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Compensated compactness for differential forms in Carnot groups and applications ☆

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Abstract

In this paper we prove a compensated compactness theorem for differential forms of the intrinsic complex of a Carnot group. The proof relies on an L^s-Hodge decomposition for these forms. Because of the lack of homogeneity of the intrinsic exterior differential, Hodge decomposition is proved using the parametrix of a suitable 0-order Laplacian on forms.

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Contents

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1. Introduction

In the last few years, so-called subriemannian structures have been largely studied in several respects, such as differential geometry, geometric measure theory, subelliptic differential equations, complex variables, optimal control theory, mathematical models in neurosciences, non-holonomic mechanics, robotics. Roughly speaking, a subriemannian structure on a manifold M is defined by a subbundle H of the tangent bundle TM, that defines the "admissible" directions at any point of M (typically, think of a mechanical system with non-holonomic constraints). Usually, H is called the *horizontal* bundle. If we endow each fiber H_x of H with a scalar product $\langle , \rangle_x$, there is a naturally associated distance d on M, defined as the Riemannian length of the horizontal curves on M, i.e. of the curves γ such that $\gamma'(t) \in H_{\gamma(t)}$. Nowadays, the distance d is called Carnot–Carathéodory distance associated with H, or control distance, since it can be viewed as the minimal cost of a control problem, with constraints given by H.

Among all subriemannian structures, a prominent position is taken by the so-called Carnot groups (simply connected Lie groups $\mathbb{G}$ with stratified nilpotent algebra $\mathfrak{g}$: see e.g. [3,25,27]), which play versus subriemannian spaces the role played by Euclidean spaces (considered as tangent spaces) versus Riemannian manifolds. In this case, the first layer of the stratification of the algebra – that can be identified with a linear subspace of the tangent space to the group at the origin – generates, by left translations, our horizontal subbundle. Moreover, through the exponential map, Carnot groups can be identified with the Euclidean space $\mathbb{R}^n$ endowed with a (non-commutative) group law, where $n = \dim \mathfrak{g}$.

In this picture, horizontal vector fields (i.e. sections of H) are the natural counterpart of the vector fields in Euclidean spaces. In the Euclidean setting, several questions in pde's and calculus of variations (like, e.g., non-periodic homogenization for second order elliptic equations or semicontinuity of variational functional in elasticity) can be reduced to the following problem: given two sequences $(E_k)_k$ and $(D_n)_n$ of vector fields weakly convergent in $L^2(\mathbb{R}^n)$, what can we say about the convergence of their scalar product? The compensated compactness (or div–curl) theorem of Murat and Tartar [18,19] provides an answer: it states basically that the scalar product $\langle E_k, D_k \rangle$ still converges in the sense of distributions, provided $\{\operatorname{div} D_k \colon k \in \mathbb{N}\}$ and $\{\operatorname{curl} E_k \colon k \in \mathbb{N}\}$ are compact in $H^{-1}_{\mathrm{loc}}(\mathbb{R}^n)$ and $(H^{-1}_{\mathrm{loc}}(\mathbb{R}^n))^{n(n-1)/2}$, respectively.

When attacking for instance the study of the non-periodic homogenization of differential operators in a Carnot group $\mathbb{G}$, it is natural to look for a similar statement for horizontal vector fields in $\mathbb{G}$. In fact, a preliminary difficulty consists in finding the appropriate notion of divergence and curl operators for horizontal vector fields in Carnot groups. To this end, it is convenient to write our problem in terms of differential forms, and to attack the more general problem of compensated compactness for sequences of differential forms. Indeed, we can identify each vector field E_k with a 1-form η_k, and each vector field D_k with the 1-form γ_k. Then, the compactness of $\operatorname{curl} E_k$ is equivalent to the compactness of $d\eta_k$. Analogously, denoting by $*$ the Hodge duality operator, the compactness of $\operatorname{div} D_k$ is equivalent to the compactness of $*d(*\gamma_k)$, and hence to the compactness of $d(*\gamma_k)$. With these notations, if φ is a smooth function with compact support and dV denotes the volume element in $\mathbb{R}^n$, then $\langle E_k, D_k\rangle\varphi\, dV = \varphi\eta_k \wedge *\gamma_k$.

Thus, a natural formulation of the compensated compactness theorem in the De Rham complex (Ω, d) reads as follows (see, e.g., [14] and [20]):

If $1 < s_i < \infty$, $0 \leqslant h_i \leqslant n$ for $i = 1, 2$, and $0 < \varepsilon < 1$, assume that $\alpha_i^\varepsilon \in L^{s_i}_{\mathrm{loc}}(\mathbb{R}^n, \Omega^{h_i})$ for $i = 1, 2$, where $\frac{1}{s_1} + \frac{1}{s_2} = 1$ and $h_1 + h_2 = n$. Assume that

$$\alpha_i^\varepsilon \rightharpoonup \alpha_i \quad \text{weakly in } L^{s_i}_{\mathrm{loc}}\big(\mathbb{R}^n, \Omega^{h_i}\big) \quad \text{as } \varepsilon \to 0, \tag{1}$$

and that

$$\big\{d\alpha_i^\varepsilon\big\} \quad \text{is pre-compact in } W^{-1,s_i}_{\mathrm{loc}}\big(\mathbb{R}^n, \Omega^{h_i+1}\big) \tag{2}$$

for $i = 1, 2$.

Then

$$\int_{\mathbb{R}^n} \varphi\alpha_1^\varepsilon \wedge \alpha_2^\varepsilon \to \int_{\mathbb{R}^n} \varphi\alpha_1 \wedge \alpha_2 \quad \text{as } \varepsilon \to 0 \tag{3}$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

Thus, when dealing with Carnot groups, we are reduced preliminarily to look for a somehow “intrinsic” notion of differential forms such that

- Intrinsic 1-forms should be horizontal 1-forms, i.e. forms that are dual of horizontal vector fields, where by duality we mean that, if v is a vector field in $\mathbb{R}^n$, then its dual form $v^\natural$ acts as $v^\natural(w) = \langle v, w\rangle$, for all $w \in \mathbb{R}^n$.
- Some “intrinsic” exterior differential should act between intrinsic forms. Again, the intrinsic differential of a smooth function, should be its horizontal differential (that is dual operator of the gradient along a basis of the horizontal bundle).
- “Intrinsic forms” and the “intrinsic differential” should define a complex that is exact and self-dual under Hodge $*$-duality.

It turns out that such a complex (in fact a sub-complex of the De Rham complex) has been defined and studied by M. Rumin in [24] and [23] ([22] for contact structures), so that we are provided with a good setting for our theory. For sake of self-consistency of the paper, we present in Section 2 the main features of this complex, that will be denoted by (E_0^*, d_c), where $d_c : E_0^h \to E_0^{h+1}$ is a suitable exterior differential. We stress now that a crucial property of d_c relies on the fact that it is in general a non-homogeneous higher order differential operator. To better understand how this feature affects the compensated compactness theorem, we begin by sketching the basic steps of the proof in the Euclidean setting. The crucial point consists in proving the following Hodge type decomposition: if $0 < \varepsilon < 1$, let α^ε be compactly supported differential h-forms such that

$$\alpha^\varepsilon \rightharpoonup \alpha \quad \text{as } \varepsilon \to 0 \quad \text{weakly in } L^s\big(\mathbb{R}^n, \Omega^h\big) \tag{4}$$

and

$$\big\{d\alpha^\varepsilon\big\} \quad \text{is compact in } W^{-1,s}_{\mathrm{loc}}\big(\mathbb{R}^n, \Omega^{h+1}\big). \tag{5}$$

Then there exist h-forms ω^ε and $(h-1)$-forms ψ^ε such that

- $\omega^\varepsilon \to \omega$ strongly in $L^s_{\mathrm{loc}}(\mathbb{R}^n, \Omega^h)$;
- $\psi^\varepsilon \to \psi$ strongly in $L^s_{\mathrm{loc}}(\mathbb{R}^n, \Omega^{h-1})$;
- $\alpha^\varepsilon = \omega^\varepsilon + d\psi^\varepsilon$.

Roughly speaking (for instance, modulo suitable cut-off functions), the proof of the decomposition can be carried out as follows (see e.g. [20]).

- let $\Delta := \delta d + d\delta$ be the Laplace operator on k-forms, where $\delta = d^*$ is the L^2 formal adjoint of d;
- we write

$$\alpha^\varepsilon = \Delta\Delta^{-1}\alpha^\varepsilon = \delta d\Delta^{-1}\alpha^\varepsilon + d\delta\Delta^{-1}\alpha^\varepsilon;$$

- we set

$$\omega^\varepsilon := \delta d\Delta^{-1}\alpha^\varepsilon = \delta\Delta^{-1}d\alpha^\varepsilon$$

  that is strongly compact in $L^s_{\mathrm{loc}}(\mathbb{R}^n)$, since $d\alpha^\varepsilon$ is strongly compact in $W^{-1,s}_{\mathrm{loc}}(\mathbb{R}^n)$;
- we set

$$\psi^\varepsilon := \delta\Delta^{-1}\alpha^\varepsilon$$

  that converges weakly in $W^{1,s}_{\mathrm{loc}}(\mathbb{R}^n)$ and hence strongly in $L^s_{\mathrm{loc}}(\mathbb{R}^n)$.

If we want to repeat a similar argument, we face several difficulties. First of all, the "naif Laplacian" associated with d_c, i.e.

$$\delta_c d_c + d_c\delta_c$$

where $\delta_c = d_c^*$, in general is not homogeneous (and therefore, as long as we know, we lack Rockland type hypoellipticity results and optimal estimates in a "natural" scale of Sobolev spaces). Even if d_c is homogeneous, as in the Heisenberg group $\mathbb{H}^n$, such a "Laplacian" is not homogeneous. For instance, on 1-forms in $\mathbb{H}^1$, $\delta_c d_c$ is a 4th order operator, while $d_c\delta_c$ is a 2nd order one. This is due to the fact that the order of d_c depends on the order of the forms on which it acts on. In fact, d_c on 1-forms in $\mathbb{H}^1$ is a 2nd order operator, as well as its adjoint δ_c (which acts on 2-form), while δ_c on 1-forms is a first order operator, since it is the adjoint of d_c on 0-forms, which is a first order operator.

Though in the particular case of 1-forms in $\mathbb{H}^1$ this difficulty can be overcame as in [2], by using the suitable homogeneous 4th order operator $\delta_c d_c + (d_c\delta_c)^2$ defined by Rumin [22] that satisfies also sharp a priori estimates, the general situation requires different arguments.

In general, the lack of homogeneity of d_c can be described through the notion of *weight* of vector fields and, by duality, of differential forms (see [24]). Elements of the jth layer of $\mathfrak{g}$ are said to have (pure) weight $w = j$; by duality, a 1-form that is dual of a vector field of (pure) weight $w = j$ will be said to have (pure) weight $w = j$. Vector fields in the direct sum of the first $j-1$ layers of $\mathfrak{g}$ are said to have weight $w < j$. Thus, a non-vanishing 1-form is said to have weight $w \geqslant j$ if it vanishes on all vectors of weight $w < j$. This procedure can be extended to h-forms. Clearly, there are forms that have no pure weight, but we can decompose E_0^h in the direct sum of orthogonal spaces of forms of pure weight, and therefore we can find a basis of E_0^h given by orthonormal forms of increasing pure weights. We refer to such a basis as to a basis adapted to the filtration of E_0^h induced by the weight.

Then, once suitable adapted bases of h-forms and $(h+1)$-forms are chosen, d_c can be viewed as a matrix-valued operator such that, if α has weight p, then the component of weight q of $d_c\alpha$ is given by a differential operator in the horizontal derivatives of order $q - p \geqslant 1$, acting on the components of α.

The following two simple examples can enlight the phenomenon. We restrict ourselves to 1-forms, and therefore we need to describe only E_0^1 and E_0^2. For more examples and proofs of the statements, see Appendix B.

Let $\mathbb{G} := \mathbb{H}^1 \equiv \mathbb{R}^3$ be the first Heisenberg group, with variables (x, y, t). Set $X := \partial_x + 2y\partial_t$, $Y := \partial_y - 2x\partial_t$, $T := \partial_t$. The dual forms are respectively dx, dy and θ, where θ is the contact form of $\mathbb{H}^1$. The stratification of the algebra $\mathfrak{g}$ is given by $\mathfrak{g} = V_1 \oplus V_2$, where $V_1 = \operatorname{span}\{X, Y\}$ and $V_2 = \operatorname{span}\{T\}$. In this case, $E_0^1 = \operatorname{span}\{dx, dy\}$ and $E_0^2 = \operatorname{span}\{dx \wedge \theta, dy \wedge \theta\}$. These forms have respectively weight 1 (1-forms) and 3 (2-forms). As for 1-forms, the exterior differential d_c acts as follows:

$$\begin{aligned} d_c(\alpha_X\,dx + \alpha_Y\,dy) &= -\frac{1}{4}\big(X^2\alpha_Y - 2XY\alpha_X + YX\alpha_X\big)\,dx \wedge \theta \\ &\quad - \frac{1}{4}\big(2YX\alpha_Y - Y^2\alpha_X - XY\alpha_Y\big)\,dy \wedge \theta \\ &:= P_1(\alpha_X, \alpha_Y)\,dx \wedge \theta + P_2(\alpha_X, \alpha_Y)\,dy \wedge \theta. \end{aligned}$$

Notice that P_1, P_2 are homogeneous operators of order 2 $(= 3 - 1)$ in the horizontal derivatives.

Consider now a slightly different setting. Let $\mathbb{G} := \mathbb{H}^1 \times \mathbb{R}$, and denote by (x, y, t) the variables in $\mathbb{H}^1$ and by s the variable in $\mathbb{R}$. Set X, Y, T as above, and $S := \partial_s$. The dual form of S is ds. The stratification of the algebra $\mathfrak{g}$ is given by $\mathfrak{g} = V_1 \oplus V_2$, where $V_1 = \operatorname{span}\{X, Y, S\}$ and $V_2 = \operatorname{span}\{T\}$. In this case $E_0^1 = \operatorname{span}\{dx, dy, ds\}$ and $E_0^2 = \operatorname{span}\{dx \wedge ds, dy \wedge ds, dx \wedge \theta, dy \wedge \theta\}$. Thus, all 1-forms have weight 1, whereas 2-forms have weight 2 ($dx \wedge ds$ and $dy \wedge ds$) and 3 ($dx \wedge \theta$ and $dy \wedge \theta$). The exterior differential d_c on 1-forms acts as follows:

$$\begin{aligned} d_c(\alpha_X\,dx + \alpha_Y\,dy + \alpha_S\,ds) &= P_1(\alpha_X, \alpha_Y)\,dx \wedge \theta \\ &\quad + P_2(\alpha_X, \alpha_Y)\,dy \wedge \theta + (X\alpha_S - S\alpha_X)\,dx \wedge ds \\ &\quad + (Y\alpha_S - S\alpha_Y)\,dy \wedge ds, \end{aligned}$$

where P_1, P_2 have been defined above. Thus, the components of d_c are homogeneous differential operators of order 2 or 1.

To overcome the difficulties arising from the lack of homogeneity of d_c, we rely on an argument introduced in [24] (when dealing with the notion of CC-elliptic complex). Let us give a non-rigorous sketch of the argument. Denote by $\Delta_{\mathbb{G}}$ the positive scalar sub-Laplacian associated with a basis of the first layer of $\mathfrak{g}$ ($\Delta_{\mathbb{G}}$ is a Hörmander's sum-of-squares operator). Remember that, once adapted bases of E_0^h and E_0^{h+1} are chosen, d_c can be viewed as a matrix-valued differential operator, whose entries are homogeneous operators in the horizontal derivatives. Then we can multiply d_c from the left and from the right by suitable diagonal matrices whose entries are positive or negative fractional powers of $\Delta_{\mathbb{G}}$, in such a way that all entries of the resulting matrix-valued operator are 0-order operators. By the way, this notion of order of an operator, as well as all combination rules that are applied, have a precise meaning only in the setting of a pseudodifferential calculus. We rely on the CGGP-calculus (see [5] and Appendix A). In such a way, we obtain a "0-order exterior differential" $\tilde{d}_c$, and eventually a "0-order Laplacian" $\tilde{d}_c(\tilde{d}_c)^* + (\tilde{d}_c)^*\tilde{d}_c$, that, thanks to [24] and [5], has both a right and a left parametrix. Thus, we can mimic the proof we have sketched above for the De Rham complex (again, working in a precise pseudodifferential calculus allows the composition of different operators).

It is worth noticing that the lack of homogeneity of the exterior differential d_c affects also the natural hypotheses we assume in order to prove Hodge decomposition and compensated compactness theorem for forms in E_0. Indeed, in the Euclidean setting, assumptions (4) and (5) are naturally correlated by the fact that the exterior differential d is a homogeneous operator of order 1, which maps continuously $L^s_{\mathrm{loc}}(\mathbb{R}^n)$ into $W^{-1,s}_{\mathrm{loc}}(\mathbb{R}^n)$. Instead, when we are dealing with the complex (E_0^*, d_c), given a sequence of h-forms α^ε that converges weakly $L^s_{\mathrm{loc}}(\mathbb{R}^n, E_0^h)$, then the different components of $d_c\alpha^\varepsilon$ converge weakly in Sobolev spaces of different negative orders, according to the weight of the different components. For instance, if we denote by $W^{-k,s}_{\mathbb{G},\mathrm{loc}}(\mathbb{R}^n)$ the Sobolev space of negative order $-k$ associated with horizontal derivatives (see Section 3), then in our model examples $\mathbb{H}^1$ and $\mathbb{H}^1 \times \mathbb{R}$, with an obvious meaning of the notations, assumption (5) for 1-forms becomes

$$\left\{P_i\left(\alpha^\varepsilon_X, \alpha^\varepsilon_Y\right)\right\} \quad \text{compact in } W^{-2,s}_{\mathbb{G},\mathrm{loc}}\left(\mathbb{R}^n\right), \quad i = 1, 2,$$

when $\mathbb{G} = \mathbb{H}^1$, and

$$\left\{P_i\left(\alpha^\varepsilon_X, \alpha^\varepsilon_Y\right)\right\} \quad \text{compact in } W^{-2,s}_{\mathbb{G},\mathrm{loc}}\left(\mathbb{R}^n\right), \quad i = 1, 2,$$

as well as

$$\left\{X\alpha^\varepsilon_S - S\alpha^\varepsilon_X\right\}, \left\{Y\alpha^\varepsilon_S - S\alpha^\varepsilon_Y\right\} \quad \text{compact in } W^{-1,s}_{\mathbb{G},\mathrm{loc}}\left(\mathbb{R}^n\right)$$

when $\mathbb{G} = \mathbb{H}^1 \times \mathbb{R}$.

Our compensated compactness result for horizontal vector fields is contained in its simplest form in Theorem 5.1, that can be derived by standard arguments from a general statement (Theorem 4.13) for intrinsic differential h-forms, that holds whenever all intrinsic h-forms have the same pure weight (this is always true if $h = 1$).

In Section 2 we establish most of the notations, and we collect more or less known results about Carnot groups and the basic ingredients of Rumin's theory. In Section 3 we introduce from the functional point of view all the function spaces we need in the sequel, with a special attention for negative order spaces (which turn out to be spaces of currents). Moreover we emphasize the connections between our function spaces and the pseudodifferential operators of the CGGP-calculus. In Section 4 we establish and we prove our main results: Hodge decomposition and compensated compactness for forms (Theorems 4.1 and 4.13). In Section 5 we apply our main results to prove a div–curl theorem for horizontal vector fields (Theorem 5.1). We illustrate several different explicit examples, and we apply the theory to the study of the H-convergence of divergence form second order differential operators in Carnot groups. In Appendix A we summarize the basic facts of the theory of pseudodifferential operators in homogeneous groups as given in [5]. Moreover, we prove representation theorems and continuity properties for pseudodifferential operators in our scale of Sobolev spaces. Finally, in Appendix B we write explicitly the structure of the intrinsic differential d_c and we analyze a list of detailed examples.

2. Preliminary results and notations

A *Carnot group* $\mathbb{G}$ *of step* κ is a simply connected Lie group whose Lie algebra $\mathfrak{g}$ has dimension n, and admits a *step* κ *stratification*, i.e. there exist linear subspaces $V_1, \dots, V_\kappa$ such that

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_\kappa, \quad [V_1, V_i] = V_{i+1}, \quad V_\kappa \neq \{0\}, \quad V_i = \{0\} \quad \text{if } i > \kappa, \tag{6}$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. Let $m_i = \dim(V_i)$, for $i = 1, \dots, \kappa$ and $h_i = m_1 + \cdots + m_i$ with $h_0 = 0$ and, clearly, $h_\kappa = n$. Choose a basis $e_1, \dots, e_n$ of $\mathfrak{g}$ adapted to the stratification, i.e. such that

$$e_{h_{j-1}+1}, \dots, e_{h_j} \quad \text{is a basis of } V_j \quad \text{for each } j = 1, \dots, \kappa.$$

Let $X = \{X_1, \dots, X_n\}$ be the family of left invariant vector fields such that $X_i(0) = e_i$. Given (6), the subset $X_1, \dots, X_{m_1}$ generates by commutations all the other vector fields; we will refer to $X_1, \dots, X_{m_1}$ as *generating vector fields* of the group. The exponential map is a one to one map from $\mathfrak{g}$ onto $\mathbb{G}$, i.e. any $p \in \mathbb{G}$ can be written in a unique way as $p = \exp(p_1 X_1 + \cdots + p_n X_n)$. Using these *exponential coordinates*, we identify p with the n-tuple $(p_1, \dots, p_n) \in \mathbb{R}^n$ and we identify $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$, where the explicit expression of the group operation $\cdot$ is determined by the Campbell–Hausdorff formula. If $p \in \mathbb{G}$ and $i = 1, \dots, \kappa$, we put $p^i = (p_{h_{i-1}+1}, \dots, p_{h_i}) \in \mathbb{R}^{m_i}$, so that we can also identify p with $(p^1, \dots, p^\kappa) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_\kappa} = \mathbb{R}^n$.

For any $x \in \mathbb{G}$, the *(left) translation* $\tau_x : \mathbb{G} \to \mathbb{G}$ is defined as

$$z \mapsto \tau_x z := x \cdot z.$$

For any $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, is defined as

$$\delta_\lambda(x_1, \dots, x_n) = \left(\lambda^{d_1} x_1, \dots, \lambda^{d_n} x_n\right), \tag{7}$$

where $d_i \in \mathbb{N}$ is called *homogeneity of the variable* x_i in $\mathbb{G}$ (see [10], Chapter 1) and is defined as

$$d_j = i \quad \text{whenever } h_{i-1} + 1 \leqslant j \leqslant h_i, \tag{8}$$

hence $1 = d_1 = \cdots = d_{m_1} < d_{m_1+1} = 2 \leqslant \cdots \leqslant d_n = \kappa$.

In addition, we remind that

$$\delta_\lambda x \cdot \delta_\lambda y = \delta_\lambda (x \cdot y)$$

and that the inverse x^{-1} of an element $x = (x_1, \dots, x_n) \in (\mathbb{R}^n, \cdot)$ has the form

$$x^{-1} = (-x_1, \dots, -x_n).$$

The Lie algebra $\mathfrak{g}$ can be endowed with a scalar product $\langle\cdot,\cdot\rangle$, making $\{X_1,\dots,X_n\}$ an orthonormal basis.

As customary, we fix a smooth homogeneous norm $|\cdot|$ in $\mathbb{G}$ such that the gauge distance $d(x,y) := |y^{-1}x|$ is a left-invariant true distance, equivalent to the Carnot–Carathéodory distance in $\mathbb{G}$ (see [25], p. 638). We set $B(p,r) = \{q \in \mathbb{G}; d(p,q) < r\}$.

The Haar measure of $\mathbb{G} = (\mathbb{R}^n,\cdot)$ is the Lebesgue measure $\mathcal{L}^n$ in $\mathbb{R}^n$. If $A \subset \mathbb{G}$ is $\mathcal{L}^n$-measurable, we write also $|A| := \mathcal{L}^n(A)$.

We denote by Q the *homogeneous dimension* of $\mathbb{G}$, i.e. we set

$$Q := \sum_{i=1}^{\kappa} i \dim(V_i).$$

Since for any $x \in \mathbb{G}$ $|B(x,r)| = |B(e,r)| = r^Q|B(e,1)|$, Q is the Hausdorff dimension of the metric space $(\mathbb{G},d)$.

The subbundle of the tangent bundle $T\mathbb{G}$ that is spanned by the vector fields $X_1,\dots,X_{m_1}$ plays a particularly important role in the theory, and it is called the *horizontal bundle* $H\mathbb{G}$; the fibers of $H\mathbb{G}$ are

$$H\mathbb{G}_x = \operatorname{span}\{X_1(x),\dots,X_{m_1}(x)\}, \quad x \in \mathbb{G}.$$

A subriemannian structure is defined on $\mathbb{G}$, endowing each fiber of $H\mathbb{G}$ with a scalar product $\langle\cdot,\cdot\rangle_x$ and with a norm $|\cdot|_x$ making the basis $X_1(x), \dots, X_{m_1}(x)$ an orthonormal basis.

The sections of $H\mathbb{G}$ are called *horizontal sections*, and a vector of $H\mathbb{G}_x$ is a *horizontal vector*.

