqslant Km^2/kl^2. \tag{4.6}$$

Note that without q we get upper and lower bounds and our upper bound does not involve $\{m/l\}^2$, as does the bound (3.10). This fact is true because we can no longer use q to fashion a double- or multiple-well situation (cf. p. 410 of [4]). However, we caution the reader that p and w can still be used to fashion multiple-well situations; these require the ratio K/k to be large and in these cases the bounds in (4.6) become very poor.

5. Improved Ratio Results Based on Our Proof of the Payne–Pólya–Weinberger conjecture

Based on our proof of the Payne–Pólya–Weinberger conjecture [5, 6] (see [16, 17] for the original papers of Payne, Pólya, and Weinberger) we proved [6]

$$\frac{\lambda_2}{\lambda_1} \leqslant 1 + \frac{BC}{bc}\,(j_{n/2,1}^2/j_{n/2-1,1}^2 - 1) \tag{5.1}$$

for the ratio of the first two eigenvalues of the n-dimensional elliptic eigenvalue problem

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + q(x) u = \lambda w(x) u \tag{5.2}$$

on a bounded domain Ω with Dirichlet boundary conditions imposed on u. Here $[a_{ij}]$ is taken as a symmetric matrix with $0 < b \leqslant [a_{ij}] \leqslant B$ (in the sense of quadratic forms), $q \geqslant 0$, and $0 < c \leqslant w \leqslant C$, where b, B, c, and C are constants. In (5.1), the notation $j_{p,k}$ represents the kth positive zero of the Bessel function $J_p(x)$.

For us here, $n = 1$ so we need only the facts $j_{1/2,1} = \pi$ and $j_{-1/2,1} = \pi/2$. We then have

$$\lambda_2/\lambda_1 \leqslant 1 + 3BC/bc, \tag{5.3}$$

but with a little more work we can do better. In particular, we obtain

THEOREM 5.1. *Consider the regular Sturm–Liouville problem* (1.1) *with Dirichlet boundary conditions and* $q \geqslant 0$. *Let* k, K *be positive constants such that* $k \leqslant p(x)\, w(x) \leqslant K$. *Then the first two eigenvalues of this problem satisfy*

$$\lambda_2/\lambda_1 \leqslant 1 + 3K/k \tag{5.4}$$

with equality if and only if $k = K$ *and* $q \equiv 0$.

Remark. Obviously, $k = bc$, $K = BC$ will work here and give back (5.3). The point, though, is that we can do better.

Proof. To obtain the improved result we simply make the change of variable

$$t = \int_{\alpha}^{x} \frac{ds}{p(s)}. \tag{5.5}$$

Since p is positive and bounded away from 0 and $1/p$ is integrable it is clear that this leads to the new regular Sturm–Liouville problem

$$-\ddot{y} + p(x(t))\, q(x(t))\, y = \lambda p(x(t))\, w(x(t))\, y, \tag{5.6}$$

again with Dirichlet boundary conditions. By applying our previous result (inequality (5.3)) to this equation we immediately obtain (5.4) since we have $k \leqslant p(x(t))\, w(x(t)) \leqslant K$ for all t in the interval under consideration.

The case of equality follows from the fact that equality in (5.3) holds if and only if $B = b = p$, $q = 0$, and $C = c = w$.

Remarks. (1) The transformation (5.5) "explains" why only $p(x)\,w(x)$ matters in our bounds. This fact also is evident from (2.2b) though our new viewpoint makes it more transparent. We could, in fact, have developed our entire argument in Section 2 with one of p or w identically one and then passed back to the general case by using the transformation (5.5). Use of the transformation (5.5) is old, being used to good effect by Leighton in [12, pp. 227–228], for example.

(2) The case of equality where $p(x)\,w(x)=\text{constant}$ and $q(x)=0$ for all x has some interesting sidelights. In this case, of course, one can give the eigenfunctions of (5.6), and hence also of (1.1), explicitly. In particular, one finds (neglecting the normalization constant)

$$y_m=\sin\left(m\pi t\Big/\int_\alpha^\beta\frac{dx}{p(x)}\right)=\sin\left(m\pi\int_\alpha^x\frac{ds}{p(s)}\Big/\int_\alpha^\beta\frac{dx}{p(x)}\right).$$

Again, some discussion of such problems may be found in [12, p. 218, problems 9, 10]. Also, McLaughlin was able to make use of an extension of this idea to obtain comparison results and inequalities for individual eigenvalues of Sturm–Liouville problems in [15].

