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Coalgebras and Modal Logic

Martin Rößiger

Institut für Algebra
Technische Universität Dresden
Germany
roessig@math.tu-dresden.de

Abstract

Coalgebras are of growing importance in theoretical computer science. To develop languages for them is significant for the specification and verification of systems. Modal logic has proved to be suitable for this purpose. So far, most approaches have presented a language to describe only deterministic coalgebras. The present paper introduces a generalization that also covers non-deterministic systems. Models for our modal language are F-coalgebras where the functor F is inductively constructed from constant sets and the identity functor using product, coproduct, exponentiation, and the power set functor. Thus, Kripke-structures constitute a special case. First we introduce a language that is based on a multisorted modal setting: here the sorts are given by the subfunctors of F. Then we consider a restricted language that still has the same expressiveness. It turns out that, for the case of Kripke-structures, the obtained language is equivalent to the "usual" modal logic for these structures. Hence this approach actually constitutes a bridge between modal languages for coalgebras and the modal logic for Kripke-structures. A well-known result from modal logic can be transfered to our setting: for so-called image-finite coalgebras bisimilarity coincides with logical equivalence. Finally, we present a sound and complete deduction calculus in case the constants in F are finite.

Introduction

Coalgebras provide a unifying view on a large variety of dynamic systems such as transition systems, automata, data structures, and objects (cf. e.g. [Jac96,Rut97]). Therefore it has been of great interest to develop some kind of language to describe them. From the theoretical point of view, this could give means to compare these systems or existing languages for them. On the other hand, such languages could, for instance, be applied to specify systems or even to verify properties of them.

In [HenR95,Jac95], equations are used to describe coalgebras. A. Corradini ([Cor97]) introduces an equational calculus for coalgebras of certain polynomial functors. H. Gumm ([Gum98]) and A. Kurz ([Kur98a]) show that covarieties are characterized by some kind of co-equations (which constitutes a dual version of Birkhoff's theorem).

L. Moss first shows that the shape of a coalgebra, given by the corresponding functor, determines in a canonical way a generalized modal language. In [Mos97] he derives a coalgebraic logic for coalgebras of a large class of functors and shows that this language is expressive enough to distinguish elements up to bisimilarity. For uniform functors he gives characterizing formulas that uniquely determine the "future behaviour" of an element, i.e. each such formula corresponds uniquely to some element of the terminal coalgebra. A. Baltag follows these ideas in [Bal00] where he defines infinitary modal logics to capture simulation and bisimulation. This leads to a new perspective on games that are used in logic.

A. Kurz ([Kur98b]) first presents a modal logic for coalgebras (of certain polynomial functors) using nexttime-operators and atomic propositions. He shows its relevance for specification purposes and also gives a complete axiomatization. A similar language is presented in [Röß98] for polynomial functors and is generalized in [Röß99] to datafunctors. Both papers also introduce a complete axiomatization. B. Jacobs ([Jac99]) first uses also lasttime-operators in addition to nexttime-operators. He investigates coalgebras that also allow to model nondeterministic systems and relates them to Galois algebras.

Here we deal with the same class of functors as in [Jac99] which are inductively constructed from constant sets and the identity functor using product, coproduct, exponentiation, and the power set functor. Therefore the corresponding coalgebras constitute a bridge to Kripke-structures which turn out to be a special case. Section 1 introduces some basic notions and terminology from coalgebra theory as well as a few examples. In Section 2 we present a language for coalgebras of the above mentioned functors on the basis of a multisorted modal logic. Here the sorts are given by the subfunctors of F. Still, this leads to a rather complex logic. Therefore, Section 3 introduces a fragment of it that still has the same expressiveness for a slightly restricted class of functors. We show that for the case of Kripke-structures, this fragment is equivalent to the "usual" modal logic. Section 4 investigates the expressiveness of the introduced language with regard to bisimilarity. It turns out that a well-known result from modal logic generalized to our setting: for so-called image-finite coalgebras, bisimilarity coincides with logical equivalence. Section 5 is devoted to stating a sound and complete deduction calculus. Eventually, Section 6 concludes with discussing the present approach and makes, in particular, suggestions how to continue and extend it.

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1 Functors and Coalgebras

In this chapter we define Kripke-polynomial functors and the corresponding coalgebras as well as some basic notions from coalgebra theory.

In the following we shall use some basic constructions in the category **Set** of (small) sets, like products, coproducts, and exponents (regarded as products). More precisely, the product of two sets S_1 and S_2 is denoted by $S_1 \times S_2$ with projections $\pi_i : S_1 \times S_2 \to S_i$ (for $i = 1, 2$). The coproduct (disjoint union) of S_1 and S_2 is written as $S_1 + S_2$ with coprojections $\kappa_i : S_i \to S_1 + S_2$ (for $i = 1, 2$). Eventually, the exponent of S_1 and S_2 is given by $S_2 \Rightarrow S_1$ or $S_1^{S_2}$ with an evaluation mapping $ev : (S_2 \Rightarrow S_1) \times S_2 \to S_1$. In particular, we shall consider mappings $\pi_s : (S_2 \Rightarrow S_1) \to S_1$ (for $s \in S_2$) where $\pi_s(t) := ev(t, s)$.

1.1. Definition. We call a functor $F : \mathbf{Set} \to \mathbf{Set}$ **Kripke-polynomial** if F is inductively constructed from

- constant functors $F_C : S \mapsto C$ (where C is some fixed non-empty set) and
- the identity functor $\mathsf{Id} : S \mapsto S$

using finitely many times

- product: $T_1 \times T_2 : S \mapsto T_1(S) \times T_2(S)$,
- coproduct: $T_1 + T_2 : S \mapsto T_1(S) + T_2(S)$,
- exponentiation: $(E \Rightarrow T) : S \mapsto (E \Rightarrow T(S))$ (where E is some fixed non-empty set), and
- the power set functor $\mathcal{P}(T) : S \mapsto \mathcal{P}\big(T(S)\big)$.

We call G a **subfunctor** of F (denoted by $G \leq F$) if G occurs as a functor during the inductive construction of F.[1] Moreover, for subfunctors T and G of F, we write $T \prec G$ to mean that G is constructed in the next step after T, i.e. if we have $G \in \{T_1 \times T_2, T_1 + T_2, E \Rightarrow T_1, \mathcal{P}(T_1)\}$ with $T \in \{T_1, T_2\}$.

Throughout this article we assume F to be a fixed Kripke-polynomial functor such that the identity functor Id is a subfunctor of F. Note that the above definition is the same as in [Jac99] for a polynomial functor.

1.2. Definition. Let $F : \mathbf{Set} \to \mathbf{Set}$ be a Kripke-polynomial functor. An F-**coalgebra** is a pair (S, α) consisting of a set S and a mapping $\alpha : S \to F(S)$. A mapping $h : S \to S'$ is a **homomorphism** between two F-coalgebras (S, α) and (S', α') if $F(h) \circ \alpha = \alpha' \circ h$. A relation $R \subseteq S \times S'$ is called a **bisimulation relation** if there exists some $\alpha_R : R \to F(R)$ such that the corresponding projections $\pi_S : R \to S$ and $\pi_{S'} : R \to S'$ are homomorphisms. Elements $s \in S$ and $s' \in S'$ are called **bisimilar** if there exists a bisimulation relation R with $(s, s') \in R$.

Coalgebras are often used to model transition systems and automata where

[1] Note that this notion differs from the notion of a subfunctor used in category theory: there a functor G is a subfunctor of a functor F if it is a subobject in the functor category.

the functor determines the kind of system that is being modelled. The following examples visualize this for Kripke-structures (cf. e.g. [Kri59,Kri63]), alternating automata ([Var97]), and deterministic transition systems (cf. [Rut97]).

1.3. Example (Kripke-structures). Kripke-structures are F-coalgebras (S, α) for the functor $F = \mathcal{P}(\mathsf{Id}) \times \{0,1\}^{\mathsf{AtProp}}$ where AtProp denotes the set of atomic propositions: for each world $s \in S$, $\alpha(s)$ gives the set of worlds accessible from s in its first component and the set of atomic propositions that hold in s in its second component.

1.4. Example (alternating automata). Let $\mathcal{B}^+(S)$ denote the set of all positive Boolean formulas over S (i.e. Boolean formulas built from elements of S using $\wedge$ and $\vee$) including the formulas $\top$ and $\perp$. Then an alternating Büchi word automaton ([Var97]) is a tuple $(\Sigma, S, s^0, \varrho, Fin)$ where Σ is a finite non-empty alphabet, S is a finite non-empty set of states, $s^0 \in S$ is an initial state, $Fin \subseteq S$ is a set of accepting states, and $\varrho : S \times \Sigma \to \mathcal{B}^+(S)$ is a partial transition function. Given a word $w = a_0 a_1 \ldots$ over Σ, a run of such an automaton on w is an S-labeled tree with root s^0 such that for each node x of depth i we have that if $\varrho(s, a_i) = \theta$ and x has children $x_1, \ldots, x_k$ then the set of labels of $\{x_1, \ldots, x_k\}$ satisfies θ. For instance, if $\varrho(s^0, a_0) = (s_1 \vee s_2) \wedge (s_3 \vee s_4)$ then the nodes of the run tree at level 1 contain the label s_1 or the label s_2 and also contain the label s_3 or the label s_4. Each such automaton $(\Sigma, S, s^0, \varrho, Fin)$ can be regarded as an F-coalgebra for

$$F = \Big(\big(\mathcal{P}(\mathcal{P}(\mathsf{Id})) + \{*\} \big)^{\Sigma} \Big) \times \{0,1\}^{\{i,f\}}.$$

Suppose, for each $s \in S$ and each $a \in \Sigma$ we write $\varrho(s, a)$ (if it is defined) in a disjoint normal form $\bigvee_{i \in I^a} \bigwedge_{j \in J_i^a} s_{i,j}^a$. Then the automaton $(\Sigma, S, s^0, \varrho, Fin)$ corresponds to an F-coalgebra (S, α) with

$$\alpha : s \mapsto \big((\overline{\varrho}(s,a))_{a \in \Sigma}, b_i, b_f \big)$$

where $\overline{\varrho}(s, a) := \kappa_1\big(\big\{\{s_{i,j}^a\}_{j \in J_i^a}\big\}_{i \in I^a}\big)$ if $\varrho(s, a)$ is defined and $\overline{\varrho}(s, a) := \kappa_2(*)$ otherwise. The elements $b_i, b_f \in \{0,1\}$ indicate whether s is an initial and an accepting state, respectively. This is an "underspecification" because, conversely, not each such F-coalgebra is in fact an alternating Büchi word automaton: for instance, it does not necessarily have a unique initial state.

