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# Generalized LIL for geometrically weighted random series in Banach spaces ☆

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ABSTRACT

Let $\{X, X_n;\ n \geqslant 0\}$ be a sequence of independent and identically distributed random variables, taking values in a separable Banach space $B$ with topological dual $B^*$. Considering the geometrically weighted series $\xi(\beta) = \sum_{n=0}^{\infty} \beta^n X_n$ for $0 < \beta < 1$, motivated by Einmahl and Li (2005, 2008), a general law of the iterated logarithm for $\xi(\beta)$ is established.



## 1. Introduction and main result

Suppose that $(B, \|\cdot\|)$ is a separable Banach space with topological dual $B^*$, and $(\Omega, \mathcal{U}, \mathrm{P})$ is a probability space. Let $\{X, X_n,\ n \geqslant 0\}$ be a sequence of $B$-valued random variables. As usual, write $S_n = \sum_{k=0}^{n} X_k$, $n \geqslant 0$ and $Lt = \log(t \vee e)$, $LLt = L(Lt)$, $t \geqslant 0$.

Consider a geometrically weighted series $\xi(\beta) = \sum_{n=0}^{\infty} \beta^n X_n$, $0 < \beta < 1$. When $B = \Re$, it is well known that if $\{X_n,\ n \geqslant 0\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with mean zero and variance one, Bovier and Picco [1] obtained a law of the iterated logarithm (LIL) as follows:

$$\lim_{\beta \nearrow 1} d\left(\gamma(\beta) \sum_{n=0}^{\infty} \beta^n X_n, [-1, 1]\right) = 0 \quad \text{a.s.}$$

and

$$C\left(\gamma(\beta) \sum_{n=0}^{\infty} \beta^n X_n\right) = [-1, 1] \quad \text{a.s.},$$

where $\gamma(\beta) = \sqrt{1-\beta^2}/\sqrt{2LL((1-\beta^2)^{-1})}$, $d(x, K)$ means the distance from the point $x$ to the set $K$, and $C(x_\beta)$ (resp. $C(x_n)$) denotes the set of limit points of $\{x_\beta\}$ as $\beta \nearrow 1$ (resp. of $\{x_n\}$ as $n \to \infty$). This result was also extended to stationary ergodic martingale difference sequence by Picco and Vares [5] with a very complicated method. Later, Zhang [6] studied the strong approximation theorems of $\xi(\beta)$ for general cases and proved the LIL for independent but not identically distributed

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cases and mixing dependent cases, respectively. Moreover, Zhang [7] extended it to the Banach space setting, and got the following theorem.

**Theorem A.** *Let $\{X, X_n;\ n \geqslant 0\}$ be a sequence of i.i.d. $B$-valued random variables with $X \in WM_0^2 := \{X\colon\ \mathsf{E}f(X) = 0,\ \mathsf{E}f^2(X) < \infty$ for all $f \in B^*\}$. Assume that*

$$\mathsf{E}\|X\|^2/LL\|X\|^2 < \infty,$$

*and*

$$\frac{S_n}{\sqrt{2nLLn}} \to 0 \quad \text{in probability}. \tag{1.1}$$

*Then we have*

$$\lim_{\beta \nearrow 1} \gamma(\beta) \left\| \sum_{n=0}^{\infty} \beta^n X_n \right\| = \sigma \quad \text{a.s.},$$

*where $\sigma^2 = \sup_{f \in B_1^*} \mathsf{E}f^2(X)$.*

Noting that Zhang [7] assumed that $\mathsf{E}f^2(X) < \infty$, it is interesting to consider the situation when $\mathsf{E}f^2(X)$ is infinite. In this paper, we aim to solve this problem by establishing a general LIL for $B$-valued geometrically weighted series, where $\mathsf{E}f^2(X)$ may be infinite. Before stating our main result, we shall present some necessary notations. Let $c_n$ be a sequence of positive real numbers satisfying

$$c_n/\sqrt{n} \nearrow \infty, \tag{1.2}$$

and

$$\forall \varepsilon > 0\ \exists m_\varepsilon > 0\colon \quad c_n/c_m \leqslant (1+\varepsilon)n/m, \quad n \geqslant m \geqslant m_\varepsilon. \tag{1.3}$$

