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Deformed BPS monopole in Ω-background

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A B S T R A C T

We study the BPS condition in the Ω-deformed $\mathcal{N} = 2$ super Yang–Mills theory when one of the ϵ-parameters of the background is zero. We obtain the deformed BPS equation for dyons and the formula for their central charge. In particular, we find that the deformed BPS monopole equation has axially-symmetric solution and is equivalent to the Ernst equation. The monopole charge is shown to be undeformed. We construct one-monopole solution explicitly and examine its profile.

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The Ω-background deformation of $\mathcal{N} = 2$ supersymmetric gauge theories [1–3] has been attracted much attentions. The Ω-background is characterized by two complex parameters ϵ_1 and ϵ_2 associated with the $U(1)^2$ action on four-dimensional spacetime $\mathbf{R}^4 = \mathbf{R}^2 \times \mathbf{R}^2$. This background breaks the super-Poincaré invariance in general. By introducing the R-symmetry gauge field Wilson line, the deformed theory has one equivariantly nilpotent supercharge. The instanton partition function can be evaluated via the localization theorem by using the supercharge. One can reproduce the Seiberg–Witten (SW) prepotential [4] from the instanton partition function by taking the limit $\epsilon_1, \epsilon_2 \to 0$ [2].

The theory has extended supersymmetries in two dimensions when one of the ϵ-parameters is set to be zero. For $\epsilon_2 = 0$, the instanton partition function gives the prepotential deformed by ϵ_1. The deformed prepotential is related to the Yang–Yang functional in a quantum integrable system, where ϵ_1 plays a role of the Planck constant [5–8]. Recently, it has been pointed out that the prepotential can be also evaluated by the period integral of the deformed SW differential obtained from the quantization of the SW curve [9]. The periods are identified with the exact Bohr–Sommerfeld integral of the 1d sine-Gordon model [10–13].

The deformation of the prepotential should be also derived from the microscopic calculation of the deformed super Yang–Mills theory. In the SW theory, the period integrals of the SW differential are the central charges of the supersymmetry algebra, which is related to the masses of the BPS states [14]. In this Letter we will investigate the BPS states in the Ω-deformed theory in order to study the deformed prepotential from the field theoretical point of view. In the previous paper [15], we have shown that the theory with $\epsilon_1 = 0$ or $\epsilon_2 = 0$ has the deformed $\mathcal{N} = (2, 2)$ or $(2, 1)$ su-

persymmetry,[1] which depends on the choice of the Wilson lines. We also derived the formula for its central charge and the BPS monopole equation which preserves part of the supersymmetries. In this Letter we will consider more general BPS configurations, which give the BPS equation for dyon. We will obtain the central charge for the BPS dyon, which includes the monopole charge as an example. We will then study the BPS monopole solution and evaluate its charge. For the construction of the solution, it is useful to refer to axially-symmetric monopole solutions in the Yang–Mills–Higgs model [16,17] based on the Manton's ansatz [18]. Forgács et al. [19] showed that the monopole equation with this ansatz reduces to the Ernst equation [20]. It can be solved by the inverse scattering method [21] and the Bäcklund transformation [22,23]. They found that one obtains multi-monopole solutions by applying multi-consecutive Bäcklund transformations to a simple vacuum solution [24]. In this Letter we will show that the deformed BPS monopole equation is equivalent to the same Ernst equation. We will also solve the BPS equation associated with one-monopole perturbatively in ϵ_1.

We consider $U(N)$ $\mathcal{N} = 2$ super Yang–Mills theory in the Ω-background. The theory contains a gauge field A_m ($m = 1, 2, 3, 4$), Weyl fermions Λ^I, $\bar{\Lambda}^I$ ($I = 1, 2$), and complex scalars φ, $\bar{\varphi}$. They belong to the adjoint representation of $U(N)$ gauge group. Here I denotes an $SU(2)_I$ R-symmetry indices. We also introduce R-symmetry gauge field Wilson lines $\mathcal{A}^I{}_J$ and $\bar{\mathcal{A}}^I{}_J$. The Lagrangian is given by [25]

$$\mathcal{L}_\Omega = \frac{1}{g^2 \kappa} \text{Tr} \left[\frac{1}{4} F_{mn} F^{mn} + \left(D_m \varphi - F_{mn} \Omega^n \right) \left(D^m \bar{\varphi} - F^{mp} \bar{\Omega}_p \right) \right.$$