If f is a real function defined in $\mathbb{G}$, we denote by ${}^{\mathrm{v}}f$ the function defined by ${}^{\mathrm{v}}f(p) := f(p^{-1})$, and, if $T \in \mathcal{D}'(\mathbb{G})$, then ${}^{\mathrm{v}}T$ is the distribution defined by $\langle {}^{\mathrm{v}}T|\varphi\rangle := \langle T|{}^{\mathrm{v}}\varphi\rangle$ for any test function φ.

Following [10], we also adopt the following multi-index notation for higher-order derivatives. If $I = (i_1,\dots,i_n)$ is a multi-index, we set $X^I = X_1^{i_1}\cdots X_n^{i_n}$. By the Poincaré–Birkhoff–Witt theorem (see, e.g. [4], I.2.7), the differential operators X^I form a basis for the algebra of left invariant differential operators in $\mathbb{G}$. Furthermore, we set $|I| := i_1 + \cdots + i_n$ the order of the differential operator X^I, and $d(I) := d_1 i_1 + \cdots + d_n i_n$ its degree of homogeneity with respect to group dilations. From the Poincaré–Birkhoff–Witt theorem, it follows, in particular, that any homogeneous linear differential operator in the horizontal derivatives can be expressed as a linear combination of the operators X^I of the special form above.

Again following e.g. [10], we can define a group convolution in $\mathbb{G}$: if, for instance, $f \in \mathcal{D}(\mathbb{G})$ and $g \in L^1_{\mathrm{loc}}(\mathbb{G})$, we set

$$f * g(p) := \int f(q) g(q^{-1}p)\, dq \quad \text{for } p \in \mathbb{G}. \tag{9}$$

We remind that, if (say) g is a smooth function and L is a left invariant differential operator, then $L(f*g) = f * Lg$.

The dual space of $\mathfrak{g}$ is denoted by $\bigwedge^1 \mathfrak{g}$. The basis of $\bigwedge^1 \mathfrak{g}$, dual of the basis $X_1,\dots,X_n$, is the family of covectors $\{\theta_1,\dots,\theta_n\}$. We indicate as $\langle\cdot,\cdot\rangle$ also the inner product in $\bigwedge^1 \mathfrak{g}$ that makes $\theta_1,\dots,\theta_n$ an orthonormal basis.

Following Federer (see [8], 1.3), the exterior algebras of $\mathfrak{g}$ and of $\bigwedge^1 \mathfrak{g}$ are the graded algebras indicated as $\bigwedge_* \mathfrak{g} = \bigoplus_{k=0}^n \bigwedge_k \mathfrak{g}$ and $\bigwedge^* \mathfrak{g} = \bigoplus_{k=0}^n \bigwedge^k \mathfrak{g}$ where $\bigwedge_0 \mathfrak{g} = \bigwedge^0 \mathfrak{g} = \mathbb{R}$ and, for $1 \leqslant k \leqslant n$,

$$\bigwedge_k \mathfrak{g} := \operatorname{span}\{X_{i_1} \wedge \cdots \wedge X_{i_k}\colon 1 \leqslant i_1 < \cdots < i_k \leqslant n\},$$
$$\bigwedge^k \mathfrak{g} := \operatorname{span}\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}\colon 1 \leqslant i_1 < \cdots < i_k \leqslant n\}.$$

The elements of $\bigwedge_k \mathfrak{g}$ and $\bigwedge^k \mathfrak{g}$ are called *k-vectors* and *k-covectors*.

We denote by Θ^k the basis $\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}\colon 1 \leqslant i_1 < \cdots < i_k \leqslant n\}$ of $\bigwedge^k \mathfrak{g}$.

The dual space $\bigwedge^1(\bigwedge_k \mathfrak{g})$ of $\bigwedge_k \mathfrak{g}$ can be naturally identified with $\bigwedge^k \mathfrak{g}$. The action of a k-covector φ on a k-vector v is denoted as $\langle \varphi | v \rangle$.

The inner product $\langle \cdot, \cdot \rangle$ extends canonically to $\bigwedge_k \mathfrak{g}$ and to $\bigwedge^k \mathfrak{g}$ making the bases $X_{i_1} \wedge \cdots \wedge X_{i_k}$ and $\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$ orthonormal.

As in [8], 1.7.8, we denote by $*$ the Hodge duality operator

$$* : \bigwedge_k \mathfrak{g} \longleftrightarrow \bigwedge_{n-k} \mathfrak{g} \quad \text{and} \quad * : \bigwedge^k \mathfrak{g} \longleftrightarrow \bigwedge^{n-k} \mathfrak{g},$$

for $1 \leqslant k \leqslant n$.

If $v \in \bigwedge_k \mathfrak{g}$ we define $v^\natural \in \bigwedge^k \mathfrak{g}$ by the identity $\langle v^\natural | w \rangle := \langle v, w \rangle$, and analogously we define $\varphi^\natural \in \bigwedge_k \mathfrak{g}$ for $\varphi \in \bigwedge^k \mathfrak{g}$.

To fix our notations, we remind the following definition.

Definition 2.1. If V, W are finite-dimensional linear vector spaces and $L : V \to W$ is a linear map, we define

$$\Lambda_h L : \bigwedge_h V \to \bigwedge_h W$$

as the linear map defined by

$$(\Lambda_h L)(v_1 \wedge \cdots \wedge v_h) = L(v_1) \wedge \cdots \wedge L(v_h)$$

for any simple h-vector $v_1 \wedge \cdots \wedge v_h \in \bigwedge_h V$

$$\Lambda^h L : \bigwedge^h W \to \bigwedge^h V$$

as the linear map defined by

$$\big\langle (\Lambda^h L)(\alpha) \big| v_1 \wedge \cdots \wedge v_h \big\rangle = \big\langle \alpha \big| (\Lambda_h L)(v_1 \wedge \cdots \wedge v_h) \big\rangle$$

for any $\alpha \in \bigwedge^h W$ and any simple h-vector $v_1 \wedge \cdots \wedge v_h \in \bigwedge_h V$.

Starting from $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$, we can define by left translation fiber bundles over $\mathbb{G}$ that we can still denote by $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$, respectively. To do this, for instance we identify $\bigwedge^h \mathfrak{g}$ with the fiber $\bigwedge_e^h \mathfrak{g}$ over the origin, and we define the fiber over $x \in \mathbb{G}$ by putting

$$\bigwedge_{h,p} \mathfrak{g} := (\Lambda_h \, d\tau_p)\Big(\bigwedge_{h,e} \mathfrak{g}\Big)$$

and, respectively,

$$\bigwedge_p^h \mathfrak{g} := \left(\Lambda^h d\tau_{p^{-1}}\right)\left(\bigwedge_e^h \mathfrak{g}\right)$$

for any $p \in \mathbb{G}$ and $h = 1, \dots, n$.

The inner products $\langle\cdot,\cdot\rangle$ on $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$ induce inner products on each fiber $\bigwedge_{h,p} \mathfrak{g}$ and $\bigwedge_p^h \mathfrak{g}$ by the identity

$$\left\langle \Lambda_h d\tau_p(v), \Lambda_h d\tau_p(w)\right\rangle_p := \langle v, w\rangle$$

and

$$\left\langle \Lambda^h d\tau_{p^{-1}}(\alpha), \Lambda^h d\tau_{p^{-1}}(\beta)\right\rangle_p := \langle \alpha, \beta\rangle.$$

Lemma 2.2. *If $p, q \in \mathbb{G}$, then*

$$\Lambda_h d\tau_q : \bigwedge_{h,p} \mathfrak{g} \to \bigwedge_{h,qp} \mathfrak{g}$$

and

$$\Lambda^h d\tau_{q^{-1}} : \bigwedge_p^h \mathfrak{g} \to \bigwedge_{qp}^h \mathfrak{g}$$

are isometries onto.

Definition 2.3. If $\alpha \in \bigwedge^1 \mathfrak{g}$, $\alpha \neq 0$, we say that α has *pure weight k*, and we write $w(\alpha) = k$, if $\alpha^\natural \in V_k$. Obviously,

$$w(\alpha) = k \quad \text{if and only if} \quad \alpha = \sum_{j=h_{k-1}+1}^{h_k} \alpha_j \theta_j,$$

with $\alpha_{h_{k-1}+1}, \dots, \alpha_{h_k} \in \mathbb{R}$. More generally, if $\alpha \in \bigwedge^h \mathfrak{g}$, we say that α has pure weight k if α is a linear combination of covectors $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ with $w(\theta_{i_1}) + \cdots + w(\theta_{i_h}) = k$.

Remark 2.4. If $\alpha, \beta \in \bigwedge^h \mathfrak{g}$ and $w(\alpha) \neq w(\beta)$, then $\langle \alpha, \beta\rangle = 0$. Indeed, it is enough to notice that, if $w(\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}) \neq w(\theta_{j_1} \wedge \cdots \wedge \theta_{j_h})$, with $i_1 < i_2 < \cdots < i_h$ and $j_1 < j_2 < \cdots < j_h$, then for at least one of the indices $\ell = 1, \dots, h$, $i_\ell \neq j_\ell$, and hence $\langle \theta_{i_1} \wedge \cdots \wedge \theta_{i_h}, \theta_{j_1} \wedge \cdots \wedge \theta_{j_h}\rangle = 0$.

We have

$$\bigwedge^h \mathfrak{g} = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \bigwedge^{h,p} \mathfrak{g}, \tag{10}$$

where $\bigwedge^{h,p} \mathfrak{g}$ is the linear span of the h-covectors of weight p and $M_h^{\min}$, $M_h^{\max}$ are respectively the smallest and the largest weight of h-covectors.

Since the elements of the basis Θ^h have pure weights, a basis of $\bigwedge^{h,p} \mathfrak{g}$ is given by $\Theta^{h,p} := \Theta^h \cap \bigwedge^{h,p} \mathfrak{g}$ (in the Introduction, we called such a basis an adapted basis).

As pointed out in Remark 2.4, the decomposition in (10) is orthogonal. We denote by $\Pi^{h,p}$ the orthogonal projection of $\bigwedge^h \mathfrak{g}$ on $\bigwedge^{h,p} \mathfrak{g}$.

The identification of $\bigwedge^h \mathfrak{g}$ and $\bigwedge_e^h \mathfrak{g}$ yields a corresponding identification of the basis Θ^h of $\bigwedge^h \mathfrak{g}$ and Θ_e^h of $\bigwedge_e^h \mathfrak{g}$. Then $\Theta_x^h := \Lambda^k(d\tau_{x^{-1}})\Theta_e^h$ is a basis of $\bigwedge_x^h \mathfrak{g}$. Notice that the Lie algebra $\mathfrak{g}$ can be identified with the Lie algebra of the left invariant vector fields on $\mathbb{G} \equiv \mathbb{R}^n$. Hence, the elements of Θ_x^h can be identified with the elements of Θ^h evaluated at the point x. Through all this paper, we make systematic use of these identifications, interchanging the roles of left invariant vector fields and elements of $\bigwedge_1 \mathfrak{g}$.

Keeping in mind the decomposition (10), we can define in the same way several fiber bundles over $\mathbb{G}$ (that we still denote with the same symbol $\bigwedge^{h,p} \mathfrak{g}$), by setting $\bigwedge_e^{h,p} \mathfrak{g} := \bigwedge^{h,p} \mathfrak{g}$ and $\bigwedge_x^{h,p} \mathfrak{g} := \Lambda^k(d\tau_{x^{-1}}) \bigwedge_e^{h,p} \mathfrak{g}$. Clearly, all previous arguments related to the basis Θ^h can be repeated for the basis $\Theta^{h,p}$.

Sections of $\bigwedge_h \mathfrak{g}$ are called h-vector fields, and sections of $\bigwedge^h \mathfrak{g}$ are called h-forms. We denote by Ω_h (Ω^h) the vector space of all smooth sections of $\bigwedge_h \mathfrak{g}$ (of $\bigwedge^h \mathfrak{g}$, respectively) and by $d : \Omega^h \to \Omega^{h+1}$ the exterior differential acting on h-forms.

We denote also by $\Omega^{h,p}$ the vector space of all smooth h-forms in $\mathbb{G}$ of pure weight p, i.e. the space of all smooth sections of $\bigwedge^{h,p} \mathfrak{g}$. We have

$$\Omega^h = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \Omega^{h,p}. \tag{11}$$

Lemma 2.5. *We have $d(\bigwedge^{h,p} \mathfrak{g}) \subset \bigwedge^{h+1,p} \mathfrak{g}$, i.e., if $\alpha \in \bigwedge^{h,p} \mathfrak{g}$ and $d\alpha \neq 0$ is a left invariant h-form of weight p, then $w(d\alpha) = w(\alpha)$.*

Proof. See [24], Section 2.1. □

Let now $\alpha \in \Omega^{h,p}$ be a (say) smooth form of pure weight p. We can write

$$\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i \theta_i^h, \quad \text{with } \alpha_i \in \mathcal{E}(\mathbb{G}).$$

Then

$$d\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_j (X_j \alpha_i)\theta_j \wedge \theta_i^h + \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i \, d\theta_i^h.$$

Hence we can write

$$d = d_0 + d_1 + \cdots + d_\kappa,$$

where

$$d_0\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i \, d\theta_i^h$$

does not increase the weight,

$$d_1\alpha = \sum_{\theta_i^h\in\Theta^{h,p}} \sum_{j=1}^{m_1} (X_j\alpha_i)\theta_j \wedge \theta_i^h$$

increases the weight of 1, and, more generally,

$$d_k\alpha = \sum_{\theta_i^h\in\Theta^{h,p}} \sum_{w(\theta_j)=k} (X_j\alpha_i)\theta_j \wedge \theta_i^h, \quad k=1,\dots,\kappa.$$

In particular, d_0 is an algebraic operator, in the sense that its action can be identified at any point with the action of an operator from $\bigwedge^h \mathfrak{g}$ to $\bigwedge^{h+1} \mathfrak{g}$ (that we denote again by d_0) through the formula

$$(d_0\alpha)(x) = \sum_{\theta_i^h\in\Theta^{h,p}} \alpha_i(x)\, d\theta_i^h = \sum_{\theta_i^h\in\Theta^{h,p}} \alpha_i(x)\, d_0\theta_i^h,$$

by Lemma 2.5.

Analogously, δ_0, the L^2-adjoint of d_0 in Ω^* defined by

$$\int \langle d_0\alpha, \beta\rangle\, dV = \int \langle \alpha, \delta_0\beta\rangle\, dV$$

for all compactly supported smooth forms $\alpha \in \Omega^h$ and $\beta \in \Omega^{h+1}$, is again an algebraic operator preserving the weight.

Definition 2.6. If $0 \leqslant h \leqslant n$ we set

$$E_0^h := \ker d_0 \cap \ker \delta_0 = \ker d_0 \cap \mathcal{R}(d_0)^\perp \subset \Omega^h.$$

Since the construction of E_0^h is left invariant, this space of forms can be viewed as the space of sections of a fiber bundle, generated by left translations and still denoted by E_0^h.

We denote by $N_h^{\min}$ and $N_h^{\max}$ the minimum and the maximum, respectively, of the weights of forms in E_0^h.

If we set $E_0^{h,p} := E_0^h \cap \Omega^{h,p}$, then

$$E_0^h = \bigoplus_{p=N_h^{\min}}^{N_h^{\max}} E_0^{h,p}.$$

Indeed, if $\alpha \in E_0^h$, by (11), we can write $\alpha = \sum_{p=N_h^{\min}}^{N_h^{\max}} \alpha_p$, with $\alpha_p \in \Omega^{h,p}$ for all p. The assertion follows by proving that $\alpha_p \in \mathbb{E}_0^h$. Indeed, by definition, $0 = d_0\alpha = \sum_{p=N_h^{\min}}^{N_h^{\max}} d_0\alpha_p$. But

$w(d_0\alpha_p) \neq w(d_0\alpha_q)$ for $p \neq q$, and hence the $d_0\alpha_p$'s are linear independent and therefore they are all 0. Analogously, $\delta_0\alpha_p = 0$ for all p, and the assertion follows.

We denote by $\Pi_{E_0}^{h,p}$ the orthogonal projection of Ω^h on $E_0^{h,p}$.

We notice that also the space of forms $E_0^{h,p}$ can be viewed as the space of smooth sections of a suitable fiber bundle generated by left translations, that we still denote by $E_0^{h,p}$.

As customary, if $\Omega \subset \mathbb{G}$ is an open set, we denote by $\mathcal{E}(\Omega, E_0^h)$ the space of smooth sections of E_0^h. The spaces $\mathcal{D}(\Omega, E_0^h)$ and $\mathcal{S}(\mathbb{G}, E_0^h)$ are defined analogously.

Since both $E_0^{h,p}$ and E_0^h are left invariant as $\bigwedge^h \mathfrak{g}$, they are subbundles of $\bigwedge^h \mathfrak{g}$ and inherit the scalar product on the fibers.

In particular, we can obtain a left invariant orthonormal basis $\Xi_0^h = \{\xi_j^h\}$ of E_0^h such that

$$\Xi_0^h = \bigcup_{p=N_h^{\min}}^{N_h^{\max}} \Xi_0^{h,p}, \tag{12}$$

where $\Xi_0^{h,p} := \Xi^h \cap \bigwedge^{h,p} \mathfrak{g}$ is a left invariant orthonormal basis of $E_0^{h,p}$. All the elements of $\Xi_0^{h,p}$ have pure weight p. Without loss of generality, the indices j of $\Xi_0^h = \{\xi_j^h\}$ are ordered once for all in increasing way with respect to the weight of the corresponding element of the basis.

Correspondingly, the set of indices $\{1, 2, \dots, \dim E_0^h\}$ can be written as the union of finite sets (possibly empty) of indices

$$\{1, 2, \dots, \dim E_0^h\} = \bigcup_{p=N_h^{\min}}^{N_h^{\max}} I_{0,p}^h,$$

where

$$j \in I_{0,p}^h \quad \text{if and only if} \quad \xi_j^h \in \Xi_0^{h,p}.$$

Without loss of generality, we can take

$$\Xi_0^1 = \Xi_0^{1,1} := \Theta^{1,1}.$$

Once the basis Θ_0^h is chosen, the spaces $\mathcal{E}(\Omega, E_0^h)$, $\mathcal{D}(\Omega, E_0^h)$, $\mathcal{S}(\mathbb{G}, E_0^h)$ can be identified with $\mathcal{E}(\Omega)^{\dim E_0^h}$, $\mathcal{D}(\Omega)^{\dim E_0^h}$, $S(\mathbb{G})^{\dim E_0^h}$, respectively.

Proposition 2.7. *(See [24].) If $0 \leqslant h \leqslant n$ and $*$ denotes the Hodge duality, then*

$$*E_0^h = E_0^{n-h}.$$

By a simple linear algebra argument we can prove the following lemma.

Lemma 2.8. *If $\beta \in \Omega^{h+1}$, then there exists a unique $\alpha \in \Omega^h \cap (\ker d_0)^\perp$ such that*

$$\delta_0 d_0 \alpha = \delta_0 \beta.$$

Definition 2.9. Let α, β be as in Lemma 2.8. Then we set

$$\alpha := d_0^{-1}\beta.$$

In particular

$$\alpha = d_0^{-1}\beta \quad \text{if and only if} \quad d_0\alpha - \beta \in \ker \delta_0.$$

Remark 2.10. We stress that d_0^{-1} is an algebraic operator, like d_0 and δ_0.

Finally, we notice that

$$d_0^{-1}\left(\bigwedge\nolimits^{h+1,p}\mathfrak{g}\right) \subset \bigwedge\nolimits^{h,p}\mathfrak{g}. \tag{13}$$

Since $d_0^{-1}d_0 = Id$ on $\mathcal{R}(d_0^{-1})$, we can write $d_0^{-1}d = Id + D$, where D is a differential operator that increases the weight. Clearly, $D : \mathcal{R}(d_0^{-1}) \to \mathcal{R}(d_0^{-1})$. As a consequence of the nilpotency of $\mathbb{G}$, $D^k = 0$ for k large enough, and the following result holds.

Lemma 2.11. *(See [24].) The map $d_0^{-1}d$ induces an isomorphism from $\mathcal{R}(d_0^{-1})$ to itself. In addition, there exists a differential operator*

$$P = \sum_{k=0}^{N}(-1)^k D^k, \quad N \in \mathbb{N} \textit{ suitable},$$

such that

$$P d_0^{-1} d = d_0^{-1} d P = Id_{\mathcal{R}(d_0^{-1})}.$$

We set $Q := P d_0^{-1}$.

Remark 2.12. If α has pure weight k, then $P\alpha$ is a sum of forms of pure weight greater or equal to k.

We state now the following key results. Some examples will be discussed in detail in Appendix B.

Theorem 2.13. *(See [24].) There exists a differential operator $d_c : E_0^h \to E_0^{h+1}$ such that*

(i) $d_c^2 = 0$;
(ii) *the complex $E_0 := (E_0^*, d_c)$ is exact;*

(iii) *the differential d_c acting on h-forms can be identified, with respect to the bases Ξ_0^h and Ξ_0^{h+1}, with a matrix-valued differential operator $L^h := (L^h_{i,j})$. If $j \in I^h_{0,p}$ and $i \in I^{h+1}_{0,q}$, then the $L^h_{i,j}$'s are homogeneous left invariant differential operators of order $q-p \geqslant 1$ in the horizontal derivatives, and $L^h_{i,j} = 0$ if $j \in I^h_{0,p}$ and $i \in I^{h+1}_{0,q}$, with $q-p<1$.*

In particular, if $h=0$ and $f \in E_0^0 = \mathcal{E}(\mathbb{G})$, then $d_c f = \sum_{i=1}^{m_1}(X_i f)\theta_i^1$ is the *horizontal differential of f*.

The proof of Theorem 2.13 relies on the following result.

Theorem 2.14. *(See [24].) The de Rham complex (Ω^*, d) splits in the direct sum of two subcomplexes (E^*, d) and (F^*, d), with*

$$E := \ker d_0^{-1} \cap \ker\bigl(d_0^{-1} d\bigr) \quad \textit{and} \quad F := \mathcal{R}\bigl(d_0^{-1}\bigr) + \mathcal{R}\bigl(d d_0^{-1}\bigr),$$

such that:

(i) *The projection Π_E on E along F is given by $\Pi_E = Id - Qd - dQ$.*
(ii) *If Π_{E_0} is the orthogonal projection from Ω^* on E_0^*, then $\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0}$ and $\Pi_E\Pi_{E_0}\Pi_E = \Pi_E$.*
(iii) *$d_c = \Pi_{E_0} d \Pi_E$.*
(iv) *$*E = F^{\perp}$ (in the sense of the $L^2(\mathbb{G}, \Omega^*)$-duality).*

Remark 2.15. We have

$$d\Pi_E = \Pi_E\, d. \tag{14}$$

Indeed, we can write $\alpha = \Pi_E\alpha + \Pi_F\alpha$, and hence $d\alpha = d\Pi_E\alpha + d\Pi_F\alpha$. But E and F are complexes, so that $d\Pi_F\alpha \in F$ and $d\Pi_E\alpha \in E$; therefore $\Pi_E\, d\alpha = \Pi_E\, d\Pi_E\alpha + \Pi_E\, d\Pi_F\alpha = d\Pi_E\alpha$, and we are done.