Note that condition (1.3) is satisfied if $c_n/n$ is non-increasing or if $c_n = c(n) : [0, \infty) \to [0, \infty)$ is regularly varying at infinity with exponent $\gamma < 1$. The reader is referred to Einmahl and Li [2,3] for more details about the choice of $c_n$. Let

$$H(t) = \sup_{f \in B_1^*} \mathsf{E}f^2(X) I\{\|X\| \leqslant t\}, \quad t \geqslant 0,$$

where $B_1^*$ is the unit ball of $B^*$, and set

$$\alpha_0 = \sup\left\{\alpha \geqslant 0\colon \sum_{n=1}^{\infty} n^{-1} \exp\left(-\frac{\alpha^2 c_n^2}{2nH(c_n)}\right) = \infty\right\}. \tag{1.4}$$

Now we are in a position to exhibit our main result.

**Theorem 1.1.** *Let $\{X, X_n;\ n \geqslant 0\}$ be a sequence of i.i.d. $B$-valued random variables with $\mathsf{E}X = 0$. Assume that*

$$\sum_{n=1}^{\infty} \mathsf{P}\big(\|X\| \geqslant c_n\big) < \infty, \tag{1.5}$$

*and*

$$\frac{S_n}{c_n} \to 0 \quad \text{in probability}, \tag{1.6}$$

*where $c_n$ is a sequence of real numbers satisfying conditions* (1.2) *and* (1.3)*. Then we have with probability one*

$$\limsup_{\beta \nearrow 1} \left\| \sum_{n=0}^{\infty} \beta^n X_n \right\| \Big/ c_{(1-\beta^2)^{-1}} = \alpha_0,$$

*where $c_{(1-\beta^2)^{-1}} = c_{[(1-\beta^2)^{-1}]}$, and $\alpha_0$ is defined in* (1.4)*.*

**Remark 1.1.** It is obvious that we extend Theorem A in two sides. On one hand, we use a general normalizing sequence $c_n$, which contains the classical $\sqrt{2nLLn}$, and on the other hand, one can get the LIL when $\sup_{f \in B_1^*} \mathsf{E}f^2(X)$ is infinite, by taking some special normalizing sequence $c_n$ (for example, $c_n = \sqrt{nh(n)}$, where $h(x)$ is a slowly varying function (cf. Einmahl and Li [2,3])). Generally condition (1.5) is necessary for the conclusion of Theorem 1.1, and (1.6) here is for getting precise value of the LIL.

## 2. Proof

In what follows we use the notation $a_n \sim b_n$ if $a_n/b_n \to 1$ as $n \to \infty$, and $[x]$ means the largest integer less than $x$. We also denote with $C, C_1, K$ etc. generic constants that may be different in each of its appearance.

We first proceed with some lemmas which will be used later.

**Lemma 2.1.** *(See Zhang [6].) Let $\{a_n;\ n \geqslant 0\}$ be a sequence of real numbers or a sequence in $B$, $\{A_n;\ n \geqslant 0\}$ a sequence of monotonically non-decreasing positive numbers satisfying $\|\sum_{k=0}^{n} a_k\| \leqslant A_n$ for $n$ large enough and $A_{kn}/A_n \leqslant C_0 k^Q$, $k \geqslant 1$, $n \geqslant 0$, for some $C_0$, $Q > 0$. Then for any $r > 0$ and $N_0 \geqslant 1$, we have*

$$\limsup_{\beta \nearrow 1} A_{N(\beta)}^{-1} \left\| \sum_{n=0}^{\infty} \beta^{nr} a_n \right\| \leqslant \frac{r}{2} + \frac{rC_0}{2} \int_1^{\infty} \exp\left(\frac{-rx}{2}\right) x^Q \, dx,$$

