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On differential Rota–Baxter algebras

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Abstract

A Rota–Baxter operator of weight λ is an abstraction of both the integral operator (when $\lambda = 0$) and the summation operator (when $\lambda = 1$). We similarly define a differential operator of weight λ that includes both the differential operator (when $\lambda = 0$) and the difference operator (when $\lambda = 1$). We further consider an algebraic structure with both a differential operator of weight λ and a Rota–Baxter operator of weight λ that are related in the same way that the differential operator and the integral operator are related by the First Fundamental Theorem of Calculus. We construct free objects in the corresponding categories. In the commutative case, the free objects are given in terms of generalized shuffles, called mixable shuffles. In the noncommutative case, the free objects are given in terms of angularly decorated rooted forests. As a byproduct, we obtain structures of a differential algebra on decorated and undecorated planar rooted forests.

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1. Introduction

1.1. Motivation

This paper studies the algebraic structure reflecting the relation between the differential operator and the integral operator as in the First Fundamental Theorem of Calculus.

As is well-known, the two principal components of calculus are the differential calculus which studies the differential operator $\mathrm{d}(f)(x) = \frac{\mathrm{d}f}{\mathrm{d}x}(x)$ and the integral calculus which studies the integral operator $P(f)(x) = \int_a^x f(t)\mathrm{d}t$. The discrete versions of these two operators are the difference operator and the summation operator. The abstraction of the differential operator and difference operator led to the development of differential algebra and difference algebra [5,21]. Likewise, the integral operator P and the summation operator have been abstracted to give the notion of Rota–Baxter operators (previously called Baxter operators) and Rota–Baxter algebras [3,24,25]. In the last few years, major progress has been made in both differential algebra and Rota–Baxter algebra, with applications in broad areas in mathematics and physics [1,2,4,6,10,11,13,14,16,26,27]. For instance, both operators played important roles in the recent developments in renormalization of quantum field theory [6,7,12].

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The differential operator and the integral operator are related by the First Fundamental Theorem of Calculus stating that (under suitable conditions)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\mathrm{d}t\right) = f(x). \tag{1}$$

Thus the integral operator is the right inverse of the differential operator, so that $(\mathrm{d} \circ P)(f) = f$. A similar relation holds for the difference operator and the summation operator (see Example 1.1(e)). It is therefore natural for us to introduce the notion of differential integral algebra, or more generally the notion of differential Rota–Baxter algebra, that provides a framework to put differential/difference algebra and Rota–Baxter algebra together in the spirit of Eq. (1).

Quite often, problems on differential equations and differential algebra are studied by translating them to integral problems. This transition uses in disguise the underlying structure of differential Rota–Baxter algebra. In fact, Baxter [3] defined his algebra and gave an algebraic proof of the Spitzer identity in probability guided by such a point of view for first order linear ODEs. This view was further stressed by Rota [25] in connection with finding q-analogues of classical identities of special functions. The framework introduced in the current paper should provide a natural setting to study such problems. The reader is also invited to consult the paper [23], which came to our attention after our paper was submitted, where a similar structure was independently defined under a different context and was applied to study boundary problems for linear ODE in differential algebras. This coincidence might suggest that a systematic study of differential Rota–Baxter algebra is now in order after the observation of Baxter several decades ago.

1.2. Definitions and preliminary examples

We now give more details of our constructions. By analogy to a Rota–Baxter operator that unifies the notions of an integral operator and a summation operator, we first unify the concepts of the differential operator and the difference operator by the concept of a λ-differential operator, where λ is a fixed element in the ground ring, that gives the differential (resp. difference) operator when λ is 0 (resp. 1). We then introduce the concept of a differential Rota–Baxter algebra of weight λ consisting of an algebra with both a λ-differential operator and a λ-Rota–Baxter operator with a compatibility condition between these two operators.

Definition 1.1. Let **k** be a unitary commutative ring. Let $\lambda \in \mathbf{k}$ be fixed.