Moreover, by Theorem 2.14(iv), if $\alpha \in \Omega^h$ and $\beta \in \Omega^{n-h}$ are compactly supported forms with $0 \leqslant h \leqslant n$, we have

$$\int_{\mathbb{G}} \alpha \wedge (\Pi_E\beta) = \int_{\mathbb{G}} (\Pi_E\alpha) \wedge (\Pi_E\beta) = \int_{\mathbb{G}} (\Pi_E\alpha) \wedge \beta. \tag{15}$$

Finally, if $\alpha \in \Omega^h$ and $\beta \in E_0^{n-h}$ with $0 \leqslant h \leqslant n$, we have

$$\alpha \wedge \beta = (\Pi_{E_0}\alpha) \wedge \beta. \tag{16}$$

Remark 2.16. If $\psi_i \in \mathcal{D}(\mathbb{G}, E_0^{h_i})$ for $i=1,2$ with $h_1 + 1 + h_2 = n$, we have

$$\int_{\mathbb{G}} d_c\psi_1 \wedge \psi_2 = (-1)^{h_1} \int_{\mathbb{G}} \psi_1 \wedge d_c\psi_2.$$

Indeed

$$\begin{aligned}
\int_{\mathbb{G}} d_c\psi_1 \wedge \psi_2 &= \int_{\mathbb{G}} (\Pi_{E_0}\, d\Pi_E\psi_1)\wedge\psi_2 \\
&= \int_{\mathbb{G}} (d\Pi_E\psi_1)\wedge\psi_2 \quad \text{(by (16))} \\
&= \int_{\mathbb{G}} d\big((\Pi_E\psi_1)\wedge\psi_2\big) + (-1)^{h_1}\int_{\mathbb{G}} (\Pi_E\psi_1)\wedge d\psi_2 \\
&= (-1)^{h_1}\int_{\mathbb{G}} (\Pi_E\psi_1)\wedge d\psi_2 \quad \text{(by Stokes theorem)} \\
&= (-1)^{h_1}\int_{\mathbb{G}} \psi_1\wedge(\Pi_E\, d\psi_2) \quad \text{(by (15))} \\
&= (-1)^{h_1}\int_{\mathbb{G}} \psi_1\wedge\big(d\Pi_E(\psi_2)\big) \quad \text{(by (14))} \\
&= (-1)^{h_1}\int_{\mathbb{G}} \psi_1\wedge(d_c\psi_2) \quad \text{(again by (16)).}
\end{aligned}$$

Proposition 2.17. *(See [24], formula (7).) For any $\alpha \in E_0^{h,p}$, if we denote by $(\Pi_E\alpha)_j$ the component of $\Pi_E\alpha$ of weight j (that is necessarily greater or equal than p, by Remark* 2.12*), then*

$$\begin{aligned}
(\Pi_E\alpha)_p &= \alpha, \\
(\Pi_E\alpha)_{p+k+1} &= -d_0^{-1}\Big(\sum_{1\leqslant \ell\leqslant k+1} d_\ell(\Pi_E\alpha)_{p+k+1-\ell}\Big).
\end{aligned} \tag{17}$$

Proposition 2.18. *Let j, p, k be fixed, $N_h^{\min} \leqslant p \leqslant N_h^{\max}$, $j \in I_{0,p}^h$, and $0 \leqslant k \leqslant M_h^{\max} - p$. Then there exist homogeneous left invariant differential operators $Q_{p,p+k,j,i}^h$, $i \in I_{p+k}^h$, of order k, such that, if $\alpha = \alpha_j\xi_j^h$, then*

$$(\Pi_E\alpha)_{p+k} = \sum_{i\in I_{p+k}^h} \big(Q_{p,p+k,j,i}^h\alpha_j\big)\theta_i^h.$$

Therefore

$$\Pi_E\alpha = \sum_{k=0}^{M_h^{\max}-p}\ \sum_{i\in I_{p+k}^h} \big(Q_{p,p+k,j,i}^h\alpha_j\big)\theta_i^h.$$

Proof. The proof can be carried out by induction on k. □

Remark 2.19. We can notice that, if $\alpha \in E_0^{h,p}$, then $d_c\alpha$ has no components of weight $j = p$. Indeed,

$$\Pi_E\alpha = \alpha + \text{terms of weight greater than } p.$$

Thus

$$d\Pi_E\alpha = d_0\alpha + \text{terms of weight greater than } p.$$

But $d_0\alpha = 0$ by the very definition of $E_0^{h,p}$, and the assertion follows.

Definition 2.20. If $\Omega \subset \mathbb{G}$ is an open set and $1 \leqslant h \leqslant n$, we say that T is an h-current on Ω if T is a continuous linear functional on $\mathcal{D}(\Omega, E_0^h)$ endowed with the usual topology. We write $T \in \mathcal{D}'(\Omega, E_0^h)$.

Any (usual) distribution $T \in \mathcal{D}'(\Omega)$ can be identified canonically with an n-current $\tilde{T} \in \mathcal{D}'(\Omega, E_0^n)$ through the formula

$$\langle \tilde{T} | \alpha \rangle := \langle T | {*\alpha} \rangle \tag{18}$$

for any $\alpha \in \mathcal{D}(\Omega, E_0^n)$.

If $\Xi_0^h = \{\xi_1^h, \dots, \xi_{\dim E_0^h}^h\}$ is a left invariant basis of E_0^h and $T \in \mathcal{D}'(\Omega, E_0^h)$, then there exist (uniquely determined) $T_1, \dots, T_{\dim E_0^h} \in \mathcal{D}'(\Omega)$ such that

$$T = \sum_j \tilde{T}_j \llcorner \left(*\xi_j^h \right),$$

where

$$\left\langle \tilde{T}_j \llcorner \left(*\xi_j^h \right) \middle| \phi \right\rangle := \left\langle \tilde{T}_j \middle| \phi \wedge *\xi_j^h \right\rangle$$

for all $\phi \in \mathcal{D}(\Omega, E_0^h)$. Currents can be viewed as forms with distributional coefficients in the following sense: if $\alpha \in E(\Omega, E_0^h)$, then α can be identified canonically with an h-current T_α through the formula

$$\langle T_\alpha | \varphi \rangle := \int_\Omega *\alpha \wedge \varphi \tag{19}$$

for any $\varphi \in \mathcal{D}(\Omega, E_0^h)$. Moreover, if $\alpha = \sum_j \alpha_j \xi_j^h$ then

$$T_\alpha = \sum_j \tilde{\alpha}_j \llcorner \left(*\xi_j^h \right)$$

(see [1], but we refer also to [7], Sections 17.3, 17.4 and 17.5).

The notion of convolution can be extended by duality to currents.

Definition 2.21. Let $\varphi \in \mathcal{D}(\mathbb{G})$ and $T \in \mathcal{E}'(\mathbb{G}, E_0^h)$ be given, and denote by ${}^{\mathrm{v}}\varphi$ the function defined by ${}^{\mathrm{v}}\varphi(p) := \varphi(p^{-1})$. Then we set

$$\langle \varphi * T | \alpha \rangle := \langle T | {}^{\mathrm{v}}\varphi * \alpha \rangle$$

for any $\alpha \in \mathcal{D}(\mathbb{G}, E_0^h)$.

We need a few definitions. We set

$$\mathcal{I}_0^h := \{p;\ I_{0,p}^h \neq \emptyset\} \quad \text{and} \quad |\mathcal{I}_0^h| = \operatorname{card} \mathcal{I}_0^h. \tag{20}$$

Let

$$\underline{m} = (m_{N_h^{\min}}, \ldots, m_{N_h^{\max}})$$

be an $|\mathcal{I}_0^h|$-dimensional vector where the components are indexed by the elements of $\mathcal{I}_0^h$ (i.e. by the possible weights) taken in increasing order. We stress that, since weights p such that $I_{0,p}^h = \emptyset$ can exist, then some consecutive indices in $\underline{m}$ can be missed. In the sequel we shall say that $\underline{m}$ is an *h-vector weight*. We say that $\underline{m} \geqslant 0$ if $m_p \geqslant 0$ for $p \in \mathcal{I}_0^h$, and that $\underline{m} \geqslant \underline{n}$ if $m_p \geqslant n_p$ for all $p \in \mathcal{I}_0^h$. We say also that $\underline{m} > \underline{n}$ if $m_p > n_p$ *for all* $p \in \mathcal{I}_0^h$. Finally, if m_0 is a real number, we identify m_0 with the h-vector weight $m_0 = (m_0, \ldots, m_0)$. In particular, we set $\underline{m} - m_0 := (m_{N_h^{\min}} - m_0, \ldots, m_{N_h^{\max}} - m_0)$.

Definition 2.22. A special h-vector weight that we shall use in the sequel is the h-vector weight $\underline{N}_h = (m_{N_h^{\min}}, \ldots, m_{N_h^{\max}})$ with

$$m_p = p \quad \text{for all } p \in I_0^h.$$

If all h-forms have *pure weight* N_h, i.e. if $N_h^{\min} = N_h^{\max} := N_h$, then an h-vector weight has only one component, i.e. $\underline{m} = (m_{N_h})$.

3. Function spaces

Through the next sections, we use notations and results contained in Appendix A and basically relying on the pseudodifferential operators and their calculus of Christ, Geller, Głowacki and Polin [5]. Briefly, we refer to their operators as to CGGP-operators, and we call CGGP-calculus the associated calculus.

Let $\{X_1, \ldots, X_{m_1}\}$ be the fixed basis of the horizontal layer $\mathfrak{g}_1$ of $\mathfrak{g}$ chosen in Section 2. We denote by $\Delta_{\mathbb{G}}$ the non-negative horizontal sub-Laplacian

$$\Delta_{\mathbb{G}} := -\sum_{j=1}^{m_1} X_j^2.$$

If $1 < s < \infty$ and $a \in \mathbb{C}$, we define $\Delta_{\mathbb{G}}^a$ in $L^s(\mathbb{G})$ following [9]. If in addition $m \geqslant 0$, again as in [9], we denote by $W_{\mathbb{G}}^{m,s}(\mathbb{G})$ the domain of the realization of $\Delta_{\mathbb{G}}^{m/2}$ in $L^s(\mathbb{G})$ endowed with the graph norm. In fact, since $s \in (1, \infty)$ is fixed through all the paper, to avoid cumbersome notations, we do not stress the explicit dependence on s of the fractional powers $\Delta_{\mathbb{G}}^{m/2}$ and of its domain.

Proposition 3.1. *The operators $\Delta_{\mathbb{G}}^{m/2}$ are left invariant on $W_{\mathbb{G}}^{m,s}(\mathbb{G})$.*

Proof. The proof is straightforward, keeping in mind the form of $\Delta_{\mathbb{G}}^{m/2}$ ([9], p. 181), and the representation of the heat semigroup associated with $\Delta_{\mathbb{G}}$ ([9], Theorem 3.1(i)). □

We remind that

Proposition 3.2. *(See [9], Corollary 4.13.) If $1 < s < \infty$ and $m \in \mathbb{N}$, then the space $W_{\mathbb{G}}^{m,s}(\mathbb{G})$ coincides with the space of all $u \in L^s(\mathbb{G})$ such that*

$$X^I u \in L^s(\mathbb{G}) \quad \textit{for all multi-index } I \textit{ with } d(I) = m,$$

endowed with the natural norm.

Proposition 3.3. *(See [9], Corollary 4.14.) If $1 < s < \infty$ and $m \geqslant 0$, then the space $W_{\mathbb{G}}^{m,s}(\mathbb{G})$ is independent of the choice of $X_1, \dots, X_{m_1}$.*

Proposition 3.4. *If $1 < s < \infty$ and $m \geqslant 0$, then $\mathcal{S}(\mathbb{G})$ and $\mathcal{D}(\mathbb{G})$ are dense subspaces of $W_{\mathbb{G}}^{m,s}(\mathbb{G})$.*

Proof. The density of $\mathcal{D}(\mathbb{G})$ is proved in [9], Theorem 4.5. If $m \in \mathbb{N} \cup \{0\}$, by Proposition 3.2, $\mathcal{S}(\mathbb{G}) \subset W_{\mathbb{G}}^{m,s}(\mathbb{G})$, since the vector fields $X_1, \dots, X_{m_1}$ have polynomial coefficients (see [11], Proposition 2.2). Thus, by [9], Proposition 4.2, $\mathcal{S}(\mathbb{G}) \subset W_{\mathbb{G}}^{m,s}(\mathbb{G})$ for $m \geqslant 0$. Moreover, since $\mathcal{D}(\mathbb{G})$ is a dense subspace of $W_{\mathbb{G}}^{m,s}(\mathbb{G})$, the assertion follows. □

Definition 3.5. Let $m \geqslant 0$, $1 < s < \infty$ be fixed indices. Let $\Omega \subset \mathbb{G}$ be a given open set with $\mathcal{L}^n(\partial\Omega) = 0$ (from now on, even if not explicitly stated, we shall assume this regularity property whenever an open set is meant to localize a statement). We denote by $\mathring{W}_{\mathbb{G}}^{m,s}(\Omega)$ the completion in $W_{\mathbb{G}}^{m,s}(\mathbb{G})$ of $\mathcal{D}(\Omega)$. More precisely, denote by $v \to r_\Omega v$ the restriction operator to Ω; we say that u belongs to $\mathring{W}_{\mathbb{G}}^{m,s}(\Omega)$ if there exists a sequence of test functions $(u_k)_{k\in\mathbb{N}}$ in $\mathcal{D}(\Omega)$ and $U \in W_{\mathbb{G}}^{m,s}(\mathbb{G})$, such that $u_k \to U$ in $W_{\mathbb{G}}^{m,s}(\mathbb{G})$ and $u = r_\Omega U$. On the other hand, since in particular $u_k \to U$ in $L^s(\mathbb{G})$, necessarily $U \equiv 0$ outside of Ω. Therefore, if $u = r_\Omega U_1 = r_\Omega U_2$ with U_1, U_2 both belonging to the completion in $W_{\mathbb{G}}^{m,s}(\mathbb{G})$ of $\mathcal{D}(\Omega)$, then $U_1 \equiv U_2$, so that, without loss of generality, we can set

$$\|u\|_{\mathring{W}_{\mathbb{G}}^{m,s}(\Omega)} := \|p_0(u)\|_{W_{\mathbb{G}}^{m,s}(\mathbb{G})},$$

where $p_0(u)$ denotes the continuation of u by zero outside of Ω.

It is well known that $W_{\mathbb{G},\mathrm{loc}}^{1,s}(\mathbb{G})$ is continuously embedded in $W_{\mathrm{loc}}^{1/(\kappa+1),s}(\mathbb{G})$ (see [21]); thus, by classical Rellich theorem and interpolation arguments ([9], Theorem 4.7 and [26], 1.16.4, Theorem 1), we have:

Lemma 3.6. *Let $\Omega \subset \mathbb{G}$ be a bounded open set. If $s > 1$, and $m > 0$, then*

$$\mathring{W}^{m,s}_{\mathbb{G}}(\Omega) \quad \text{is compactly embedded in } L^s(\Omega).$$

Proposition 3.7. *If $m \geqslant 0$, $1 < s < \infty$ and $\Omega \subset \mathbb{G}$ is a bounded open set, then*

$$\|u\|_{\mathring{W}^{m,s}_{\mathbb{G}}(\Omega)} \approx \big\| \Delta_{\mathbb{G}}^{m/2} p_0(u) \big\|_{L^s(\mathbb{G})}$$

when $u \in \mathring{W}^{m,s}_{\mathbb{G}}(\Omega)$ and $p_0(u)$ denotes its continuation by zero outside of Ω.

Proof. By Definition 3.5,

$$\|u\|_{\mathring{W}^{m,s}_{\mathbb{G}}(\Omega)} = \big\| p_0(u) \big\|_{W^{m,s}_{\mathbb{G}}(\mathbb{G})} \geqslant \big\| \Delta_{\mathbb{G}}^{m/2} p_0(u) \big\|_{L^s(\mathbb{G})},$$

so that we have only to prove the reverse estimate.

We want to show preliminarily that the map $u \to \Delta_{\mathbb{G}}^{m/2} p_0(u)$ from $\mathring{W}^{m,s}_{\mathbb{G}}(\Omega)$ to $L^s(\mathbb{G})$ is injective. Let $u \in \mathring{W}^{m,s}_{\mathbb{G}}(\Omega)$ be such that $\Delta_{\mathbb{G}}^{m/2} p_0(u) = 0$. If $(\rho_\varepsilon)_{\varepsilon>0}$ are group mollifiers, by the left invariance of $\Delta_{\mathbb{G}}^{m/2}$, we have $\rho_\varepsilon * p_0(u) \in \mathcal{D}(\mathbb{G})$ and $\Delta_{\mathbb{G}}^{m/2}(\rho_\varepsilon * p_0(u)) = 0$ for $\varepsilon > 0$. By [9], Theorem 3.15(iii), keeping in mind that $\mathcal{D}(\mathbb{G}) \subset \mathrm{Dom}(\Delta_{\mathbb{G}}^{\alpha})$ for all $\alpha \geqslant 0$, if N is an integer number such that $N \geqslant m/2$, then $\Delta_{\mathbb{G}}^{N}(\rho_\varepsilon * p_0(u)) = \Delta_{\mathbb{G}}^{N-m/2} \Delta_{\mathbb{G}}^{m/2}(\rho_\varepsilon * p_0(u)) = 0$, so that $\rho_\varepsilon * p_0(u) = 0$, e.g. by Bony's maximum principle. Then, taking the limit as $\varepsilon \to 0$, $p_0(u) = 0$, and eventually $u = 0$.

We can achieve now the proof by using a simple form of the following Peetre–Tartar lemma (see, e.g., [6], p. 126):

Lemma 3.8 *(Peetre–Tartar). Let V, V_1, V_2, W be Banach spaces, and let $A_i \in \mathcal{L}(V, V_i)$ be continuous linear maps for $i = 1, 2$, the map A_1 being compact. Suppose there exists $c_0 > 0$ such that*

$$\|v\|_V \leqslant c_0 \big(\|A_1 v\|_{V_1} + \|A_2 v\|_{V_2} \big) \tag{21}$$

for any $v \in V$. In addition, let $L \in \mathcal{L}(V, W)$ be a continuous linear map such that

$$L|_{\ker A_2} \equiv 0. \tag{22}$$

Then there exists $C > 0$ such that

$$\|Lv\|_W \leqslant C \|A_2 v\|_{V_2} \tag{23}$$

for any $v \in V$.

For our purposes, we choose $V = \mathring{W}^{m,s}_{\mathbb{G}}(\Omega)$, $V_1 = V_2 = L^s(\mathbb{G})$, $W = W^{m,s}(\mathbb{G})$, $A_1 = p_0$, $A_2 = \Delta_{\mathbb{G}}^{m/2} \circ p_0$, $L = p_0$. Indeed, $A_1 := p_0$ is a compact map from $\mathring{W}^{m,s}_{\mathbb{G}}(\Omega)$ to $L^s(\mathbb{G})$, by Lemma 3.6. On the other hand, we have already pointed out in Definition 3.5 that $p_0(u) \in W^{m,s}(\mathbb{G})$, so that $\Delta_{\mathbb{G}}^{m/2} p_0(u) \in L^s(\mathbb{G})$, and $\|\Delta_{\mathbb{G}}^{m/2} p_0(u)\|_{L^s(\mathbb{G})} \leqslant \|p_0(u)\|_{W^{m,s}(\mathbb{G})} := \|u\|_{\mathring{W}^{m,s}_{\mathbb{G}}(\Omega)}$

(again by Definition 3.5). Thus $A_2 := \Delta_{\mathbb{G}}^{m/2} \circ p_0 : \mathring{W}_{\mathbb{G}}^{m,s}(\Omega) \to L^s(\mathbb{G})$ continuously. The same argument shows that (21) holds. On the other hand, we have shown that $\ker A_2 = \{0\}$, so that (22) holds.

Then (23) reads as

$$\|u\|_{\mathring{W}_{\mathbb{G}}^{m,s}(\Omega)} = \|p_0(u)\|_{W^{m,s}(\mathbb{G})} \leqslant C \|\Delta_{\mathbb{G}}^{m/2} p_0(u)\|_{L^s(\mathbb{G})},$$

achieving the proof of the proposition. □

Lemma 3.9. *If $m > 0$ let $P_m \in \mathbf{K}^{-m-Q}$ be the kernel defined in Theorem* A.16 *and Remark* A.17. *If $\Omega \subset\subset \mathbb{G}$ is an open set, $R > R_0(s, \mathbb{G}, m, \Omega)$ is sufficiently large, and $u \in \mathcal{D}(\Omega)$, then*

$$\|u\|_{W_{\mathbb{G}}^{m,s}(\mathbb{G})} \approx \|\mathcal{O}((P_m)_R)u\|_{L^s(\mathbb{G})} = \|\Delta_{\mathbb{G},R}^{m/2} u\|_{L^s(\mathbb{G})},$$

with equivalence constants depending on $s, \mathbb{G}, m, \Omega$.

Proof. By Proposition 3.7, there exists $c_\Omega > 0$ such that (keeping in mind that we can think $p_0(u) = u$)

$$\|\Delta_{\mathbb{G}}^{m/2} u\|_{L^s(\mathbb{G})} \leqslant \|u\|_{W_{\mathbb{G}}^{m,s}(\mathbb{G})} \leqslant c_\Omega \|\Delta_{\mathbb{G}}^{m/2} u\|_{L^s(\mathbb{G})}.$$

By Remark A.17, we have

$$\Delta_{\mathbb{G}}^{m/2} u = \mathcal{O}((P_m)_R)u + Su,$$

where $Su = u * (1 - \psi_R) P_m$. Hence

$$\|\Delta_{\mathbb{G}}^{m/2} u\|_{L^s(\mathbb{G})} \leqslant \|\mathcal{O}((P_m)_R)u\|_{L^s(\mathbb{G})} + \|u * (1 - \psi_R) P_m\|_{L^s(\mathbb{G})}.$$

On the other hand, by [9], Proposition 1.10, and a standard argument (see e.g. [15,16])

$$\begin{aligned}\|u * (1 - \psi_R) P_m\|_{L^s(\mathbb{G})} &\leqslant C_s \|u\|_{L^s(\mathbb{G})} \cdot \|(1 - \psi_R) P_m\|_{L^1(\mathbb{G})} \\ &\leqslant C(s, \mathbb{G}, m) R^{-m} \|u\|_{L^s(\mathbb{G})} \leqslant C(s, \mathbb{G}, m) R^{-m} c_\Omega \|\Delta_{\mathbb{G}}^{m/2} u\|_{L^s(\mathbb{G})} \\ &\leqslant \frac{1}{2} \|\Delta_{\mathbb{G}}^{m/2} u\|_{L^s(\mathbb{G})},\end{aligned}$$

provided $R > R_0(s, \mathbb{G}, m, \Omega)$. Therefore

$$\|\Delta_{\mathbb{G}}^{m/2} u\|_{L^s(\mathbb{G})} \leqslant 2 \|\mathcal{O}((P_m)_R)u\|_{L^s(\mathbb{G})}$$

and hence

$$\|u\|_{W_{\mathbb{G}}^{m,s}(\mathbb{G})} \leqslant 2c_\Omega \|\mathcal{O}((P_m)_R)u\|_{L^s(\mathbb{G})}.$$

Conversely,

$$
\begin{aligned}
&\big\|\mathcal{O}\big((P_m)_R\big)u\big\|_{L^s(\mathbb{G})} \\
&\quad \leqslant \big\|\Delta_{\mathbb{G}}^{m/2}u\big\|_{L^s(\mathbb{G})} + \big\|u*(1-\psi_R)P_m\big\|_{L^s(\mathbb{G})} \\
&\quad \leqslant \frac{3}{2}\big\|\Delta_{\mathbb{G}}^{m/2}u\big\|_{L^s(\mathbb{G})} \leqslant \frac{3}{2}\|u\|_{W_{\mathbb{G}}^{m,s}(\mathbb{G})}.
\end{aligned}
$$

This achieves the proof of the lemma. □

Definition 3.10. Let $\Omega \subset \mathbb{G}$ be an open set. If $m \geqslant 0$ and $1 < s < \infty$, $W_{\mathbb{G}}^{-m,s}(\Omega)$ is the dual space of $\mathring{W}_{\mathbb{G}}^{m,s'}(\Omega)$, where $1/s + 1/s' = 1$. It is well known that, if $m \in \mathbb{N}$ and Ω is bounded, then

$$
W_{\mathbb{G}}^{-m,s}(\Omega) = \left\{ \sum_{d(I)=m} X^I f_I,\ f_I \in L^s(\Omega) \text{ for any } I \text{ such that } d(I)=m \right\}
$$

and

$$
\|u\|_{W_{\mathbb{G}}^{-m,s}(\Omega)} \approx \inf\left\{ \sum_I \|f_I\|_{L^s(\Omega)};\ d(I)=m,\ \sum_{d(I)=m} X^I f_I = u \right\}.
$$

Proposition 3.11. *If* $1 < s < \infty$ *and* $m, m' \geqslant 0$, $m' < m$, *then*

$$
W_{\mathbb{G}}^{m,s}(\mathbb{G}) \hookrightarrow W_{\mathbb{G}}^{m',s}(\mathbb{G}) \quad \textit{and} \quad W_{\mathbb{G}}^{-m',s}(\mathbb{G}) \hookrightarrow W_{\mathbb{G}}^{-m,s}(\mathbb{G})
$$

algebraically and topologically.

In addition, if Ω *is a bounded open set,* $1 < s < \infty$ *and* $m, m' \geqslant 0$, $m' < m$, *then*

$$
\mathring{W}_{\mathbb{G}}^{m,s}(\Omega) \quad \textit{is compactly embedded in } W_{\mathbb{G}}^{m',s}(\Omega)
$$

and

$$
W_{\mathbb{G}}^{-m',s}(\Omega) \quad \textit{is compactly embedded in } W_{\mathbb{G}}^{-m,s}(\Omega).
$$

Proof. The first assertion is nothing but [9], Proposition 4.2. As for the second assertion, take first $R > 0$, and let Ω_0 be a sufficiently large bounded open neighborhood of $\bar{\Omega}$. If $u \in \mathring{W}_{\mathbb{G}}^{m,s}(\Omega)$, by Lemma A.22, we can write

$$
u = \Delta_{\mathbb{G},R}^{-m'/2} \circ \Delta_{\mathbb{G},R}^{m'/2} u + \varphi S u,
$$

where $\varphi \in \mathcal{D}(\Omega_0)$ and $S \in \mathcal{OC}^{-\infty}$. By Lemma A.11, the map $u \to \varphi S u$ is compact from $\mathring{W}_{\mathbb{G}}^{m,s}(\Omega)$ to $W_{\mathbb{G}}^{m',s}(\Omega_0)$. As for the first term, the same property follows from Proposition A.18, Lemma A.7, and Lemma 3.6.

Finally, the third assertion of the proposition follows by duality. □

Remark 3.12. The compactness result of Proposition 3.11 can be improved as in the Euclidean space (see e.g. [17], Section 1.4.6). For sake of simplicity, let us restrict ourselves to the case $m \in \mathbb{N}$ and $m' = 0$. We have

$$\mathring{W}^{m,s}_{\mathbb{G}}(\Omega) \quad \text{is compactly embedded in } L^{\sigma}(\Omega)$$

and

$$L^{\sigma'}(\Omega) \quad \text{is compactly embedded in } W^{-m,s'}_{\mathbb{G}}(\Omega),$$

if s, s' and σ, σ' are Hölder conjugate exponents, provided $\sigma(m - Q/s) + Q > 0$.