*and*

$$\limsup_{\beta \nearrow 1} A_{N(\beta)}^{-1} \left\| \sum_{n=N_0 N(\beta)}^{\infty} \beta^{nr} a_n \right\| \leqslant \frac{rC_0}{2} \int_{N_0}^{\infty} \exp\left(\frac{-rx}{2}\right) x^Q \, dx + C_0 \exp\left(\frac{-rN_0}{2}\right) N_0^Q,$$

*where $N(\beta) = [\frac{1}{1-\beta^2}]$.*

**Lemma 2.2.** *(See Zhang [7].) Let $\{u_n;\ n \geqslant 0\}$ be a non-increasing sequence of positive numbers and $\{x_n;\ n \geqslant 0\}$ be a sequence in $B$. Then for each $x > 0$,*

$$\left\| \sum_{i=0}^{n} u_i x_i \right\| \leqslant u_0 \max_{i \leqslant n} \left\| \sum_{j=0}^{i} x_j \right\|.$$

Now we begin to prove Theorem 1.1.

**Proof.** For $\alpha > 0$ small enough, set $X_i' = X_i I(\|X_i\| \leqslant \alpha c_i)$, $X_i'' = X_i - X_i'$, $i \geqslant 0$, and denote the sums of the first $n$ of these variables by $S_n'$ and $S_n''$, $n \geqslant 1$, respectively. Observe that (1.5) implies for any $\varepsilon > 0$, $\sum_{n=1}^{\infty} \mathsf{P}(\|X\| > \varepsilon c_n) < \infty$ (cf. Lemma 1 of Einmahl and Li [2]), which guarantees that

$$\sum_{n=1}^{\infty} \mathsf{P}\big(X_n \neq X_n'\big) < \infty,$$

and then we have with probability one $X_n = X_n'$ eventually. Thus for completing the proof, what we need to do is to demonstrate that

$$\limsup_{\beta \nearrow 1} \frac{\|\sum_{n=0}^{\infty} \beta^n X_n'\|}{c_{(1-\beta^2)^{-1}}} = \alpha_0 \quad \text{a.s.} \tag{2.1}$$

We first show the upper bound, i.e.,

$$\limsup_{\beta \nearrow 1} \frac{\|\sum_{n=0}^{\infty} \beta^n X_n'\|}{c_{(1-\beta^2)^{-1}}} \leqslant \alpha_0 \quad \text{a.s.} \tag{2.2}$$

Writing $\tau(\beta) = 1/(1-\beta^2)$, it is easily seen that $\tau(\beta)$ is a monotonically increasing continuous function of $\beta$ and $\tau(\beta) \nearrow \infty$ as $\beta \nearrow 1$. Set

$$\beta_k := \sqrt{1 - (1+\theta)^{-k}}, \quad k \geqslant 1, \tag{2.3}$$

for any $\theta > 0$ (without loss of generality, we can assume that $\theta$ is small enough). Then we have $\beta_k \nearrow 1$ and $\tau(\beta_k) = (1+\theta)^k$ as $k \nearrow \infty$.

Note that for $\beta_{k-1} \leqslant \beta \leqslant \beta_k$,

$$\left\| \sum_{n=0}^{\infty} \beta^n X_n' \right\| \Big/ c_{\tau(\beta)} \leqslant \sup_{0 \leqslant \beta \leqslant \beta_k} \left\| \sum_{n=0}^{\infty} \beta^n X_n' \right\| \Big/ c_{\tau(\beta_{k-1})},$$

and thus for proving (2.2), it is enough to present that

$$\limsup_{k\to\infty} \frac{\sup_{0\leqslant\beta\leqslant\beta_k} \|\sum_{n=0}^{\infty}\beta^n X_n'\|}{c_{\tau(\beta_{k-1})}} \leqslant \alpha_0 \quad \text{a.s.} \tag{2.4}$$