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[1] We denote $\mathcal{N} = (p, q)$ by supersymmetry with p chiral and q anti-chiral supercharges.

$$+ \Lambda^I \sigma^m D_m \bar{\Lambda}_I - \frac{i}{\sqrt{2}} \Lambda^I [\bar{\varphi}, \Lambda_I] + \frac{i}{\sqrt{2}} \bar{\Lambda}_I [\varphi, \bar{\Lambda}^I]$$

$$+ \frac{1}{\sqrt{2}} \bar{\Omega}^m \Lambda^I D_m \Lambda_I - \frac{1}{2\sqrt{2}} \bar{\Omega}^{mn} \Lambda^I \sigma_{mn} \Lambda_I$$

$$- \frac{1}{\sqrt{2}} \Omega^m \bar{\Lambda}_I D_m \bar{\Lambda}^I + \frac{1}{2\sqrt{2}} \Omega^{mn} \bar{\Lambda}_I \bar{\sigma}_{mn} \bar{\Lambda}^I$$

$$+ \frac{1}{2} \big([\varphi, \bar{\varphi}] + i \Omega^m D_m \bar{\varphi} - i \bar{\Omega}^m D_m \varphi + i \bar{\Omega}^m \Omega^n F_{mn} \big)^2$$

$$- \frac{1}{\sqrt{2}} \bar{\mathcal{A}}^J_I \Lambda^I \Lambda_J - \frac{1}{\sqrt{2}} \mathcal{A}^J_I \bar{\Lambda}^I \bar{\Lambda}_J \Big], \tag{1}$$

where $F_{mn} = \partial_m A_n - \partial_n A_m + i[A_m, A_n]$ is the gauge field strength, $D_m = \partial_m + i[A_m, *]$ is the gauge covariant derivative, g is the gauge coupling constant and the constant κ normalizes basis of $U(N)$. We will consider the Lagrangian in Euclidean spacetime. We then define the Dirac matrices $\sigma_m = (i\tau^1, i\tau^2, i\tau^3, 1)$ and $\bar{\sigma}_m = (-i\tau^1, -i\tau^2, -i\tau^3, 1)$, where τ^c $(c = 1, 2, 3)$ are the Pauli matrices. σ_{mn} and $\bar{\sigma}_{mn}$ are the Lorentz generators. We set the vacuum theta-angle to zero for simplicity. The Ω-background is parametrized by Ω_{mn} and $\bar{\Omega}_{mn}$ which take the form

$$\Omega^{mn} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & i\epsilon_1 & 0 & 0 \\ -i\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\epsilon_2 \\ 0 & 0 & i\epsilon_2 & 0 \end{pmatrix},$$

$$\bar{\Omega}^{mn} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & -i\bar{\epsilon}_1 & 0 & 0 \\ i\bar{\epsilon}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\bar{\epsilon}_2 \\ 0 & 0 & -i\bar{\epsilon}_2 & 0 \end{pmatrix}. \tag{2}$$

The vector fields Ω^m and $\bar{\Omega}^m$ are defined by $\Omega^m = \Omega^{mn} x_n$ and $\bar{\Omega}^m = \bar{\Omega}^{mn} x_n$. They generate $U(1) \times U(1)$ actions on $\mathbf{R}^4$. Two parameters ϵ_1, ϵ_2 break the super-Poincaré invariance in general. However, when one of the ϵ-parameters ϵ_1 or ϵ_2 becomes zero, the super-Poincaré invariance over the (x^1, x^2)-plane or the (x^3, x^4)-plane recovers respectively [5].