Definition 3.13. If $\underline{m} \geqslant 0$ is an h-vector weight, $0 \leqslant h \leqslant n$, and $s > 1$, we say that a measurable section α of E_0^h, $\alpha := \sum_p \sum_{j \in I^h_{0,p}} \alpha_j \xi_j^h$ belongs to $W^{\underline{m},s}_{\mathbb{G}}(\mathbb{G}, E_0^h)$ if, for all $p \in \mathcal{I}_0^h$, i.e. for all p, $N_h^{\min} \leqslant p \leqslant N_h^{\max}$, such that $I^h_{0,p} \neq \emptyset$,

$$\alpha_j \in W^{m_p,s}_{\mathbb{G}}(\mathbb{G})$$

for all $j \in I^h_{0,p}$, endowed with the natural norm.

The spaces $W^{\underline{m},s}_{\mathbb{G}}(\Omega, E_0^h)$, where Ω is an open set in $\mathbb{G}$, as well as the local spaces $W^{\underline{m},s}_{\mathbb{G},\mathrm{loc}}(\Omega, E_0^h)$ are defined in the obvious way.

Since

$$W^{\underline{m},s}_{\mathbb{G}}\big(\Omega, E_0^h\big) \quad \text{is isometric to } \prod_{p \in \mathcal{I}_0^h} \big(W^{m_p,s}_{\mathbb{G}}(\mathbb{G})\big)^{\operatorname{card} \mathcal{I}^h_{0,p}},$$

then

- $W^{\underline{m},s}_{\mathbb{G}}(\Omega, E_0^h)$ is a reflexive Banach space (remember $s > 1$);
- $\mathbf{C}^\infty(\Omega, E_0^h) \cap W^{\underline{m},s}_{\mathbb{G}}(\Omega, E_0^h)$ is dense in $W^{\underline{m},s}_{\mathbb{G}}(\Omega, E_0^h)$.

The spaces $\mathring{W}^{\underline{m},s}_{\mathbb{G}}(\Omega, E_0^h)$ are defined in the obvious way.

We can define and characterize the dual spaces of Sobolev spaces of forms.

Proposition 3.14. *If* $1 < s < \infty$, $1/s + 1/s' = 1$, $0 \leqslant h \leqslant n$, $\underline{m}$ *is an* h*-vector weight, and* $\Omega \subset \mathbb{G}$ *is a bounded open set, then the dual space* $(\mathring{W}^{\underline{m},s'}_{\mathbb{G}}(\Omega, E_0^h))^*$ *coincides with the set of all currents* $T \in D'(\Omega, E_0^h)$ *of the form (with the notations of* (18)*)*

$$T = \sum_p \sum_{j \in I^h_{0,p}} \tilde{T}_j \llcorner \big(*\xi_j^h\big) \tag{24}$$

with $T_j \in W_{\mathbb{G}}^{-m_p,s}(\Omega)$ *for all* $j \in I_{0,p}^h$ *and for* $p \in \mathcal{I}_0^h$*. The action of* T *on the form* $\alpha = \sum_p \sum_{j\in I_{0,p}^h} \alpha_j \xi_j^h \in \mathring{W}_{\mathbb{G}}^{\underline{m},s'}(\Omega, E_0^h)$ *is given by the identity*

$$T(\alpha) = \sum_p \sum_{j\in I_{0,p}^h} \langle T_j | \alpha_j \rangle. \tag{25}$$

In particular, it is natural to set

$$W_{\mathbb{G}}^{-\underline{m},s}\big(\Omega, E_0^h\big) := \big(\mathring{W}_{\mathbb{G}}^{\underline{m},s'}\big(\Omega, E_0^h\big)\big)^*.$$

Moreover, if T *is as in* (24)

$$\|T\|_{W_{\mathbb{G}}^{-\underline{m},s}(\Omega,E_0^h)} \approx \sum_p \sum_{j\in I_{0,p}^h} \|T_j\|_{W_{\mathbb{G}}^{-m_p,s}(\Omega)}.$$

Proof. Suppose (24) holds. If $\alpha = \sum_q \sum_{i\in I_{0,q}^h} \alpha_i \xi_i^h$ is smooth and compactly supported in Ω, then (keeping in mind that the basis $\{\xi_j^h\}$ is orthonormal, so that $\xi_i^h \wedge *\xi_j^h = \delta_{ij}\, dV$)

$$\begin{aligned}\Big\langle \sum_p \sum_{j\in I_{0,p}^h} \tilde T_j \llcorner \big(*\xi_j^h\big) \Big| \alpha \Big\rangle &= \sum_p \sum_{j\in I_{0,p}^h} \sum_q \sum_{i\in I_{0,q}^h} \big\langle \tilde T_j \llcorner \big(*\xi_j^h\big) \big| \alpha_i \xi_i^h \big\rangle \\ &= \sum_p \sum_{j\in I_{0,p}^h} \sum_q \sum_{i\in I_{0,q}^h} \big\langle \tilde T_j \big| \alpha_i \big(\xi_i^h\big) \wedge *\xi_j^h \big\rangle = \sum_q \sum_{i\in I_{0,q}^h} \langle T_i | \alpha_i \rangle.\end{aligned}$$

Thus, clearly, if $T_i \in W_{\mathbb{G}}^{-m_q,s}(\Omega)$ for all $i \in I_{0,q}^h$ and for $q \in \mathcal{I}_0^h$, then the map $\alpha \to \sum_p \sum_{j\in I_{0,p}^h} \langle T_j | \alpha_j \rangle$ belongs to $(\mathring{W}_{\mathbb{G}}^{\underline{m},s'}(\Omega, E_0^h))^*$.

Suppose now $T \in (\mathring{W}_{\mathbb{G}}^{\underline{m},s'}(\Omega, E_0^h))^*$. Since $\mathcal{D}(\Omega, E_0^h) \hookrightarrow \mathring{W}_{\mathbb{G}}^{\underline{m},s'}(\Omega, E_0^h)$, then T can be identified with a current that we still denote by T. Thus, by (19), we can write

$$T = \sum_p \sum_{j\in I_{0,p}^h} \tilde T_j \llcorner \big(*\xi_j^h\big).$$

If $i \in I_{0,p}^h$ for some $p \in \mathcal{I}_0^h$ and $f \in \mathcal{D}(\Omega)$, we can consider the map

$$f \to \big\langle T \big| f \xi_i^h \big\rangle = \sum_p \sum_{j\in I_{0,p}^h} \big\langle \tilde T_j \big| f \xi_i^h \wedge \big(*\xi_j^h\big) \big\rangle = \langle T_i | f \rangle.$$

Because of the boundedness of T, we get

$$\big|\langle T_i | f \rangle\big| \leqslant C \big\| f \xi_i^h \big\|_{\mathring{W}_{\mathbb{G}}^{\underline{m},s'}(\Omega,E_0^h)} = C \|f\|_{\mathring{W}_{\mathbb{G}}^{m_p,s'}(\Omega)},$$

that yields $T_i \in W_{\mathbb{G}}^{-m_p,s}(\Omega)$. This achieve the proof. □

4. Hodge decomposition and compensated compactness

In this section we state and we prove our main results, i.e. a Hodge decomposition theorem for forms in E_0^* and – as a consequence – our compensated compactness theorem in E_0^*. Through this section, we assume that h, the degree of the forms we are dealing with, is fixed once and for all, $1 \leqslant h \leqslant n$, even if it is not mentioned explicitly in the statements.

From now on, we always assume that an orthonormal left invariant basis $\{\xi_j^\ell\}$ of E_0^ℓ has been fixed for all $\ell = 1, \dots, n$, and therefore pseudodifferential operators acting on intrinsic forms or current and matrix-valued pseudodifferential operators can be identified. We use this identification without referring explicitly to it.

Theorem 4.1. *Let $s > 1$ and $h = 1, \dots, n$ be fixed, and suppose h-forms have pure weight N_h. Let $\Omega \subset\subset \mathbb{G}$ a given open set, and let $\alpha^\varepsilon \in L^s(\mathbb{G}, E_0^h) \cap \mathcal{E}'(\Omega, E_0^h)$ be compactly supported differential h-forms such that*

$$\alpha^\varepsilon \rightharpoonup \alpha \quad \textit{as } \varepsilon \to 0 \quad \textit{weakly in } L^s\big(\mathbb{G}, E_0^h\big)$$

and

$$\big\{d_c\alpha^\varepsilon\big\} \quad \textit{is pre-compact in } W_{\mathbb{G}}^{-(\underline{N}_{h+1}-N_h),s}\big(\mathbb{G}, E_0^h\big).$$

Then there exist h-forms $\omega^\varepsilon \in L^s_{\mathrm{loc}}(\mathbb{G}, E_0^h)$ and $(h-1)$-forms $\psi^\varepsilon \in L^s_{\mathrm{loc}}(\mathbb{G}, E_0^{h-1})$ such that

(i) *$\omega^\varepsilon \to \omega$ strongly in $L^s_{\mathrm{loc}}(\mathbb{G}, E_0^h)$;*
(ii) *$\psi^\varepsilon \to \psi$ strongly in $L^s_{\mathrm{loc}}(\mathbb{G}, E_0^{h-1})$;*
(iii) *$\alpha^\varepsilon = \omega^\varepsilon + d_c\psi^\varepsilon$.*

We can choose ω^ε and ψ^ε supported in a fixed suitable neighborhood of Ω. Moreover, if the α^ε are smooth, then ω^ε and ψ^ε are smooth.

Remark 4.2. We stress that $d_c : L^s(\mathbb{G}, E_0^h) \to W_{\mathbb{G}}^{-(\underline{N}_{h+1}-N_h),s}(\mathbb{G}, E_0^h)$. Indeed, if $\alpha = \sum_{j \in I_{0,N_h}^h} \alpha_j \xi_j^h \in L^s(\mathbb{G}, E_0^h)$ and $(d_c\alpha)_i$ is a component of weight q of $d_c\alpha$, then (keeping in mind that h-forms have pure weight N_h) $(d_c\alpha)_i = \sum_j L_{i,j}^h \alpha_j$, where $L_{i,j}^h$ is a homogeneous differential operator in the horizontal vector fields of order $q - N_h \geqslant 1$, so that $(d_c\alpha)_i \in W_{\mathbb{G}}^{-(q-N_h),s}(\mathbb{G})$. On the other hand $(\underline{N}_{h+1} - N_h)_q = q - N_h$, and the assertion follows.

The proof of Theorem 4.1 entails several preliminary statements.

Definition 4.3. Let $R > 0$ be fixed. If $0 \leqslant h \leqslant n$, following Rumin we define the "0-order differential" acting on compactly supported h-currents belonging to $\mathcal{E}'(B(e, R), E_0^h)$ by

$$\tilde{d}_c := \Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2} d_c \Delta_{\mathbb{G},R}^{\underline{N}_h/2},$$

where $\underline{N}_h$ is defined in Definition 2.22. By Lemma A.13, the definition is well posed, and

$$\tilde{d}_c : \mathcal{E}'\big(B(e, R), E_0^h\big) \to \mathcal{E}'\big(B(e, 3R), E_0^h\big).$$

Analogously, we define the following "0-order codifferential" acting on compactly supported $(h+1)$-currents belonging to $\mathcal{E}'(B(e,R), E_0^{h+1})$:

$$\widetilde{\delta_c} := \Delta_{\mathbb{G},R}^{N_h/2} \delta_c \Delta_{\mathbb{G},R}^{-N_{h+1}/2}.$$

Again the definition is well posed, and

$$\widetilde{\delta_c} : \mathcal{E}'\big(B(e,R), E_0^{h+1}\big) \to \mathcal{E}'\big(B(e,3R), E_0^{h}\big).$$

By Theorem A.8(a) in Appendix A,

$$\widetilde{\delta_c} = (\tilde{d}_c)^*.$$

Notice also that

$$\tilde{d}_c^2 = 0, \quad \tilde{\delta}_c^2 = 0 \quad \big(\mathrm{mod}\ \mathcal{OC}^{-\infty}\big).$$

Let now $T = \sum_p \sum_{j\in I_{0,p}^h} \tilde{T}_j \llcorner (*\xi_j^h) \in \mathcal{E}'(B(e,R), E_0^h)$ be given.

By Theorem 2.13, the differential d_c acting on h-forms can be identified with a matrix-valued differential operator $L^h := (L_{i,j}^h)$, where the $L_{i,j}^h$'s are homogeneous left invariant differential operator of order $q-p$ if $j \in I_{0,p}^h$ and $i \in I_{0,q}^{h+1}$. Thus, by Definition A.19, we have

$$\tilde{d}_c T = \sum_q \sum_{i\in I_{0,q}^{h+1}} \sum_{p<q} \sum_{j\in I_{0,p}^h} \big(\widetilde{\Delta_{\mathbb{G},R}^{-q/2} L_{i,j}^h \Delta_{\mathbb{G},R}^{p/2}} T_j\big) \llcorner \big(*\xi_i^{h+1}\big).$$

Analogously, if $T = \sum_p \sum_{j\in I_{0,p}^{h+1}} \tilde{T}_j \llcorner (*\xi_j^{h+1}) \in \mathcal{E}'(B(e,R), E_0^{h+1})$, then

$$\widetilde{\delta_c} T = \sum_q \sum_{i\in I_{0,q}^{h}} \sum_{q<p} \sum_{j\in I_{0,p}^{h+1}} \big(\widetilde{\Delta_{\mathbb{G},R}^{q/2}\, {}^t L_{j,i}^h \Delta_{\mathbb{G},R}^{-p/2}} T_j\big) \llcorner \big(*\xi_i^{h}\big).$$

Proposition 4.4. *Both $\tilde{d}_c$ and $\widetilde{\delta_c}$ are matrix-valued pseudodifferential operators of the CGGP-calculus, acting respectively on $\mathcal{E}'(\mathbb{G}, E_0^h)$ and $\mathcal{E}'(\mathbb{G}, E_0^{h+1})$. Moreover $\tilde{d}_c \sim P^h := (P_{ij}^h)$, where*

$$P_{ij}^h = P_{-q} \,\underline{*}\, \big(L_{i,j}^h P_p\big) \quad \text{if } i \in I_{0,q}^{h+1} \text{ and } j \in I_{0,p}^h, \tag{26}$$

and $\widetilde{\delta_c} \sim Q^h := (Q_{ij}^h)$, where

$$Q_{ij}^h = P_q \,\underline{*}\, \big({}^t L_{j,i}^h P_{-p}\big) \quad \text{if } i \in I_{0,q}^{h} \text{ and } j \in I_{0,p}^{h+1}. \tag{27}$$

Proof. Take $i \in I_{0,q}^{h+1}$ and $j \in I_{0,p}^h$. Statement (26) follows by proving that

$$\Delta_{\mathbb{G},R}^{-q/2} L_{i,j}^h \Delta_{\mathbb{G},R}^{p/2} \sim P_{-q} \,\underline{*}\, \big(L_{i,j}^h P_p\big).$$

The proof of (27) is analogous. Thus, notice first that, by (49) and Lemma A.12, the cores of $L^h_{i,j}\Delta_{\mathbb{G},R}^{-p/2}$ and $\Delta_{\mathbb{G},R}^{-q/2}$ are, respectively, $L^h_{i,j}P_p$ and P_{-q}. Hence the assertion follows by Theorem A.8(c). □

Remark 4.5. With Rumin's notations (see [23,24]), when acting on $\mathcal{S}_0(\mathbb{G}, E_0^h)$,

$$\mathcal{O}_0\big(P^h\big) \equiv d_c^\nabla.$$

An analogous assertion holds for $\mathcal{O}_0(Q^h)$.

We set

$$\Delta^{(0)}_{\mathbb{G},R} := \widetilde{\delta_c}\tilde{d}_c + \tilde{d}_c\widetilde{\delta_c}.$$

The following assertion is a straightforward consequence of Theorem A.8 and Proposition 4.4.

Proposition 4.6. *$\Delta^{(0)}_{\mathbb{G},R}$ is a matrix-valued 0-order pseudodifferential operator of the CGGP-calculus acting on $\mathcal{E}'(\mathbb{G}, E_0^h)$, and*

$$\Delta^{(0)}_{\mathbb{G},R} \sim \Delta^{(0)}_{\mathbb{G}} := \big(\Delta^{(0)}_{\mathbb{G},ij}\big),$$

where

$$\Delta^{(0)}_{\mathbb{G},ij} = \sum_\ell \big(Q^h_{i\ell} \mathbin{\underline{*}} P^h_{\ell j} + P^{h-1}_{i\ell} \mathbin{\underline{*}} Q^{h-1}_{\ell j}\big).$$

Remark 4.7. As in Remark 4.5, with the notations of [23,24], when acting on $\mathcal{S}_0(\mathbb{G}, E_0^h)$,

$$\begin{aligned}\mathcal{O}_0\big(\Delta^{(0)}_{\mathbb{G}}\big) &= \mathcal{O}_0\big(Q^h\big) \circ d_c\mathcal{O}_0\big(P^h\big) + \mathcal{O}_0\big(P^{h-1}\big) \circ \delta_c\mathcal{O}_0\big(Q^{h-1}\big) \\ &= \delta_c^\nabla d_c^\nabla + d_c^\nabla \delta_c^\nabla = \Box_{d_c}.\end{aligned}$$

Theorem 4.8. *For any $R > 0$ there exists a (matrix-valued) CGGP-pseudodifferential operator $(\Delta^{(0)}_{\mathbb{G},R})^{-1}$ such that*

$$\big(\Delta^{(0)}_{\mathbb{G},R}\big)^{-1}\Delta^{(0)}_{\mathbb{G},R} = Id \quad on \quad \mathcal{E}'\big(\mathbb{G}, E_0^h\big) \quad \big(\mathrm{mod}\, \mathcal{OC}^{-\infty}\big) \tag{28}$$

and

$$\Delta^{(0)}_{\mathbb{G},R}\big(\Delta^{(0)}_{\mathbb{G},R}\big)^{-1} = Id \quad on \quad \mathcal{E}'\big(\mathbb{G}, E_0^h\big) \quad \big(\mathrm{mod}\, \mathcal{OC}^{-\infty}\big). \tag{29}$$

Proof. Keeping in mind [5], Theorems 5.1 and 5.11, it follows from Rockland's condition (see Theorem A.4), that is satisfied by [23], that there exists $(\Delta^{(0)}_{\mathbb{G}})^{-1} \in \mathbf{K}^{-Q}$ such that

$$\big(\Delta^{(0)}_{\mathbb{G}}\big)^{-1} \mathbin{\underline{*}} \Delta^{(0)}_{\mathbb{G}} = \Delta^{(0)}_{\mathbb{G}} \mathbin{\underline{*}} \big(\Delta^{(0)}_{\mathbb{G}}\big)^{-1} = \delta.$$

The assertion follows taking now $(\Delta^{(0)}_{\mathbb{G},R})^{-1} := \mathcal{O}((\Delta^{(0)}_{\mathbb{G}})^{-1}_R)$ for $R > 0$. □

Remark 4.9. If $\alpha \in \mathcal{E}'(B(e,r), E_0^h)$, then, by Lemma A.13, both

$$\operatorname{supp}\big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1}\Delta_{\mathbb{G},R}^{(0)}\alpha \quad \text{and} \quad \operatorname{supp}\big(\Delta_{\mathbb{G},R}^{(0)}\Delta_{\mathbb{G},R}^{(0)}\big)^{-1}\alpha$$

are contained in a fixed ball B depending only on r, R. Thus, we can multiply the identities (28) and (29) by a suitable test function φ that is identically one on B, and then we can replace the smoothing operators S appearing in (28) and (29) by operators of the form φS, that maps $\mathcal{E}'(\mathbb{G}, E_0^h)$ in $\mathcal{D}(\mathbb{G}, E_0^h)$.

Proposition 4.10. *For any* $R > 0$

$$\big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1}\tilde{d}_c = \tilde{d}_c\big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1} \quad \textit{on} \quad \mathcal{E}'\big(\mathbb{G}, E_0^h\big) \quad \big(\operatorname{mod} \mathcal{OC}^{-\infty}\big), \tag{30}$$

and

$$\big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1}\widetilde{\delta_c} = \widetilde{\delta_c}\big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1} \quad \textit{on} \quad \mathcal{E}'\big(\mathbb{G}, E_0^h\big) \quad \big(\operatorname{mod} \mathcal{OC}^{-\infty}\big). \tag{31}$$

Proof. By duality, it is enough to prove (30). In the sequel, S will always denote a smoothing operator belonging to $\mathcal{OC}^{-\infty}$ that may change from formula to formula, and, with the same convention, we shall denote by S_0 an operator of the form φS, with $S \in \mathcal{OC}^{-\infty}$ and $\varphi \in \mathcal{D}(\mathbb{G})$. Keeping in mind Remark 4.9, we have

$$\begin{aligned}
\tilde{d}_c\big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1} &= \big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1}\Delta_{\mathbb{G},R}^{(0)}\tilde{d}_c\big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1} + S_0 \\
&= \big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1}(\widetilde{\delta_c}\tilde{d}_c + \tilde{d}_c\widetilde{\delta_c})\tilde{d}_c\big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1} + S_0 \\
&= \big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1}\tilde{d}_c\widetilde{\delta_c}\tilde{d}_c\big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1} + S_0 \\
&= \big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1}\tilde{d}_c(\widetilde{\delta_c}\tilde{d}_c + \tilde{d}_c\widetilde{\delta_c})\big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1} \\
&= \big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1}\tilde{d}_c\Delta_{\mathbb{G},R}^{(0)}\big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1} + S_0 \\
&= \big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1}\tilde{d}_c + S_0. \qquad \square
\end{aligned}$$

Remark 4.11. We can repeat the arguments of Remark 4.9 also for (30) and (31).