1.5. Example (transition systems). Deterministic transition systems with output Σ are represented by coalgebras (S, α) of the functor $F = (\Sigma \times \mathsf{Id}) + \{*\}$. In each state s, such a transition system can either perform a transition $s \stackrel{a}{\longrightarrow} s'$ or terminates. That corresponds to the cases $\alpha : s \mapsto \kappa_1(a, s')$ and $\alpha : s \mapsto \kappa_2(*)$, respectively. If, additionally, the transition system allows for some input I then the corresponding functor F' is $F' := (I \Rightarrow F)$.

2 The Language

This section defines a language for F-coalgebras and gives the corresponding semantics. Moreover, we show that homomorphisms preserve formulas.

For a given set X, let $\mathcal{B}(X)$ denote the set of all boolean formulas over X, i.e. boolean formulas built from elements of X and $\bot$ using $\rightarrow$.

Whenever we have a mapping $f : X \rightarrow Y$ and $X' \subseteq X$, $Y' \subseteq Y$, then $f(X')$ and $f^{-1}(Y')$ denote the sets $\{f(x) \mid x \in X'\}$ and $\{x \in X \mid f(x) \in Y'\}$, respectively.

2.1. Remark. A multisorted modal setting proves to be suitable for defining a language for F-coalgebras, following [Ven99]. However, there are only rather few approaches that deal with multisorted modal languages (cf. e.g. [MonR97,Ven98]) and there does not exist a standard reference for it. Usually, models in a multisorted setting are based on Kripke-frames $\big((S_i)_{i\in I}, (R_{ij})_{i,j\in I}\big)$ where I is an indexing set, S_i denotes the i-th sort, and $R_{ij} \subseteq S_i \times S_j$ for all $i, j \in I$. A family of languages $(\mathcal{L}_i)_{i\in I}$ is defined by a simultaneous induction. Each $\mathcal{L}_i$ is given by

$$\varphi_i ::= \bot \mid \varphi_i \rightarrow \psi_i \mid p_i \mid \langle ij\rangle\varphi_j$$

where p_i is a variable of sort i and $\varphi_j \in \mathcal{L}_j$. Now a model is a frame $\mathcal{F} = \big((S_i)_{i\in I}, (R_{ij})_{i,j\in I}\big)$ equipped with a valuation V that takes each variable p_i of sort i to a subset of S_i. The semantics $(\vDash_i \subseteq S_i \times \mathcal{L}_i)_{i\in I}$ is defined sortwise by induction on formulas. For $s_i \in S_i$, we have

$$(\mathcal{F}, V), s_i \vDash_i p_i \quad :\Leftrightarrow s_i \in V(p_i) \text{ and}$$
$$(\mathcal{F}, V), s_i \vDash_i \langle ij\rangle\varphi_j :\Leftrightarrow \exists s_j \in S_j \text{ with } (s_i, s_j) \in R_{ij} \text{ and } (\mathcal{F}, V), s_j \vDash_j \varphi_j.$$

A family of complete calculi $(\vdash_i)_{i\in I}$ for the family $(\mathcal{L}_i)_{i\in I}$ of languages is then defined by a simultaneous induction on all sorts $i \in I$. For all $i, j \in I$, we have

(Taut)$_i$ all substitution instances of boolean tautologies in $\mathcal{L}_i$,

(MP)$_i$ $\dfrac{\vdash_i \varphi_i,\ \vdash_i \varphi_i \rightarrow \psi_i}{\vdash_i \psi_i}$,

(K)$_{ij}$ $[ij](\varphi_j \rightarrow \psi_j) \rightarrow ([ij]\varphi_j \rightarrow [ij]\psi_j)$,

(N)$_{ij}$ $\dfrac{\vdash_j \varphi_j}{\vdash_i [ij]\varphi_j}$

where $[ij]\varphi_j$ abbreviates $\neg\langle ij\rangle\neg\varphi_j$.

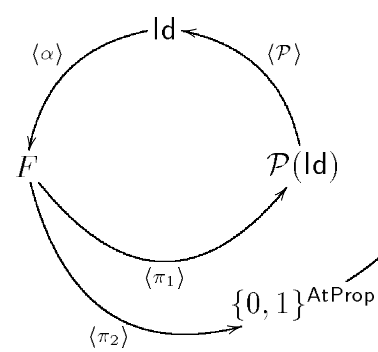
In the following we shall use a similar approach to define a language for F-coalgebras. The sorts shall be indexed by subfunctors of F. Connections between the sorts shall be given by "neighbourhood", that is to say we shall relate only those sorts with each other that are indexed by subfunctors G and T of F with $T \prec G$. Moreover, we also shall relate the sort Id with the sort F.

2.2. Definition. Let F be a Kripke-polynomial functor. We define a family $(\mathcal{L}_G)_{G \leq F}$ of languages by a simultaneous induction as follows:

$$
\begin{array}{ll}
G = F_C : & \varphi ::= \bot \mid \varphi \to \psi \mid c \ \text{ where } c \in C, \\
G = \mathsf{Id} : & \varphi ::= \bot \mid \varphi \to \psi \mid \langle \alpha \rangle \psi \ \text{ where } \psi \in \mathcal{L}_F, \\
G = T_1 \times T_2 : & \varphi ::= \bot \mid \varphi \to \psi \mid \langle \pi_i \rangle \psi \ \text{ where } i = 1,2 \text{ and } \psi \in \mathcal{L}_{T_i}, \\
G = T_1 + T_2 : & \varphi ::= \bot \mid \varphi \to \psi \mid \langle \kappa_i \rangle \psi \ \text{ where } i = 1,2 \text{ and } \psi \in \mathcal{L}_{T_i}, \\
G = (E \Rightarrow T) : & \varphi ::= \bot \mid \varphi \to \psi \mid \langle \pi_e \rangle \psi \ \text{ where } e \in E \text{ and } \psi \in \mathcal{L}_T, \\
G = \mathcal{P}(T) : & \varphi ::= \bot \mid \varphi \to \psi \mid \langle \mathcal{P} \rangle \psi \ \text{ where } \psi \in \mathcal{L}_T.
\end{array}
$$

We use $\top$, $\neg$, $\wedge$, $\vee$, and $\leftrightarrow$ as defined as usual from $\bot$ and $\to$. Also, let $\varphi \dot{\vee} \psi$ be an abbreviation for $(\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$. For each operator $\langle \sigma \rangle$, we shall use $[\sigma]\psi$ to abbreviate $\neg\langle \sigma \rangle \neg \psi$.

For visualizing the connections between the sorts of our models, one can view them as a directed graph whose nodes are given by the sorts. We draw an edge from sort G to sort T if and only if $T \prec G$ or $T = F$ and $G = \mathsf{Id}$. These edges are then labeled with the corresponding modal operators. For instance, for the functor $F = \mathcal{P}(\mathsf{Id}) \times \{0,1\}^{\mathsf{AtProp}}$ (cf. Example 1.3) we obtain the following directed graph:



The above construction of modal operators "along mappings" is akin to the construction of the generic model in [Rei98]. This approach uses nested sketches to canonically describe models and their languages on a high level of abstraction.

Note that the mappings π_i, π_e, and κ_i in the definition below are the corresponding projections and injections of the respective products and coproducts.

2.3. Definition. Let (S, α) be an F-coalgebra. The semantics for the languages $(\mathcal{L}_G)_{G \leq F}$ is defined following the inductive structure of formulas. Whenever $G \leq F$ and $\varphi \in \mathcal{L}_G$ we define the subset $\|\varphi\|^S_G \subseteq G(S)$ containing all elements of $G(S)$ that satisfy φ as follows (the semantics of boolean connectives is omitted here for the sake of simplicity):

$$
\begin{array}{ll}
G = F_C : & \|c\|^S_{F_C} := \{c\}, \\
G = \mathsf{Id} : & \|\langle \alpha \rangle \psi\|^S_{\mathsf{Id}} := \alpha^{-1}\big(\|\psi\|^S_F\big),
\end{array}
$$

$G = T_1 \times T_2$: $\|\langle\pi_i\rangle\psi\|^S_{T_1\times T_2} := \pi_i^{-1}(\|\psi\|^S_{T_i})$,
$G = T_1 + T_2$: $\|\langle\kappa_i\rangle\psi\|^S_{T_1+T_2} := \kappa_i(\|\psi\|^S_{T_i})$,
$G = (E\Rightarrow T)$: $\|\langle\pi_e\rangle\psi\|^S_{(E\Rightarrow T)} := \pi_e^{-1}(\|\psi\|^S_T)$,
$G = \mathcal{P}(T)$: $\|\langle\mathcal{P}\rangle\psi\|^S_{\mathcal{P}(T)} := \{t \in \mathcal{P}(T(S)) \mid \exists u \in t : u \in \|\psi\|^S_T\}$.

For $G \leq F$ and $t \in G(S)$, we write $(S,\alpha), t \vDash_G \varphi$ to mean that $t \in \|\varphi\|^S_G$. Moreover, $(S,\alpha) \vDash_G \varphi$ expresses that $(S,\alpha), t \vDash_G \varphi$ for each $t \in G(S)$ (i.e. $\|\varphi\|^S_G = G(S)$) and $\vDash_G \varphi$ denotes that $(S,\alpha) \vDash_G \varphi$ for each F-coalgebra (S,α).

Let (S,α) and (S',α') be F-coalgebras and $G \leq F$. We say that elements $t \in G(S)$ and $t' \in G(S')$ are **logically equivalent** w.r.t. $\mathcal{L}_G$ if they satisfy exactly the same formulas of $\mathcal{L}_G$.

Note that, for the case $G = \mathcal{P}(T) \leq F$ in the above definition, we have, in particular, that $\|[\mathcal{P}]\psi\|^S_{\mathcal{P}(T)} = \mathcal{P}(\|\psi\|^S_T)$.

Viewed the semantics in the context of Remark 2.1, the relations between the sorts of a model are given by the graphs of the mappings $\alpha : S \to F(S)$, $\pi_i : (T_1 \times T_2)(S) \to T_i(S)$, $\pi_e : (E\Rightarrow T)(S) \to T(S)$ and the inverse graphs of the mappings $\kappa_i : T_i(S) \to (T_1 + T_2)(S)$.

The following proposition checks a basic property of $(\mathcal{L}_G)_{G\leq F}$ – that homomorphisms preserve formulas:

2.4. Proposition. *Let $h : (S,\alpha) \to (S',\alpha')$ be a homomorphism, $G \leq F$, and $\varphi \in \mathcal{L}_G$. Then we have*

$$\|\varphi\|^S_G = G(h)^{-1}(\|\varphi\|^{S'}_G).$$

Proof. By induction on the structure of formulas. □

3 Simplifying the Language

The previous section introduced the languages $(\mathcal{L}_G)_{G\leq F}$ to describe F-coalgebras for Kripke-polynomial functors F. Actually, we are only interested in the language $\mathcal{L}_{\mathsf{Id}}$. However, this language seems to be rather complex since, for each $G \leq F$, $\mathcal{L}_G$ features boolean connectives. For most subfunctors, this can be omitted without loosing expressiveness for the language $\mathcal{L}_{\mathsf{Id}}$. The present section introduces a family $(\overline{\mathcal{L}}_G)_{G\leq F}$ of languages where each $\overline{\mathcal{L}}_G$ is a fragment of $\mathcal{L}_G$. We shall show that $\mathcal{L}_{\mathsf{Id}}$ still embeds into $\overline{\mathcal{L}}_{\mathsf{Id}}$ provided we have the following: whenever there is a constant functor F_C with $F_C \leq T_1 + T_2 \leq F$ such that we do not have $F_C < \mathcal{P}(T) \leq T_1 + T_2$ then the constant set C is finite.