Supersymmetries in the limit ϵ_1 or $\epsilon_2 \to 0$ have been studied in [15]. They depend on the choice of the Wilson lines. To see them, it is convenient to introduce the topological twist [26]. We identify the $SU(2)_I$ R-symmetry index with the $SU(2)_R$ spinor index in the Lorentz group $SO(4) = SU(2)_L \times SU(2)_R$. The twisted supercharges are defined as

$$Q_m = (\bar{\sigma}_m)^{I\alpha} Q_{\alpha I}, \qquad \bar{Q} = \delta^{\dot{\alpha}}_I \bar{Q}^I_{\dot{\alpha}},$$

$$\bar{Q}_{mn} = -(\bar{\sigma}_{mn})^{\dot{\alpha}}_I \bar{Q}^I_{\dot{\alpha}}, \tag{3}$$

where $Q_{\alpha I}$ and $\bar{Q}^I_{\dot{\alpha}}$ are supercharges associated with the $\mathcal{N} = (4, 4)$ supersymmetry in the undeformed theory. The $\mathcal{N} = (4, 4)$ supersymmetry algebra reads

$$\{Q_m, \bar{Q}\} = 4P_m,$$

$$\{Q_m, \bar{Q}_{pq}\} = 2\big(\delta^{nq}\delta^{mp} - \delta^{np}\delta^{mq} - \varepsilon^{mnpq}\big) P_n,$$

$$\{\bar{Q}, \bar{Q}\} = 4\sqrt{2}\bar{Z},$$

$$\{Q_m, Q_n\} = -4\sqrt{2}\delta_{mn} Z,$$

$$\{\bar{Q}_{mn}, \bar{Q}\} = 0,$$

$$\{\bar{Q}_{mn}, \bar{Q}_{pq}\} = \sqrt{2}\big(\delta^{mp}\delta^{nq} - \delta^{mq}\delta^{np} - \varepsilon^{mnpq}\big) \bar{Z}, \tag{4}$$

where P^m is the four-momentum and $Z, \bar{Z}$ are the central charges. When the Wilson line is

$$\mathcal{A}^I_J = -\frac{1}{2} \Omega_{mn} (\bar{\sigma}^{mn})^I_J, \qquad \bar{\mathcal{A}}^I_J = -\frac{1}{2} \bar{\Omega}_{mn} (\bar{\sigma}^{mn})^I_J, \tag{5}$$

the theory has $\mathcal{N} = (2, 1)$ supersymmetry which is generated by Q^1, Q^2, $\bar{Q}$ in the case of $\epsilon_1 = 0$ and by Q^3, Q^4, $\bar{Q}$ in the case of $\epsilon_2 = 0$. On the other hand, when the Wilson line is

$$\mathcal{A}^I_J = -\frac{1}{2} \bar{\Omega}_{mn} (\bar{\sigma}^{mn})^I_J, \qquad \bar{\mathcal{A}}^I_J = -\frac{1}{2} \Omega_{mn} (\bar{\sigma}^{mn})^I_J, \tag{6}$$

the theory has $\mathcal{N} = (2, 2)$ supersymmetry which is generated by Q^3, Q^4, $\bar{Q}^{13}$, $\bar{Q}^{14}$ in the case of $\epsilon_1 = 0$ and by Q^1, Q^2, $\bar{Q}^{13}$, $\bar{Q}^{14}$ in the case of $\epsilon_2 = 0$. We note that the $\mathcal{N} = (2, 1)$ and $\mathcal{N} = (2, 2)$ transformations for the fields are deformed by the remaining ϵ-parameter.