(a) A **differential k-algebra of weight** λ (also called a λ-**differential k-algebra**) is an associative **k**-algebra R together with a linear operator $\mathrm{d} : R \to R$ such that

$$\mathrm{d}(xy) = \mathrm{d}(x)y + x\mathrm{d}(y) + \lambda \mathrm{d}(x)\mathrm{d}(y), \quad \forall x, y \in R, \tag{2}$$

and

$$\mathrm{d}(1) = 0. \tag{3}$$

Such an operator is called a **differential operator of weight** λ or a **derivation of weight** λ. It is also called a λ-**differential operator** or a λ-**derivation**. The category of differential algebras (resp. commutative differential algebras) of weight λ is denoted by $\mathbf{Dif}_\lambda$ (resp. $\mathbf{CDif}_\lambda$).

(b) A **Rota–Baxter k-algebra of weight** λ is an associative **k**-algebra R together with a linear operator $P : R \to R$ such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \quad \forall x, y \in R. \tag{4}$$

Such an operator is called a **Rota–Baxter operator of weight** λ or a λ-**Rota–Baxter operator**. The category of Rota–Baxter algebras (resp. commutative Rota–Baxter algebras) of weight λ is denoted by $\mathbf{RB}_\lambda$ (resp. $\mathbf{CRB}_\lambda$).

(c) A **differential Rota–Baxter k-algebra of weight** λ (also called a λ-**differential Rota–Baxter k-algebra**) is an associative **k**-algebra R together with a differential operator d of weight λ and a Rota–Baxter operator P of weight λ such that

$$\mathrm{d} \circ P = \mathrm{id}_R. \tag{5}$$

The category of differential Rota–Baxter algebras (resp. commutative differential Rota–Baxter algebras) of weight λ is denoted by $\mathbf{DRB}_\lambda$ (resp. $\mathbf{CDRB}_\lambda$).

We also use $\mathbf{Alg} = \mathbf{Alg_k}$ to denote the category of $\mathbf{k}$-algebras. When there is no danger of confusion, we will suppress λ and $\mathbf{k}$ from the notations. We will also denote the set of nonnegative integers by $\mathbb{N}$ and the set of positive integers by $\mathbb{N}_+$.

Note that we require that a differential operator d satisfies $d(1) = 0$. A linear operator d satisfying Eq. (2) is called a **weak differential operator of weight** λ. A weak differential operator of weight λ with $d(1) \neq 0$ is called a **degenerate differential operator** of weight λ for the reason given in Remark 2.4, and will be discussed in Section 2.1.

We next give some simple examples of differential, Rota–Baxter and differential Rota–Baxter algebras. Further examples will be given in later sections. In particular, the algebras of λ-Hurwitz series are differential Rota–Baxter algebras (Proposition 2.10). By Theorem 3.2 and Theorem 4.2, every differential algebra naturally gives rise to a differential Rota–Baxter algebra. We also obtain the structure of a differential Rota–Baxter algebra on rooted forests (Theorem 5.1).

Example 1.1. (a) A 0-derivation and a 0-differential algebra is a derivation and differential algebra in the usual sense [21].

(b) Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Let $R = \mathrm{Cont}(\mathbb{R})$ denote the $\mathbb{R}$-algebra of continuous functions $f : \mathbb{R} \to \mathbb{R}$, and consider the usual "difference quotient" operator d_λ on R defined by

$$(d_\lambda(f))(x) = (f(x+\lambda) - f(x))/\lambda. \tag{6}$$

Then it is immediate that d_λ is a λ-derivation on R. When $\lambda = 1$, we obtain the usual difference operator on functions. Further, the usual derivation is $d_0 := \lim_{\lambda\to 0} d_\lambda$.

(c) A difference algebra [5] is defined to be a commutative algebra R together with an injective algebra endomorphism ϕ on R. It is simple to check that $\phi - \mathrm{id}$ is a differential operator of weight 1.

(d) By the First Fundamental Theorem of Calculus in Eq. (1), $(\mathrm{Cont}(\mathbb{R}), d/dx, \int_0^x)$ is a differential Rota–Baxter algebra of weight 0.

(e) Let $0 < \lambda \in \mathbb{R}$. Let R be an $\mathbb{R}$-subalgebra of $\mathrm{Cont}(\mathbb{R})$ that is closed under the operators

$$P_0(f)(x) = -\int_x^\infty f(t)\,dt, \qquad P_\lambda(f)(x) = -\lambda \sum_{n\geq 0} f(x+n\lambda).$$

For example, R can be taken to be the $\mathbb{R}$-subalgebra generated by e^{-x}: $R = \sum_{k\geq 1} \mathbb{R}e^{-kx}$. Then P_λ is a Rota–Baxter operator of weight λ and, for the d_λ in Eq. (6),

$$d_\lambda \circ P_\lambda = \mathrm{id}_R, \quad \forall 0 \neq \lambda \in \mathbb{R},$$

reducing to the fundamental theorem $d_0 \circ P_0 = \mathrm{id}_R$ when λ goes to 0. So $(R, d_\lambda, P_\lambda)$ is a differential Rota–Baxter algebra of weight λ.