From Lemma 2.2, we have that for any $0\leqslant\beta\leqslant\beta_k$ and any integer $M\geqslant 0$,

$$\left\|\sum_{n=0}^{M}\beta^n X_n'\right\| = \left\|\sum_{n=0}^{M}(\beta/\beta_k)^n\beta_k^n X_n'\right\| \leqslant (\beta/\beta_k)^0 \sup_{0\leqslant m\leqslant M}\left\|\sum_{n=0}^{m}\beta_k^n X_n'\right\| = \sup_{0\leqslant m\leqslant M}\left\|\sum_{n=0}^{m}\beta_k^n X_n'\right\|,$$

and this, coupled with letting $M\to\infty$, implies

$$\sup_{0\leqslant\beta\leqslant\beta_k}\left\|\sum_{n=0}^{\infty}\beta^n X_n'\right\| \leqslant \sup_{0\leqslant m\leqslant\infty}\left\|\sum_{n=0}^{m}\beta_k^n X_n'\right\|.$$

Notice that if we can show for any $\varepsilon>0$ and $\theta>0$

$$\sum_{k=1}^{\infty}\mathsf{P}\left(\sup_{0\leqslant m\leqslant\infty}\left\|\sum_{n=0}^{m}\beta_k^n X_n'\right\| \geqslant (\alpha_0+2\varepsilon)(1+\varepsilon)(1+\theta)c_{\tau(\beta_k)}\right) < \infty, \tag{2.5}$$

then (2.4) follows from the Borel–Cantelli lemma, (1.3) and the fact $\tau(\beta_k)/\tau(\beta_{k-1}) = 1+\theta$, immediately.

It is readily seen that under the assumption (1.5), (1.6) implies $\lim_{n\to\infty}\mathsf{E}\|S_n\|/c_n = 0$, and this of course entails that

$$\lim_{\beta\nearrow 1}\mathsf{E}\left\|\sum_{n=0}^{\infty}\beta^n X_n\right\|\Big/ c_{\tau(\beta)} = \lim_{k\to\infty}\mathsf{E}\left\|\sum_{n=0}^{\infty}\beta_k^n X_n\right\|\Big/ c_{\tau(\beta_k)} = 0.$$

Then for any $\varepsilon>0$, we have for $k$ large enough, $\mathsf{E}\|\sum_{n=0}^{M}\beta_k^n X_n'\| \leqslant \varepsilon c_{\tau(\beta_k)}$. Also notice that for any $n$, there exists an $N_0$ such that $N_0 N(\beta_k)\leqslant n\leqslant (N_0+1)N(\beta_k)$, and thus we have that for $n$ large enough

$$c_n \leqslant c_{(N_0+1)N(\beta_k)} \leqslant (N_0+1)c_{N(\beta_k)} \leqslant C c_{\tau(\beta_k)}. \tag{2.6}$$

Hence, by applying Theorem 3.1 of Einmahl and Li [3] with (2.6), for any $M\geqslant 0, \varepsilon>0,\ \theta>0,\ k$ sufficiently large and $\alpha$ small enough, it follows that