(3) It should be noted that the result of this section improves only upon our earlier upper bound for λ_2/λ_1. Eigenvalue ratios such as λ_m/λ_1 for $m\geqslant 3$ do not seem to be amenable to the approach used in our proof of the Payne–Pólya–Weinberger conjecture. However, one can push (5.4) to apply to the ratio λ_{2m}/λ_m. That is, we have

$$\lambda_{2m}/\lambda_m\leqslant 1+3K/k \tag{5.7}$$

for $m=1,2,3,\ldots$, which improves upon the $j=2$ case of (3.9) above. This follows as in our proof of Proposition 3.2 in [4, p. 409], using the fact that the zeros of the solution $y(x;\lambda)$ to (1.1) satisfying initial conditions $y(\alpha;\lambda)=0$, $y'(\alpha;\lambda)=1$ are decreasing functions of λ (see [7, Theorem 4, p. 270] or [8, pp. 454–455]). Similar arguments are to be found in [14].

6. The Stretched String: Comparison with the Ratio Results of Keller and of Mahar and Willner

The papers of Keller [11] and Mahar and Wilner [14] (see also [18, 19] and the paper of Gentry and Banks [10]) solve the problem of minimizing and maximizing λ_2/λ_1 and other eigenvalue ratios for the stretched string

$$-y''=\lambda w(x)y \tag{6.1}$$

on a finite interval with Dirichlet boundary conditions and positive weight function w bounded by $0 < c \leqslant w(x) \leqslant C$ (in our notation). We set $\gamma = c/C$ ([11] and [14] use α for this variable) and consider the full range $0 < \gamma \leqslant 1$.

Our best bounds developed above for the eigenvalue ratio λ_2/λ_1 for (6.1) are

$$\max\{1, 4\gamma\} \leqslant \lambda_2/\lambda_1 \leqslant 1 + 3/\gamma \leqslant 4\gamma. \tag{6.2}$$

These bounds are depicted in Fig. 1.

The optimal bounds of [11, 14] (which were computed numerically in those papers) are also sketched in Fig. 1. These have the asymptotic form near $\gamma = 0$ of

$$\mu(\gamma) \equiv \min_{c,\gamma} \frac{\lambda_2}{\lambda_1} \sim 1 + \frac{8}{\pi}\gamma^{1/2} + \cdots \tag{6.3}$$

(Keller [11, p. 491]) and

$$\nu(\gamma) \equiv \max_{c,\gamma} \frac{\lambda_2}{\lambda_1} \sim \frac{\pi^2}{2\gamma^{1/2}} + O(\gamma^{-1/4}) \tag{6.4}$$

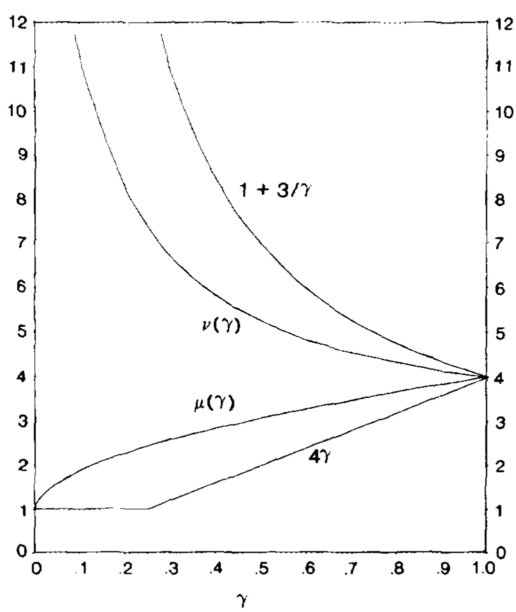

FIG. 1. Graph showing upper and lower bounds from (6.2) and the optimal upper and lower bounds $\nu(\gamma)$ and $\mu(\gamma)$ as computed by Mahar and Willner [14] and by Keller [11], respectively.

(Mahar and Willner [14, p. 520]). Here C_γ denotes the class of weight functions for which bounds c and C exist with $0 < c \leqslant w(x) \leqslant C$ and $\gamma = c/C$ (obviously $c/C \geqslant \gamma$ is enough).

While it is clear that the exact bounds of Keller and of Mahar and Willner are better than ours, especially for γ near 0, it is interesting that our bounds give a reasonable qualitative picture of the situation. Moreover, our bounds are explicit and can be determined with a minimum of effort for any given problem.