3.1. Definition. Let F be a Kripke-polynomial functor. For each subfunctor G of F, we define the fragment $\overline{\mathcal{L}}_G$ of $\mathcal{L}_G$ as follows:

$G = F_C$: $\varphi ::= c$ where $c \in C$,

$$
\begin{array}{ll}
G = \mathsf{Id}: & \varphi ::= \bot \mid \varphi \to \psi \mid \langle\alpha\rangle\psi \ \text{ where } \psi \in \overline{\mathcal{L}}_F, \\
G = T_1 \times T_2: & \varphi ::= \langle\pi_i\rangle\psi \ \text{ where } i = 1,2 \text{ and } \psi \in \overline{\mathcal{L}}_{T_i}, \\
G = T_1 + T_2: & \varphi ::= \langle\kappa_i\rangle\psi \ \text{ where } i = 1,2 \text{ and } \psi \in \overline{\mathcal{L}}_{T_i}, \\
G = (E \Rightarrow T): & \varphi ::= \langle\pi_e\rangle\psi \ \text{ where } e \in E \text{ and } \psi \in \overline{\mathcal{L}}_T, \\
G = \mathcal{P}(T): & \varphi ::= [\mathcal{P}]\psi \ \text{ where } \psi \in \mathcal{B}(\overline{\mathcal{L}}_T),
\end{array}
$$

i.e. we first close $\overline{\mathcal{L}}_T$ under boolean connectives and then apply $[\mathcal{P}]$ to the resulting formulas.

Note that, for polynomial functors F, the languages for F-coalgebras given in [Röß98] and [Röß99] as well as the language $\overline{\mathcal{L}}_{\mathsf{Id}}$ are all equivalent. Moreover, for those functors considered in [Kur98b], the language $\overline{\mathcal{L}}_{\mathsf{Id}}$ is also equivalent to the corresponding language introduced in [Kur98b].

3.2. Example (1.3. continued). For F-coalgebras with $F = \mathcal{P}(\mathsf{Id}) \times \{0,1\}^{\mathsf{AtProp}}$, we obtain a language $\overline{\mathcal{L}}_{\mathsf{Id}}$ given by

$$\varphi ::= \bot \mid \varphi \to \psi \mid \langle\alpha\rangle\langle\pi_1\rangle[\mathcal{P}]\varphi \mid \langle\alpha\rangle\langle\pi_2\rangle\langle\pi_p\rangle 0 \mid \langle\alpha\rangle\langle\pi_2\rangle\langle\pi_p\rangle 1$$

where $p \in \mathsf{AtProp}$. Let (S,α) be an F-coalgebra and $s \in S$ such that $\alpha(s) = (S', V_s)$ where $S' \subseteq S$ and $V_s : \mathsf{AtProp} \to \{0,1\}$. Then a formula $\langle\alpha\rangle\langle\pi_1\rangle[\mathcal{P}]\varphi$ holds in s if φ holds in all $s' \in S'$. Moreover, $\langle\alpha\rangle\langle\pi_2\rangle\langle\pi_p\rangle 1$ holds in s if for the atomic proposition p we have $V_s(p) = 1$, that is to say if the atomic proposition p holds in s. The formula $\langle\alpha\rangle\langle\pi_2\rangle\langle\pi_p\rangle 0$ expresses that p does not hold in s.

Let us consider the usual finitary (mono-)modal logic $\mathcal{L}$ for Kripke-structures which is given by

$$\varphi ::= \bot \mid \varphi \to \psi \mid p \mid \Box\varphi$$

where $p \in \mathsf{AtProp}$. Thus, we obtain that $\overline{\mathcal{L}}_{\mathsf{Id}}$ is equivalent to $\mathcal{L}$ where a corresponding translation $T : \overline{\mathcal{L}}_{\mathsf{Id}} \to \mathcal{L}$ is given by

$$
\begin{aligned}
T &: \langle\alpha\rangle\langle\pi_1\rangle[\mathcal{P}]\varphi \ \mapsto \Box T(\varphi), \\
T &: \langle\alpha\rangle\langle\pi_2\rangle\langle\pi_p\rangle 0 \mapsto \neg p, \\
T &: \langle\alpha\rangle\langle\pi_2\rangle\langle\pi_p\rangle 1 \mapsto p.
\end{aligned}
$$

3.3. Example (1.4. continued). Assume we deal with alternating automata that are represented by coalgebras of the functor $F = \big((\mathcal{P}(\mathcal{P}(\mathsf{Id})) + \{*\})^{\Sigma}\big) \times \{0,1\}^{\{i,f\}}$. Then we obtain the following language $\overline{\mathcal{L}}_{\mathsf{Id}}$:

$$
\begin{aligned}
\varphi ::= \bot \mid \varphi \to \psi \mid \langle\alpha\rangle\langle\pi_1\rangle\langle\pi_a\rangle\langle\kappa_1\rangle[\mathcal{P}]\psi \text{ where } \psi \in \mathcal{B}(\overline{\mathcal{L}}_{\mathcal{P}(\mathsf{Id})}) \\
\mid \langle\alpha\rangle\langle\pi_1\rangle\langle\pi_a\rangle\langle\kappa_2\rangle * \mid \langle\alpha\rangle\langle\pi_2\rangle\langle\pi_i\rangle 0 \mid \langle\alpha\rangle\langle\pi_2\rangle\langle\pi_i\rangle 1 \mid \\
\langle\alpha\rangle\langle\pi_2\rangle\langle\pi_f\rangle 0 \mid \langle\alpha\rangle\langle\pi_2\rangle\langle\pi_f\rangle 1.
\end{aligned}
$$

Now, for a given F-coalgebra (S, α), consider some $s \in S$ such that $\alpha(s) = \big((\overline{\varrho}(s,a))_{a\in\Sigma}, b_i, b_f\big)$ with $\overline{\varrho}(s,a) := \kappa_1\big(\big\{\{s^a_{i,j}\}_{j\in J^a_i}\big\}_{i\in I^a}\big)$ if $\varrho(s,a)$ is defined and $\overline{\varrho}(s,a) := \kappa_2(*)$ otherwise. Then the formulas $\langle\alpha\rangle\langle\pi_2\rangle\langle\pi_i\rangle 1$ and $\langle\alpha\rangle\langle\pi_2\rangle\langle\pi_f\rangle 1$ indicate whether s is an initial and a final state, respectively, in other words, whether we have $b_i = 1$ and $b_f = 1$, respectively. For some given $a \in \Sigma$, the formula $\langle\alpha\rangle\langle\pi_1\rangle\langle\pi_a\rangle\langle\kappa_2\rangle *$ expresses that $\varrho(a,s)$ is not defined. Now, let $\psi \in \mathcal{B}(\overline{\mathcal{L}}_{\mathcal{P}(\mathsf{Id})})$ be, for instance, of the form $[\mathcal{P}]\varphi \to [\mathcal{P}]\theta$. Then the formula $\langle\alpha\rangle\langle\pi_1\rangle\langle\pi_a\rangle\langle\kappa_1\rangle[\mathcal{P}]\psi$ is satisfied if, for all $i \in I^a$, we have that, whenever φ holds for all $s^a_{i,j}$ with $j \in J^a_i$, then also θ holds for all $s^a_{i,j}$ with $j \in J^a_i$. Note that the formulas in $\mathcal{B}^+(S)$ given by $\varrho(s,a)$ do not have anything to do with the language $\overline{\mathcal{L}}_{\mathsf{Id}}$ since a model of $\mathcal{B}^+(S)$ is the set of all children of some node in a run tree whereas models of $\overline{\mathcal{L}}_{\mathsf{Id}}$ are F-coalgebras.

3.4. Example (1.5. continued). Let us consider F-coalgebras of the functor $F = (\Sigma \times \mathsf{Id}) + \{*\}$ that represent deterministic transition systems with output alphabet Σ. The language $\overline{\mathcal{L}}_{\mathsf{Id}}$ is given by

$$\varphi ::= \bot \mid \varphi \to \psi \mid \langle\alpha\rangle\langle\kappa_1\rangle\langle\pi_1\rangle a \mid \langle\alpha\rangle\langle\kappa_1\rangle\langle\pi_2\rangle\varphi \mid \langle\alpha\rangle\langle\kappa_2\rangle *$$

where $a \in \Sigma$. Let (S, α) be an F-coalgebra and $s \in S$. A formula $\langle\alpha\rangle\langle\kappa_1\rangle\langle\pi_1\rangle a$ holds in s if $\alpha(s) = \kappa_1(a, s')$ for some $s' \in S$, in other words, if (S, α) does not terminate in s yielding an output a. The formula $\langle\alpha\rangle\langle\kappa_1\rangle\langle\pi_2\rangle\varphi$ expresses that (S, α) performs a transition in s such that φ holds in the successor state. Finally, $\langle\alpha\rangle\langle\kappa_2\rangle *$ is satisfied if $\alpha(s) = \kappa_2(*)$, that means if (S, α) terminates in s.

The remainder of this section discusses how $\mathcal{L}_{\mathsf{Id}}$ embeds into $\overline{\mathcal{L}}_{\mathsf{Id}}$.

3.5. Definition. For the following subfunctors T and G of F we define an embedding $\mathsf{emb}_{\langle\sigma\rangle}$ that maps $\mathcal{B}(\overline{\mathcal{L}}_T)$ into $\mathcal{B}(\overline{\mathcal{L}}_G)$. We distinguish the following cases:

(a) $G = T_1 \times T_2$, $T = T_i$, $\sigma = \pi_i$,

(b) $G = T_1 + T_2$, $T = T_i$, $\sigma = \kappa_i$,

(c) $G = (E \Rightarrow T')$, $T = T'$, $\sigma = \pi_e$ (where $e \in E$),

(d) $G = \mathsf{Id}$, $T = F$, $\sigma = \alpha$.