$$\begin{aligned}
&\mathsf{P}\left(\sup_{1\leqslant m\leqslant M}\left\|\sum_{n=0}^{m}\beta_k^n X_n'\right\| \geqslant (\alpha_0+2\varepsilon)(1+\varepsilon)(1+\theta)c_{\tau(\beta_k)}\right)\\
&\quad\leqslant \mathsf{P}\left(\sup_{1\leqslant m\leqslant M}\left\|\sum_{n=0}^{m}\beta_k^n X_n'\right\| \geqslant (1+\varepsilon)(1+\theta)\mathsf{E}\left\|\sum_{n=0}^{M}\beta_k^n X_n'\right\| + (\alpha_0+\varepsilon)(1+\varepsilon)(1+\theta)c_{\tau(\beta_k)}\right)\\
&\quad\leqslant \exp\left(-\frac{(\alpha_0+\varepsilon)^2(1+\varepsilon)^2(1+\theta)^2 c_{\tau(\beta_k)}^2}{(2+\varepsilon)\sup_{f\in B_1^*}(\sum_{n=0}^{M}\beta_k^{2n}\mathsf{E}f^2(X_n'))}\right) + C\frac{\sum_{n=0}^{M}\mathsf{E}\|\beta_k^n X_n'\|^3}{(\alpha_0+\varepsilon)^3(1+\varepsilon)^3(1+\theta)^3 c_{\tau(\beta_k)}^3}\\
&\quad\leqslant \exp\left(-\frac{(\alpha_0+\varepsilon)^2 c_{\tau(\beta_k)}^2}{2H(\alpha c_M)\sum_{n=0}^{M}\beta_k^{2n}}\right) + C\frac{\sum_{n=0}^{M}\beta_k^{2n}\mathsf{E}\|X\|^3 I(\|X\|\leqslant\alpha c_M)}{(\alpha_0+\varepsilon)^3(1+\varepsilon)^3 c_{\tau(\beta_k)}^3}\\
&\quad\leqslant \exp\left(-\frac{(\alpha_0+\varepsilon)^2 c_{\tau(\beta_k)}^2}{2\tau(\beta_k)H(c_{\tau(\beta_k)})}\right) + C\frac{\tau(\beta_k)\mathsf{E}\|X\|^3 I(\|X\|\leqslant c_{\tau(\beta_k)})}{(\alpha_0+\varepsilon)^3(1+\varepsilon)^3 c_{\tau(\beta_k)}^3}.
\end{aligned}$$

Recalling that $\tau(\beta_k)\sim(1+\theta)^k$, it is easily seen that (2.5) follows from the relations (4.7) and (4.8) of Einmahl and Li [3] by letting $M\to\infty$. Therefore, the proof of the upper bound, i.e. (2.2), is complete.

Then for proving (2.1), it is sufficient to prove the lower bound

$$\limsup_{\beta\nearrow 1}\frac{\|\sum_{n=0}^{\infty}\beta^n X_n'\|}{c_{(1-\beta^2)^{-1}}} \geqslant \alpha_0 \quad \text{a.s.} \tag{2.7}$$

Also note that by virtue of Theorem 4.1 in Einmahl and Li [3], we have with probability one

$$\limsup_{n\to}\frac{\|S_n'\|}{c_n} \leqslant \limsup_{n\to\infty}\frac{\|S_n\|}{c_n} \leqslant \alpha_0 < \infty,$$

and then it follows from Lemma 2.1 that

$$\limsup_{\beta\nearrow 1}\frac{\|\sum_{n=N_0N(\beta)+1}^{\infty}\beta^n X_n'\|}{c_{\tau(\beta)}}\leqslant \limsup_{\beta\nearrow 1}\frac{\|\sum_{n=N_0N(\beta)+1}^{\infty}\beta^n X_n'\|}{c_{N(\beta)}}$$
$$\leqslant \alpha_0\left(\frac{C_0}{2}\int_{N_0}^{\infty}\exp\left(\frac{-x}{2}x^Q\right)dx+C_0\exp\left(\frac{-N_0}{2}\right)N_0^Q\right)\to 0 \quad \text{as } N_0\to\infty.$$

Thus for proving (2.7), we need only to show that for $0<\alpha<\alpha_0$ and $N_0$ large enough, with probability one

$$\limsup_{\beta\nearrow 1}\left\|\sum_{n=0}^{N_0N(\beta)}\beta^n X_n'\right\|\Big/c_{\tau(\beta)}\geqslant\alpha. \tag{2.8}$$

Now we start to demonstrate (2.8) holds. First we may assume that

$$\limsup_{\beta\nearrow 1}\mathsf{P}\left(\left\|\sum_{n=0}^{N_0N(\beta)}\beta^n X_n'\right\|\geqslant\alpha c_{\tau(\beta)}\right)\leqslant 1/2.$$