1.3. Main results and outline of the paper

Our main purpose in this paper is to construct free objects in the various categories of λ-differential algebras and λ-differential Rota–Baxter algebras.

In Section 2, we first prove basic properties of λ-differential algebras. We then construct the free objects in $\mathbf{Dif}_\lambda$ in Theorem 2.5 and cofree objects in $\mathbf{Dif}_\lambda$ in Corollary 2.9. The construction of free objects in $\mathbf{CDRB}_\lambda$ is carried out in Section 3 (Theorem 3.2) and the construction of free objects in $\mathbf{DRB}_\lambda$ is carried out in Section 4 (Theorem 4.2). Both constructions rely on the explicit construction of free Rota–Baxter algebras, in the commutative case in [16,17] and in the noncommutative case in [2,10,11]. Consequently, we obtain a structure of a differential algebra on the mixable shuffle and shuffle algebras, and on angularly decorated rooted trees. We further obtain the structure of a λ-differential algebra on planar rooted forests in Section 5 (Theorem 5.1). Grossman and Larson [15] have also obtained a differential algebra structure on trees. There are differences between their construction and ours. The multiplications on the trees are different. Further our trees form a differential algebra while theirs form a ring of differential operators. It would be interesting to further explore their relation.

2. Differential algebras of weight λ

We first give some basic properties of λ-differential algebras, followed by a study of free and cofree λ-differential algebras.

2.1. Basic properties and degenerate differential operators

Some basic properties of differential operators can be easily generalized to λ-differential operators. The following proposition generalizes the power rule in differential calculus and the well-known result of Leibniz [21, p.60]. It holds without the assumption that $\mathrm{d}(1) = 0$.

Proposition 2.1. *Let (R, d) be a differential* $\mathbf{k}$*-algebra of weight λ.*
(a) *Let $x \in R$ and $n \in \mathbb{N}_+$. Then*

$$\mathrm{d}(x^n) = \sum_{i=1}^{n} \binom{n}{i} \lambda^{i-1} x^{n-i} \mathrm{d}(x)^i.$$

(b) *Let $x, y \in R$, and let $n \in \mathbb{N}$. Then*

$$\mathrm{d}^n(xy) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k \mathrm{d}^{n-j}(x) \mathrm{d}^{k+j}(y). \tag{7}$$

Proof. (a) The proof is similar to the inductive proof on n for the usual power rule, using an index shift and Pascal's rule.

(b) The proof is again similar to the case for differential operators. Proceeding by induction on n, the case $n = 0$ is trivial, so assume that Eq. (7) holds for n, and consider

$$\mathrm{d}^{n+1}(xy) = \mathrm{d}^n(\mathrm{d}(xy)) = \mathrm{d}^n(\mathrm{d}(x)y) + \mathrm{d}^n(x\mathrm{d}(y)) + \lambda \mathrm{d}^n(\mathrm{d}(x)\mathrm{d}(y)). \tag{8}$$

Applying the induction hypothesis to the first term gives

$$\begin{aligned} &\sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k \mathrm{d}^{n+1-j}(x) \mathrm{d}^{k+j}(y) \\ &\quad = \sum_{k=0}^{n} \sum_{j=1}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k \mathrm{d}^{n+1-j}(x) \mathrm{d}^{k+j}(y) + \sum_{k=0}^{n} \binom{n}{k} \lambda^k \mathrm{d}^{n+1}(x) \mathrm{d}^k(y). \end{aligned}$$

Doing the same to the second term in Eq. (8) followed by an index shift gives

$$\begin{aligned} &\sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k \mathrm{d}^{n-j}(x) \mathrm{d}^{k+j+1}(y) \\ &\quad = \sum_{k=0}^{n} \sum_{j=1}^{n+1-k} \binom{n}{k} \binom{n-k}{j-1} \lambda^k \mathrm{d}^{n+1-j}(x) \mathrm{d}^{k+j}(y) \\ &\quad = \sum_{k=0}^{n} \sum_{j=1}^{n-k} \binom{n}{k} \binom{n-k}{j-1} \lambda^k \mathrm{d}^{n+1-j}(x) \mathrm{d}^{k+j}(y) + \sum_{k=0}^{n} \binom{n}{k} \lambda^k \mathrm{d}^k(x) \mathrm{d}^{n+1}(y). \end{aligned}$$