The embedding $\mathsf{emb}_{\langle\sigma\rangle}$ is given by $\mathsf{emb}_{\langle\sigma\rangle} : \varphi \mapsto \langle\sigma\rangle\varphi$ for $\varphi \in \overline{\mathcal{L}}_T$ and then continued on $\mathcal{B}(\overline{\mathcal{L}}_T)$ in the canonical way (in case $\overline{\mathcal{L}}_T \neq \mathcal{B}(\overline{\mathcal{L}}_T)$). In other words, $\mathsf{emb}_{\langle\sigma\rangle} : \mathcal{B}(\overline{\mathcal{L}}_T) \to \mathcal{B}(\overline{\mathcal{L}}_G)$ is defined as follows:

if $T = \mathsf{Id}$: $\mathsf{emb}_{\langle\sigma\rangle} : \varphi \mapsto \langle\sigma\rangle\varphi$ and

if $T \neq \mathsf{Id}$: $\mathsf{emb}_{\langle\sigma\rangle} : \bot \mapsto \bot$,

$$\mathsf{emb}_{\langle\sigma\rangle} : (\varphi \to \psi) \mapsto \big(\mathsf{emb}_{\langle\sigma\rangle}(\varphi) \to \mathsf{emb}_{\langle\sigma\rangle}(\psi)\big),$$

$$\mathsf{emb}_{\langle\sigma\rangle} : \varphi \mapsto \langle\sigma\rangle\varphi \quad \text{if } \varphi \in \overline{\mathcal{L}}_T.$$

If $T \neq \mathsf{Id}$, the embedding $\mathsf{emb}_{\langle\sigma\rangle}$ does nothing but to take a boolean connection of formulas in $\overline{\mathcal{L}}_T$ and put $\langle\sigma\rangle$ in front of each modal operator that occurs in it. Thus, the mapping $\mathsf{emb}_{\langle\sigma\rangle}$ "pushes" the boolean connection part of a formula in $\mathcal{B}(\overline{\mathcal{L}}_T)$ one level further to the "next" subfunctor G of F. It is now immediate that the semantics is preserved (note that $\mathcal{B}(\overline{\mathcal{L}}_T)$ is a fragment of $\mathcal{L}_T$):

3.6. Lemma. *Let (S, α) be an F-coalgebra and let T, G, and σ be as in one of the cases* (a), (c), *or* (d) *of Definition 3.5. Then, for every $\varphi \in \mathcal{B}(\overline{\mathcal{L}}_T)$, we have that*

$$\|\mathsf{emb}_{\langle\sigma\rangle}(\varphi)\|_G^S = \|\langle\sigma\rangle\varphi\|_G^S.$$

In case (b) *of Definition 3.5 we have that*

$$\|\mathsf{emb}_{\langle\kappa_i\rangle}(\varphi)\|_{T_1+T_2}^S \cap \kappa_i\big(T_i(S)\big) = \|\langle\kappa_i\rangle\varphi\|_{T_1+T_2}^S.$$

Proof. In case that $T = \mathsf{Id}$ the claim is trivial. If $T \neq \mathsf{Id}$ the proof is straighforward using induction on the structure of φ. □

Obviously, the language $\mathcal{L}_{\mathsf{Id}}$ is at least as expressive as $\overline{\mathcal{L}}_{\mathsf{Id}}$ since $\overline{\mathcal{L}}_{\mathsf{Id}}$ is a fragment of $\mathcal{L}_{\mathsf{Id}}$. In order to show that the converse also holds we need to restrict the functor F: throughout the remainder of this section we assume the following: whenever there is a constant functor F_C with $F_C \leq T_1 + T_2 \leq F$ such that we do not have $F_C < \mathcal{P}(T) \leq T_1 + T_2$ then the constant set C is finite. That means if we regard F-coalgebras as transition systems then some of its sets of output values are required to be finite.

In order to define a translation from $\mathcal{L}_{\mathsf{Id}}$ to $\overline{\mathcal{L}}_{\mathsf{Id}}$ we need to find a formula of $\overline{\mathcal{L}}_{T_1+T_2}$ that expresses $\langle\kappa_i\rangle\top \in \mathcal{L}_{T_1+T_2}$ in case $T_1 + T_2 \leq F$. For that purpose, we first define a formula $\Delta_G \in \mathcal{B}(\overline{\mathcal{L}}_G)$ with $\|\Delta_G\|_G^S = G(S)$:

3.7. Definition. Let G be a subfunctor of F such that whenever $F_C \leq G$ and we do not have $F_C < \mathcal{P}(T) \leq G$ then the constant set C is finite. We define a formula $\Delta_G \in \mathcal{B}(\overline{\mathcal{L}}_G)$ as follows:

$G = F_C$: $\quad \Delta_{F_C} := \bigvee_{c \in C} c$,

$G = \mathsf{Id}$: $\quad \Delta_{\mathsf{Id}} := \top$,

$G = T_1 \times T_2$: $\quad \Delta_{T_1 \times T_2} := \mathsf{emb}_{\langle\pi_1\rangle}(\Delta_{T_1})$,

$G = T_1 + T_2$: $\quad \Delta_{T_1+T_2} := \mathsf{emb}_{\langle\kappa_1\rangle}(\Delta_{T_1}) \vee \mathsf{emb}_{\langle\kappa_2\rangle}(\Delta_{T_2})$,

$G = (E \Rightarrow T)$: $\quad \Delta_{(E\Rightarrow T)} := \mathsf{emb}_{\langle\pi_{e_E}\rangle}(\Delta_T)$ for some fixed $e_E \in E$,

$G = \mathcal{P}(T)$: $\quad \Delta_{\mathcal{P}(T)} := [\mathcal{P}]\top$.

3.8. Lemma. *If (S, α) is an F-coalgebra and $T_1 + T_2 \leq F$ then we have $\|\mathsf{emb}_{\langle\kappa_i\rangle}(\Delta_{T_i})\|_{T_1+T_2}^S = \kappa_i\big(T_i(S)\big)$.*

Proof. Assume G to be a subfunctor of F as in Definition 3.7. Then it is straightforward to show by induction on the structure of F that $\|\Delta_G\|_G^S = G(S)$ using Lemma 3.6.

In case $T_i = \mathsf{Id}$ we have $\mathsf{emb}_{\langle\kappa_i\rangle}(\Delta_{T_i}) = \langle\kappa_i\rangle\top$ and we are done. If $T_i \neq \mathsf{Id}$, the formula Δ_{T_i} is of the form $\bigvee_{j=1}^{n} \psi_j$ with $\psi_j \in \overline{\mathcal{L}}_{T_i}$ and we have that $\bigcup_{j=1}^{n} \|\psi_j\|_{T_i}^{S} = T_i(S)$. Thus, by Definition 3.5, we get

$$\begin{aligned}\|\mathsf{emb}_{\langle\kappa_i\rangle}(\Delta_{T_i})\|_{T_1+T_2}^{S} &= \textstyle\bigcup_{j=1}^{n} \|\langle\kappa_i\rangle\psi_j\|_{T_1+T_2}^{S} = \bigcup_{j=1}^{n} \kappa_i\big(\|\psi_j\|_{T_i}^{S}\big)\\ &= \kappa_i\big(\textstyle\bigcup_{j=1}^{n} \|\psi_j\|_{T_i}^{S}\big) = \kappa_i\big(T_i(S)\big).\end{aligned}$$ □

3.9. Definition. For each subfunctor G of F, we define a translation $\mathsf{T}_G : \mathcal{L}_G \to \mathcal{B}(\overline{\mathcal{L}}_G)$ by a simultaneous induction as follows (we only give T_G explicitly for the non-boolean-connection-part of $\mathcal{L}_G$ and then assume T_G to be continued in the canonical way):

$G = F_C$: $\mathsf{T}_{F_C} : c \mapsto c,$

$G = \mathsf{Id}$: $\mathsf{T}_{\mathsf{Id}} : \langle\alpha\rangle\psi \mapsto \mathsf{emb}_{\langle\alpha\rangle}\big(\mathsf{T}_F(\psi)\big),$

$G = T_1 \times T_2$: $\mathsf{T}_{T_1\times T_2} : \langle\pi_i\rangle\psi \mapsto \mathsf{emb}_{\langle\pi_i\rangle}\big(\mathsf{T}_{T_i}(\psi)\big),$

$G = T_1 + T_2$: $\mathsf{T}_{T_1+T_2} : \langle\kappa_i\rangle\psi \mapsto \mathsf{emb}_{\langle\kappa_i\rangle}\big(\mathsf{T}_{T_i}(\psi)\big) \wedge \mathsf{emb}_{\langle\kappa_i\rangle}(\Delta_{T_i}),$

$G = (E\Rightarrow T)$: $\mathsf{T}_{(E\Rightarrow T)} : \langle\pi_e\rangle\psi \mapsto \mathsf{emb}_{\langle\pi_e\rangle}\big(\mathsf{T}_T(\psi)\big),$

$G = \mathcal{P}(T)$: $\mathsf{T}_{\mathcal{P}(T)} : \langle\mathcal{P}\rangle\psi \mapsto \neg[\mathcal{P}]\neg\mathsf{T}_T(\psi).$

Now it follows immediately from Lemmas 3.6 and 3.8 that T_{Id} indeed embeds $\mathcal{L}_{\mathsf{Id}}$ into $\mathcal{B}(\overline{\mathcal{L}}_{\mathsf{Id}}) = \overline{\mathcal{L}}_{\mathsf{Id}}$:

3.10. Proposition. *Let F be a Kripke-polynomial functor such that whenever $F_C \leq T_1 + T_2 \leq F$ and we do not have $F_C < \mathcal{P}(T) \leq T_1 + T_2$ then the constant C is finite. Let (S,α) be an F-coalgebra. Then, for each $G \leq F$ and each $\varphi \in \mathcal{L}_G$, we have that*

$$\|\varphi\|_G^S = \|\mathsf{T}_G(\varphi)\|_G^S.$$ □

4 Expressiveness

In order to distinguish elements of F-coalgebras up to bisimilarity we do not need the full expressiveness of $\mathcal{L}_{\mathsf{Id}}$: it is sufficient to consider a fragment of it. Thus, we define a restricted family $(\tilde{\mathcal{L}}_G)_{G\leq F}$ of languages and prove that $\tilde{\mathcal{L}}_{\mathsf{Id}}$ is powerful enough to distinguish elements up to bisimilarity for so-called image-finite F-coalgebras.

4.1. Definition. Let F be a Kripke-polynomial functor. For each subfunctor G of F, we define a fragment $\tilde{\mathcal{L}}_G$ of $\mathcal{L}_G$ as follows:

$G = F_C$: $\varphi ::= c$ where $c \in C$,

$G = \mathsf{Id}$: $\varphi ::= \bot \mid \varphi \to \psi \mid \langle\alpha\rangle\psi$ where $\psi \in \tilde{\mathcal{L}}_F$,

$G = T_1 \times T_2$: $\varphi ::= \langle\pi_i\rangle\psi$ where $i = 1, 2$ and $\psi \in \tilde{\mathcal{L}}_{T_i}$,

$G = T_1 + T_2$: $\varphi ::= \langle\kappa_i\rangle\psi$ where $i = 1, 2$ and $\psi \in \tilde{\mathcal{L}}_{T_i}$,

$G = (E \Rightarrow T): \varphi ::= \langle \pi_e \rangle \psi$ where $e \in E$ and $\psi \in \tilde{\mathcal{L}}_T$,

$G = \mathcal{P}(T): \quad \varphi ::= \langle \mathcal{P} \rangle \bigwedge \Phi \mid [\mathcal{P}] \bigvee \Phi$ where $\Phi \subseteq \tilde{\mathcal{L}}_T$, Φ finite.