Otherwise, we would have

$$\mathsf{P}\left(\limsup_{\beta\nearrow 1}\left\|\sum_{n=0}^{N_0N(\beta)}\beta^n X_n'\right\|\geqslant\alpha c_{\tau(\beta)}\right)\geqslant\limsup_{\beta\nearrow 1}\mathsf{P}\left(\left\|\sum_{n=0}^{N_0N(\beta)}\beta^n X_n'\right\|\geqslant\alpha c_{\tau(\beta)}\right)>1/2,$$

which implies (2.8) holds by the Kolmogorov's 0–1 law. Thus to prove (2.8), it suffices to exhibit that for any subsequence $\{\beta_k\}$ ($\beta_k\nearrow 1$ as $k\nearrow\infty$),

$$\mathsf{P}\left(\left\|\sum_{n=0}^{N_0N(\beta_k)}\beta_k^n X_n'\right\|\geqslant\alpha c_{\tau(\beta_k)}\text{ i.o.}\right)=1. \tag{2.9}$$

To that end, for any $k$, we choose a functional $f_k\in B_1^*$ such that

$$\mathsf{E}f_k^2(X)I\big(\|X\|\leqslant c_{\tau(\beta_k)}\big)\geqslant(1-\varepsilon)H(c_{\tau(\beta_k)}),$$

where $0<\varepsilon<1$ to be specified later on. For any $k,n\geqslant 0$, define

$$\xi_{k,n}=\beta_k^n f_k(X_n)I\big(\|X_n\|\leqslant c_{\tau(\beta_k)}\big),\qquad \xi_{k,n}'=\xi_{k,n}-\mathsf{E}\xi_{k,n}.$$

Since $|\mathsf{E}\xi_{k,n}|\leqslant\beta_k^n\mathsf{E}\|X\|I(\|X\|\leqslant c_{\tau(\beta_k)})$ and

$$\mathsf{P}\left(\left\|\sum_{n=0}^{N_0N(\beta_k)}\beta_k^n X_n'\right\|\geqslant\alpha c_{\tau(\beta_k)}\right)\geqslant\mathsf{P}\left(\sum_{n=0}^{N_0N(\beta_k)}\xi_{k,n}\geqslant\alpha c_{\tau(\beta_k)}\right),$$

it is enough to show that for a suitable $0<\varepsilon<1$,

$$\sum_{k=1}^{\infty}\mathsf{P}\left(\sum_{n=0}^{N_0N(\beta_k)}\xi_{k,n}'\geqslant(1+\varepsilon)\alpha c_{\tau(\beta_k)}\right)=\infty. \tag{2.10}$$

Also using a non-uniform bound on the rate of convergence in the central limit theorem (cf. Petrov [4]), we have

$$\mathsf{P}\left(\sum_{n=0}^{N_0N(\beta_k)}\xi_{k,n}'\geqslant(1+\varepsilon)\alpha c_{\tau(\beta_k)}\right)\geqslant\mathsf{P}\left(\sum_{n=0}^{N_0N(\beta_k)}\sigma_{k,n}\pounds\geqslant(1+\varepsilon)\alpha c_{\tau(\beta_k)}\right)-A\alpha^{-3}(1+\varepsilon)^{-3}c_{\tau(\beta_k)}^{-3}\sum_{n=0}^{N_0N(\beta_k)}\mathsf{E}\big|\xi_{k,n}'\big|^3,$$

where £ is a standard normal random variable, $\sigma_{k,n}=\mathsf{Var}(\xi_{k,n})$ and $A$ is an absolute constant. Observing that $\mathsf{E}|\xi_{k,n}'|^3\leqslant 8\mathsf{E}|\xi_{k,n}|^3\leqslant 8\beta_k^{2n}\mathsf{E}\|X\|^3I(\|X\|\leqslant c_{\tau(\beta_k)})$, we then conclude from the relation (4.8) of Einmahl and Li [3] again that

$$\sum_{k=1}^{\infty}c_{\tau(\beta_k)}^{-3}\sum_{n=0}^{N_0N(\beta_k)}\mathsf{E}\big|\xi_{k,n}'\big|^3\leqslant C\sum_{k=1}^{\infty}c_{\tau(\beta_k)}^{-3}\tau(\beta_k)\mathsf{E}\|X\|^3I\big(\|X\|\leqslant c_{\tau(\beta_k)}\big)<\infty.$$