Proof of Theorem 4.1. In the sequel, S will always denote a smoothing operator belonging to $\mathcal{OC}^{-\infty}$ that may change from formula to formula, and, with the same convention, we shall denote by S_0 an operator of the form φS, with $S \in \mathcal{OC}^{-\infty}$ and $\varphi \in \mathcal{D}(\mathbb{G})$. Moreover, without loss of generality, we may assume $\alpha^\varepsilon \in \mathcal{D}(\Omega, E_0^h)$. Take now $R > 0$ such that $\Omega \subset B(e,R)$; by Lemma A.13, $\Delta_{\mathbb{G},R}^{-N_h/2}\alpha^\varepsilon \in \mathcal{D}(B(e,2R), E_0^h)$ and therefore, by (29),

$$\Delta_{\mathbb{G},R}^{(0)}\big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1}\Delta_{\mathbb{G},R}^{-N_h/2}\alpha^\varepsilon - \Delta_{\mathbb{G},R}^{-N_h/2}\alpha^\varepsilon = S\Delta_{\mathbb{G},R}^{-N_h/2}\alpha^\varepsilon, \tag{32}$$

with $S \in \mathcal{OC}^{-\infty}$. Since $\operatorname{supp} \Delta^{(0)}_{\mathbb{G},R}(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta^{-N_h/2}_{\mathbb{G},R}\alpha^\varepsilon \subset B(e,4R)$, we can multiply the previous identity by a cut-off function $\varphi \equiv 1$ on $B(e,4R)$ without affecting the left-hand side of the identity. Thus, we can write (32) as

$$\Delta^{(0)}_{\mathbb{G},R}\big(\Delta^{(0)}_{\mathbb{G},R}\big)^{-1}\Delta^{-N_h/2}_{\mathbb{G},R}\alpha^\varepsilon - \Delta^{-N_h/2}_{\mathbb{G},R}\alpha^\varepsilon = \varphi S \Delta^{-N_h/2}_{\mathbb{G},R}\alpha^\varepsilon = S_0\alpha^\varepsilon, \tag{33}$$

by Lemma A.10. From (33), it follows easily that

$$\Delta^{N_h/2}_{\mathbb{G},R}\Delta^{(0)}_{\mathbb{G},R}\big(\Delta^{(0)}_{\mathbb{G},R}\big)^{-1}\Delta^{-N_h/2}_{\mathbb{G},R}\alpha^\varepsilon = \Delta^{N_h/2}_{\mathbb{G},R}\Delta^{-N_h/2}_{\mathbb{G},R}\alpha^\varepsilon + \Delta^{N_h/2}_{\mathbb{G},R}S_0\alpha^\varepsilon, \tag{34}$$

so that, by Lemma A.22 and arguing as above,

$$\Delta^{N_h/2}_{\mathbb{G},R}\Delta^{(0)}_{\mathbb{G},R}\big(\Delta^{(0)}_{\mathbb{G},R}\big)^{-1}\Delta^{-N_h/2}_{\mathbb{G},R}\alpha^\varepsilon = \alpha^\varepsilon + S_0\alpha^\varepsilon. \tag{35}$$

If we write explicitly $\Delta^{(0)}_{\mathbb{G},R}$ in (35), we get

$$\begin{aligned}\alpha^\varepsilon = {}& \Delta^{N_h/2}_{\mathbb{G},R}\Delta^{N_h/2}_{\mathbb{G},R}\delta_c\Delta^{-N_{h+1}/2}_{\mathbb{G},R}\Delta^{-N_{h+1}/2}_{\mathbb{G},R}d_c\Delta^{N_h/2}_{\mathbb{G},R}\big(\Delta^{(0)}_{\mathbb{G},R}\big)^{-1}\Delta^{-N_h/2}_{\mathbb{G},R}\alpha^\varepsilon \\ &+ \Delta^{N_h/2}_{\mathbb{G},R}\Delta^{-N_h/2}_{\mathbb{G},R}d_c\Delta^{N_{h-1}/2}_{\mathbb{G},R}\Delta^{N_{h-1}/2}_{\mathbb{G},R}\delta_c\Delta^{-N_h/2}_{\mathbb{G},R}\big(\Delta^{(0)}_{\mathbb{G},R}\big)^{-1}\Delta^{-N_h/2}_{\mathbb{G},R}\alpha^\varepsilon \\ &+ S_0\alpha^\varepsilon := I_1 + I_2 + S_0\alpha^\varepsilon.\end{aligned} \tag{36}$$

By Lemma A.22,

$$\begin{aligned} I_2 &= d_c\Delta^{N_{h-1}/2}_{\mathbb{G},R}\Delta^{N_{h-1}/2}_{\mathbb{G},R}\delta_c\Delta^{-N_h/2}_{\mathbb{G},R}\big(\Delta^{(0)}_{\mathbb{G},R}\big)^{-1}\Delta^{-N_h/2}_{\mathbb{G},R}\alpha^\varepsilon + S_0\alpha^\varepsilon \\ &:= d_c\psi^\varepsilon + S_0\alpha^\varepsilon. \end{aligned} \tag{37}$$

Thus (36) becomes

$$\begin{aligned}\alpha^\varepsilon = {}& \Delta^{N_h/2}_{\mathbb{G},R}\Delta^{N_h/2}_{\mathbb{G},R}\delta_c\Delta^{-N_{h+1}/2}_{\mathbb{G},R}\Delta^{-N_{h+1}/2}_{\mathbb{G},R}d_c\Delta^{N_h/2}_{\mathbb{G},R}\big(\Delta^{(0)}_{\mathbb{G},R}\big)^{-1}\Delta^{-N_h/2}_{\mathbb{G},R}\alpha^\varepsilon \\ &+ S_0\alpha^\varepsilon + d_c\psi^\varepsilon := \omega^\varepsilon + d_c\psi^\varepsilon.\end{aligned} \tag{38}$$

We want to show that $(\psi^\varepsilon)_{\varepsilon>0}$ and $(\omega^\varepsilon)_{\varepsilon>0}$ converge strongly in $L^s_{\mathrm{loc}}(\mathbb{G}, E_0^{h-1})$ and $L^s_{\mathrm{loc}}(\mathbb{G}, E_0^h)$, respectively. As for $(\psi^\varepsilon)_{\varepsilon>0}$, by Proposition A.21, $(\Delta^{-N_h/2}_{\mathbb{G},R}\alpha^\varepsilon)_{\varepsilon>0}$ converges weakly in $W^{N_h,s}_{\mathbb{G}}(\mathbb{G}, E_0^h)$. On the other hand, by Proposition A.18, $((\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta^{-N_h/2}_{\mathbb{G},R}\alpha^\varepsilon)_{\varepsilon>0}$ converges weakly in $W^{N_h,s}_{\mathbb{G}}(\mathbb{G}, E_0^h)$. Thus, again by Proposition A.18, $(\Delta^{-N_h/2}_{\mathbb{G},R}(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta^{-N_h/2}_{\mathbb{G},R}\alpha^\varepsilon)_{\varepsilon>0}$ converges weakly in $W^{2N_h,s}_{\mathbb{G}}(\mathbb{G}, E_0^h)$. We remind that all intrinsic h-forms have the same weight N_h, so that all the components of a form in E_0^h belonging to $W^{2N_h,s}_{\mathbb{G}}(\mathbb{G}, E_0^h)$ belong to the same Sobolev space $W^{2N_h,s}_{\mathbb{G}}(\mathbb{G}, E_0^h)$.

For sake of simplicity, denote now by β_j^ε, $j \in I^h_{0,N_h}$, a generic component of $\Delta^{-N_h/2}_{\mathbb{G},R}(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta^{-N_h/2}_{\mathbb{G},R}\alpha^\varepsilon$ that converges weakly in $W^{2N_h,s}_{\mathbb{G}}(\mathbb{G}, E_0^{h-1})$. If $i \in I^{h-1}_{0,q}$ $(q < N_h)$, then the ith component of $\delta_c\beta_j^\varepsilon$ is given by ${}^tL_{j,i}\beta_j^\varepsilon$. Keeping in mind that $L_{j,i}$ is a homogeneous differential operator in the horizontal vector fields of order $N_h - q$, then $({}^tL_{j,i}\beta_j^\varepsilon)_{\varepsilon>0}$ converges weakly in $W^{N_h+q,s}_{\mathbb{G}}(\mathbb{G})$, so that, eventually, the ith component of $(\psi^\varepsilon)_{\varepsilon>0}$ converges weakly in $W^{N_h-q,s}_{\mathbb{G}}(\mathbb{G})$. Then the assertion follows by Rellich theorem (Proposition 3.11), since $\operatorname{supp}\psi^\varepsilon$ is contained in a fixed neighborhood of Ω, and $q < N_h$.

Let us consider now $(\omega^\varepsilon)_{\varepsilon>0}$. By Lemma A.11, we can forget the smoothing operator S_0. By Proposition 4.10 and Remark 4.11, we can write

$$\begin{aligned}
&\Delta_{\mathbb{G},R}^{\frac{N_h}{}}\Delta_{\mathbb{G},R}^{\frac{N_h}{}/2}\delta_c\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}d_c\Delta_{\mathbb{G},R}^{\frac{N_h}{}/2}\big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1}\Delta_{\mathbb{G},R}^{-\frac{N_h}{}/2}\alpha^\varepsilon \\
&\quad= \Delta_{\mathbb{G},R}^{\frac{N_h}{}/2}\Delta_{\mathbb{G},R}^{\frac{N_h}{}/2}\delta_c\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}\big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1}\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}d_c\alpha^\varepsilon + S_0\alpha^\varepsilon \\
&\quad= \Delta_{\mathbb{G},R}^{\frac{N_h}{}/2}\big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1}\Delta_{\mathbb{G},R}^{\frac{N_h}{}/2}\delta_c\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}d_c\alpha^\varepsilon + S_0\alpha^\varepsilon. \qquad (39)
\end{aligned}$$

By Proposition A.21,

$$\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}d_c\alpha^\varepsilon \quad \text{is pre-compact in } W^{\frac{N}{}_{h+1}+N_h,s}_{\mathbb{G},\mathrm{loc}}\big(\mathbb{G},E_0^h\big).$$

Arguing as above, denote now by β_j^ε, $j \in I_{0,p}^{h+1}$, a generic component of $\beta^\varepsilon := \Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}d_c\alpha^\varepsilon$. We know that β_j^ε is pre-compact in $W^{p+N_h,s}_{\mathbb{G},\mathrm{loc}}(\mathbb{G},E_0^{h+1})$. Moreover notice that $\delta_c\beta_\varepsilon$ is an h-form, and therefore, by assumption, has pure weight N_h. If $i \in I_{0,N_h}^h$ ($N_h < p$), then the ith component of $\delta_c\beta_j^\varepsilon$ is given by ${}^tL_{j,i}\beta_j^\varepsilon$. Keeping in mind that $L_{j,i}$ is a homogeneous differential operator in the horizontal vector fields of order $j-i = p-N_h$, then $(\delta_c\beta_j^\varepsilon)_i$ is pre-compact in $W^{2N_h,s}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$. Thus, $\delta_c\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}d_c\alpha^\varepsilon$ is pre-compact in $W^{2\frac{N}{}_h,s}_{\mathbb{G},\mathrm{loc}}(\mathbb{G},E_0^h)$. Again, by Proposition A.21, $\Delta_{\mathbb{G},R}^{\frac{N_h}{}/2}\delta_c\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}d_c\alpha^\varepsilon$ is pre-compact in $W^{\frac{N}{}_h,s}_{\mathbb{G},\mathrm{loc}}(\mathbb{G},E_0^h)$. As above, we can rely now on the fact that all components of $\Delta_{\mathbb{G},R}^{\frac{N_h}{}/2}\delta_c\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}d_c\alpha^\varepsilon$ have the same weight and hence belong to the same Sobolev space, to conclude that

$$\big(\Delta_{\mathbb{G},R}^{(0)}\big)^{-1}\Delta_{\mathbb{G},R}^{\frac{N_h}{}/2}\delta_c\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\frac{N}{}_{h+1}/2}d_c\alpha^\varepsilon$$

is pre-compact in $W^{\frac{N}{}_h,s}_{\mathbb{G},\mathrm{loc}}(\mathbb{G},E_0^h)$. Then, we achieve the proof of the theorem using again Proposition A.21.

Finally, the last statement follows by Lemma A.13 and Theorem A.8(b). □

Lemma 4.12. *If $\alpha \in \mathcal{E}(\mathbb{G},E_0^h)$ with $2 \leqslant h \leqslant n$ and $\beta \in \mathcal{E}(\mathbb{G},E_0^{n-h-2})$, then*

$$dd_c\alpha \wedge (\Pi_E\beta) = 0.$$

Proof. By Remark 2.15, we have

$$\begin{aligned}
dd_c\alpha \wedge (\Pi_E\beta) &= (\Pi_E dd_c\alpha) \wedge \beta = (d\Pi_E d_c\alpha) \wedge \beta \\
&= (\Pi_{E_0} d\Pi_E d_c\alpha) \wedge \beta = (d_c d_c\alpha) \wedge \beta = 0. \qquad \square
\end{aligned}$$

Theorem 4.13. *If $1 < s_i < \infty$, $0 \leqslant h_i \leqslant n$ for $i = 1, 2$, and $0 < \varepsilon < 1$, assume that $\alpha_i^\varepsilon \in L^{s_i}_{\mathrm{loc}}(\mathbb{G}, E_0^{h_i})$ for $i = 1, 2$, where $\frac{1}{s_1} + \frac{1}{s_2} = 1$ and $h_1 + h_2 = n$. Suppose h_1-forms have pure weight N_{h_1} (by Hodge duality, this implies that also h_2-forms have pure weight $N_{h_2} = Q - N_{h_1}$). Assume that, for any open set $\Omega_0 \subset\subset \mathbb{G}$,*

$$\alpha_i^\varepsilon \to \alpha_i \quad \text{weakly in } L^{s_i}\big(\Omega_0, E_0^{h_i}\big), \tag{40}$$

and that

$$\big\{d_c\alpha_i^\varepsilon\big\} \quad \text{is pre-compact in } W^{-(\underline{N}_{h_i+1}-N_{h_i}),s_i}_{\mathbb{G},\mathrm{loc}}\big(\mathbb{G}, E_0^{h_i}\big) \tag{41}$$

for $i = 1, 2$.

Then

$$\int_{\mathbb{G}} \varphi\alpha_1^\varepsilon \wedge \alpha_2^\varepsilon \to \int_{\mathbb{G}} \varphi\alpha_1 \wedge \alpha_2 \tag{42}$$

for any $\varphi \in \mathcal{D}(\mathbb{G})$.

Proof. By Remark 4.2, without loss of generality we can assume that both α_1^ε and α_2^ε are smooth forms. In addition, let us prove that, if Ω is an open neighborhood of $\operatorname{supp}\varphi$, then

$$d_c\big(\varphi\alpha_1^\varepsilon\big) \quad \text{is pre-compact in } W^{-(\underline{N}_{h_i+1}-N_{h_i}),s_1}_{\mathbb{G},}\big(\Omega, E_0^h\big). \tag{43}$$

An analogous argument can be repeated for $\psi\alpha_2^\varepsilon$, where $\psi \in \mathcal{D}(\Omega)$ is identically 1 on $\operatorname{supp}\varphi$. Thus, without loss of generality, we could restrict ourselves to prove that

$$\int_{\mathbb{G}} \alpha_1^\varepsilon \wedge \alpha_2^\varepsilon \to \int_{\mathbb{G}} \alpha_1 \wedge \alpha_2 \tag{44}$$

when (40) and (41) hold and $\alpha_i \in \mathcal{D}(\Omega, E_0^{h_i})$ for $i = 1, 2$.

In order to prove (43), set $\beta^\varepsilon := d_c(\varphi\alpha_1^\varepsilon)$, with $\beta^\varepsilon = \sum_q \sum_{i \in I_{0,q}^{h_1+1}} \beta_i^\varepsilon \xi_i^{h+1}$. If $\alpha_1^\varepsilon = \sum_p \sum_{j \in I_{0,p}^{h_1}} (\alpha_1^\varepsilon)_j \xi_j^h$, then, by Theorem 2.13, when $i \in I_{0,q}^{h_1+1}$, we have

$$\begin{aligned}
\beta_i &= \sum_{p<q} \sum_{j \in I_{0,p}^h} L_{i,j}^h\big(\varphi\big(\alpha_1^\varepsilon\big)_j\big) = \varphi \sum_{p<q} \sum_{j \in I_{0,p}^h} L_{i,j}^h\big(\alpha_1^\varepsilon\big)_j + \sum_{p<q} \sum_{j \in I_{0,p}^h} \sum_{1 \leqslant |\gamma| \leqslant q-p} (P_\gamma\varphi)\big(Q_\gamma\big(\alpha_1^\varepsilon\big)_j\big) \\
&= \varphi\big(d_c\big(\alpha_1^\varepsilon\big)\big)_i + \sum_{p<q} \sum_{j \in I_{0,p}^h} \sum_{1 \leqslant |\gamma| \leqslant q-p} (P_\gamma\varphi)\big(Q_\gamma\big(\alpha_1^\varepsilon\big)_j\big),
\end{aligned}$$

where P_γ and Q_γ are homogeneous left invariant differential operators of order $|\gamma|$ and $q - p - |\gamma|$, respectively, in the horizontal derivatives. By (41), $\varphi(d_c(\alpha_1^\varepsilon))_i$ is compact in $W_{\mathbb{G}}^{-(q-p),s}(\Omega)$. On the other hand $Q_\gamma(\alpha_1^\varepsilon)_j$ is bounded in $W_{\mathbb{G}}^{-(q-p-|\gamma|),s}(\Omega)$, and therefore compact in $W_{\mathbb{G}}^{-(q-p),s}(\Omega)$ by Proposition 3.11, since $|\gamma| > 0$. This proves (43).

We can proceed now to prove (44). By Theorem 4.1 we can write

$$\alpha_i^\varepsilon = d_c\psi_i^\varepsilon + \omega_i^\varepsilon, \quad i = 1, 2,$$

with ψ_i^ε and ω_i^ε supported in a suitable neighborhood Ω_0 of $\bar{\Omega}$ and converging strongly in $L^{s_i}(\Omega_0, E_0^{h_i})$. Thus the integral of $\alpha_1^\varepsilon \wedge \alpha_2^\varepsilon$ in (44) splits into the sum of 4 terms. Clearly, 3 of them are easy to deal with, since they are the integral of the wedge product of two sequences of forms, at least one of them converging strongly. Thus, we are left with the term

$$\int_{\mathbb{G}} d_c\psi_1^\varepsilon \wedge d_c\psi_2^\varepsilon,$$

with $\psi_i^\varepsilon \in \mathcal{D}(\Omega_0, E_0^{k_i})$ for $i = 1, 2$. By Remark 2.16, we have

$$\int_{\mathbb{G}} d_c\psi_1^\varepsilon \wedge d_c\psi_2^\varepsilon = 0,$$

since $d_c^2 = 0$. This achieves the proof of the theorem. □

5. Div–curl theorem and H-convergence

We state some dual formulations of our main theorem for horizontal vector fields in $\mathbb{G}$, i.e. for sections of $H\mathbb{G}$. Since in this case the compensated compactness theorem takes a form akin to the original form of the theorem proved by Murat and Tartar, we can refer to it as to the div–curl theorem for Carnot groups. In this case, our compensated compactness theorem applies for any Carnot group $\mathbb{G}$, since, as pointed out in Example B.1, E_0^1 consists precisely of all forms of pure weight 1. In addition, as in [12] and [2], the div–curl theorem makes possible to develop a theory of the H-convergence for second order divergence form elliptic differential operators in Carnot groups of the form

$$\begin{cases} \mathcal{L}u := \sum_{i,j=1}^{m_1} X_i^*(a_{i,j}(x)X_j u) = f \in W_{\mathbb{G}}^{-1,2}(\Omega), \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{45}$$

with application for instance to non-periodic homogenization. Here $\mathcal{A}(x) := (a_{i,j}(x))_{i,j=1,\dots,m_1}$ is an $m_1 \times m_1$ elliptic matrix with measurable entries.

We stress again that $\mathcal{L}$ is elliptic with respect to the structure of the group $\mathbb{G}$, but is degenerate elliptic as a usual differential operator in $\mathbb{R}^n$.

If V is a horizontal vector field, i.e. if V is a section of $H\mathbb{G}$, as customary we set

$$\operatorname{div}_{\mathbb{G}} V := \big(*d_c\big(*V^\natural\big)\big)^\natural$$

and

$$\operatorname{curl}_G V := \big(d_c V^\natural\big)^\natural.$$

Moreover, if f is a function, we denote by $\nabla_{\mathbb{G}} f$ the horizontal vector field $\nabla_{\mathbb{G}} f := (X_1 f, \dots, X_{m_1} f)$. Set now $E_{0,h} := (E_0^h)^\natural$ (with the induced scalar product). An orthonormal basis of $E_{0,1}$

is given by $X_1, \dots, X_{m_1}$, and hence the horizontal vector field V can be written in the form $V := \sum_{j=1}^{m_1} V_j X_j$ and therefore identified with the vector-valued function $(V_1, \dots, V_{m_1})$. In the sequel, we write also $(V_{X_1}, \dots, V_{X_{m_1}})$. Thus $\mathrm{div}_{\mathbb{G}} V = \sum_{j=1}^{m_1} X_j V_j$. The Dirichlet problem (45) takes the form

$$\begin{cases} \mathcal{L}u := -\mathrm{div}_{\mathbb{G}}(\mathcal{A}(x)\nabla_{\mathbb{G}} u) = f \in W_{\mathbb{G}}^{-1,2}(\Omega), \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{46}$$

If we refer to the examples of Appendix B, the operator curl_G on a horizontal vector field V takes the following forms:

- Example B.2: if $V = (V_X, V_Y)$, then

$$\mathrm{curl}_G V = P_1(V_X, V_Y) X \wedge T + P_2(V_X, V_Y) Y \wedge T.$$

Let D be another horizontal vector field. In this case, assumption (41) of Theorem 4.13, with $\alpha_1 := V^\natural$ and $*\alpha_2 := D^\natural$, becomes

$$P_i(V_X, V_Y) \quad \text{compact in } W_{\mathbb{G},\mathrm{loc}}^{-2,s_1}(\mathbb{G}), \quad i = 1, 2$$

and

$$\mathrm{div}_{\mathbb{G}} D \quad \text{compact in } W_{\mathbb{G},\mathrm{loc}}^{-1,s_2}(\mathbb{G}).$$

- Example B.3: if $V = (V_X, V_Y, V_S)$, then

$$\mathrm{curl}_G V = P_1(V_X, V_Y) X \wedge T + P_2(V_X, V_Y) Y \wedge T \\ + (X V_S - S V_X) X \wedge S + (Y V_S - S V_Y) Y \wedge S.$$

As above, (41) of Theorem 4.13 becomes

$$P_i(V_X, V_Y) \quad \text{compact in } W_{\mathbb{G},\mathrm{loc}}^{-2,s_1}(\mathbb{G}), \quad i = 1, 2,$$
$$X V_S - S V_X, Y V_S - S V_Y \quad \text{compact in } W_{\mathbb{G},\mathrm{loc}}^{-1,s_1}(\mathbb{G}),$$

and

$$\mathrm{div}_{\mathbb{G}} D \quad \text{compact in } W_{\mathbb{G},\mathrm{loc}}^{-1,s_2}(\mathbb{G}).$$

- Example B.4: if $V = (V_{X_1}, V_{X_2}, V_{Y_1}, V_{Y_2}, V_S)$, then

$$\begin{aligned} \mathrm{curl}_G V = {} & (X_1\alpha_{X_2} - X_2\alpha_{X_1}) X_1 \wedge X_2 + (Y_1\alpha_{Y_2} - Y_2\alpha_{Y_1}) Y_1 \wedge Y_2 \\ & + (X_1\alpha_{Y_2} - Y_2\alpha_{X_1}) X_1 \wedge Y_2 + (X_2\alpha_{Y_1} - Y_1\alpha_{X_2}) X_2 \wedge Y_1 \\ & + (X_1\alpha_S - S\alpha_{X_1}) X_1 \wedge S + (X_2\alpha_S - S\alpha_{X_2}) X_2 \wedge S \\ & + (Y_1\alpha_S - S\alpha_{Y_1}) Y_1 \wedge S + (Y_2\alpha_S - S\alpha_{Y_2}) Y_2 \wedge S \\ & + \frac{X_1\alpha_{Y_1} - Y_1\alpha_{X_1} - X_2\alpha_{Y_2} + Y_2\alpha_{X_2}}{\sqrt{2}} \frac{1}{\sqrt{2}} (X_1 \wedge Y_1 - X_2 \wedge Y_2). \end{aligned}$$

Here, assumption (41) requires that all the coefficients of $\mathrm{curl}_G V$, as well as $\mathrm{div}_{\mathbb{G}} D$, are compact in $W^{-1,s_1}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$, and $W^{-1,s_2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$, respectively.

- Example B.5: if $V = (V_{X_1}, V_{X_2}, V_{X_3}, V_{X_4}, V_{X_5}, V_{X_6})$, then

$$\begin{aligned}\mathrm{curl}_G V = {} & (X_1\alpha_{X_3} - X_3\alpha_{X_1})X_1 \wedge X_3 \\ & + \big(X_1(X_1\alpha_{X_2} - X_2\alpha_{X_1}) - X_4\alpha_{X_1}\big)X_1 \wedge X_4 \\ & + \big(X_2(X_1\alpha_{X_2} - X_2\alpha_{X_1}) - X_4\alpha_{X_2}\big)X_2 \wedge X_4 \\ & + \big(X_2(X_2\alpha_{X_3} - X_3\alpha_{X_2}) - X_5\alpha_{X_2}\big)X_2 \wedge X_5 \\ & + \big(X_3(X_2\alpha_{X_3} - X_3\alpha_{X_2}) - X_5\alpha_{X_3}\big)X_3 \wedge X_5.\end{aligned}$$

As above, (41) of Theorem 4.13 becomes

$$\begin{gathered} X_1\alpha_{X_3} - X_3\alpha_{X_1} \quad \text{compact in } W^{-1,s_1}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}), \\ X_1(X_1\alpha_{X_2} - X_2\alpha_{X_1}) - X_4\alpha_{X_1},\, X_2(X_1\alpha_{X_2} - X_2\alpha_{X_1}) - X_4\alpha_{X_2}, \\ X_2(X_2\alpha_{X_3} - X_3\alpha_{X_2}) - X_5\alpha_{X_2} \quad \text{compact in } W^{-2,s_1}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}), \\ X_3(X_2\alpha_{X_3} - X_3\alpha_{X_2}) - X_5\alpha_{X_3} \quad \text{compact in } W^{-3,s_1}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}), \end{gathered}$$

and

$$\mathrm{div}_{\mathbb{G}} D \quad \text{compact in } W^{-1,s_2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}).$$

- Example B.6: if $V = (V_1, V_2)$, then

$$\begin{aligned}\mathrm{curl}_G V = {} & \big(X_2(X_1V_2 - X_2V_1) - X_3V_2\big)X_2 \wedge X_3 \\ & + \big(X_1\big(X_1^2V_2 - (X_1X_2 + X_3)V_1\big) - X_4V_1\big)X_1 \wedge X_4.\end{aligned}$$

As above, (41) of Theorem 4.13 becomes

$$\begin{gathered} X_2(X_1V_2 - X_2V_1) - X_3V_2 \quad \text{compact in } W^{-2,s_1}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}), \\ X_1\big(X_1^2V_2 - (X_1X_2 + X_3)V_1\big) - X_4V_1 \quad \text{compact in } W^{-3,s_1}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}), \end{gathered}$$

and

$$\mathrm{div}_{\mathbb{G}} D \quad \text{compact in } W^{-1,s_2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}).$$

- Example B.7: if $V = (V_1, V_2)$, then

$$\begin{aligned}\mathrm{curl}_G V = {} & \big(X_1\big(X_1^2V_2 - X_1X_2V_1 - X_3V_1\big) - X_4V_1\big)X_1 \wedge X_4 \\ & + \big(X_2\big(X_2X_1V_2 - X_2^2V_1 - X_3V_2\big) - X_5V_2\big)X_2 \wedge X_5 \\ & + \frac{1}{2}\big(X_1\big(X_2X_1V_2 - X_2^2V_1 - X_3V_2\big) - X_5V_1\end{aligned}$$

$$+ X_2\big(X_1^2V_2 - X_1X_2V_1 - X_3V_1\big) - X_4V_2\big)(X_1 \wedge X_5 + X_2 \wedge X_4).$$

Here, assumption (41) requires that all the coefficients of $\mathrm{curl}_G V$ are compact in $W^{-3,s_1}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$, and that $\mathrm{div}_{\mathbb{G}} D$ is compact in $W^{-1,s_2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$.