The languages $(\tilde{\mathcal{L}}_G)_{G \leq F}$ are usually less expressive than $(\mathcal{L}_G)_{G \leq F}$. For instance, let $F = \mathsf{Id} \times \mathcal{P}(F_C)$ where C is a countable set. Then we cannot give a formula $\varphi \in \tilde{\mathcal{L}}_{\mathcal{P}(F_C)}$ such that φ holds for any F-coalgebra (S, α) in all $t \in \big(\mathcal{P}(F_C)\big)(S) = \mathcal{P}(C)$ since t might be empty or countable. On the other hand, $[\mathcal{P}]\top \in \mathcal{L}_{\mathcal{P}(F_C)}$ satisfies this property.

In the following we prove that the family $(\tilde{\mathcal{L}}_G)_{G \leq F}$ is in fact expressive enough to distinguish elements up to bisimilarity. That requires an equivalent definition of bisimulation (see Definition 1.2) by induction on subfunctors of F. The following definition is equivalent to the notion of the lifting of a relation given in [Jac95].

4.2. Definition. Let $R \subseteq S \times S'$. For $G \leq F$ we define $R_G \subseteq G(S) \times G(S')$ as follows:

$G = F_C: \quad t R_{F_C} t' :\Leftrightarrow t = t'$,

$G = \mathsf{Id}: \quad t R_{\mathsf{Id}} t' :\Leftrightarrow t R t'$,

$G = T_1 \times T_2: \quad t R_{T_1 \times T_2} t' :\Leftrightarrow \forall i = 1,2: \pi_i(t) R_{T_i} \pi_i(t')$,

$G = T_1 + T_2: \quad t R_{T_1 + T_2} t' :\Leftrightarrow \forall i = 1,2:$ if $t \in \kappa_i\big(T_i(S)\big)$ then $t' \in \kappa_i\big(T_i(S')\big)$ and $\kappa_i^{-1}(t) R_{T_i} \kappa_i^{-1}(t')$, [2]

$G = (E \Rightarrow T): \quad t R_{(E \Rightarrow T)} t' :\Leftrightarrow \forall e \in E: \pi_e(t) R_T \pi_e(t')$,

$G = \mathcal{P}(T): \quad t R_{\mathcal{P}(T)} t' :\Leftrightarrow \forall x \in t: \exists y \in t': x R_T y$ and $\forall y \in t': \exists x \in t: x R_T y$.

4.3. Lemma. *Let (S, α) and (S', α') be F-coalgebras and $R \subseteq S \times S'$. Then R is a bisimulation between (S, α) and (S', α') if and only if, for all $(s, s') \in R$, we have $\alpha(s) R_F \alpha'(s')$.* □

Similarly as for Kripke-structures, we obtain that bisimilarity coincides with logical equivalence for so-called image-finite structures. Here this concept is defined as follows:

4.4. Definition. Let F be a Kripke-polynomial functor and S be a set. An element $t \in F(S)$ is called **image-finite** if we have $t \in F'(S)$ where F' is the functor that is constructed as F but only using the *finite* power set functor $\mathcal{P}_{fin}$ instead of the power set functor $\mathcal{P}$. An F-coalgebra (S, α) is called **image-finite** if, for each $s \in S$, $\alpha(s) \in F(S)$ is image-finite.

Lemma 4.7 requires a formula $\Delta_G(t) \in \tilde{\mathcal{L}}_G$ that can be constructed for $G \leq F$ and some image-finite $t \in G(S)$ such that $(S, \alpha), t \models_G \Delta_G(t)$.

[2] Note that κ_i is an injective mapping and therefore κ_i^{-1} is a partial mapping from $(T_1 + T_2)(S)$ to $T_i(S)$ with its domain being $\kappa_i(T_i(S))$.

4.5. Definition. Let $G \leq F$ and $t \in G(S)$ be image-finite. We define the formula $\Delta_G(t) \in \tilde{\mathcal{L}}_G$ as follows:

$G = F_C$: $\Delta_{F_C}(t) := t \in C$,

$G = \mathsf{Id}$: $\Delta_{\mathsf{Id}}(t) := \top$,

$G = T_1 \times T_2$: $\Delta_{T_1 \times T_2}(t) := \langle \pi_1 \rangle \Delta_{T_1}\big(\pi_1(t)\big)$,

$G = T_1 + T_2$: $\Delta_{T_1+T_2}(t) := \langle \kappa_i \rangle \Delta_{T_i}\big(\kappa_i^{-1}(t)\big)$ where $t \in \kappa_i\big(T_i(S)\big)$,

$G = (E \Rightarrow T)$: $\Delta_{(E \Rightarrow T)}(t) := \langle \pi_{e_E} \rangle \Delta_T\big(\pi_{e_E}(t)\big)$ for some fixed $e_E \in E$,

$G = \mathcal{P}(T)$: $\Delta_{\mathcal{P}(T)}(t) := [\mathcal{P}] \bigvee_{i=1}^{n} \Delta_T(x_i)$ where $t = \{x_1, \ldots, x_n\}$.

4.6. Lemma. *Let (S, α) be an F-coalgebra, $G \leq F$, and $t \in G(S)$ image-finite. Then we have*

$$(S, \alpha), t \vDash_G \Delta_G(t).$$

Proof. By induction on the structure of $\Delta_G(t)$. □

4.7. Lemma. *Let (S, α) and (S', α') be image-finite F-coalgebras and let $\approx \subseteq S \times S'$ denote logical equivalence w.r.t. $\tilde{\mathcal{L}}_{\mathsf{Id}}$. Let $G \leq F$ and $t \in G(S)$, $t' \in G(S')$ with $t \not\approx_G t'$. Then there exists a formula $\theta_G(t, t') \in \tilde{\mathcal{L}}_G$ such that*

$$(S, \alpha), t \vDash_G \theta_G(t, t') \text{ and } (S', \alpha'), t' \nvDash_G \theta_G(t, t').$$

Proof. By induction on subfunctors G of F:

$G = F_C$: We set $\theta_G(t, t') := t \in C$.

$G = \mathsf{Id}$: By assumption there exists some $\varphi \in \tilde{\mathcal{L}}_{\mathsf{Id}}$ such that $(S, \alpha), t \vDash \varphi$ and $(S', \alpha'), t' \nvDash \varphi$ since $\tilde{\mathcal{L}}_{\mathsf{Id}}$ is closed under negation. We set $\theta_G(t, t') := \varphi$.

$G = T_1 \times T_2$: There is some $i \in \{1, 2\}$ with $\pi_i(t) \not\approx_{T_i} \pi_i(t')$. We set $\theta_G(t, t') := \langle \pi_i \rangle \theta_{T_i}\big(\pi_i(t), \pi_i(t')\big)$.

$G = T_1 + T_2$: Let $t \in \kappa_i\big(T_i(S)\big)$. If $t' \notin \kappa_i\big(T_i(S')\big)$ then we set $\theta_G(t, t') := \Delta_G(t)$. By Lemma 4.6 we automatically get that $(S, \alpha), t \vDash_G \theta_G(t, t')$ and $(S', \alpha'), t' \nvDash_G \theta_G(t, t')$. In case $t' \in \kappa_i\big(T_i(S')\big)$ we have $\kappa_i^{-1}(t) \not\approx_{T_i} \kappa_i^{-1}(t')$. Thus, we obtain some $\theta_{T_i}\big(\kappa_i^{-1}(t), \kappa_i^{-1}(t')\big)$ by the induction hypothesis and we put $\theta_G(t, t') := \langle \kappa_i \rangle \theta_{T_i}\big(\kappa_i^{-1}(t), \kappa_i^{-1}(t')\big)$.

$G = (E \Rightarrow T)$: There exists some $e \in E$ with $\pi_e(t) \not\approx_T \pi_e(t')$ and thus we set $\theta_G(t, t') := \langle \pi_e \rangle \theta_T\big(\pi_e(t), \pi_e(t')\big)$.

$G = \mathcal{P}(T)$: Assume that there is some $x \in t$ such that, for all $y_i \in t' = \{y_1, \ldots, y_n\}$, we have $x \not\approx_T y_i$. Hence, for each $1 \leq i \leq n$, we obtain some $\theta_T(x, y_i)$ with $(S, \alpha), x \vDash_T \theta_T(x, y_i)$ and $(S', \alpha'), y_i \nvDash_T \theta_T(x, y_i)$. We define $\theta_G(t, t') := \langle \mathcal{P} \rangle \bigwedge_{i=1}^{n} \theta_T(x, y_i)$. In the dual case there exists some $y \in t'$ such that, for all $x_j \in t = \{x_1, \ldots, x_m\}$, we have $x_j \not\approx_T y$. Thus, we obtain formulas $\theta_T(x_j, y)$ with $(S, \alpha), x_j \vDash_T \theta_T(x_j, y)$ and $(S', \alpha'), y \nvDash_T \theta_T(x_j, y)$. We put $\theta_G(t, t') := [\mathcal{P}] \bigvee_{j=1}^{m} \theta_T(x_j, y)$. □

4.8. Proposition. *Let (S, α) and (S', α') be image-finite F-coalgebras. Then the largest bisimulation relation $\sim \subseteq S \times S'$ between (S, α) and (S', α') and* *the logical equivalence relation* $\approx \subseteq S \times S'$ *w.r.t.* $\tilde{\mathcal{L}}_{\mathsf{Id}}$ *coincide.*

Proof. "$\subseteq$": Assume $s \in S$ and $s' \in S'$ with $s \sim s'$. The corresponding projections π_S and $\pi_{S'}$ of the bisimulation relation $\sim$ are homomorphisms. Therefore, by Proposition 2.4, we have $s \approx s'$.

"$\supseteq$": Assume that $\approx$ is not a bisimulation relation. Hence, by Lemma 4.3, there exist some $s \in S$ and $s' \in S'$ with $s \approx s'$ and $\alpha(s) \not\approx_F \alpha'(s')$. Lemma 4.7 yields some $\theta_F\big(\alpha(s), \alpha'(s')\big) \in \tilde{\mathcal{L}}_F$ such that we have $(S, \alpha), \alpha(s) \vDash_F \theta_F\big(\alpha(s), \alpha'(s')\big)$ and $(S', \alpha'), \alpha'(s') \nvDash_F \theta_F\big(\alpha(s), \alpha'(s')\big)$. Therefore the formula $\langle\alpha\rangle\theta_F\big(\alpha(s), \alpha'(s')\big) \in \tilde{\mathcal{L}}_{\mathsf{Id}}$ distinguishes s and s' which contradicts with $s \approx s'$. □

It is not surprising that we need to assume the coalgebras in Proposition 4.8 to be image-finite. This restriction is already needed for the analogous result in the case of Kripke-structures.

5 A Complete Calculus

This section presents a complete calculus that is defined – as to be expected – by a simultaneous induction on the subfunctors of F.