Thus by Pascal's rule,

$$\begin{aligned} \mathrm{d}^n(\mathrm{d}(x)y) + \mathrm{d}^n(x\mathrm{d}(y)) &= \sum_{k=0}^{n} \sum_{j=1}^{n-k} \binom{n}{k} \binom{n+1-k}{j} \lambda^k \mathrm{d}^{n+1-j}(x) \mathrm{d}^{k+j}(y) \\ &\quad + \sum_{k=0}^{n} \binom{n}{k} \lambda^k \mathrm{d}^{n+1}(x) \mathrm{d}^k(y) + \sum_{k=0}^{n} \binom{n}{k} \lambda^k \mathrm{d}^k(x) \mathrm{d}^{n+1}(y) \\ &= \sum_{k=0}^{n} \sum_{j=0}^{n+1-k} \binom{n}{k} \binom{n+1-k}{j} \lambda^k \mathrm{d}^{n+1-j}(x) \mathrm{d}^{k+j}(y) \\ &= \sum_{k=1}^{n} \sum_{j=0}^{n+1-k} \binom{n}{k} \binom{n+1-k}{j} \lambda^k \mathrm{d}^{n+1-j}(x) \mathrm{d}^{k+j}(y) + \sum_{j=0}^{n+1} \binom{n+1}{j} \mathrm{d}^{n+1-j}(x) \mathrm{d}^j(y). \end{aligned}$$

For the same reason, the third term in Eq. (8) gives

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}\lambda^{k+1}\mathrm{d}^{n+1-j}(x)\mathrm{d}^{k+j+1}(y)\\
&\quad=\sum_{k=1}^{n+1}\sum_{j=0}^{n+1-k}\binom{n}{k-1}\binom{n+1-k}{j}\lambda^{k}\mathrm{d}^{n+1-j}(x)\mathrm{d}^{k+j}(y)\\
&\quad=\sum_{k=1}^{n}\sum_{j=0}^{n+1-k}\binom{n}{k-1}\binom{n+1-k}{j}\lambda^{k}\mathrm{d}^{n+1-j}(x)\mathrm{d}^{k+j}(y)+\lambda^{n+1}\mathrm{d}^{n+1}(x)\mathrm{d}^{n+1}(y).
\end{aligned}$$

Therefore another application of Pascal's rule gives

$$\begin{aligned}
\mathrm{d}^n(\mathrm{d}(x)y)+\mathrm{d}^n(x\mathrm{d}(y))+\lambda\mathrm{d}^n(\mathrm{d}(x)\mathrm{d}(y)) &= \sum_{k=1}^{n}\sum_{j=0}^{n+1-k}\binom{n+1}{k}\binom{n+1-k}{j}\lambda^{k}\mathrm{d}^{n+1-j}(x)\mathrm{d}^{k+j}(y)\\
&\quad+\sum_{j=0}^{n+1}\binom{n+1}{j}\mathrm{d}^{n+1-j}(x)\mathrm{d}^{j}(y)+\lambda^{n+1}\mathrm{d}^{n+1}(x)\mathrm{d}^{n+1}(y)\\
&=\sum_{k=0}^{n+1}\sum_{j=0}^{n+1-k}\binom{n+1}{k}\binom{n+1-k}{j}\lambda^{k}\mathrm{d}^{n+1-j}(x)\mathrm{d}^{k+j}(y).
\end{aligned}$$

This completes the induction. □

We now briefly study degenerate λ-differential operators, that is, weak differential operators d for which $\mathrm{d}(1)\neq 0$. We first note that, for any $\lambda\in\mathbf{k}$ and any $\mathbf{k}$-algebra R, the zero map

$$\mathrm{d}: R\to R,\qquad \mathrm{d}(r)=0,\quad \forall r\in R$$

is a differential operator of weight λ, called the **zero differential operator** of weight λ.