Theorem 4.13 yields the following result that generalizes to arbitrary Carnot groups Theorem 3.3 of [12] and Theorem 5.5 of [2], extending to the setting of Carnot groups Theorem 5.3 and its Corollary 5.4 of [14].

Theorem 5.1. *Let $\Omega \subset \mathbb{G}$ be an open set, and let $s, \sigma > 1$ be a Hölder conjugate pair. Moreover, with the notations of* (20)*, if $p \in \mathcal{I}_0^2$ (i.e. if $p \geqslant 2$ is the weight of an intrinsic 2-form), let $a(p) > 1$ and $b > 1$ be such that*

$$a(p) > \frac{Qs}{Q + (p-1)s} \quad \text{and} \quad b > \frac{Q\sigma}{Q+\sigma}.$$

Let now $G^k \in L^s_{\mathrm{loc}}(\Omega, H\mathbb{G})$ and $D^k \in L^\sigma_{\mathrm{loc}}(\Omega, H\mathbb{G})$ be horizontal vector fields for $k \in \mathbb{N}$, weakly convergent to G and D in $L^s_{\mathrm{loc}}(\Omega, H\mathbb{G})$ and in $L^\sigma_{\mathrm{loc}}(\Omega, H\mathbb{G})$, respectively.

If the components of $\{\mathrm{curl}_{\mathbb{G}} G^k\}$ of weight p are bounded in $L^{a(p)}_{\mathrm{loc}}(\Omega, H\mathbb{G})$ for $p \in \mathcal{I}_0^2$ and $\{\mathrm{div}_{\mathbb{G}} D^k\}$ is bounded in $L^b_{\mathrm{loc}}(\Omega, H\mathbb{G})$, then

$$\big\langle G^k, D^k\big\rangle \to \langle G, D\rangle \quad \text{in } \mathcal{D}'(\Omega),$$

i.e.

$$\int_\Omega \big\langle G^k(x), D^k(x)\big\rangle_x \varphi(x)\,dx \to \int_\Omega \big\langle G(x), D(x)\big\rangle_x \varphi(x)\,dx$$

for any $\varphi \in \mathcal{D}(\Omega)$.

Proof. We want to apply Theorem 4.13 (with its notations) to the forms

$$\alpha_1^k := \big(G^k\big)^\natural \quad \text{and} \quad \alpha_2^k := *\big(D^k\big)^\natural,$$

taking $h_1 = 1$, $h_2 = n-1$, $s_1 = s$, $s_2 = \sigma$.

The assertion will follow by showing that $\{\mathrm{div}_{\mathbb{G}} D^k\}$ is compact in $W^{-1,\sigma}_{\mathbb{G},\mathrm{loc}}(\Omega)$ and the components of $\{\mathrm{curl}_{\mathbb{G}} G^k\}$ of weight p are compact in $W^{1-p,s}_{\mathbb{G},\mathrm{loc}}(\Omega)$. Indeed, $p-1$ is precisely the component of index p of $\underline{N}_2 - 1 = \underline{N_2} - N_1$.

But this follows by a simple computation from Remark 3.12, since

(i) $L^b_{\mathrm{loc}}(\Omega, H\mathbb{G})$ is compactly embedded in $W^{-1,\sigma}_{\mathbb{G},\mathrm{loc}}(\Omega)$;
(ii) $L^{a(p)}_{\mathrm{loc}}(\Omega, H\mathbb{G})$ is compactly embedded in $W^{1-p,s}_{\mathbb{G},\mathrm{loc}}(\Omega)$.

Indeed, in order to prove (i), it is enough to notice that

$$b'(1 - Q/s) + Q > b'\big(1 - Q/s + Q(1 - 1/\sigma - 1/Q)\big) = 0,$$

whereas, to prove (ii) we notice that

$$a(p)'(p-1-Q/\sigma)+Q > a(p)'\left(p-1-\frac{Q}{\sigma}+Q\left(1-\frac{Q+(p-1)s}{Qs}\right)\right)=0. \qquad \square$$

In particular, as we pointed out above, Theorem 5.1 makes possible to extend the notion of Murat–Tartar H-convergence (see e.g. [19]), given in [12] and [2] for $\mathbb{G}=\mathbb{H}^n$, to an arbitrary Carnot group $\mathbb{G}$. In fact, the definitions given in [12] and [2] are naturally stated in general Carnot groups as follows.

Definition 5.2. If $0<\alpha\leqslant\beta<\infty$ and Ω is an open subset of $\mathbb{G}$, we denote by $M(\alpha,\beta;\Omega)$ the set of $(m\times m)$-matrix-valued measurable functions in Ω such that

$$\langle \mathcal{A}(x)\xi,\xi\rangle_{\mathbb{R}^m}\geqslant\frac{1}{\beta}\left|\mathcal{A}(x)\xi\right|^2_{\mathbb{R}^m}\quad\text{and}\quad \langle \mathcal{A}(x)\xi,\xi\rangle_{\mathbb{R}^m}\geqslant\alpha|\xi|^2_{\mathbb{R}^m}$$

for all $\xi\in\mathbb{R}^m$ and for a.e. $x\in\Omega$.

Definition 5.3. We say that a sequence of matrices $\mathcal{A}^k\in M(\alpha,\beta;\Omega)$ H-converges to the matrix $\mathcal{A}^{\mathrm{eff}}\in M(\alpha',\beta';\Omega)$ for some $0<\alpha'\leqslant\beta'<\infty$, if for every $f\in W^{-1,2}_{\mathbb{G}}(\Omega)$, called u_k the solutions in $\mathring{W}^{1,2}_{\mathbb{G}}(\Omega)$ of the problems $-\mathrm{div}_{\mathbb{G}}(\mathcal{A}^k\nabla_{\mathbb{G}}u_k)=f$, the following convergences hold:

$$\begin{gathered} u_k\to u_\infty\quad\text{in } \mathring{W}^{1,2}_{\mathbb{G}}(\Omega)\text{-weak},\\ A^k\nabla_{\mathbb{G}}u_k\to\mathcal{A}^{\mathrm{eff}}\nabla_{\mathbb{G}}u_\infty\quad\text{in } L^2(\Omega;H\mathbb{G})\text{-weak}. \end{gathered}$$

Therefore u_∞ is solution of the problem $-\mathrm{div}_{\mathbb{G}}(\mathcal{A}^{\mathrm{eff}}\nabla_{\mathbb{G}}u_\infty)=f$ in Ω.

Repeating verbatim the arguments of Theorem 4.4 of [12], we can show now that the sets $M(\alpha,\beta;\Omega)$ are compact in the topology of the H-convergence.

Theorem 5.4. *If $0<\alpha\leqslant\beta<\infty$ and Ω is a bounded open subset of $\mathbb{G}$, then for any sequence of matrices $\mathcal{A}^n\in M(\alpha,\beta;\Omega)$ there exists a subsequence A^{k_i} and a matrix $A^{\mathrm{eff}}\in M(\alpha,\beta;\Omega)$ such that A^{k_i} H-converges to A^{eff}.*

Appendix A. Pseudodifferential operators

To keep the paper as much self-contained as possible, we open this appendix by reminding some basic definitions and results taken from [5] on pseudodifferential operators on homogeneous groups.

We set

$$\mathcal{S}_0:=\left\{u\in\mathcal{S}\colon \int_{\mathbb{G}}x^\alpha u(x)\,dx=0\right\}$$

for all monomials x^α.

If $\alpha \in \mathbb{R}$ and $\alpha \notin \mathbb{Z}^+ := \mathbb{N} \cup \{0\}$, then we denote by $\mathbf{K}^\alpha$ the set of the distributions in $\mathbb{G}$ that are smooth away from the origin and homogeneous of degree α, whereas, if $\alpha \in \mathbb{Z}^+$, we say that $K \in \mathcal{D}'(\mathbb{G})$ belongs to $\mathbf{K}^\alpha$ if has the form

$$K = \tilde{K} + p(x)\ln|x|,$$

where $\tilde{K}$ is smooth away from the origin and homogeneous of degree α, and p is a homogeneous polynomial of degree α.

Kernels of type α according to Folland [9] belong to $\mathbf{K}^{\alpha-Q}$. In particular, if $0 < \alpha < Q$, and $h(t,x)$ is the heat kernel associated with the sub-Laplacian $\Delta_{\mathbb{G}}$, then ([9], Proposition 3.17) the kernel $R_\alpha \in L^1_{\mathrm{loc}}(\mathbb{G})$ defined by

$$R_\alpha(x) := \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{(\alpha/2)-1} h(x,t)\,dt$$

belongs to $\mathbf{K}^{\alpha-Q}$.

If $K \in \mathbf{K}^\alpha$, we denote by $\mathcal{O}_0(K)$ the operator defined on $\mathcal{S}_0$ by $\mathcal{O}_0(K)u := u * K$.

Proposition A.1. *(See [5], Proposition 2.2.)* $\mathcal{O}_0(K) : \mathcal{S}_0 \to \mathcal{S}_0$.

Theorem A.2. *(See [15,16].) If* $K \in \mathbf{K}^{-Q}$, *then* $\mathcal{O}_0(K) : L^2(\mathbb{G}) \to L^2(\mathbb{G})$.

Remark A.3. We stress that we have also

$$\mathcal{S}_0(\mathbb{G}) \subset \mathrm{Dom}\big(\Delta_{\mathbb{G}}^{-\alpha/2}\big) \quad \text{with } \alpha > 0.$$

Indeed, take $M \in \mathbb{N}$, $M > \alpha/2$. If $u \in \mathcal{S}_0(\mathbb{G})$, we can write $u = \Delta_{\mathbb{G}}^M v$, where

$$v := \big(\mathcal{O}_0(R_2) \circ \mathcal{O}_0(R_2) \circ \cdots \circ \mathcal{O}_0(R_2)\big)u \in \mathcal{S}_0(\mathbb{G})$$

(M times). Since $v \in \mathrm{Dom}(\Delta_{\mathbb{G}}^M) \cap \mathrm{Dom}(\Delta_{\mathbb{G}}^{M-\alpha/2})$ (by Proposition 3.4), then $u = \Delta_{\mathbb{G}}^M v \in \mathrm{Dom}(\Delta_{\mathbb{G}}^{-\alpha/2})$, and $\Delta_{\mathbb{G}}^{M-\alpha/2} v = \Delta_{\mathbb{G}}^{-\alpha/2}\Delta_{\mathbb{G}}^M v$, by [9], Proposition 3.15(iii).

Theorem A.4. *(See [13] and [5], Theorem 5.11.) If* $K \in \mathbf{K}^{-Q}$, *and let the following Rockland condition hold: for every non-trivial irreducible unitary representation* π *of* $\mathbb{G}$, *the operator* $\overline{\pi_K}$ *is injective on* $\mathbf{C}^\infty(\pi)$, *the space of smooth vectors of the representation* π. *Then the operator* $\mathcal{O}_0(K) : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ *is left invertible.*

Obviously, if $\mathcal{O}_0(K)$ is formally self-adjoint, i.e. if $K = {}^{\mathrm{v}}K$, then $\mathcal{O}_0(K)$ is also right invertible.

Proposition A.5. *(See [5], Proposition 2.3.) If* $K_i \in \mathbf{K}^{\alpha_i}$, $i = 1,2$, *then there exists at least one* $K \in \mathbf{K}^{\alpha_1+\alpha_2+Q}$ *such that*

$$\mathcal{O}_0(K_2) \circ \mathcal{O}_0(K_1) = \mathcal{O}_0(K).$$

It is possible to provide a standard procedure yielding such a K *(see* [5], *p.* 42*). Following* [5], *we write* $K = K_2 \,\underline{*}\, K_1$.

We can give now a (simplified) definition of pseudodifferential operator on $\mathbb{G}$, following [5], Definition 2.4.

Definition A.6. If $\alpha \in \mathbb{R}$, we say that $\mathcal{K}$ is a pseudodifferential operator of order α on $\mathbb{G}$ with core K if

(1) $K \in \mathcal{D}'(\mathbb{G} \times \mathbb{G})$.
(2) Let $\beta := -Q - \alpha$. There exist $K^m = K_x^m \in \mathbf{K}^{\beta+m}$ depending smoothly on $x \in \mathbb{G}$ such that for each $N \in \mathbb{N}$ there exists $M \in \mathbb{Z}^+$ such that, if we set

$$K_x - \sum_{m=0}^{M} K_x^m := E_M(x, \cdot),$$

then $E_M \in \mathbf{C}^N(\mathbb{G} \times \mathbb{G})$.
(3) For some finite $R \geqslant 0$, $\operatorname{supp} K_x \subset B(e, R)$ for all $x \in \mathbb{G}$.
(4) If $u \in \mathcal{D}(\mathbb{G})$ and $x \in \mathbb{G}$, then

$$\mathcal{K}u(x) = (u * K_x)(x).$$

We write $K \sim \sum_m K^m$, $\mathcal{K} = \mathcal{O}(K)$, and $r(K) = r(\mathcal{K}) = \inf\{R > 0 \text{ such that (3) holds}\}$.

We let

$$\mathcal{OC}^\alpha(\mathbb{G}) := \{\text{pseudodifferential operators of order } \alpha \text{ on } \mathbb{G}\}.$$

Clearly, if $\mathcal{K} \in \mathcal{OC}^\alpha(\mathbb{G})$, then $\mathcal{K} : \mathcal{D}(\mathbb{G}) \to \mathcal{E}(\mathbb{G})$. Moreover, $\mathcal{K}$ can be extended to an operator $\mathcal{K} : \mathcal{E}'(\mathbb{G}) \to \mathcal{D}'(\mathbb{G})$.

Lemma A.7. *If* $\operatorname{supp} u \subset B(e, \rho)$*, then* $\operatorname{supp} \mathcal{K}u \subset B(e, \rho + r(\mathcal{K}))$.

If $\gamma = (\gamma_1, \dots, \gamma_n) \in (\mathbb{Z}^+)^n$, for any $f \in \mathcal{D}'(\mathbb{G})$ we set

$$M_\gamma f = x^\gamma f,$$

and, if $X = (X_1, \dots, X_n)$ is our fixed basis of $\mathfrak{g}$, we denote by $\sigma_\gamma(X)$ the coefficient of x^γ in the expansion of $(\gamma!/|\gamma|!)(x \cdot X)^{d(\gamma)}$.

Theorem A.8. *(See [5], Theorem 2.5.) We have:*

(a) *If* $\mathcal{K} := \mathcal{O}(K) \in \mathcal{OC}^\alpha(\mathbb{G})$*, then there exists a core* K^* *such that* $\mathcal{O}(K^*) \in \mathcal{OC}^\alpha(\mathbb{G})$ *and*

$$\langle v, \mathcal{K}u \rangle_{L^2(\mathbb{G})} = \langle \mathcal{O}(K^*) v, u \rangle_{L^2(\mathbb{G})}$$

for all $u, v \in \mathcal{D}(\mathbb{G})$.
(b) *If* $\mathcal{K} \in \mathcal{OC}^\alpha(\mathbb{G})$*,* $V \subset \mathbb{G}$ *is an open set, and* $u \in \mathcal{E}'(\mathbb{G})$ *is smooth on* V*, then* $\mathcal{K}u$ *is smooth on* V*.*

(c) *If $\mathcal{K}_i \in \mathcal{OC}_i^{\alpha}(\mathbb{G})$, $K_i \sim \sum_m K_i^m$, $i = 1, 2$, then $\mathcal{K} := \mathcal{K}_2 \circ \mathcal{K}_1$ (that is well defined by Lemma* A.7*) belongs to $\mathcal{OC}^{\alpha_1+\alpha_2}(\mathbb{G})$. Moreover $K \sim \sum_m K^m$, where*

$$K_x^m = \sum_{d(\gamma)+j+\ell=m} \frac{1}{\gamma!}\big[(-M)^\gamma \big(K_2^\ell\big)_x\big] \underline{*} \big[\sigma_\gamma(X)\big(K_1^j\big)_x\big],$$

where $\sigma_\gamma(X)$ acts in the x-variable.

Theorem A.9. *(See [5], p. 63 (3).) If $\mathcal{K} \in \mathcal{OC}^0(\mathbb{G})$, then $\mathcal{O}(K) : L^p_{\mathrm{loc}}(\mathbb{G}) \to L^p_{\mathrm{loc}}(\mathbb{G})$ is continuous. In particular, by Lemma* A.7*, $\mathcal{O}(K) : L^p(\mathbb{G}) \cap \mathcal{E}'(B(e,\rho)) \to L^p(\mathbb{G})$ continuously.*

We say that a convolution operator $u \to u * E(x,\cdot)$ from $\mathcal{E}'$ to $\mathcal{D}'$ belongs to $\mathcal{OC}^{-\infty}(\mathbb{G})$ if E is smooth on $\mathbb{G} \times \mathbb{G}$. We notice that, properly speaking, $\mathcal{OC}^{-\infty}(\mathbb{G})$ is not contained in $\mathcal{OC}^{\alpha}(\mathbb{G})$ for $\alpha \in \mathbb{R}$, since $E(x,\cdot)$ is not assumed to be compactly supported.

If $\mathcal{T}, \mathcal{S} \in \mathcal{OC}^{\ell}(\mathbb{G})$, we say that $\mathcal{S} = \mathcal{T}$ mod $\mathcal{OC}^{-\infty}$ if $\mathcal{S} - \mathcal{T} \in \mathcal{OC}^{-\infty}(\mathbb{G})$.

A straightforward computation proves the following results

Lemma A.10. *If $\mathcal{S} \in \mathcal{OC}^{-\infty}(\mathbb{G})$, $\varphi \in \mathcal{D}(\mathbb{G})$, and $\mathcal{O}(K) \in \mathcal{OC}^m(\mathbb{G})$ for $m \in \mathbb{R}$, then both $(\varphi\mathcal{S}) \circ \mathcal{O}(K)$ and $\mathcal{O}(K) \circ (\varphi\mathcal{S})$ belong to $\mathcal{OC}^{-\infty}(\mathbb{G})$.*

Lemma A.11. *If $\Omega \subset \mathbb{G}$ is a bounded open set, $m, m' \in \mathbb{R}$, $1 < s < \infty$, and $\mathcal{T} \in \mathcal{OC}^{-\infty}(\mathbb{G})$, then, if $\varphi \in \mathcal{D}(\mathbb{G})$, the map*

$$\varphi T : W^{m,s}_{\mathbb{G}}(\mathbb{G}) \cap \mathcal{E}'(\Omega) \to W^{m',s}_{\mathbb{G}}(\mathbb{G})$$

is compact.

From now on, let $\psi \in \mathcal{D}(\mathbb{G})$ be a fixed non-negative function such that

$$\operatorname{supp} \psi \subset B(e,1) \quad \text{and} \quad \psi \equiv 1 \quad \text{on } B\left(e, \frac{1}{2}\right).$$

We set

$$\psi_R := \psi \circ \delta_{1/R}.$$

If $K \in \mathbf{K}^m$, then $K_R := \psi_R K$ is a core satisfying (1)–(3) of Definition A.6. In addition, $K_R \sim K$, since we can write $K_R = K + (\psi_R - 1)K$, with $(\psi_R - 1)K \in \mathcal{E}(\mathbb{G})$. Thus $\mathcal{O}(K_R) \in \mathcal{OC}^{-m-Q}(\mathbb{G})$.

Thus, if K is a Folland kernel of type $\alpha \in \mathbb{R}$, then K_R is a core of a pseudodifferential operator $\mathcal{O}(K_R) \in \mathcal{OC}^{-\alpha}(\mathbb{G})$. In particular, if $0 < \alpha < Q$, then $\mathcal{O}((R_\alpha)_R)$ belongs to $\mathcal{OC}^{-\alpha}(\mathbb{G})$ (see [9], Proposition 3.17).

Lemma A.12. *If $K \in \mathbf{K}^m$, and X^I is a left invariant homogeneous differential operator, then*

$$X^I \mathcal{O}(K_R) \in \mathcal{OC}^{-m+d(I)-Q}(\mathbb{G}).$$

Moreover, the core $K_{R,I}$ of $X^I \mathcal{O}(K_R)$ satisfies

$$K_{R,I} \sim X^I K$$

and

$$X^I \mathcal{O}(K_R) = \mathcal{O}\big(\big(X^I K\big)_R\big) \quad \operatorname{mod} \mathcal{OC}^{-\infty}.$$

Lemma A.13. *If $u \in \mathcal{E}'(\mathbb{G})$ and $\operatorname{supp} u \subset B(0,\rho)$ then $\operatorname{supp} \mathcal{O}(K_R)u \subset B(0, R+\rho)$. Moreover, if $\rho = R$, then*

$$\mathcal{O}(K_{4R})u \equiv u * K \quad \text{on } B(0,R).$$

Proposition A.14. *Let $K_i \in \mathbf{K}^i$ be given cores for $i = 1, 2$, and let $R > 0$ be fixed. Then*

$$\mathcal{O}\big((K_2 \underline{*} K_1)_R\big) = \mathcal{O}\big((K_1)_R\big) \circ \mathcal{O}\big((K_2)_R\big) \quad \operatorname{mod} \mathcal{OC}^{-\infty}.$$

In particular, $\mathcal{O}((K_1)_R) \circ \mathcal{O}((K_2)_R) = \mathcal{O}(K)$ for a suitable core K with $K \sim K_2 \underline{*} K_1$.

Remark A.15. As in Remark 5 at p. 63 of [5], the previous calculus can be formulated for matrix-valued operators and hence, once left invariant bases $\{\xi_j^h\}$ of E_0^h are chosen, we obtain pseudodifferential operators acting on h-forms and h-currents, together with the related calculus.

In particular, let $K := (K_{ij})_{i=1,\dots,N,\, j=1,\dots,M}$ an $M \times N$ matrix whose entries K_{ij} belong to $\mathbf{K}^{m_{ij}}$. Then K acts between $\mathcal{S}_0(\mathbb{G})^N$ and $\mathcal{S}_0(\mathbb{G})^M$ as follows: if $T = (T_1, \dots, T_M)$, then

$$\mathcal{O}_0(K)T := T * K := \Big(\sum_j T_j * K_{1j}, \dots, \sum_j T_j * K_{Mj}\Big).$$

When $K_{ij} \in \mathbf{K}^m$ for all i, j, we write shortly that $K \in \mathbf{K}^m$.

If $K := (K_{ij})_{i=1,\dots,N,\, j=1,\dots,M'}$ and $K' := (K'_{ij})_{i=1,\dots,M',\, j=1,\dots,M}$, we write

$$K' \underline{*} K := \Big(\sum_\ell K'_{i\ell} \underline{*} K_{\ell j}\Big).$$

Notice that

$$\mathcal{O}_0\big(K'\big) \circ \mathcal{O}_0(K) = O_0\big(K' \underline{*} K\big). \tag{47}$$

In addition, if $\tilde{K} = (\tilde{K}_{ij})$ is a matrix-valued pseudodifferential operator of the CGGP-calculus, and $K = (K_{ij})$ is a matrix-valued core as above with $\tilde{K}_{ij} \sim K_{ij}$ for all i, j, we write $\tilde{K} \sim K$, and $\tilde{K} - K$ is a matrix-valued smoothing operator. As above, if all the K_{ij}'s are pseudodifferential operators of the same order α, we refer to α as to the order of the matrix-valued pseudodifferential operator K.

Finally, we prove that the fractional powers of $\Delta_{\mathbb{G}}$, when acting on suitable function spaces, can be written as suitable convolution operators. This is more or less known (see for instance [5], Section 6), though not explicitly stated in the form we need.

Theorem A.16. *If $m \in \mathbb{R}$ and $1 < s < \infty$, then $\mathcal{S}_0(\mathbb{G}) \subset \mathrm{Dom}(\Delta_{\mathbb{G}}^{m/2})$, and there exists $P_m \in \mathbf{K}^{-m-Q}$ such that*

$$\Delta_{\mathbb{G}}^{m/2} u = u * P_m \quad \text{for all } u \in \mathcal{S}_0(\mathbb{G}).$$

Moreover, if $R > 0$ then

$$\mathcal{O}\big((P_m)\big)_R \in \mathcal{OC}^m(\mathbb{G}). \tag{48}$$

Coherently, in the sequel we shall write

$$\Delta_{\mathbb{G},R}^{m/2} := \mathcal{O}\big((P_m)\big)_R. \tag{49}$$

Remark A.17. The same argument shows that, if $m \geqslant 0$, then $\mathcal{D}(\mathbb{G}) \subset \mathrm{Dom}(\Delta_{\mathbb{G}}^{m/2})$, and

$$\Delta_{\mathbb{G}}^{m/2} u = u * P_m \quad \text{for all } u \in \mathcal{D}(\mathbb{G}).$$

Proposition A.18. *If $\Omega \subset \mathbb{G}$ is a bounded open set, $m, \alpha \in \mathbb{R}$, $1 < s < \infty$, and $\mathcal{T} \in \mathcal{OC}^\alpha(\mathbb{G})$, then*

$$\mathcal{T} : W_{\mathbb{G}}^{m+\alpha,s}(\mathbb{G}) \cap \mathcal{E}'(\Omega) \to W_{\mathbb{G}}^{m,s}(\mathbb{G})$$

continuously.