We shall state this calculus for the language $\overline{\mathcal{L}}_{\mathsf{Id}}$ instead of $\mathcal{L}_{\mathsf{Id}}$. The reason is that the language $\mathcal{L}_{\mathsf{Id}}$ is more complex than necessary: Section 2 shows that, for most functors, its fragment $\overline{\mathcal{L}}_{\mathsf{Id}}$ is as expressive as $\mathcal{L}_{\mathsf{Id}}$. Moreover, the "classical" special case of the (usual) modal logic for Kripke-structures is, syntactically, an instance of $\overline{\mathcal{L}}_{\mathsf{Id}}$ (cf. Example 3.2).

Defining a complete calculus for $\mathcal{L}_{\mathsf{Id}}$, however, would be rather straightforward using Remark 2.1. For each $G \leq F$, one would have $(\mathrm{Taut})_G$ and $(\mathrm{MP})_G$ as well as $(\mathrm{K})_{T,G}$ and $(\mathrm{N})_{T,G}$ where $T \prec G$ or $T = F$ and $G = \mathsf{Id}$. Furthermore, some additional axioms would be needed to capture the local structure of the functor (cf. Definition 5.1). That would yield a family of calculi indexed by subfunctors of F such that the G-th calculus is complete w.r.t. $\mathcal{L}_G$.

The family $(\vdash_G)_{G\leq F}$ of calculi that shall actually be defined here is somewhat simpler but not complete w.r.t. *every* $G \leq F$. As we are aiming at a description language for F-coalgebras (i.e. at $\overline{\mathcal{L}}_{\mathsf{Id}}$) it is only necessary to make the calculus $\vdash_{\mathsf{Id}}$ complete w.r.t. $\overline{\mathcal{L}}_{\mathsf{Id}}$. This shall be outlined in the remainder of the present section.

Similarly to Section 3 we assume all constant sets C that occur in F to be finite in the remainder of this section. This restriction is not surprising as it is also required in [Kur98b,Röß98] in order to define a complete calculus.

5.1. Definition. We define a family $(\vdash_G)_{G\leq F}$ of calculi for $\big(\mathcal{B}(\overline{\mathcal{L}}_G)\big)_{G\leq F}$ by a simultaneous induction on all subfunctors G of F:

$G = F_C$: (Det) $\vdash_{F_C} \dot{\bigvee}_{c\in C} c$,

$G = \mathsf{Id}$: (Taut) all substitution instances of boolean tautologies in $\overline{\mathcal{L}}_{\mathsf{Id}}$, (MP) $\dfrac{\vdash_{\mathsf{Id}} \varphi,\ \vdash_{\mathsf{Id}} \varphi \to \psi}{\vdash_{\mathsf{Id}} \psi}$,

(N) $\dfrac{\vdash_F \varphi}{\vdash_{\mathsf{Id}} \langle\alpha\rangle\varphi}$,

(Det) $\vdash_{T_1\times T_2} \langle\pi_i\rangle\varphi \leftrightarrow [\pi_i]\varphi$ if $\mathsf{Id} = T_i \prec T_1 \times T_2$,

$\vdash_{T_1+T_2} \mathsf{emb}_{\langle\kappa_i\rangle}(\Delta_{T_i}) \to (\langle\kappa_i\rangle\varphi \leftrightarrow [\kappa_i]\varphi)$ if $\mathsf{Id} = T_i \prec T_1 + T_2$,

$\vdash_{(E\Rightarrow T)} \langle\pi_e\rangle\varphi \leftrightarrow [\pi_e]\varphi$ if $\mathsf{Id} = T \prec (E\Rightarrow T)$,

$\vdash_{\mathsf{Id}} \langle\alpha\rangle\varphi \leftrightarrow [\alpha]\varphi$ if $\mathsf{Id} = F$,

(K) $\vdash_{T_1\times T_2} \langle\pi_i\rangle(\varphi \to \psi) \to (\langle\pi_i\rangle\varphi \to \langle\pi_i\rangle\psi)$ if $\mathsf{Id} = T_i \prec T_1 \times T_2$,

$\vdash_{T_1+T_2} \langle\kappa_i\rangle(\varphi \to \psi) \to (\langle\kappa_i\rangle\varphi \to \langle\kappa_i\rangle\psi)$ if $\mathsf{Id} = T_i \prec T_1 + T_2$,

$\vdash_{(E\Rightarrow T)} \langle\pi_e\rangle(\varphi \to \psi) \to (\langle\pi_e\rangle\varphi \to \langle\pi_e\rangle\psi)$ if $\mathsf{Id} = T \prec (E\Rightarrow T)$,

$\vdash_{\mathsf{Id}} \langle\alpha\rangle(\varphi \to \psi) \to (\langle\alpha\rangle\varphi \to \langle\alpha\rangle\psi)$ if $\mathsf{Id} = F$,

$G = T_1 \times T_2$: (N) $\dfrac{\vdash_{T_i} \varphi}{\vdash_{T_1\times T_2} \mathsf{emb}_{\langle\pi_i\rangle}(\varphi)}$ for $i = 1, 2$,

$G = T_1 + T_2$: (Copr) $\vdash_{T_1+T_2} \mathsf{emb}_{\langle\kappa_1\rangle}(\Delta_{T_1}) \dot\vee \mathsf{emb}_{\langle\kappa_2\rangle}(\Delta_{T_2})$,

(N) $\dfrac{\vdash_{T_i} \varphi}{\vdash_{T_1+T_2} \mathsf{emb}_{\langle\kappa_i\rangle}(\Delta_{T_i}) \to \mathsf{emb}_{\langle\kappa_i\rangle}(\varphi)}$ for $i = 1, 2$,

(In) $\vdash_{T_1+T_2} \langle\kappa_i\rangle\varphi \to \mathsf{emb}_{\langle\kappa_i\rangle}(\Delta_{T_i})$ for $i = 1, 2$,

$G = (E\Rightarrow T)$: (N) $\dfrac{\vdash_T \varphi}{\vdash_{(E\Rightarrow T)} \mathsf{emb}_{\langle\pi_e\rangle}(\varphi)}$ for $e \in E$,

$G = \mathcal{P}(T)$: (Taut) all substitution instances of boolean tautologies in $\mathcal{B}(\overline{\mathcal{L}}_T)$,

(MP) $\dfrac{\vdash_T \varphi,\ \vdash_T \varphi \to \psi}{\vdash_T \psi}$,

(K) $\vdash_{\mathcal{P}(T)} [\mathcal{P}](\varphi \to \psi) \to ([\mathcal{P}]\varphi \to [\mathcal{P}]\psi)$,

(N) $\dfrac{\vdash_T \varphi}{\vdash_{\mathcal{P}(T)} [\mathcal{P}]\varphi}$.

5.2. Example (1.3. continued). In case our models are Kripke-structures we deal with a functor $F = \mathcal{P}(\mathsf{Id}) \times \{0,1\}^{\mathsf{AtProp}}$. Hence we obtain the following axioms and rules for the subfunctors G of F:

$G = \mathsf{Id}$: (Taut) all substitution instances of boolean tautologies in $\overline{\mathcal{L}}_{\mathsf{Id}}$,

(MP) $\dfrac{\vdash_{\mathsf{Id}} \varphi,\ \vdash_{\mathsf{Id}} \varphi \to \psi}{\vdash_{\mathsf{Id}} \psi}$,

(N) $\dfrac{\vdash_F \varphi}{\vdash_{\mathsf{Id}} \langle\alpha\rangle\varphi}$,

$G = \mathcal{P}(\mathsf{Id})$: (K) $\vdash_{\mathcal{P}(\mathsf{Id})} [\mathcal{P}](\varphi \to \psi) \to ([\mathcal{P}]\varphi \to [\mathcal{P}]\psi)$,

(N) $\dfrac{\vdash_{\mathsf{Id}} \varphi}{\vdash_{\mathcal{P}(\mathsf{Id})} [\mathcal{P}]\varphi}$,

$G = F_{\{0,1\}}$: (Det) $\vdash_{F_{\{0,1\}}} 0 \dot\vee 1$,

$G = (\mathsf{AtProp}{\Rightarrow}F_{\{0,1\}})$: (N) $\dfrac{\vdash_{F_{\{0,1\}}} \varphi}{\vdash_{(\mathsf{AtProp}\Rightarrow F_{\{0,1\}})} \mathsf{emb}_{\langle\pi_p\rangle}\varphi}$ for $p \in \mathsf{AtProp}$,

$F = \mathcal{P}(\mathsf{Id}) \times \{0,1\}^{\mathsf{AtProp}}$: (N) $\dfrac{\vdash_{\mathcal{P}(\mathsf{Id})} \varphi}{\vdash_F \mathsf{emb}_{\langle\pi_1\rangle}\varphi}$, $\dfrac{\vdash_{\{0,1\}^{\mathsf{AtProp}}} \varphi}{\vdash_F \mathsf{emb}_{\langle\pi_2\rangle}\varphi}$.

That means, for $G = \mathsf{Id}$, the calculus $\vdash_{\mathsf{Id}}$ is given as follows:

(Taut) all substitution instances of boolean tautologies in $\overline{\mathcal{L}}_{\mathsf{Id}}$,

(MP) $\dfrac{\vdash_{\mathsf{Id}} \varphi,\ \vdash_{\mathsf{Id}} \varphi \to \psi}{\vdash_{\mathsf{Id}} \psi}$,

(N) $\dfrac{\vdash_{\mathsf{Id}} \varphi}{\vdash_{\mathsf{Id}} \langle\alpha\rangle\langle\pi_1\rangle[\mathcal{P}]\varphi}$,

(K) $\vdash_{\mathsf{Id}} \langle\alpha\rangle\langle\pi_1\rangle[\mathcal{P}](\varphi \to \psi) \to (\langle\alpha\rangle\langle\pi_1\rangle[\mathcal{P}]\varphi \to \langle\alpha\rangle\langle\pi_1\rangle[\mathcal{P}]\psi)$,

(Det) $\vdash_{\mathsf{Id}} \langle\alpha\rangle\langle\pi_2\rangle\langle\pi_p\rangle 1 \dot{\vee} \langle\alpha\rangle\langle\pi_2\rangle\langle\pi_p\rangle 0$.

Up to the last clause, this is exactly the complete calculus for $\mathcal{L}$ (cf. Example 3.2) known from modal logic for Kripke-structures (cf. e.g. [Gol87,Pop94]) modulo the translation given in Example 3.2. The last axiom states that $\langle\alpha\rangle\langle\pi_2\rangle\langle\pi_p\rangle 0$ does not contribute to the expressiveness of $\overline{\mathcal{L}}_{\mathsf{Id}}$ and therefore we can also dispense with this formula. Hence this restricted language is even syntactically equivalent to $\mathcal{L}$.