We now examine the BPS equation in the Ω-background for the dyonic state from the energy bound. To find the bound, we go back to the Minkowski spacetime and use the phase transformation of φ and Ω to set those to $\varphi = -\bar{\varphi}$, $\Omega = -\bar{\Omega}$ and define $\hat{\Omega} = \sqrt{2}i\Omega$, $\phi = \sqrt{2}i\varphi$ such that ϕ and Ω are real values. Performing the Bogomol'nyi completion, the energy becomes

$$E = \int d^3x \frac{1}{\kappa} \text{Tr}\Big[\frac{1}{2} \big\{ E_i \pm \big(D_i\phi + \hat{\Omega}^j F_{ji}\big) \sin\theta \big\}^2$$

$$+ \frac{1}{2} \big\{ B_i \pm \big(D_i\phi + \hat{\Omega}^j F_{ji}\big) \cos\theta \big\}^2$$

$$+ \frac{1}{2} \big(D_0\phi + \hat{\Omega}^j F_{j0}\big)^2 \Big]$$

$$\mp \frac{1}{\kappa} \int d^3x \text{Tr}\big[B^i D_i\phi\big] \cos\theta$$

$$\mp \frac{1}{\kappa} \int d^3x \text{Tr}\big[E^i \big(D_i\phi + \hat{\Omega}^j F_{ji}\big) \sin\theta\big], \tag{7}$$

where $E_i = F_{i0}$ $(i = 1, 2, 3)$ is the electric field, $B_i = \frac{1}{2}\epsilon_{ijk} F^{jk}$ is the magnetic field and θ is an arbitrary parameter. Here $x_0 = -ix_4$ and other vectors are defined in a similar way.

The energy bound is given by the last two terms in (7). The energy is saturated provided that the following BPS conditions are satisfied:

$$E_i \pm \big(D_i\phi + \hat{\Omega}^j F_{ji}\big) \sin\theta = 0,$$

$$B_i \pm \big(D_i\phi + \hat{\Omega}^j F_{ji}\big) \cos\theta = 0,$$

$$D_0\phi + \hat{\Omega}^j F_{j0} = 0. \tag{8}$$

The last equation in (8) and the equations of motion for the electric field imply Gauss' law $D_i E_i = 0$. Using the Bianchi identity $D_i B_i = 0$ and Gauss' law, the energy bound is rewritten as

$$E = \mp \big\{ Q_m \cos\theta + (Q_e + \delta Q_e) \sin\theta \big\}, \tag{9}$$

where Q_m, Q_e are the undeformed magnetic and electric charges defined by

$$Q_m = \frac{1}{\kappa} \int d^3x \, \partial_i \text{Tr}[B_i\phi], \qquad Q_e = \frac{1}{\kappa} \int d^3x \, \partial_i \text{Tr}[E_i\phi], \tag{10}$$

while δQ_e denotes the correction to the electric charge:

$$\delta Q_e = \frac{1}{\kappa} \int d^3x \text{Tr}\big[\hat{\Omega}^j F_{ji} E^i\big]. \tag{11}$$

The energy bound is minimized when the parameter θ satisfies the following condition

$$\sin\theta = \frac{Q_m}{\sqrt{Q_m^2 + Q_e'^2}}, \tag{12}$$

where we have defined the deformed electric charge $Q_e' = Q_e + \delta Q_e$. When this condition is satisfied, the energy is given by the mass of the BPS state, namely, the dyon mass,

$$M_{\text{dyon}} = \sqrt{Q_{\text{m}}^2 + Q_{\text{e}}'^2}. \tag{13}$$

To see the relation between the energy bound and the central charge in the supersymmetry algebra, we evaluate the deformed central charge by calculating the anti-commutation relation of the Noether charges associated with the deformed supersymmetry transformation in Euclidean space [15]. We have found that

$$Z = \int d^3x \, \frac{1}{\sqrt{2}\kappa} \, \text{Tr}\big[iD_i\phi B_i - D_i\phi E_i - \hat{\Omega}^j F_{ji}E_i\big], \tag{14}$$

where we have used the BPS conditions (8) and the parameter (12). The mass of the BPS state defined by the deformed BPS equations (8) is given by the deformed central charge:

$$M_{\text{dyon}} = \sqrt{Q_{\text{m}}^2 + Q_{\text{e}}'^2} = \sqrt{2}|Z|. \tag{15}$$

We note that in the BPS monopole state $B_i \neq 0$, $E_i = 0$, the expression of central charge is not deformed by the Ω-background. However, this can be deformed through the deformed solution of the monopole equation.