Proof. Suppose first $m, m+\alpha \geqslant 0$. Let $u \in W_{\mathbb{G}}^{m+\alpha,s}(\mathbb{G}) \cap \mathcal{E}'(\Omega)$ be given. Without loss of generality, we can assume $u \in \mathcal{D}(\Omega_1)$, where Ω_1 is a given bounded open neighborhood of Ω, since $\mathcal{D}(\Omega_1)$ is dense in $W_{\mathbb{G}}^{m+\alpha,s}(\mathbb{G}) \cap \mathcal{E}'(\Omega)$. Indeed, by Proposition 3.4, if $\varepsilon > 0$, we can find $u_\varepsilon \in \mathcal{D}(\mathbb{G})$ such that $\|u - u_\varepsilon\|_{W_{\mathbb{G}}^{m+\alpha,s}(\mathbb{G})} < \varepsilon$. Let now $\psi \in \mathcal{D}(\Omega_1)$ be such that $\psi \equiv 1$ on Ω. Then, by [9], Corollary 4.15,

$$\|u - \psi u_\varepsilon\|_{W_{\mathbb{G}}^{m+\alpha,s}(\mathbb{G})} = \|\psi u - \psi u_\varepsilon\|_{W_{\mathbb{G}}^{m+\alpha,s}(\mathbb{G})} \leqslant C_\psi \|u - u_\varepsilon\|_{W_{\mathbb{G}}^{m+\alpha,s}(\mathbb{G})} < C_\psi \varepsilon.$$

By definition, there exists a bounded open set $\Omega_{\mathcal{T}}$ (depending only on Ω_1 and $\mathcal{T}$) such that $\mathcal{T}u \in \mathcal{D}(\Omega_{\mathcal{T}})$. If $R > 0$ is fixed (sufficiently large), by Proposition 3.9, we have

$$\|\mathcal{T}u\|_{W_{\mathbb{G}}^{m,s}(\mathbb{G})} \approx \big\|\Delta_{\mathbb{G},R}^{m/2} \mathcal{T}u\big\|_{L^s(\mathbb{G})}.$$

On the other hand,

$$\Delta_{\mathbb{G},R}^{m/2} \mathcal{T}u = \Delta_{\mathbb{G},R}^{m/2} \mathcal{T} \Delta_{\mathbb{G},R}^{-(m+\alpha)/2} \Delta_{\mathbb{G},R}^{(m+\alpha)/2} u + \varphi_0 S u,$$

with $S \in \mathcal{OC}^{-\infty}$ and $\varphi_0 \in \mathcal{D}(\mathbb{G})$ with $\varphi_0 \equiv 1$ on $\Omega_1 \cdot B(e, 2R)$, since $\Delta_{\mathbb{G},R}^{-(m+\alpha)/2} \Delta_{\mathbb{G},R}^{(m+\alpha)/2} u$ is supported in $\Omega_1 \cdot B(e, 2R)$. Then the assertion follows by Proposition A.18, since

$$\Delta_{\mathbb{G},R}^{m/2} \mathcal{T} \Delta_{\mathbb{G},R}^{-(m+\alpha)/2} \in \mathcal{OC}^0(\mathbb{G}),$$

by Theorem A.8 and by Lemma A.11.

This accomplishes the proof when $m, m+\alpha \geqslant 0$. Remaining cases can be dealt by duality. □

Definition A.19. Let $T \in \mathcal{E}'(\mathbb{G}, E_0^h)$ be a compactly supported h-current on $\mathbb{G}$ of the form

$$T = \sum_p \sum_{j \in I_{0,p}^h} \tilde{T}_j \llcorner \left(*\xi_j^h\right) \quad \text{with } T_j \in \mathcal{E}'(\mathbb{G}) \quad \text{for } j = 1, \dots, \dim E_0^h.$$

Let $\underline{m}$ be an h-vector weight, and let $R > 0$ be fixed. We set (with the notation of (49))

$$\Delta_{\mathbb{G},R}^{\underline{m}/2} T := \sum_p \sum_{j \in I_{0,p}^h} \left(\widetilde{\Delta_{\mathbb{G},R}^{m_p/2} T_j}\right) \llcorner \left(*\xi_j^h\right).$$

In particular, if T can be identified with a compactly supported h-form $\alpha = \sum_p \sum_{j \in I_{0,p}^h} \alpha_j \xi_j^h$, then our previous definition becomes

$$\Delta_{\mathbb{G},R}^{\underline{m}/2} \alpha = \sum_p \sum_{j \in I_{0,p}^h} \left(\alpha_j * (P_{m_p})_R\right) \xi_j^h.$$

Remark A.20. As in Definition A.19, if $\underline{m}$ is an h-vector weight, we define the operator

$$\mathcal{O}_0(P_{\underline{m}}) : \mathcal{S}_0\left(\mathbb{G}, E_0^h\right) \to \mathcal{S}_0\left(\mathbb{G}, E_0^h\right)$$

as follows: if $\alpha = \sum_p \sum_{j \in I_{0,p}^h} \alpha_j \xi_j^h$ with $\alpha_j \in \mathcal{S}_0(\mathbb{G})$, then

$$\mathcal{O}_0(P_{\underline{m}}) \alpha := \sum_p \sum_{j \in I_{0,p}^h} (\alpha_j * P_{m_p}) \xi_j^h.$$

In other words, $P_{\underline{m}}$ can be identified with the matrix $((P_{\underline{m}})_{ij})$, where

$$(P_{\underline{m}})_{ij} = 0 \quad \text{if } i \neq j \quad \text{and} \quad (P_{\underline{m}})_{jj} = m_p \quad \text{if } j \in I_{0,p}^h.$$

We can write

$$\Delta_{\mathbb{G},R}^{\underline{m}/2} \sim P_{\underline{m}}.$$

The following result is a straightforward consequence of Proposition A.18, thanks to "diagonal form" of the operator $\Delta_{\mathbb{G},R}^{\underline{m}/2}$.

Proposition A.21. *Let $\Omega \subset \mathbb{G}$ be a bounded open set. If $\underline{m}$ and $\underline{\alpha}$ are h-vector weights, and $1 < s < \infty$, then for any $R > 0$*

$$\Delta_{\mathbb{G},R}^{\underline{\alpha}/2} : W_{\mathbb{G}}^{\underline{m}+\underline{\alpha},s}\left(\mathbb{G}, E_0^h\right) \cap \mathcal{E}'\left(\Omega, E_0^h\right) \to W_{\mathbb{G}}^{\underline{m},s}\left(\mathbb{G}, E_0^h\right)$$

continuously.

Lemma A.22. *If $\underline{m}$ is an h-vector weight, then for any $R > 0$*

$$\Delta_{\mathbb{G},R}^{\underline{m}/2} \circ \Delta_{\mathbb{G},R}^{-\underline{m}/2} = Id \mod \mathcal{OC}^{-\infty}$$

and

$$\Delta_{\mathbb{G},R}^{-\underline{m}/2} \circ \Delta_{\mathbb{G},R}^{\underline{m}/2} = Id \mod \mathcal{OC}^{-\infty}.$$

Appendix B. Differential forms in Carnot groups

In this appendix, we provide a list of explicit examples of the complex (E_0, d_c) for some significant groups.

Example B.1. First of all, we stress that in any Carnot group $\mathbb{G}$ the space E_0^1 consists precisely of all horizontal forms, i.e. of all forms of weight 1. Indeed, notice first that on 0-forms $d_0 = 0$. On the other hand, if X_i, X_j are left invariant vector fields, and $\theta_\ell \in \Theta^1$, by the identity

$$d_0\theta_\ell(X_i, X_j) = d\theta_\ell(X_i, X_j) = -\theta_\ell\big([X_i, X_j]\big),$$

it follows that $d_0\theta_\ell = 0$ if and only if θ_ℓ has weight one, since $[X_i, X_j]$ belongs to $V_2 \oplus \cdots \oplus V_\kappa$.

Example B.2. Let $\mathbb{G} := \mathbb{H}^1 \equiv \mathbb{R}^3$ be the first Heisenberg group, with variables (x, y, t). Set $X := \partial_x + 2y\partial_t$, $Y := \partial_y - 2x\partial_t$, $T := \partial_t$. We have $X^\natural = dx$, $Y^\natural = dy$, $T^\natural = \theta$ (the contact form of $\mathbb{H}^1$). The stratification of the algebra $\mathfrak{g}$ is given by $\mathfrak{g} = V_1 \oplus V_2$, where $V_1 = \operatorname{span}\{X, Y\}$ and $V_2 = \operatorname{span}\{T\}$. In this case

$$\begin{aligned} E_0^1 &= \operatorname{span}\{dx, dy\};\\ E_0^2 &= \operatorname{span}\{dx \wedge \theta, dy \wedge \theta\};\\ E_0^3 &= \operatorname{span}\{dx \wedge dy \wedge \theta\}. \end{aligned}$$

Moreover

$$\begin{aligned} &d_c(\alpha_1 dx + \alpha_2 dy)\\ &\quad = \Pi_{E_0} d\left(\alpha_1\, dx + \alpha_2\, dy - \frac{1}{4}(X\alpha_2 - Y\alpha_1)\theta\right)\\ &\quad = D(\alpha_1\, dx + \alpha_2\, dy), \end{aligned}$$

where D is the second order differential of horizontal 1-forms in $\mathbb{H}^1$ that has the form

$$\begin{aligned} &D(\alpha_1\, dx + \alpha_2\, dy)\\ &\quad = -\frac{1}{4}\big(X^2\alpha_2 - 2XY\alpha_1 + YX\alpha_1\big)\, dx \wedge \theta - \frac{1}{4}\big(2YX\alpha_2 - Y^2\alpha_1 - XY\alpha_2\big)\, dy \wedge \theta\\ &\quad := P_1(\alpha_1, \alpha_2)\, dx \wedge \theta + P_2(\alpha_1, \alpha_2)\, dy \wedge \theta. \end{aligned}$$

On the other hand, if

$$\alpha = +\alpha_{13}\, dx \wedge \theta + \alpha_{23}\, dy \wedge \theta \in E_0^2,$$

then

$$d_c\alpha = (X\alpha_{23} - Y\alpha_{13})\, dx \wedge dy \wedge \theta.$$

Example B.3. Let $\mathbb{G} := \mathbb{H}^1 \times \mathbb{R}$, and denote by (x, y, t) the variables in $\mathbb{H}^1$ and by s the variable in $\mathbb{R}$. Set X, Y, T as above, and $S := \partial_s$. We have $X^\natural = dx$, $Y^\natural = dy$, $S^\natural = ds$, $T^\natural = \theta$. The stratification of the algebra $\mathfrak{g}$ is given by $\mathfrak{g} = V_1 \oplus V_2$, where $V_1 = \operatorname{span}\{X, Y, S\}$ and $V_2 = \operatorname{span}\{T\}$. In this case

$$\begin{aligned}
E_0^1 &= \operatorname{span}\{dx, dy, ds\};\\
E_0^2 &= \operatorname{span}\{dx \wedge ds, dy \wedge ds, dx \wedge \theta, dy \wedge \theta\};\\
E_0^3 &= \operatorname{span}\{dx \wedge dy \wedge \theta, dx \wedge ds \wedge \theta, dy \wedge ds \wedge \theta\}.
\end{aligned}$$

Moreover

$$\begin{aligned}
&d_c(\alpha_1\, dx + \alpha_2\, dy + \alpha_3\, ds)\\
&\quad = D(\alpha_1\, dx + \alpha_2\, dy) + (X\alpha_3 - S\alpha_1)\, dx \wedge ds + (Y\alpha_3 - S\alpha_2)\, dy \wedge ds,
\end{aligned}$$

where D is the second order differential of horizontal 1-forms in $\mathbb{H}^1$ that has the form $D(\alpha_1\, dx + \alpha_2\, dy) = P_1(\alpha_1, \alpha_2)\, dx \wedge \theta + P_2(\alpha_1, \alpha_2)\, dy \wedge \theta$.

On the other hand, if

$$\alpha = \alpha_{13}\, dx \wedge ds + \alpha_{23}\, dy \wedge ds + \alpha_{14}\, dx \wedge \theta + \alpha_{24}\, dy \wedge \theta \in E_0^2,$$

then

$$\begin{aligned}
d_c\alpha = {} & (X\alpha_{24} - Y\alpha_{14})\, dx \wedge dy \wedge \theta\\
& + \left(T\alpha_{13} - S\alpha_{14} - \frac{1}{4}\big(X^2\alpha_{23} - XY\alpha_{13}\big)\right) dx \wedge ds \wedge \theta\\
& + \left(T\alpha_{23} - S\alpha_{24} - \frac{1}{4}\big(YX\alpha_{23} - Y^2\alpha_{13}\big)\right) dy \wedge ds \wedge \theta.
\end{aligned}$$

Example B.4. Let now $\mathbb{G} := \mathbb{H}^2 \times \mathbb{R}$, and denote by (x_1, x_2, y_1, y_2, t) the variables in $\mathbb{H}^2$ and by s the variable in $\mathbb{R}$. Set $X_i := \partial_{x_i} + 2y_i\partial_t$, $Y_i := \partial_{x_i} - 2x_i\partial_t$, $i = 1, 2$, $T := \partial_t$, and $S := \partial_s$. We have $X_i^\natural = dx_i$, $Y_i^\natural = dy_i$, $i = 1, 2$, $S^\natural = ds$, $T^\natural = \theta$ (the contact form of $\mathbb{H}^2$). The stratification of the algebra $\mathfrak{g}$ is given by $\mathfrak{g} = V_1 \oplus V_2$, where $V_1 = \operatorname{span}\{X_1, X_2, Y_1, Y_2, S\}$ and $V_2 = \operatorname{span}\{T\}$.

Let us restrict ourselves to show the structure of the intrinsic differential on E_0^1, i.e. on horizontal 1-forms. Using the notations of (10), we can chose an orthonormal basis of $\bigwedge^h \mathfrak{g}$, $h = 1, 2, 3$ as follows:

h = 1: $\Theta^{1,1} = (\theta_1^1, \dots, \theta_5^1) = (dx_1, dx_2, dy_1, dy_2, ds)$, and $\Theta^{1,2} = (\theta_6^1) = (\theta)$.
h = 2: $\Theta^{2,2} = (\theta_1^2, \dots, \theta_{10}^2) = (dx_1 \wedge dx_2, dy_1 \wedge dy_2, dx_1 \wedge dy_1, dx_1 \wedge dy_2, dx_2 \wedge dy_1, dx_2 \wedge dy_2, dx_1 \wedge ds, dx_2 \wedge ds, dy_1 \wedge ds, dy_2 \wedge ds)$, $\Theta^{2,3} = (\theta_{11}^2, \dots, \theta_{15}^2) = (dx_1 \wedge \theta, dx_2 \wedge \theta, dy_1 \wedge \theta, dy_2 \wedge \theta, ds \wedge \theta)$.
h = 3: $\Theta^{3,3} = (\theta_1^3, \dots, \theta_{10}^3) = (dx_1 \wedge dx_2 \wedge dy_1, dx_1 \wedge dx_2 \wedge dy_2, dx_1 \wedge dx_2 \wedge ds, dx_1 \wedge dy_1 \wedge dy_2, dx_1 \wedge dy_1 \wedge ds, dx_2 \wedge dy_1 \wedge dy_2, dy_1 \wedge dy_2 \wedge ds, dx_1 \wedge dy_2 \wedge ds, dx_2 \wedge dy_2 \wedge ds, dy_1 \wedge dy_2 \wedge ds)$, $\Theta^{3,4} = (\theta_{11}^3, \dots, \theta_{20}^3) = (dx_1 \wedge dx_2 \wedge \theta, dy_1 \wedge dy_2 \wedge \theta, dx_1 \wedge dy_1 \wedge \theta, dx_1 \wedge dy_2 \wedge \theta, dx_2 \wedge dy_1 \wedge \theta, dx_2 \wedge dy_2 \wedge \theta, dx_1 \wedge ds \wedge \theta, dx_2 \wedge ds \wedge \theta, dy_1 \wedge ds \wedge \theta, dy_2 \wedge ds \wedge \theta)$.

We have:

$$d_0\theta_i^1 = 0 \quad \text{when } i = 1, \dots, 5, \qquad d_0\theta_6^1 = 4\big(\theta_3^2 + \theta_6^2\big);$$
$$d_0\theta_i^2 = 0 \quad \text{when } i = 1, \dots, 10, \qquad d_0\theta_{11}^2 = 4\theta_2^3, \qquad d_0\theta_{12}^2 = -4\theta_1^3,$$
$$d_0\theta_{13}^2 = -4\theta_6^3, \qquad d_0\theta_{14}^2 = 4\theta_4^3, \qquad d_0\theta_{15}^2 = 4\big(\theta_5^3 + \theta_{10}^3\big).$$

As usual, E_0^1 is the space of left invariant horizontal 1-forms, and an orthonormal basis of E_0^1 is given by $\{dx_1, dx_2, dy_1, dy_2, ds\}$. In addition, the left invariant form $\alpha = \sum_j \alpha_j \theta_j^2$ belongs to E_0^2 if and only if

$$\alpha_6 = -\alpha_3$$

and

$$\alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{15} = 0.$$

Hence an orthonormal basis of E_0^2 is given by $\{\xi_1^2, \xi_2^2, \frac{1}{\sqrt{2}}(\xi_3^2 - \xi_6^2), \xi_4^2, \xi_5^2, \xi_7^2, \xi_8^2, \xi_9^2, \xi_{10}^2\} = \{dx_1 \wedge dx_2, dy_1 \wedge dy_2, \frac{1}{\sqrt{2}}(dx_1 \wedge dy_1 - dx_2 \wedge dy_2), dx_1 \wedge dy_2, dx_2 \wedge dy_1, dx_1 \wedge ds, dx_2 \wedge ds, dy_1 \wedge ds, dy_2 \wedge ds\}$. In particular, the orthogonal projection $\Pi_{E_0}\alpha$ of α on E_0 has the form

$$\Pi_{E_0}\alpha = \sum_{\substack{j=1 \\ j\neq 3,6}}^{10} \alpha_j \xi_j^2 + \frac{\alpha_3 - \alpha_6}{2}\big(\xi_3^2 - \xi_6^2\big). \tag{50}$$

We want now to write explicitly d_c acting on forms $\alpha = \alpha(x) = \sum_{j=1}^5 \alpha_j(x)\xi_j^1$. To this end, let us write first $\Pi_{E^1}\alpha$. Because of the structure of $\bigwedge^1 \mathfrak{g}$, by Proposition 2.17,

$$\Pi_{E^1}\alpha = \alpha + \gamma\theta,$$

for a smooth function γ, with $\gamma\theta = -d_0^{-1}(d_1\alpha)$, i.e.

$$d_0(\gamma\theta) + d_1\alpha \in \ker\delta_0, \tag{51}$$

by Definition 2.9. We can write (51) in the form

$$
\begin{aligned}
&4\gamma(dx_1\wedge dy_1+dx_2\wedge dy_2)\\
&\quad+(X_1\alpha_2-X_2\alpha_1)\,dx_1\wedge dx_2+(Y_1\alpha_4-Y_2\alpha_3)\,dy_1\wedge dy_2\\
&\quad+(X_1\alpha_3-Y_1\alpha_1)\,dx_1\wedge dy_1+(X_1\alpha_4-Y_2\alpha_1)\,dx_1\wedge dy_2\\
&\quad+(X_2\alpha_3-Y_1\alpha_2)\,dx_2\wedge dy_1+(X_2\alpha_4-Y_2\alpha_2)\,dx_2\wedge dy_2\\
&\quad+(X_1\alpha_5-S\alpha_1)\,dx_1\wedge ds+(X_2\alpha_5-S\alpha_2)\,dx_2\wedge ds\\
&\quad+(Y_1\alpha_5-S\alpha_3)\,dy_1\wedge ds+(Y_2\alpha_5-S\alpha_4)\,dy_2\wedge ds\in\ker\delta_0.
\end{aligned}\tag{52}
$$

Because of the form of ${}^tM_1$ above, this gives

$$8\gamma+X_1\alpha_3-Y_1\alpha_1+X_2\alpha_4-Y_2\alpha_2=0,$$

i.e.

$$\gamma=-\frac{1}{8}(X_1\alpha_3-Y_1\alpha_1+X_2\alpha_4-Y_2\alpha_2).$$

However, the explicit form of γ does not matter in the final expression of $d_c\alpha$. Indeed, keeping in mind that $d_0\alpha=0$, and that $\Pi_{E_0}(d_1(\gamma\theta))=\Pi_{E_0}(d\gamma\wedge\theta)=0$, and $\Pi_{E_0}(d_2(\alpha+\gamma\theta))=0$, since Π_{E_0} vanishes on forms of weight 3, by our previous computation (52), we have

$$
\begin{aligned}
d_c\alpha&=\Pi_{E_0}\big(d(\alpha+\gamma\theta)\big)\\
&=\Pi_{E_0}\big(d_0(\alpha+\gamma\theta)+d_1(\alpha+\gamma\theta)\big)+\Pi_{E_0}\big(d_2(\alpha+\gamma\theta)\big)\\
&=\Pi_{E_0}\big(d_0(\gamma\theta)+d_1\alpha\big)\\
&=\Pi_{E_0}\big((X_1\alpha_2-X_2\alpha_1)\,dx_1\wedge dx_2+(Y_1\alpha_4-Y_2\alpha_3)\,dy_1\wedge dy_2\\
&\quad+(X_1\alpha_3-Y_1\alpha_1+4\gamma)\,dx_1\wedge dy_1+(X_1\alpha_4-Y_2\alpha_1)\,dx_1\wedge dy_2\\
&\quad+(X_2\alpha_3-Y_1\alpha_2)\,dx_2\wedge dy_1+(X_2\alpha_4-Y_2\alpha_2+4\gamma)\,dx_2\wedge dy_2\\
&\quad+(X_1\alpha_5-S\alpha_1)\,dx_1\wedge ds+(X_2\alpha_5-S\alpha_2)\,dx_2\wedge ds\\
&\quad+(Y_1\alpha_5-S\alpha_3)\,dy_1\wedge ds+(Y_2\alpha_5-S\alpha_4)dy_2\wedge ds\big)\\
&=(X_1\alpha_2-X_2\alpha_1)\,dx_1\wedge dx_2+(Y_1\alpha_4-Y_2\alpha_3)\,dy_1\wedge dy_2\\
&\quad+(X_1\alpha_4-Y_2\alpha_1)\,dx_1\wedge dy_2+(X_2\alpha_3-Y_1\alpha_2)\,dx_2\wedge dy_1\\
&\quad+(X_1\alpha_5-S\alpha_1)\,dx_1\wedge ds+(X_2\alpha_5-S\alpha_2)\,dx_2\wedge ds\\
&\quad+(Y_1\alpha_5-S\alpha_3)\,dy_1\wedge ds+(Y_2\alpha_5-S\alpha_4)\,dy_2\wedge ds\\
&\quad+\frac{X_1\alpha_3-Y_1\alpha_1-X_2\alpha_4+Y_2\alpha_2}{\sqrt{2}}\frac{1}{\sqrt{2}}(dx_1\wedge dy_1-dx_2\wedge dy_2),
\end{aligned}
$$

by (50).