7. Ratio Results Following from Transformation Theory

If the coefficients in (1.1) are sufficiently differentiable then the results we established in the preceding sections can be combined with changes of variables to produce other interesting results. Specifically, the change of independent and dependent variables defined by (cf. Courant and Hilbert, Vol. I [8, p. 292])

$$\frac{dt}{dx} = \left(\frac{w}{p}\right)^{1/2} \tag{7.1}$$

and

$$y(x) = u(x)\, z(t), \tag{7.2}$$

where

$$u(x) = [p(x)\, w(x)]^{-1/4} \tag{7.3}$$

transforms Eq. (1.1) into the Schrödinger normal form

$$-\ddot{z} + \left[\frac{q}{w} + (pw)^{-1/4} \frac{d^2}{dt^2} (pw)^{1/4}\right] z = \lambda z. \tag{7.4}$$

This is a regular Sturm–Liouville problem since (1.1) was assumed to be. Also, by (7.2), Dirichlet boundary conditions remain in effect.

The second part of the potential in (7.4) can also be written in terms of derivatives with respect to x. We give two particularly useful forms.

$$\begin{aligned}(pw)^{-1/4} \frac{d^2}{dt^2} (pw)^{1/4} &= -(w/p)^{-3/4} \frac{d^2}{dx^2} (w/p)^{-1/4} + \frac{1}{w} p^{1/2} \frac{d^2}{dx^2} p^{1/2} \\ &= \frac{1}{3} w^{-5/4} p^{1/4} \frac{d^2}{dx^2} (p^{3/4} w^{1/4}) - \frac{1}{3} w^{-1/2} p \frac{d^2}{dx^2} w^{-1/2}. \end{aligned} \tag{7.5}$$

In particular these equations show the following:

1. If $(d^2/dx^2)(w^{-1/4}) \leqslant 0$ or equivalently $(d^2/dt^2)(w^{1/4}) \geqslant 0$ then the Dirichlet eigenvalues of the equation $-y'' = \lambda wy$ satisfy $\lambda_m/\lambda_1 \leqslant m^2$ for $m = 2, 3, 4, \ldots$ and also $\lambda_m/\lambda_l \leqslant \{m/l\}^2$ for $m > l \geqslant 1$.

2. If $(d^2/dx^2)(p^{3/4}) \geqslant 0$ or equivalently $(d^2/dt^2)(p^{1/4}) \geqslant 0$ then the ratios of the Dirichlet eigenvalues of the equation $-(py')' = \lambda y$ satisfy the same inequalities as those in 1.

3. The conclusions in items 1 and 2 also hold when a potential $q \geqslant 0$ is included. More generally, $q \geqslant 0$ and $(d^2/dt^2)(pw)^{1/4} \geqslant 0$ yield the same bounds for the ratios of the Dirichlet eigenvalues of (1.1) as those stated in item 1 above.

Finally, we note that we can use the transformation defined by (7.1) and (7.2) to pass from an equation of the form (7.4) to one of the form (1.1). One particular consequence is that if $\tilde{M}(t)$ is such that

$$\frac{d^2\tilde{M}}{dt^2} + (q + \mu)\tilde{M} = 0 \tag{7.6}$$

and $\tilde{M}(t) > 0$ for all t's in a closed interval then the equation

$$-\ddot{z} = (\lambda + \mu)z \tag{7.7}$$

is equivalent to

$$-(\tilde{M}^4 y')' + qy = \lambda y \tag{7.8}$$

(here primes denote differentiation with respect to x) if the transformation that we use is defined by

$$\frac{dx}{dt} = \tilde{M}(t)^2 \tag{7.9}$$

and (7.2) with $y(x) = z(t)/\tilde{M}(t)$. We could even let μ represent an arbitrary function here.

Another consequence (following Magnus and Winkler [13, pp. 51–52] but correcting some misprints) is that if $M(t)$ satisfies

$$\frac{d^2M}{dt^2} + Q(t)M = 0 \tag{7.10}$$

and is positive in some interval of interest then the equation

$$-\ddot{z} - Q(t)z = \lambda z \tag{7.11}$$

is equivalent to

$$-y'' = \lambda M^4 y \tag{7.12}$$

under the transformation defined by

$$\frac{dx}{dt} = M(t)^{-2} \tag{7.13}$$

with $y(x) = z(t)/M(t)$. Under our assumptions here the original finite interval will transform into a new finite interval on which the transformed equation is again regular.

Additionally, much general information concerning the transformation theory of general differential operators is to be found in the paper of Ahlbrandt, Hinton, and Lewis [1].