The dyon BPS state preserves parts of deformed supersymmetries. Substituting the BPS conditions (8) into the deformed supersymmetry transformation of fermions in [15], we get the following condition

$$e^{-i\theta}\xi_m(\sigma^m)_{\alpha I} \mp i(\sigma^{mn})^{\dot{\alpha}}{}_I\bar{\xi}_{mn} \mp i\delta^{\dot{\alpha}}{}_I\bar{\xi} = 0, \tag{16}$$

where ξ_m, $\bar{\xi}_{mn}$, $\bar{\xi}$ are transformation parameters associated with the supercharges Q_m, $\bar{Q}_{mn}$, and $\bar{Q}$. This condition is the same as the monopole case provided that ξ_m is replaced by $e^{-i\theta}\xi_m$.

We consider the case $\epsilon_2 = 0$ where the BPS equation preserves at least one supercharge. Our purpose is to evaluate the ϵ_1-correction to the central charge. In this case the central charge (14) becomes

$$Z = \int d^3x \, \frac{1}{\sqrt{2}\kappa} \, \text{Tr}\big[i(\boldsymbol{D}\phi)\cdot\boldsymbol{B} - (\boldsymbol{D}\phi)\cdot\boldsymbol{E} - \epsilon\big(\boldsymbol{x}\times(\boldsymbol{E}\times\boldsymbol{B})\big)_3\big], \tag{17}$$

where we have defined $\epsilon = -\text{Re}(\epsilon_1)/2$ and introduced the three-vectors $\boldsymbol{x} = (x^1, x^2, x^3)$, etc. In order to evaluate it, it is necessary to solve the dyon equation and substitute the solution into the central charge. In this Letter, we consider the deformed BPS monopole equation which is the simplest example. The deformed BPS monopole equation is obtained by setting $\theta = 0$, $A_0 = 0$, $\partial_0\phi = \partial_0 A_i = 0$ in Eqs. (8):

$$B_i \pm \big(D_i\phi + \hat{\Omega}^j F_{ji}\big) = 0. \tag{18}$$

Hereafter we consider the $SU(2)$ gauge group for simplicity. For $\epsilon_2 = 0$, we see that the deformed BPS equation has the axial symmetry around the x^3-axis. Hence we use Manton's ansatz [18] for the fields:

$$A_i^a = \left\{\eta_1\hat{\rho}^a + \left(\eta_2 + \frac{1}{g\rho}\right)\hat{z}^a\right\}\hat{\varphi}^i + W_1\hat{\rho}^i\hat{\varphi}^a + W_2\hat{z}^i\hat{\varphi}^a,$$
$$\phi^a = \phi_1\hat{\rho}^a + \phi_2\hat{z}^a, \tag{19}$$

where (ρ, φ, z) are the cylindrical coordinates for the spatial direction (x^1, x^2, x^3), $\hat{\rho} = (\cos\varphi, \sin\varphi, 0)$, $\hat{\varphi} = (-\sin\varphi, \cos\varphi, 0)$, $\hat{z} = (0, 0, 1)$ and the $SU(2)$ gauge index a runs 1, 2, 3. η_α, W_α and ϕ_α ($\alpha = 1, 2$) are functions of (ρ, z). Substituting (19) into the deformed BPS equation (18), we obtain

$$-\partial_3\eta_1 + \eta_2 W_2 = \partial_\rho\phi_1 - W_1\phi_2 + \epsilon\rho\left(\partial_\rho\eta_1 + \frac{\eta_1}{\rho} - W_1\eta_2\right), \tag{20}$$

$$-\partial_3\eta_2 - \eta_1 W_2 = \partial_\rho\phi_2 + W_1\phi_1 + \epsilon\rho\left(\partial_\rho\eta_2 + \frac{\eta_2}{\rho} + \eta_1 W_1\right), \tag{21}$$

$$\partial_\rho\eta_1 + \frac{\eta_1}{\rho} - W_1\eta_2 = \partial_3\phi_1 - W_2\phi_2 + \epsilon\rho(\partial_3\eta_1 - \eta_2 W_2), \tag{22}$$

