TWO CONJECTURES ON EDGE-COLOURING

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Chetwynd and Hilton have elsewhere posed two conjectures, one a general statement on edge-colouring simple graphs $G$ with $\Delta(G) > \frac{1}{3}|V(G)|$, and a second to the effect that a regular simple graph $G$ with $d(G) \geq \frac{1}{3}|V(G)|$ is 1-factorizable. We set out the evidence for both these conjectures and show that the first implies the second.

1. Introduction

We are concerned here with simple graphs, that is finite graphs without loops or multiple edges. An edge-colouring of a graph $G$ is a map $\phi: E(G) \rightarrow \mathcal{C}$, where $\mathcal{C}$ is a set of colours, such that no two vertices with the same colour have a common vertex. The chromatic index $\chi'(G)$ is the least value of $|\mathcal{C}|$ for which an edge-colouring exists. A well-known theorem of Vizing [17] states that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1,$$

where $\Delta(G)$ denotes the maximum degree of $G$. If $\chi'(G) = \Delta(G)$, then $G$ is said to be Class 1, and otherwise $G$ is Class 2. The question of deciding whether or not a graph is Class 1 was shown by Holyer [14] to be NP-complete. However, for certain types of graph, the problem of classifying Class 2 graphs seems to be tractable.

If $G$ satisfies the inequality

$$|E(G)| > \Delta(G)\left\lceil \frac{1}{2} |V(G)| \right\rceil,$$

then $G$ is overfull. Clearly if $G$ is overfull, then $|V(G)|$ is odd. An overfull graph has to be Class 2, since no colour class of $G$ can have more than $\left\lceil \frac{1}{2} |V(G)| \right\rceil$ edges. In [6], Chetwynd and Hilton made the following conjecture (now slightly modified).

Conjecture 1. Let $G$ be a simple graph with $\Delta(G) > \frac{1}{3}|V(G)|$. Then $G$ is Class 2 if and only if $G$ contains an overfull subgraph $H$ with $\Delta(H) = \Delta(G)$.

The graph $G$ obtained from Petersen’s graph by removing one vertex is Class 2, but contains no subgraph $H$ with $\Delta(H) = \Delta(G)$; this shows that the figure $\frac{1}{3}$ in Conjecture 1 cannot be lowered.

At the time of writing, Conjecture 1 has been proved in a number of cases. Plantholt ([15, 16]) and Chetwynd and Hilton ([3–6]) have between them established the following.

**Theorem 1.** Conjecture 1 is true if $\Delta(G) \geq |V(G)| - 3$.

In [1], Chetwynd and Hilton posed the following conjecture about regular graphs of even order. First note that a Class 1 regular graph is often called 1-factorizable, as it is the union of edge-disjoint 1-factors. Also note that a regular graph of odd order is overfull, and so is Class 2. If a graph $G$ is regular, let $d(G)$ denote its degree.

**Conjecture 2.** Let $G$ be a regular simple graph of even order satisfying

\[ d(G) \geq \frac{1}{2} |V(G)|. \]

Then $G$ is 1-factorizable.

This conjecture seems to have been known however long before being posed by Chetwynd and myself. When I told Dirac of it, he said it was "going around" in the early 1950s. The figure $\frac{1}{2} |V(G)|$ in the conjecture cannot be lowered, as is shown by the example of a graph $G$ consisting of two $K_n$'s, when $n$ is odd.

Chetwynd and Hilton ([1, 7, 8]) have proved this conjecture in a number of special cases.

**Theorem 2.** Conjecture 2 is true if either

\[ d(G) \geq \frac{1}{2}(\sqrt{7} - 1) |V(G)| \]

or

\[ d(G) \geq |V(G)| - 4. \]

The object of this note is to prove the following theorem.

**Theorem 3.** If Conjecture 1 is true, then Conjecture 2 is true.

2. **Proof of Theorem 3**

Let $G$ be a regular graph with $|V(G)| = 2n$ and $d(G) \geq n$. Suppose that Conjecture 1 is true and that $G$ is Class 2.

Let $H$ be an overfull subgraph of $G$ with $\Delta(H) = d(G)$. Since $H$ is overfull, it follows that $|V(H)|$ is odd, so $H \neq G$. Let

\[ \text{def}(H) = \sum_{v \in V(H)} (d(G) - d_H(V)). \]
It is shown in [2] that, if $H$ is overfull, then
\[ \text{def}(H) \leq \Delta(H) - 2 = d(G) - 2. \]

It follows that $G$ has an edge-cut $S$ with $|S| \leq d(G) - 2$ such that $G \setminus S = H \cup J$, where $V(H) \cap V(J) = \emptyset$.

Since $\Delta(H) = d(G) \geq n$, it follows that $H$ has at least $n + 1$ vertices. Consequently $J$ has at most $n - 1$ vertices. Thus $d(G) + 1 > |V(J)|$. Since $G$ is regular, the number of edges joining vertices of $J$ to vertices of $H$ is at least $(d(G) - |V(J)| + 1)|V(J)|$. For fixed $d(G)$, $(d(G) - |V(J)| + 1)|V(J)|$ is a quadratic in $|V(J)|$. In the range $1 \leq |V(J)| \leq n - 1$, it has two minima, one at each end point, with values $d(G)$ and $(d(G) - n + 2)(n - 1)$. But $d(G) > |S|$, and $(d(G) - n + 2)(n - 1) \geq 2n - 2 \geq d(G) - 1 > |S|$, contradicting the definition of $S$. Thus $G$ has no overfull subgraph $H$, and so, by Conjecture 1, is Class 1, or in other words is 1-factorizable. Thus Conjecture 2 is true. This proves Theorem 3. \[
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3. A final remark

Conjecture 1 has many other implications. Some of these are discussed in [11–13] by Hilton and Johnson. A survey of the main implications is given in [10]. See also [9].

Note added in proof

A.G. Chetwynd and I have recently proved Conjecture 1 in the case when $\Delta G \geq \frac{1}{3}(\sqrt{21} - 1)(|V(G)| + 1) + 1$ and $|E(G)| = \Delta(G)\lfloor \frac{1}{2} |V(G)| \rfloor$. See [18].

References