Example B.5. Let $\mathbb{G} \equiv \mathbb{R}^6$ be the Carnot group associated with the vector fields

$$X_1 = \partial_1,$$
$$X_2 = \partial_2 + x_1\partial_4,$$
$$X_3 = \partial_3 + x_2\partial_5 + x_4\partial_6$$

and

$$X_4 = \partial_4,$$
$$X_5 = \partial_5 + x_1\partial_6,$$
$$X_6 = \partial_6.$$

Only non-trivial commutation rules are

$$[X_1, X_2] = X_4, \qquad [X_2, X_3] = X_5, \qquad [X_1, X_5] = X_6, \qquad [X_4, X_3] = X_6.$$

The X_j's are left invariant and coincide with the elements of the canonical basis of $\mathbb{R}^6$ at the origin. The Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ admits the stratification

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3,$$

where $\mathfrak{g}_1 = \operatorname{span}\{X_1, X_2, X_3\}$, $\mathfrak{g}_2 = \operatorname{span}\{X_4, X_5\}$, and $\mathfrak{g}_3 = \operatorname{span}\{X_6\}$. We set also

$$\theta_5 = dx_5 - x_2\,dx_3,$$
$$\theta_4 = dx_4 - x_1\,dx_2,$$
$$\theta_6 = dx_6 - x_1\,dx_5 + (x_1x_2 - x_4)\,dx_3$$

and

$$\theta_1 = dx_1, \qquad \theta_2 = dx_2, \qquad \theta_3 = dx_3.$$

Clearly

$$\theta_i = X_i^\natural \quad \text{for } i, j = 1, \dots, 6.$$

Moreover

$$d\theta_4 = -\theta_1 \wedge \theta_2, \qquad d\theta_5 = -\theta_2 \wedge \theta_3, \qquad d\theta_6 = \theta_3 \wedge \theta_4 - \theta_1 \wedge \theta_5.$$

As in Example B.4, let us restrict ourselves to show the structure of the intrinsic differential on E_0^1, i.e. on horizontal 1-forms. Using the notations of (10), we can chose an orthonormal basis of $\bigwedge^h \mathfrak{g}$, $h = 1, 2, 3$ as follows:

h = 1: $\Theta^{1,1} = \{\theta_1, \theta_2, \theta_3\}$, $\Theta^{1,2} = \{\theta_4, \theta_5\}$, and $\Theta^{1,3} = \{\theta_6\}$.

h = 2: $\Theta^{2,2} = \{\theta_1^2, \theta_2^2, \theta_3^2\} = \{\theta_1 \wedge \theta_2, \theta_1 \wedge \theta_3, \theta_2 \wedge \theta_3\}$, $\Theta^{2,3} = \{\theta_4^2, \dots, \theta_9^2\} = \{\theta_1 \wedge \theta_4, \theta_1 \wedge \theta_5, \theta_2 \wedge \theta_4, \theta_2 \wedge \theta_5, \theta_3 \wedge \theta_4, \theta_3 \wedge \theta_5\}$, $\Theta^{2,4} = \{\theta_{10}^2, \dots, \theta_{13}^2\} = \{\theta_1 \wedge \theta_6, \theta_2 \wedge \theta_6, \theta_3 \wedge \theta_6, \theta_4 \wedge \theta_5\}$, $\Theta^{2,5} = \{\theta_{14}^2, \theta_{15}^2\} = \{\theta_4 \wedge \theta_6, \theta_5 \wedge \theta_6\}$.

h = 3: $\Theta^{3,3} = \{\theta_1^3\} = \{\theta_1 \wedge \theta_2 \wedge \theta_3\}$, $\Theta^{3,4} = \{\theta_2^3, \dots, \theta_7^3\} = \{\theta_1 \wedge \theta_2 \wedge \theta_4, \theta_1 \wedge \theta_2 \wedge \theta_5, \theta_1 \wedge \theta_3 \wedge \theta_4, \theta_1 \wedge \theta_3 \wedge \theta_5, \theta_2 \wedge \theta_3 \wedge \theta_4, \theta_2 \wedge \theta_3 \wedge \theta_5\}$, $\Theta^{3,5} = \{\theta_8^3, \dots, \theta_{13}^3\} = \{\theta_1 \wedge \theta_2 \wedge \theta_6, \theta_1 \wedge \theta_3 \wedge \theta_6, \theta_2 \wedge \theta_3 \wedge \theta_6, \theta_1 \wedge \theta_4 \wedge \theta_5, \theta_2 \wedge \theta_4 \wedge \theta_5, \theta_3 \wedge \theta_4 \wedge \theta_5\}$, $\Theta^{3,6} = \{\theta_{14}^3, \dots, \theta_{19}^3\} = \{\theta_1 \wedge \theta_4 \wedge \theta_6, \theta_1 \wedge \theta_5 \wedge \theta_6, \theta_2 \wedge \theta_4 \wedge \theta_6, \theta_2 \wedge \theta_5 \wedge \theta_6, \theta_3 \wedge \theta_4 \wedge \theta_6, \theta_3 \wedge \theta_5 \wedge \theta_6\}$, $\Theta^{3,7} = \{\theta_{20}^3\} = \{\theta_4 \wedge \theta_5 \wedge \theta_6\}$.

We notice that an orthonormal basis of $\bigwedge^h \mathfrak{g}$, $h = 4, 5, 6$ can be obtained by Hodge duality.

As usual, E_0^1 is the space of left invariant horizontal 1-forms, i.e. an orthonormal basis of E_0^1 is given by $\{\theta_1, \theta_2, \theta_3\}$. In addition, the left invariant form $\alpha = \sum_j \alpha_j \theta_j^2$ belongs to E_0^2 if and only if

$$\alpha_5 = -\alpha_8, \quad \alpha_{10} = \alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{15} = 0$$

and

$$\alpha_5 = \alpha_8, \quad \alpha_3 = \alpha_1 = 0.$$

Therefore, an orthonormal basis $\{\xi_1^2, \dots, \xi_5^2\}$ of $E_0^2 = E_0^{2,2} \oplus E_0^{2,3}$ is given by

$$\{\theta_1 \wedge \theta_3\} \cup \{\theta_1 \wedge \theta_4, \theta_2 \wedge \theta_4, \theta_2 \wedge \theta_5, \theta_3 \wedge \theta_5\}.$$

In particular, the orthogonal projection $\Pi_{E_0}\alpha$ of $\alpha \in \bigwedge^2 \mathfrak{g}$ on E_0^2 has the form

$$\Pi_{E_0}\alpha = \alpha_2 \theta_1 \wedge \theta_3 + \alpha_4 \theta_1 \wedge \theta_4 + \alpha_6 \theta_2 \wedge \theta_4 + \alpha_7 \theta_2 \wedge \theta_5 + \alpha_9 \theta_3 \wedge \theta_5. \tag{53}$$

We want now to write explicitly d_c acting on forms $\alpha = \alpha(x) = \sum_{j=1}^3 \alpha_j(x)\theta_j$. To this end, let us write first $\Pi_{E^1}\alpha$. We have

$$\begin{aligned} \Pi_{E^1}\alpha &= (\Pi_{E^1}\alpha)_1 + (\Pi_{E^1}\alpha)_2 + (\Pi_{E^1}\alpha)_3 \\ &= \alpha + (\Pi_{E^1}\alpha)_2 + (\Pi_{E^1}\alpha)_3 \\ &:= \alpha + (\gamma_4\theta_4 + \gamma_5\theta_5) + \gamma_6\theta_6, \end{aligned}$$

with

$$\begin{aligned} \gamma_4\theta_4 + \gamma_5\theta_5 &= -d_0^{-1}\big(d_1(\alpha_1\theta_1 + \alpha_2\theta_2 + \alpha_3\theta_3)\big) \\ &= -d_0^{-1}\big((X_1\alpha_2 - X_2\alpha_1)\theta_1 \wedge \theta_2 + (X_1\alpha_3 - X_3\alpha_1)\theta_1 \wedge \theta_3 \\ &\quad + (X_2\alpha_3 - X_3\alpha_2)\theta_2 \wedge \theta_3\big), \end{aligned} \tag{54}$$

and

$$\gamma_6\theta_6 = -d_0^{-1}\big(d_1(\gamma_4\theta_4 + \gamma_5\theta_5) + d_2\alpha\big). \tag{55}$$

Now (54) is equivalent to

$$d_0(\gamma_4\theta_4+\gamma_5\theta_5)+(X_1\alpha_2-X_2\alpha_1)\theta_1\wedge\theta_2+(X_1\alpha_3-X_3\alpha_1)\theta_1\wedge\theta_3 \\ +(X_2\alpha_3-X_3\alpha_2)\theta_2\wedge\theta_3\in\ker{}^tM_1, \tag{56}$$

i.e.

$$(-\gamma_4+X_1\alpha_2-X_2\alpha_1)\theta_1\wedge\theta_2+(X_1\alpha_3-X_3\alpha_1)\theta_1\wedge\theta_3 \\ +(-\gamma_5+X_2\alpha_3-X_3\alpha_2)\theta_2\wedge\theta_3\in\ker{}^tM_1, \tag{57}$$

that gives eventually

$$\gamma_4=X_1\alpha_2-X_2\alpha_1\quad\text{and}\quad\gamma_5=X_2\alpha_3-X_3\alpha_2.$$

Consider now (55), that is equivalent to

$$\begin{aligned}
&d_0(\gamma_6\theta_6)+d_1\big((X_1\alpha_2-X_2\alpha_1)\theta_4+(X_2\alpha_3-X_3\alpha_2)\theta_5+d_2\alpha\big)\\
&\quad=\gamma_6(\theta_3\wedge\theta_4-\theta_1\wedge\theta_5)+X_1(X_1\alpha_2-X_2\alpha_1)\theta_1\wedge\theta_4\\
&\qquad+X_2(X_1\alpha_2-X_2\alpha_1)\theta_2\wedge\theta_4\\
&\qquad+X_3(X_1\alpha_2-X_2\alpha_1)\theta_3\wedge\theta_4+X_1(X_2\alpha_3-X_3\alpha_2)\theta_1\wedge\theta_5\\
&\qquad+X_2(X_2\alpha_3-X_3\alpha_2)\theta_2\wedge\theta_5\\
&\qquad+X_3(X_2\alpha_3-X_3\alpha_2)\theta_3\wedge\theta_5-X_4\alpha_1\theta_1\wedge\theta_4\\
&\qquad-X_4\alpha_2\theta_2\wedge\theta_4-X_4\alpha_3\theta_3\wedge\theta_4-X_5\alpha_1\theta_1\wedge\theta_5\\
&\qquad-X_5\alpha_2\theta_2\wedge\theta_5-X_5\alpha_3\theta_3\wedge\theta_5\\
&\quad=X_1(X_1\alpha_2-X_2\alpha_1)\theta_1\wedge\theta_4+X_2(X_1\alpha_2-X_2\alpha_1)\theta_2\wedge\theta_4\\
&\qquad+\big(X_3(X_1\alpha_2-X_2\alpha_1)+\gamma_6\big)\theta_3\wedge\theta_4+\big(X_1(X_2\alpha_3-X_3\alpha_2)-\gamma_6\big)\theta_1\wedge\theta_5\\
&\qquad+X_2(X_2\alpha_3-X_3\alpha_2)\theta_2\wedge\theta_5\\
&\qquad+X_3(X_2\alpha_3-X_3\alpha_2)\theta_3\wedge\theta_5-X_4\alpha_1\theta_1\wedge\theta_4-X_4\alpha_2\theta_2\wedge\theta_4\\
&\qquad-X_4\alpha_3\theta_3\wedge\theta_4-X_5\alpha_1\theta_1\wedge\theta_5\\
&\qquad-X_5\alpha_2\theta_2\wedge\theta_5-X_5\alpha_3\theta_3\wedge\theta_5\\
&\quad=\big(X_1(X_1\alpha_2-X_2\alpha_1)-X_4\alpha_1\big)\theta_4^2+\big(X_1(X_2\alpha_3-X_3\alpha_2)-\gamma_6-X_5\alpha_1\big)\theta_5^2\\
&\qquad+\big(X_2(X_1\alpha_2-X_2\alpha_1)-X_4\alpha_2\big)\theta_6^2+\big(X_2(X_2\alpha_3-X_3\alpha_2)-X_5\alpha_2\big)\theta_7^2\\
&\qquad+\big(X_3(X_1\alpha_2-X_2\alpha_1)+\gamma_6-X_4\alpha_3\big)\theta_8^2+\big(X_3(X_2\alpha_3-X_3\alpha_2)-X_5\alpha_3\big)\theta_9^2\in\ker{}^tM_1,
\end{aligned}$$

i.e. to

$$X_1(X_2\alpha_3-X_3\alpha_2)-\gamma_6-X_5\alpha_1-\big(X_3(X_1\alpha_2-X_2\alpha_1)+\gamma_6-X_4\alpha_3\big)=0.$$

This yields

$$\gamma_6 = \frac{1}{2}\big(X_1(X_2\alpha_3 - X_3\alpha_2) - X_5\alpha_1 - X_3(X_1\alpha_2 - X_2\alpha_1) + X_4\alpha_3\big).$$

Thus

$$\begin{aligned}\Pi_{E^1}\alpha = {} & \alpha_1\theta_1 + \alpha_2\theta_2 + \alpha_3\theta_3 \\ & + (X_1\alpha_2 - X_2\alpha_1)\theta_4 + (X_2\alpha_3 - X_3\alpha_2)\theta_5 \\ & + \frac{1}{2}\big(X_1(X_2\alpha_3 - X_3\alpha_2) - X_5\alpha_1 - X_3(X_1\alpha_2 - X_2\alpha_1) + X_4\alpha_3\big)\theta_6.\end{aligned}$$

Then

$$\begin{aligned}d_c\alpha = {} & (X_1\alpha_3 - X_3\alpha_1)\theta_1\wedge\theta_3 + \big(X_1(X_1\alpha_2 - X_2\alpha_1) - X_4\alpha_1\big)\theta_1\wedge\theta_4 \\ & + \big(X_2(X_1\alpha_2 - X_2\alpha_1) - X_4\alpha_2\big)\theta_2\wedge\theta_4 \\ & + \big(X_2(X_2\alpha_3 - X_3\alpha_2) - X_5\alpha_2\big)\theta_2\wedge\theta_5 \\ & + \big(X_3(X_2\alpha_3 - X_3\alpha_2) - X_5\alpha_3\big)\theta_3\wedge\theta_5.\end{aligned}$$

Example B.6. Let $\mathbb{G} = (\mathbb{R}^4, \cdot)$ be the Carnot group whose Lie algebra is $\mathfrak{g} = V_1 \oplus V_2 \oplus V_3$ with $V_1 = \mathrm{span}\{X_1, X_2\}$, $V_2 = \mathrm{span}\{X_3\}$, and $V_3 = \mathrm{span}\{X_4\}$, the only non-zero commutation relations being

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4.$$

The group $\mathbb{G}$ is called Engel group. In exponential coordinates an explicit representation of the vector fields is

$$X_1 = \partial_1 - \frac{x_2}{2}\partial_3 - \left(\frac{x_3}{2} + \frac{x_1x_2}{12}\right)\partial_4, \qquad X_2 = \partial_2 + \frac{x_1}{2}\partial_3 + \frac{x_1^2}{12}\partial_4,$$
$$X_3 = \partial_3 + \frac{x_1}{2}\partial_4, \qquad X_4 = \partial_4.$$

Denote by $\theta_1, \dots, \theta_4$ the dual left invariant forms. The following result is proved in [24]: as in Remark B.1, an orthonormal basis of E_0^1 is given by $\{\theta_1, \theta_2\}$; an orthonormal basis of $E_0^2 = E_0^{2,3} \oplus E_0^{2,4}$ is given by $\{\theta_2 \wedge \theta_3\} \cup \{\theta_1 \wedge \theta_4\}$. Moreover, bases of E_0^3, E_0^4 can be written by Hodge duality.

If $\alpha = \alpha_1\theta_1 + \alpha_2\theta_2 \in E_0^1$, then

$$\begin{aligned}d_c\alpha = {} & \big(X_2(X_1\alpha_2 - X_2\alpha_1) - X_3\alpha_2\big)\theta_2\wedge\theta_3 \\ & + \big(X_1\big(X_1^2\alpha_2 - (X_1X_2 + X_3)\alpha_1\big) - X_4\alpha_1\big)\theta_1\wedge\theta_4.\end{aligned}$$

Example B.7. Let us consider now the free group $\mathbb{G}$ of step 3 with 2 generators, i.e. the Carnot group whose Lie algebra is $\mathfrak{g} = V_1 \oplus V_2 \oplus V_3$ with $V_1 = \operatorname{span}\{X_1, X_2\}$, $V_2 = \operatorname{span}\{X_3\}$, and $V_3 = \operatorname{span}\{X_4, X_5\}$, the only non-zero commutation relations being

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_2, X_3] = X_5.$$

In exponential coordinates, the group $\mathbb{G}$ can be identified with $\mathbb{R}^5$, and an explicit representation of the vector fields is

$$X_1 = \partial_1, \qquad X_2 = \partial_2 + x_1\partial_3 + \frac{x_1^2}{2}\partial_4 + x_1x_2\partial_5,$$
$$X_3 = \partial_3 + x_1\partial_4 + x_2\partial_5, \qquad X_4 = \partial_4, \qquad X_5 = \partial_5.$$

Denote by $\theta_1, \dots, \theta_5$ the dual left invariant forms. As in Remark B.1, an orthonormal basis of E_0^1 is given by $\{\theta_1, \theta_2\}$.

We have $d\theta_1 = d\theta_2 = 0$ and

$$d\theta_3 = -\theta_1 \wedge \theta_2, \qquad d\theta_4 = -\theta_1 \wedge \theta_3, \qquad d\theta_5 = -\theta_2 \wedge \theta_3.$$

Using the notations of (10), we can chose an orthonormal basis of $\bigwedge^h \mathfrak{g}$, $h = 1, 2, 3$ as follows (notice that an orthonormal basis of $\bigwedge^h \mathfrak{g}$, $h = 4, 5$ can be obtained by Hodge duality):

h = 1: $\Theta^{1,1} = \{\theta_1, \theta_2\}$, $\Theta^{1,2} = \{\theta_3\}$, and $\Theta^{1,3} = \{\theta_4, \theta_5\}$.
h = 2: $\Theta^{2,2} = \{\theta_1^2\} = \{\theta_1 \wedge \theta_2\}$, $\Theta^{2,3} = \{\theta_4^2, \theta_5^2\} = \{\theta_1 \wedge \theta_3, \theta_2 \wedge \theta_3\}$, $\Theta^{2,4} = \{\theta_4^2, \dots, \theta_7^2\} = \{\theta_1 \wedge \theta_4, \theta_1 \wedge \theta_5, \theta_2 \wedge \theta_4, \theta_2 \wedge \theta_5\}$, $\Theta^{2,5} = \{\theta_8^2, \theta_9^2\} = \{\theta_3 \wedge \theta_4, \theta_3 \wedge \theta_5\}$, $\Theta^{2,6} = \{\theta_{10}^2\} = \{\theta_4 \wedge \theta_5\}$.
h = 3: $\Theta^{3,4} = \{\theta_1^3\} = \{\theta_1 \wedge \theta_2 \wedge \theta_3\}$, $\Theta^{3,5} = \{\theta_2^3, \theta_3^3\} = \{\theta_1 \wedge \theta_2 \wedge \theta_4, \theta_1 \wedge \theta_2 \wedge \theta_5\}$, $\Theta^{3,6} = \{\theta_4^3, \dots, \theta_7^3\} = \{\theta_1 \wedge \theta_3 \wedge \theta_4, \theta_1 \wedge \theta_3 \wedge \theta_5, \theta_2 \wedge \theta_3 \wedge \theta_4, \theta_2 \wedge \theta_3 \wedge \theta_5\}$, $\Theta^{3,7} = \{\theta_8^3, \theta_9^3\} = \{\theta_1 \wedge \theta_4 \wedge \theta_5, \theta_2 \wedge \theta_4 \wedge \theta_5\}$, $\Theta^{3,8} = \{\theta_{10}^3\} = \{\theta_3 \wedge \theta_4 \wedge \theta_5\}$.

Thus, $\alpha = \alpha_1\theta_1^2 + \cdots + \alpha_{10}\theta_{10}^2 \in E_0^2$ if and only if

$$\alpha_1 = \alpha_2 = \alpha_3 = 0$$

and

$$\alpha_5 = \alpha_6, \qquad \alpha_8 = \alpha_9 = \alpha_{10} = 0.$$

Therefore, an orthonormal basis of E_0^2 is given by

$$\left\{\theta_4^2, \frac{1}{\sqrt{2}}\big(\theta_5^2 + \theta_6^2\big), \theta_7^2\right\}.$$

We want to show how d_c acts on 1-forms $\alpha = \alpha_1\theta_1 + \alpha_2\theta_2 \in E_0^1$. To this end, let us write $\Pi_E\alpha = \alpha + \gamma_3\theta_3 + \gamma_4\theta_4 + \gamma_5\theta_5$. We apply Proposition 2.17. We get first

$$\gamma_3\theta_3 = -d_0^{-1}(d_1\alpha) = -d_0^{-1}\big((X_1\alpha_2 - X_2\alpha_1)\theta_1 \wedge \theta_2\big),$$

i.e.

$$-\gamma_3\theta_1 \wedge \theta_2 + (X_1\alpha_2 - X_2\alpha_1)\theta_1 \wedge \theta_2$$
$$= d_0(\gamma_3\theta_3) + (X_1\alpha_2 - X_2\alpha_1)\theta_1 \wedge \theta_2 \in \ker {}^tM_1.$$

Therefore

$$\gamma_3 = X_1\alpha_2 - X_2\alpha_1.$$

Analogously,

$$\gamma_4\theta_4 + \gamma_5\theta_5 = -d_0^{-1}\big(d_1(\gamma_3\theta_3) + d_2\alpha\big).$$

This gives

$$\gamma_4 = X_1^2\alpha_2 - X_1X_2\alpha_1 - X_3\alpha_1,$$
$$\gamma_5 = X_2X_1\alpha_2 - X_2^2\alpha_1 - X_3\alpha_2.$$

Eventually, we get

$$d_c\alpha = (X_1\gamma_4 - X_4\alpha_1)\theta_1 \wedge \theta_4 + (X_2\gamma_5 - X_5\alpha_2)\theta_2 \wedge \theta_5$$
$$+ \frac{1}{2}(X_1\gamma_5 - X_5\alpha_1 + X_2\gamma_4 - X_4\alpha_2)(\theta_1 \wedge \theta_5 + \theta_2 \wedge \theta_4).$$

References

[1] Annalisa Baldi, Bruno Franchi, Maria Carla Tesi, Fundamental solution and sharp L^p estimates for Laplace operators in the contact complex of Heisenberg groups, Ric. Mat. 55 (1) (2006) 119–144.

[2] Annalisa Baldi, Bruno Franchi, Maria Carla Tesi, Compensated compactness in the contact complex of Heisenberg groups, Indiana Univ. Math. J. 57 (2008) 133–186.

[3] Andrea Bonfiglioli, Ermanno Lanconelli, Francesco Uguzzoni, Stratified Lie Groups and Potential Theory for Their Sub-Laplacians, Springer Monogr. Math., Springer-Verlag, Berlin, 2007.

[4] Nicolas Bourbaki, Éléments de mathématique. XXVI. Groupes et algèbres de Lie, Chapitre 1 : Algèbres de Lie, Actualités Sci. Ind., vol. 1285, Hermann, Paris, 1960.

[5] Michael Christ, Daryl Geller, Paweł Głowacki, Larry Polin, Pseudodifferential operators on groups with dilations, Duke Math. J. 68 (1) (1992) 31–65.

[6] Philippe G. Ciarlet, The Finite Element Method for Elliptic Problems, Stud. Math. Appl., vol. 4, North-Holland Publishing Co., Amsterdam, 1978.

[7] Jean Dieudonné, Éléments d'analyse. Tome III, Chapitres XVI et XVII, Cahiers Scientifiques, Fasc. XXXIII, Gauthier-Villars Éditeur, Paris, 1970.

[8] Herbert Federer, Geometric Measure Theory, Grundlehren Math. Wiss., Band 153, Springer-Verlag New York Inc., New York, 1969.

[9] Gerald B. Folland, Subelliptic estimates and function spaces on nilpotent Lie groups, Ark. Mat. 13 (2) (1975) 161–207.

[10] Gerald B. Folland, Elias M. Stein, Hardy Spaces on Homogeneous Groups, Math. Notes, vol. 28, Princeton University Press, Princeton, NJ, 1982.

[11] Bruno Franchi, Raul Serapioni, Francesco Serra Cassano, On the structure of finite perimeter sets in step 2 Carnot groups, J. Geom. Anal. 13 (3) (2003) 421–466.

[12] Bruno Franchi, Nicoletta Tchou, Maria Carla Tesi, Div–curl type theorem, H-convergence and Stokes formula in the Heisenberg group, Commun. Contemp. Math. 8 (1) (2006) 67–99.

[13] Paweł Głowacki, The Rockland condition for nondifferential convolution operators. II, Studia Math. 98 (2) (1991) 99–114.

[14] Tadeusz Iwaniec, Adam Lutoborski, Integral estimates for null Lagrangians, Arch. Ration. Mech. Anal. 125 (1) (1993) 25–79.

[15] A.W. Knapp, E.M. Stein, Intertwining operators for semisimple groups, Ann. of Math. (2) 93 (1971) 489–578.

[16] Adam Korányi, Stephen Vági, Singular integrals on homogeneous spaces and some problems of classical analysis, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 25 (1972) 575–648, 1971.

[17] Vladimir G. Maz'ja, Sobolev Spaces, Springer Ser. Soviet Math., Springer-Verlag, Berlin, 1985. Translated from the Russian by T.O. Shaposhnikova.

[18] François Murat, Compacité par compensation, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 5 (3) (1978) 489–507.

[19] François Murat, Luc Tartar, H-convergence, in: Topics in the Mathematical Modelling of Composite Materials, in: Progr. Nonlinear Differential Equations Appl., vol. 31, Birkhäuser Boston, Boston, MA, 1997, pp. 21–43.

[20] Joel W. Robbin, Robert C. Rogers, Blake Temple, On weak continuity and the Hodge decomposition, Trans. Amer. Math. Soc. 303 (2) (1987) 609–618.

[21] Linda Preiss Rothschild, Elias M. Stein, Hypoelliptic differential operators and nilpotent groups, Acta Math. 137 (3–4) (1976) 247–320.

[22] Michel Rumin, Formes différentielles sur les variétés de contact, J. Differential Geom. 39 (2) (1994) 281–330.

[23] Michel Rumin, Differential geometry on C–C spaces and application to the Novikov–Shubin numbers of nilpotent Lie groups, C. R. Acad. Sci. Paris Sér. I Math. 329 (11) (1999) 985–990.

[24] Michel Rumin, Around heat decay on forms and relations of nilpotent Lie groups, Année 2000–2001, in: Sémin. Théor. Spectr. Géom., vol. 19, Univ. Grenoble I, Saint, 2001, pp. 123–164.

[25] Elias M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[26] Hans Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland Math. Library, vol. 18, North-Holland Publishing Co., Amsterdam, 1978.

[27] Nicholas Th. Varopoulos, Laurent Saloff-Coste, Thierry Coulhon, Analysis and Geometry on Groups, Cambridge Tracts in Math., vol. 100, Cambridge University Press, Cambridge, 1992.