$$\partial_\rho\eta_2 + \frac{\eta_2}{\rho} + \eta_1 W_1 = \partial_3\phi_2 + W_2\phi_1 + \epsilon\rho(\partial_3\eta_2 + \eta_1 W_2), \tag{23}$$

$$\partial_\rho W_2 - \partial_3 W_1 = -\eta_1\phi_2 + \eta_2\phi_1, \tag{24}$$

where $\partial_\rho = \frac{\partial}{\partial\rho}$, $\partial_3 = \frac{\partial}{\partial z}$. We have chosen the minus sign in the BPS equation (18). These equations are invariant under the gauge transformations [27]:

$$W_1' = W_1 + \partial_\rho\Lambda, \qquad W_2' = W_2 + \partial_3\Lambda,$$
$$\phi_1' = \cos\Lambda\phi_1 + \sin\Lambda\phi_2, \qquad \phi_2' = \cos\Lambda\phi_2 - \sin\Lambda\phi_1,$$
$$\eta_1' = \cos\Lambda\eta_1 + \sin\Lambda\eta_2, \qquad \eta_2' = \cos\Lambda\eta_2 - \sin\Lambda\eta_1, \tag{25}$$

where Λ is a function of (ρ, z). For $\epsilon = 0$, these equations were shown to be equivalent to the Ernst equation and solved by Forgács et al. [19] by using the Bäcklund transformation technique. We modify their gauge-fixing conditions to

$$W_1 = \eta_1, \qquad W_2 = -\phi_1 - \epsilon\rho\eta_1. \tag{26}$$

Then Eqs. (20) and (24) become equivalent. Eq. (21) becomes

$$-\partial_3\eta_2 = \partial_\rho(\phi_2 + \epsilon\rho\eta_2). \tag{27}$$

This can be solved by the following ansatz:

$$\eta_2 = \frac{\partial_\rho f}{f}, \qquad \phi_2 + \epsilon\rho\eta_2 = -\frac{\partial_3 f}{f}, \tag{28}$$

where f is a function of (ρ, z) to be determined by other equations. Under this ansatz, (20) and (24) reduce to

$$\partial_\rho(f\phi_1 + \epsilon\rho f\eta_1) = -\partial_3(f\eta_1). \tag{29}$$

This also can be solved as

$$\eta_1 = \frac{\partial_\rho\psi}{f}, \qquad \phi_1 + \epsilon\rho\eta_1 = \frac{\partial_3\psi}{f}, \tag{30}$$

where ψ is also a function of (ρ, z). The remaining Eqs. (22) and (23) give the equations for f and ψ:

$$f\left(\partial_\rho^2 f + \frac{1}{\rho}\partial_\rho f + \partial_3^2 f\right)$$
$$- (\partial_\rho f)^2 - (\partial_3 f)^2 + (\partial_\rho\psi)^2 + (\partial_3\psi)^2 = 0, \tag{31}$$

$$f\left(\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi + \partial_3^2\psi\right) - 2\partial_\rho f\partial_\rho\psi - 2\partial_3 f\partial_3\psi = 0. \tag{32}$$

They do not include the parameter ϵ and coincide with the Ernst equation [20]. Therefore we can construct the deformed BPS monopole solution from the potentials f and ψ.

In summary, we have obtained the following solution of the deformed BPS monopole equation:

$$\eta_1 = \frac{\partial_\rho\psi}{f}, \qquad \eta_2 = \frac{\partial_\rho f}{f}, \qquad W_1 = \frac{\partial_\rho\psi}{f}, \qquad W_2 = \frac{\partial_3\psi}{f},$$
$$\phi_1 = -\frac{\partial_3\psi + \epsilon\rho\partial_\rho\psi}{f}, \qquad \phi_2 = -\frac{\partial_3 f + \epsilon\rho\partial_\rho f}{f}, \tag{33}$$
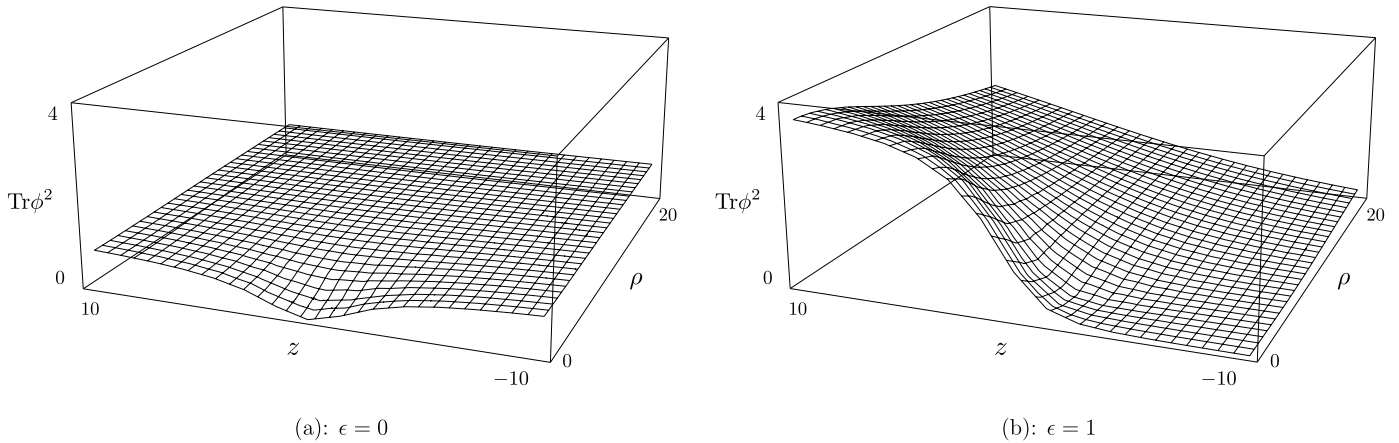
(a): $\epsilon = 0$

(b): $\epsilon = 1$

Fig. 1. Profile of $\mathrm{Tr}\,\phi^2$ for $v = 1$.

where f and ψ are the solutions of the Ernst equations (31), (32). We note that the gauge field in this solution is not deformed by ϵ. Its magnetic charge can be rewritten without using the scalar field:

$$Q_{\mathrm m} = \int d^3x\, B_i^a B_i^a. \tag{34}$$

Therefore the magnetic charge is not deformed by ϵ. To demonstrate this fact explicitly, we construct deformed one-monopole solution and evaluate its magnetic charge. The deformed solution can be constructed from the Ernst potential for the undeformed one-monopole solution [28,19]

$$f_{1\mathrm{MP}} = \rho/G, \qquad \psi_{1\mathrm{MP}} = P/G,$$
$$G = \frac{r}{\sinh vr} + r\cosh vz\coth vr - z\sinh vz,$$
$$P = z\cosh vz - r\sinh vz\coth vr, \tag{35}$$

where $r = \sqrt{\rho^2 + z^2}$ and $v = \langle\phi\rangle$ is the vacuum expectation value of the scalar field in the undeformed theory with spherical symmetry. In the deformed theory, the asymptotic behavior of the solution depends on the direction of the point at infinity. Here we fix the value of v as that of undeformed one. Substituting (35) into (33), we obtain the deformed one-monopole solution. We plot the gauge invariant quantity $\mathrm{Tr}\,\phi^2$ for $\epsilon = 0$ and $\epsilon = 1$ in Fig. 1. We see that for $\epsilon = 0$, it is spherical symmetric and thus isotropic in the ρ- and the z-direction. For $\epsilon = 1$, it receives the deformation which breaks the spherical symmetry. It increases in the region $z > 0$ and thus becomes anisotropic. As we mentioned above, its magnetic charge remains undeformed:

$$Q_{\mathrm m} = 4\pi v. \tag{36}$$

One can also solve the deformed monopole equation in a different approach. For $\epsilon = 0$, we get the solution of the form [16,17]

$$\phi_1^{(0)} = \frac{v\rho}{r}H(r), \qquad \phi_2^{(0)} = \frac{vz}{r}H(r),$$
$$\eta_1^{(0)} = -\frac{z}{r^2}F(r), \qquad \tilde{\eta}_2^{(0)} = \frac{\rho}{r^2}F(r),$$
$$W_1^{(0)} = \frac{z}{r^2}F(r), \qquad W_2^{(0)} = -\frac{\rho}{r^2}F(r), \tag{37}$$

where $\tilde{\eta}_2 = \eta_2 + 1/\rho$, and F, H are defined by

$$F(r) = 1 - \frac{vr}{\sinh vr}, \qquad H(r) = \coth vr - \frac{1}{vr}. \tag{38}$$