8. Concluding Remarks

It should be noted that all our results concerning the case $q \geqslant 0$ could be extended to cases where q is bounded below by a constant times w. One need only shift the eigenvalues by this constant and incorporate the constant shift into the potential. That is, if $q(x) \geqslant Aw(x)$ we regroup (1.1) as

$$-(py')' + (q - Aw)y = (\lambda - A)wy \tag{8.1}$$

and proceed as before. All our bounds for the case $q \geqslant 0$ will now apply but with λ's replaced by $(\lambda - A)$'s. This comment applies as well to the case where $A > 0$ since in that case, even though our bounds already apply as given, one will get better bounds by performing the shift by A. In fact, for optimal results one should first shift by

$$A = \min_{x \in \Omega} [q(x)/w(x)].$$

The results presented in this paper could probably be extended to singular problems where the product $p(x)w(x)$ is still nice in the sense that there are constants $K \geqslant k > 0$ such that $k \leqslant p(x)w(x) \leqslant K$. One point of departure for such an extension would be the transformation (5.5) and the resulting equation (5.6).

References

1. C. D. Ahlbrandt, D. B. Hinton, and R. T. Lewis, The effect of variable change on oscillation and disconjugacy criteria with applications to spectral theory and asymptotic theory, *J. Math. Anal. Appl.* **81** (1981), 234–277.

2. M. S. Ashbaugh and R. D. Benguria, Best constant for the ratio of the first two eigenvalues of one-dimensional Schrödinger operators with positive potentials, *Proc. Amer. Math. Soc.* **99** (1987), 598–599.
3. M. S. Ashbaugh and R. D. Benguria, On the ratio of the first two eigenvalues of Schrödinger operators with positive potentials, *in* "Differential Equations and Mathematical Physics" (I. W. Knowles and Y. Saitō, Eds.), pp. 16–25, Lecture Notes in Mathematics, Vol. 1285, Springer-Verlag, Berlin, 1987.
4. M. S. Ashbaugh and R. D. Benguria, Optimal bounds for ratios of eigenvalues of one-dimensional Schrödinger operators with Dirichlet boundary conditions and positive potentials, *Comm. Math. Phys.* **124** (1989), 403–415.
5. M. S. Ashbaugh and R. D. Benguria, Proof of the Payne–Pólya–Weinberger conjecture, *Bull. Amer. Math. Soc.* **25** (1991), 19–29.
6. M. S. Ashbaugh and R. D. Benguria, A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions, *Ann. of Math.* **135** (1992), 601–628.
7. G. Birkhoff and G.-C. Rota, "Ordinary Differential Equations," 3rd ed., Wiley, New York, 1978.
8. R. Courant and D. Hilbert, "Methods of Mathematical Physics," Vol. I, Interscience, New York, 1953.
9. R. E. Crandall and M. H. Reno, Ground state energy bounds for potentials $|x|^\nu$, *J. Math. Phys.* **23** (1982), 64–70.
10. R. D. Gentry and D. O. Banks, Bounds for functions of eigenvalues of vibrating systems, *J. Math. Anal. Appl.* **51** (1975), 100–128.
11. J. B. Keller, The minimum ratio of two eigenvalues, *SIAM J. Appl. Math.* **31** (1976), 485–491.
12. W. Leighton, "Ordinary Differential Equations," 3rd ed., Wadsworth, Belmont, CA, 1970.
13. W. Magnus and S. Winkler, "Hill's Equation," Wiley, New York, 1966; reprinted by Dover, New York, 1979.
14. T. J. Mahar and B. E. Willner, An extremal eigenvalue problem, *Comm. Pure Appl. Math.* **29** (1976), 517–529.
15. J. R. McLaughlin, Upper and lower bounds on eigenvalues of second-order Sturm–Liouville systems, *J. Differential Equations* **19** (1975), 201–213.
16. L. E. Payne, G. Pólya, and H. F. Weinberger, Sur le quotient de deux fréquences propres consécutives, *C. R. Acad. Sci. Paris* **241** (1955), 917–919.
17. L. E. Payne, G. Pólya, and H. F. Weinberger, On the ratio of consecutive eigenvalues, *J. Math. and Phys.* **35** (1956), 289–298.
18. B. E. Willner and T. J. Mahar, Extrema of functions of eigenvalues, *J. Math. Anal. Appl.* **72** (1979), 730–739.
19. B. E. Willner and T. J. Mahar, The two-dimensional eigenvalue range and extremal eigenvalue problems, *SIAM J. Math. Anal.* **13** (1982), 621–631.

Printed in Belgium
Uitgever: Academic Press, Inc.
Verantwoordelijke uitgever voor België:
Hubert Van Maele
Altenastraat 20, B-8310 Sint-Kruis