5.3. Proposition (Soundness). Whenever $G \leq F$ and $\varphi \in \mathcal{B}(\overline{\mathcal{L}}_G)$ then we have

$$\vdash_G \varphi \quad \Longrightarrow \quad \models_G \varphi.$$

Proof. By induction on the length of the proof. □

5.4. Definition. For each subfunctor G of F, we define a syntactical calculus $\Vdash_G$ that extends the calculus $\vdash_G$ for formulas in $\mathcal{B}(\overline{\mathcal{L}}_G)$ as follows:

(Ext) $\dfrac{\vdash_G \varphi}{\Vdash_G \varphi}$,

(Taut) all substitution instances of boolean tautologies in $\mathcal{B}(\overline{\mathcal{L}}_G)$,

(MP) $\dfrac{\Vdash_G \varphi,\ \Vdash_G \varphi \to \psi}{\Vdash_G \psi}$.

Note that only for $G = \mathsf{Id}$ and for $G = T$ with $\mathcal{P}(T) \leq F$, the caculi $\vdash_G$ and $\Vdash_G$ coincide. In the following we introduce the notion of a canonical F-coalgebra which is – as usual – constructed on maximal consistent sets of formulas.

5.5. Definition. Let G be a subfunctor of F. A subset Φ of $\mathcal{B}(\overline{\mathcal{L}}_G)$ is **consistent** if there are no formulas $\varphi_1, \ldots, \varphi_n \in \Phi$ such that

$$\Vdash_G \varphi_1 \wedge \ldots \wedge \varphi_n \to \bot.$$

A subset Φ of $\mathcal{B}(\overline{\mathcal{L}}_G)$ is called **maximal** if it is consistent and for every $\varphi \in$

$\mathcal{B}(\overline{\mathcal{L}}_G)$ we have

$$\varphi \in \Phi \text{ or } \neg\varphi \in \Phi.$$

We set $S_G := \{\Phi \subseteq \mathcal{B}(\overline{\mathcal{L}}_G) \mid \Phi \text{ is maximal}\}$.

5.6. Lemma.

(a) *Whenever* $T_1 \times T_2 \leq F$, $i \in \{1,2\}$, *and* $\varphi \in \mathcal{B}(\overline{\mathcal{L}}_{T_i})$ *then* $\Vdash_{T_i} \varphi$ *implies* $\Vdash_{T_1 \times T_2} \mathsf{emb}_{\langle \pi_i \rangle}(\varphi)$.

(b) *Whenever* $T_1 + T_2 \leq F$, $i \in \{1,2\}$, *and* $\varphi \in \mathcal{B}(\overline{\mathcal{L}}_{T_i})$ *then* $\Vdash_{T_i} \varphi$ *implies* $\Vdash_{T_1 + T_2} \mathsf{emb}_{\langle \kappa_i \rangle}(\Delta_{T_i}) \to \mathsf{emb}_{\langle \kappa_i \rangle}(\varphi)$.

(c) *Whenever* $(E \Rightarrow T) \leq F$, $e \in E$, *and* $\varphi \in \mathcal{B}(\overline{\mathcal{L}}_T)$ *then* $\Vdash_T \varphi$ *implies* $\Vdash_{(E \Rightarrow T)} \mathsf{emb}_{\langle \pi_e \rangle}(\varphi)$.

(d) *Whenever* $\varphi \in \mathcal{B}(\overline{\mathcal{L}}_F)$ *then* $\Vdash_F \varphi$ *implies* $\Vdash_{\mathsf{Id}} \mathsf{emb}_{\langle \alpha \rangle}(\varphi)$.

Proof. Depending on the definition of $\mathsf{emb}_{\langle \sigma \rangle}$, the claim is immediate or can be shown easily by induction on the length of the proof. □

5.7. Lemma. *Let G be a subfunctor of F and $\Gamma \in S_G$. Then we have, for the following cases:*

$G = F_C$: *there is exactly one* $c \in C$ *such that* $c \in \Gamma$,

$G = \mathsf{Id}$: *we have* $\Gamma_{\langle \alpha \rangle} := \mathsf{emb}^{-1}_{\langle \alpha \rangle}(\Gamma) \in S_F$,

$G = T_1 \times T_2$: *for* $i = 1, 2$, *we have* $\Gamma_{\langle \pi_i \rangle} := \mathsf{emb}^{-1}_{\langle \pi_i \rangle}(\Gamma) \in S_{T_i}$,

$G = T_1 + T_2$: *there is exactly one* $i \in \{1, 2\}$ *such that* $\mathsf{emb}_{\langle \kappa_i \rangle}(\Delta_{T_i}) \in \Gamma$. *Moreover, then we have* $\Gamma_{\langle \kappa_i \rangle} := \mathsf{emb}^{-1}_{\langle \kappa_i \rangle}(\Gamma) \in S_{T_i}$,

$G = (E \Rightarrow T)$: *for each* $e \in E$, *we have* $\Gamma_{\langle \pi_e \rangle} := \mathsf{emb}^{-1}_{\langle \pi_e \rangle}(\Gamma) \in S_T$.

Proof.

$G = F_C$: By Axiom (Det).

$G = \mathsf{Id}$: First, let $F = \mathsf{Id}$. Then we have $\Gamma_{\langle \alpha \rangle} = \{\varphi \in \overline{\mathcal{L}}_F \mid \langle \alpha \rangle \varphi \in \Gamma\}$. Assume that there exist $\varphi_1, \ldots, \varphi_n \in \Gamma_{\langle \alpha \rangle}$ with $\Vdash_F \varphi_1 \wedge \ldots \wedge \varphi_n \to \bot$. Using (Taut) and (MP) we conclude $\vdash_F \varphi_1 \wedge \ldots \wedge \varphi_n \to \bot$. By applying Rule (N) and Axiom (K), we obtain $\Vdash_{\mathsf{Id}} \langle \alpha \rangle \varphi_1 \wedge \ldots \wedge \langle \alpha \rangle \varphi_n \to \langle \alpha \rangle \bot$. Now Axiom (Det) yields $\neg \langle \alpha \rangle \top \in \Gamma$ which contradicts with $\langle \alpha \rangle \top \in \Gamma$. Now assume $\varphi, \neg\varphi \notin \Gamma_{\langle \alpha \rangle}$. Thus, $\langle \alpha \rangle \varphi, \langle \alpha \rangle \neg\varphi \notin \Gamma$ and therefore $\neg\langle \alpha \rangle \varphi, \neg\langle \alpha \rangle \neg\varphi \in \Gamma$. We finally get a contradiction by $\neg\langle \alpha \rangle \varphi, \langle \alpha \rangle \varphi \in \Gamma$ using Axiom (Det). This proves $\Gamma_{\langle \alpha \rangle} \in S_F$.

In case $F \neq \mathsf{Id}$ the maximality of $\Gamma_{\langle \alpha \rangle}$ follows from Lemma 5.6 (d) and Definition 3.5.

$G = T_1 \times T_2$: In analogy to the case $G = \mathsf{Id}$.

$G = T_1 + T_2$: Axiom (Copr) ensures that there is exactly one $i \in \{1, 2\}$ with $\mathsf{emb}_{\langle \kappa_i \rangle}(\Delta_{T_i}) \in \Gamma$. Now, for $\mathsf{emb}_{\langle \kappa_i \rangle}(\Delta_{T_i}) \in \Gamma$, the maximality of $\Gamma_{\langle \kappa_i \rangle}$ is proved as in the case $G = \mathsf{Id}$.

$G = (E \Rightarrow T)$: In analogy to the case $G = \mathsf{Id}$. □

5.8. Definition. Following the structure of F, we define, for each $G \leq F$, a mapping $\alpha_G : S_G \to G(S_F)$ as follows:

$G = F_C :$ $\quad \alpha_{F_C} : \Gamma \mapsto c$ with $c \in \Gamma$,

$G = \mathsf{Id} :$ $\quad \alpha_{\mathsf{Id}} : \Gamma \mapsto \Gamma_{\langle \alpha \rangle}$,

$G = T_1 \times T_2 :$ $\quad \alpha_{T_1 \times T_2} : \Gamma \mapsto \big(\alpha_{T_1}(\Gamma_{\langle \pi_1 \rangle}), \alpha_{T_2}(\Gamma_{\langle \pi_2 \rangle})\big)$,

$G = T_1 + T_2 :$ $\quad \alpha_{T_1 + T_2} : \Gamma \mapsto \kappa_i\big(\alpha_{T_i}(\Gamma_{\langle \kappa_i \rangle})\big)$ where $\mathsf{emb}_{\langle \kappa_i \rangle}(\Delta_{T_i}) \in \Gamma$,

$G = (E \Rightarrow T) :$ $\quad \alpha_{(E \Rightarrow T)} : \Gamma \mapsto (\alpha_T(\Gamma_{\langle \pi_e \rangle}))_{e \in E}$,

$G = \mathcal{P}(T) :$ $\quad \alpha_{\mathcal{P}(T)} : \Gamma \mapsto \{\alpha_T(\Gamma') \mid \Gamma' \in S_T$ and $\forall \psi \in \mathcal{B}(\overline{\mathcal{L}}_T) :$
$[\mathcal{P}]\psi \in \Gamma \Rightarrow \psi \in \Gamma'\}$.

We define (S_F, α_F) to be the **canonical F-coalgebra**.

Lemma 5.7 guarantees that (S_F, α_F) is indeed well-defined. The following lemma contains two standard results (cf. e.g. [Pop94]) and is not proved here.

5.9. Lemma. *Let $\mathcal{L}$ be a language containing boolean connectives and let $\vdash$ be a syntactical calculus for $\mathcal{L}$ including substitution instances of boolean tautologies and modus ponens. Let Φ be a consistent subset of $\mathcal{L}$, i.e. there are no members $\varphi_1, \ldots, \varphi_n$ of Φ with $\vdash \varphi_1 \wedge \ldots \wedge \varphi_n \to \bot$. Then there exists a maximal subset Γ of $\mathcal{L}$ (i.e. Γ is consistent and $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ for each $\varphi \in \mathcal{L}$) such that $\Phi \subseteq \Gamma$.*

Moreover, whenever $\Psi \subseteq \mathcal{L}$ and $\psi \in \mathcal{L}$, the following are equivalent:

(i) $\exists \psi_1, \ldots, \psi_n \in \Psi : \vdash \psi_1 \wedge \ldots \wedge \psi_n \to \psi$,

(ii) $\forall \Gamma \subseteq \mathcal{L}$ *with* Γ *maximal:* $\Psi \subseteq \Gamma \Rightarrow \psi \in \Gamma$.