The superscript (0) stands for the undeformed solution. This is gauge-equivalent to the solution associated with (35). The deformed solution can be obtained perturbatively by introducing ϵ-correction to this undeformed solution as in the case of monopoles in non-commutative field theories [29,30]. Thus we expand the deformed solution as

$$X_\alpha = X_\alpha^{(0)} + \epsilon X_\alpha^{(1)} + \cdots, \tag{39}$$

where $X_\alpha = \eta_1, \tilde{\eta}_2, W_\alpha, \phi_\alpha$ and $\alpha = 1, 2$. We assume that the ϵ-correction has the following form:

$$X_\alpha^{(1)} = \frac{1}{r^{n_1}} P_{n_2}(\rho, z) P_{n_3}\big(H(r), F(r)\big), \tag{40}$$

where we denote $P_n(s, t)$ by a polynomial of s and t which has degree n. When $n_1 = 3$, $n_2 = 3$, $n_3 = 1$ and we impose the regularity at the origin and finiteness at the infinity on the solution, we find that the corrections

$$\phi_1^{(1)} = \frac{z\rho F}{r^2}, \qquad \phi_2^{(1)} = 1 - \frac{\rho^2 F}{r^2}, \tag{41}$$
$$\eta_1^{(1)} = \tilde{\eta}_2^{(1)} = W_1^{(1)} = W_2^{(1)} = 0, \tag{42}$$

satisfy the deformed BPS equation and there are no higher-order corrections. This solution has the same profile as in Fig. 1. In this approach the gauge field is not deformed by ϵ. The charge remains undeformed and coincides with (36).

We now discuss the central charge in the SW theory. In [10,11], the deformed prepotential has been obtained by the deformation of the SW theory, which implies that the central charge is expressed by the period integrals on the deformed SW curve. The relation becomes

$$Z = n_e a + n_m a_D, \qquad a_D = \frac{\partial \mathcal{F}}{\partial a}, \tag{43}$$

where n_e and n_m are the electric charge number and the magnetic charge number respectively and $\mathcal{F}$ is the deformed prepotential and $a = v/\sqrt{2}$. We have shown that the magnetic charge of the BPS monopole is not deformed by ϵ. One can consider the BPS state with purely electric charge: the W-bosons. By examining their mass under the condition $\varphi = -\bar{\varphi}$ and $\Omega = -\bar{\Omega}$, we see that the mass is not deformed by ϵ. Therefore, in the Manton ansatz and the perturbative approaches, the central charge for the purely magnetic or electric BPS state is not deformed by ϵ. It is not clear whether there are ϵ-corrections for the dyon state because the central charge formula (17) contains the ϵ-dependent term. It is important to study the BPS dyon solution and its central charge in

order to determine the classical prepotential. It is also an interesting problem to investigate the perturbative corrections to the BPS mass since the perturbative part of the prepotential receives the ϵ-corrections [10].

In this Letter we have studied the BPS monopole equations using the approach of Forgács et al. It is interesting to study the Nahm construction [31,32] for monopoles in the Ω-background and its relation to the present approaches. In the string theory, the undeformed Nahm construction is naturally understood by the brane configuration [33] and the Ω-background is realized as a certain $\mathcal{N} = 2$ supergravity background [34,25]. Hence if we find the deformed Nahm construction, we may obtain the insight of the stringy realization of the monopole in the Ω-background [35,36]. The deformed monopole solutions that we have derived would be helpful to find its construction.

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