Hence, to prove (2.10), what we need to do now is to show

$$\sum_{k=1}^{\infty}\mathsf{P}\left(\sum_{n=0}^{N_0N(\beta_k)}\sigma_{k,n}\pounds\geqslant(1+\varepsilon)\alpha c_{\tau(\beta_k)}\right)=\infty. \tag{2.11}$$

Denote $\Xi := \{k \geqslant 1;\ H(\tau(\beta_k)) \leqslant (c_{\tau(\beta_k)}/\tau(\beta_k))^2\}$, and thus it leads to that for any $\eta > 0$

$$\sum_{k\in\Xi} \exp\left(-\frac{\eta c^2_{\tau(\beta_k)}}{2\tau(\beta_k)H(c_{\tau(\beta_k)})}\right) < \infty. \tag{2.12}$$

As to $k \notin \Xi$, observe

$$\begin{aligned}\sigma^2_{k,n} &= \beta_k^{2n}\left(\mathsf{E} f_k^2(X)I\left(\|X\| \leqslant c_{\tau(\beta_k)}\right) - \left(\mathsf{E} f_k(X)I\left(\|X\| \leqslant c_{\tau(\beta_k)}\right)\right)^2\right)\\ &= \beta_k^{2n}\left(\mathsf{E} f_k^2(X)I\left(\|X\| \leqslant c_{\tau(\beta_k)}\right) - \left(\mathsf{E} f_k(X)I\left(\|X\| > c_{\tau(\beta_k)}\right)\right)^2\right)\\ &\geqslant \beta_k^{2n}\left((1-\varepsilon)H(c_{\tau(\beta_k)}) - \left(\mathsf{E}\|X\|I\left(\|X\| > c_{\tau(\beta_k)}\right)\right)^2\right)\\ &\geqslant \beta_k^{2n}(1-2\varepsilon)H(c_{\tau(\beta_k)}),\end{aligned}$$

and for $0 < \beta < 1$, as $N_0 \to \infty$,

$$\frac{\sum_{n=N_0N(\beta)+1}^{\infty} \beta^{2n}}{\tau(\beta)} = \beta^{2(N_0N(\beta)+1)} \to 0,$$

where we use the fact that for $n$ large enough $\mathsf{E}\|X\|I(\|X\| \geqslant c_n) = o(c_n/n)$ deduced from the assumption (1.5). Then an application of a standard lower bound for the tail probabilities of normal random variables, yields that for large $N_0$ and large $k \notin \Xi$

$$\sum_{k\notin\Xi} \mathsf{P}\left(\sum_{n=0}^{N_0N(\beta_k)} \sigma_{k,n}\pounds \geqslant (1+\varepsilon)\alpha c_{\tau(\beta_k)}\right) \geqslant \sum_{k\notin\Xi} \exp\left(-\frac{(1+\varepsilon)^2\alpha^2 c^2_{\tau(\beta_k)}}{2(1-3\varepsilon)\tau(\beta_k)H(c_{\tau(\beta_k)})}\right). \tag{2.13}$$

Hence, by choosing $\varepsilon$ so small that $\alpha(1+\varepsilon)/\sqrt{1-3\varepsilon} < \alpha_0$ in (2.13) and applying the relation (4.7) of Einmahl and Li [3] again, we have

$$\sum_{k\notin\Xi} \mathsf{P}\left(\sum_{n=0}^{N_0N(\beta_k)} \sigma_{k,n}\pounds \geqslant (1+\varepsilon)\alpha c_{\tau(\beta_k)}\right) = \infty,$$

which, coupled with (2.12), ensures that (2.11) holds, as desired. Therefore, the proof of the lower bound is finished. □

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