5.10. Lemma. *Whenever $G \leq F$, $\Gamma \in S_G$, and $\varphi \in \mathcal{B}(\overline{\mathcal{L}}_G)$ then we have that*

$$\alpha_G(\Gamma) \in \|\varphi\|_G^{S_F} \iff \varphi \in \Gamma.$$

Proof. By a simultaneous induction on all $G \leq F$ following the structure of φ. The case that φ is a boolean connection is obvious. For the rest we shall distinguish the following cases:

$G = F_C$: By Definition 5.8 and Lemma 5.7 we have, for $\varphi = c \in \overline{\mathcal{L}}_{F_C}$, that

$$\alpha_{F_C}(\Gamma) \in \|c\|_{F_C}^{S_F} \Leftrightarrow \alpha_{F_C}(\Gamma) = c \Leftrightarrow c \in \Gamma.$$

$G = \mathsf{Id}$: Using the induction hypothesis we get, for $\varphi = \langle \alpha \rangle \psi \in \overline{\mathcal{L}}_{\mathsf{Id}}$, that

$$\begin{aligned} &\alpha_{\mathsf{Id}}(\Gamma) \in \|\langle \alpha \rangle \psi\|_{\mathsf{Id}}^{S_F} = \alpha_F^{-1}\big(\|\psi\|_F^{S_F}\big) \\ &\Leftrightarrow \alpha_F\big(\alpha_{\mathsf{Id}}(\Gamma)\big) = \alpha_F(\Gamma_{\langle \alpha \rangle}) \in \|\psi\|_F^{S_F} \\ &\Leftrightarrow \psi \in \Gamma_{\langle \alpha \rangle} \\ &\Leftrightarrow \mathsf{emb}_{\langle \alpha \rangle}(\psi) = \langle \alpha \rangle \psi \in \Gamma. \end{aligned}$$

$G = T_1 \times T_2$: By Definition 5.8 and by the induction hypothesis we have, for $\varphi = \langle \pi_i \rangle \psi \in \overline{\mathcal{L}}_{T_1 \times T_2}$, that

$$\begin{aligned} &\alpha_{T_1 \times T_2}(\Gamma) \in \|\langle \pi_i \rangle \psi\|^{S_F}_{T_1 \times T_2} = \pi_i^{-1}\big(\|\psi\|^{S_F}_{T_i}\big) \\ \Leftrightarrow\ & \pi_i\big(\alpha_{T_1 \times T_2}(\Gamma)\big) = \alpha_{T_i}(\Gamma_{\langle \pi_i \rangle}) \in \|\psi\|^{S_F}_{T_i} \\ \Leftrightarrow\ & \psi \in \Gamma_{\langle \pi_i \rangle} \\ \Leftrightarrow\ & \mathsf{emb}_{\langle \pi_i \rangle}(\psi) = \langle \pi_i \rangle \psi \in \Gamma. \end{aligned}$$

$G = T_1 + T_2$: Again, by Definition 5.8, by the induction hypothesis, and by Axiom (In), we have, for $\varphi = \langle \kappa_i \rangle \psi \in \overline{\mathcal{L}}_{T_1 + T_2}$, that

$$\begin{aligned} &\alpha_{T_1 + T_2}(\Gamma) \in \|\langle \kappa_i \rangle \psi\|^{S_F}_{T_1 + T_2} = \kappa_i\big(\|\psi\|^{S_F}_{T_i}\big) \\ \Leftrightarrow\ & \mathsf{emb}_{\langle \kappa_i \rangle}(\Delta_{T_i}) \in \Gamma \text{ and } \alpha_{T_i}(\Gamma_{\langle \kappa_i \rangle}) \in \|\psi\|^{S_F}_{T_i} \\ \Leftrightarrow\ & \mathsf{emb}_{\langle \kappa_i \rangle}(\Delta_{T_i}) \in \Gamma \text{ and } \psi \in \Gamma_{\langle \kappa_i \rangle} \\ \Leftrightarrow\ & \mathsf{emb}_{\langle \kappa_i \rangle}(\psi) = \langle \kappa_i \rangle \psi \in \Gamma. \end{aligned}$$

$G = (E \Rightarrow T)$: Analogous to the case $G = T_1 \times T_2$.

$G = \mathcal{P}(T)$: "$\Rightarrow$": Let $\varphi = [\mathcal{P}]\psi \in \overline{\mathcal{L}}_{\mathcal{P}(T)}$ and $\alpha_{\mathcal{P}(T)}(\Gamma) \in \|[\mathcal{P}]\psi\|^{S_F}_{\mathcal{P}(T)}$. Whenever $\Gamma' \in S_T$ with $\Gamma_{[\mathcal{P}]} := \{\theta \in \mathcal{B}(\overline{\mathcal{L}}_T) \mid [\mathcal{P}]\theta \in \Gamma\} \subseteq \Gamma'$ then we have $\alpha_T(\Gamma') \in \|\psi\|^{S_F}_T$. By the induction hypothesis, the latter is equivalent to $\psi \in \Gamma'$. Now Lemma 5.9 gives $\theta_1, \ldots, \theta_n \in \Gamma_{[\mathcal{P}]}$ with

$$\Vdash_T \theta_1 \wedge \ldots \wedge \theta_n \to \psi.$$

We conclude $\vdash_T \theta_1 \wedge \ldots \wedge \theta_n \to \psi$ by (Taut) and (MP) for $\vdash_T$ and, thus, we get $\vdash_{\mathcal{P}(T)} [\mathcal{P}](\theta_1 \wedge \ldots \wedge \theta_n \to \psi)$ by Rule (N). Axiom (K) finally yields $\Vdash_{\mathcal{P}(T)} [\mathcal{P}]\theta_1 \wedge \ldots \wedge [\mathcal{P}]\theta_n \to [\mathcal{P}]\psi$ which proves $[\mathcal{P}]\psi \in \Gamma$.

"$\Leftarrow$": Let $[\mathcal{P}]\psi \in \overline{\mathcal{L}}_{\mathcal{P}(T)}$ and assume that $[\mathcal{P}]\psi \in \Gamma$. Then $\psi \in \Gamma_{[\mathcal{P}]}$ and, for all $\Gamma' \in S_T$, we have $\Gamma_{[\mathcal{P}]} \subseteq \Gamma' \Rightarrow \psi \in \Gamma'$. The induction hypothesis now gives

$$\forall \Gamma' \in S_T : \Gamma_{[\mathcal{P}]} \subseteq \Gamma' \Rightarrow \alpha_T(\Gamma') \in \|\psi\|^{S_F}_T$$

which eventually proves $\alpha_{\mathcal{P}(T)}(\Gamma) \in \|[\mathcal{P}]\psi\|^{S_F}_{\mathcal{P}(T)}$. □

5.11. Theorem. *Let F be a Kripke-polynomial functor such that Id is a subfunctor of F and all constant sets that occur in F are finite. Then, for every $\varphi \in \overline{\mathcal{L}}_{\mathsf{Id}}$, the following are equivalent:*

(i) $\vdash_{\mathsf{Id}} \varphi$,

(ii) $\models_{\mathsf{Id}} \varphi$,

(iii) $(S_F, \alpha_F) \models_{\mathsf{Id}} \varphi$.

Proof. (i)$\Rightarrow$(ii). By Proposition 5.3.

(ii)$\Rightarrow$(iii). Obvious.

(iii)⇒(i). Observe that $\{\neg\varphi\}$ is not consistent (otherwise there existed some $\Gamma \in S_{\mathsf{Id}}$ with $\neg\varphi \in \Gamma$ by Lemma 5.9 and hence $(S_F, \alpha_F), \alpha_{\mathsf{Id}}(\Gamma) \models_{\mathsf{Id}} \neg\varphi$ by Lemma 5.10). Therefore we get $\vdash_{\mathsf{Id}} \neg\varphi \to \bot$ which proves $\vdash_{\mathsf{Id}} \varphi$. □

6 Conclusion

The present approach shows how to generalize both modal logic for Kripke-structures (see e.g. [Gol87,Pop94]) and modal languages for coalgebras that represent deterministic systems (cf. [Kur98b,Röß98]). We introduced a language $\mathcal{L}_{\mathsf{Id}}$ that, for a given Kripke-polynomial functor F, describes the corresponding F-coalgebras. For a slightly restricted class of functors F, the fragment $\overline{\mathcal{L}}_{\mathsf{Id}}$ of $\mathcal{L}_{\mathsf{Id}}$ turned out to be as expressive as $\mathcal{L}_{\mathsf{Id}}$. In case $\mathcal{P}(T) \leq F$, formulas of $\overline{\mathcal{L}}_{\mathsf{Id}}$ might still become rather complex since then we have $[\mathcal{P}]\varphi \in \overline{\mathcal{L}}_{\mathcal{P}(T)}$ where $\varphi \in \mathcal{B}(\overline{\mathcal{L}}_T)$. Using a still simpler language (cf. [Jac99]) could possibly be of greater interest for specifying and verifying systems. But then one would have to pay the price of a reduced expressiveness: bisimilarity would probably not equal logical equivalence for image-finite systems.

For application purposes, it might be of interest to build different languages, e.g. for modelling the methods of an object by one single modal operator. The multisorted structure makes that rather easy. For instance, in cases $G = T_1 \times T_2$ and $G = T_1 + T_2$ in Definition 2.2, one could additionally use formulas $\langle \pi_1, \pi_2 \rangle(\varphi_1, \varphi_2) \in \overline{\mathcal{L}}_{T_1 \times T_2}$ and $\langle \kappa_1, \kappa_2 \rangle(\varphi_1, \varphi_2) \in \overline{\mathcal{L}}_{T_1 + T_2}$, respectively, where $\varphi_1 \in \overline{\mathcal{L}}_{T_1}$ and $\varphi_2 \in \overline{\mathcal{L}}_{T_2}$. The corresponding semantics would then be given by

$$\|\langle \pi_1, \pi_2 \rangle(\varphi_1, \varphi_2)\|^S_{T_1 \times T_2} := \|\langle \pi_1 \rangle \varphi_1\|^S_{T_1 \times T_2} \cap \|\langle \pi_2 \rangle \varphi_2\|^S_{T_1 \times T_2} \text{ and}$$
$$\|\langle \kappa_1, \kappa_2 \rangle(\varphi_1, \varphi_2)\|^S_{T_1 + T_2} := \|\langle \kappa_1 \rangle \varphi_1\|^S_{T_1 + T_2} \cup \|\langle \kappa_2 \rangle \varphi_2\|^S_{T_1 + T_2}.$$

Similarly, for a subfunctor $(E \Rightarrow T)$ of F one could consider formulas $\langle \pi_E \rangle \varphi$ with

$$\|\langle \pi_E \rangle \varphi\|^S_{(E \Rightarrow T)} := \bigcap_{e \in E} \|\langle \pi_e \rangle \varphi\|^S_{(E \Rightarrow T)}.$$

Another opportunity is to build modal operators capturing the whole structure of F: this would then correspond to the coalgebraic logic presented in [Mos97].

It might also be of interest whether a (possibly simpler) language can distinguish elements up to similarity (cf. [Bal00]). Another option of altering the language is to add always- and pasttime-operators (cf. [Jac99]) in order to gain more expressiveness. Even more general, one could add arbitrary fixed points to the language as done in the modal μ-calculus (cf. [Sti96]) and possibly derive a generalization of the modal μ-calculus for a coalgebraic setting.

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