We next note that for any $\lambda\in\mathbf{k}$ that is invertible and for any $\mathbf{k}$-algebra R, the map

$$\mathrm{d}: R\to R,\qquad \mathrm{d}(r)=-\lambda^{-1}r,\quad \forall r\in R, \tag{9}$$

is a weak differential operator of weight λ. We call such an operator (resp. algebra) a **scalar differential operator (resp. algebra) of weight** λ. We remark that by our definition, the zero map is not a scalar differential operator even though the zero map is given by a scalar multiplication.

For $\lambda\in\mathbf{k}$ invertible, it is also easy to check that

$$P_\lambda: R\to R,\qquad P_\lambda(r)=-\lambda r,\quad \forall r\in R,$$

is a Rota–Baxter operator of weight λ. Further $\mathrm{d}_\lambda\circ P_\lambda=\mathrm{id}$. This gives an instance of a degenerate differential Rota–Baxter algebra of weight λ.

Proposition 2.2. *Let $\lambda\in\mathbf{k}$. Let (R,d) be a weak differential $\mathbf{k}$-algebra of weight λ with no zero divisors. Then the following statements are equivalent.*

(a) λ *is invertible and* d *is a scalar differential operator of weight* λ.
(b) λ *is invertible and* $\mathrm{d}(1)=-\lambda^{-1}$.
(c) $\mathrm{d}(1)\neq 0$.
(d) *For every $r\in R$, $\mathrm{d}(r)$ is a nonzero $\mathbf{k}$-multiple of r.*

Proof. We clearly have (a) ⇒ (b) ⇒ (c) and (a) ⇒ (d). So we only need to prove (c) ⇒ (a) and (d) ⇒ (c).
(c) ⇒ (a): By Eq. (2), for any $x\in R$, we have

$$\mathrm{d}(x)=\mathrm{d}(1)x+\mathrm{d}(x)+\lambda\mathrm{d}(1)\mathrm{d}(x).$$

Thus $\mathrm{d}(1)\,(x+\lambda \mathrm{d}(x)) = 0$. Since R has no zero divisors, if $\mathrm{d}(1) \neq 0$, then we have

$$x + \lambda \mathrm{d}(x) = 0. \tag{10}$$

Letting $x = 1$, we have $0 = 1+\lambda \mathrm{d}(1) = 1+\mathrm{d}(1)\lambda$ since $\lambda \in \mathbf{k}$. Thus λ is invertible and Eq. (10) gives $\mathrm{d}(x) = -\lambda^{-1}x$, as needed.
(d) $\Rightarrow$ (c): Taking $r = 1$ in (d) we have $\mathrm{d}(1) = \alpha(1) \neq 0$. Thus we have (c). $\square$

Corollary 2.3. *Let (R, d) be a weak differential algebra of weight λ that has no zero divisors. If d is not a scalar differential operator, then it is a differential operator of weight λ.*

Remark 2.4. Since a scalar differential algebra is just an algebra with a fixed scalar multiplication, its study can be reduced to the study of algebras. By Corollary 2.3, a nonscalar weak differential algebra is a differential algebra under a mild restriction. This justifies the requirement in our Definition 1.1 that a λ-differential algebra be nondegenerate. A more careful study of degenerate differential operators will be carried out in the future.

2.2. Free differential algebras of weight λ

Using the same construction as for free differential algebras (of weight 0), we obtain free differential algebras of weight λ in both the commutative and noncommutative case.

Theorem 2.5. *Let X be a set. Let*

$$\Delta(X) = X \times \mathbb{N} = \{x^{(n)} | x \in X, n \geq 0\}.$$

(a) *Let $\mathbf{k}\{X\}$ be the free commutative algebra $\mathbf{k}[\Delta X]$ on the set ΔX. Define $\mathrm{d}_X : \mathbf{k}\{X\} \to \mathbf{k}\{X\}$ as follows. Let $w = u_1 \cdots u_k, u_i \in \Delta X, 1 \leq i \leq k$, be a commutative word from the alphabet set $\Delta(X)$. If $k = 1$, so that $w = x^{(n)} \in \Delta(X)$, define $\mathrm{d}_X(w) = x^{(n+1)}$. If $k > 1$, recursively define*

$$\mathrm{d}_X(w) = \mathrm{d}_X(u_1)u_2 \cdots u_k + u_1 \mathrm{d}_X(u_2 \cdots u_k) + \lambda \mathrm{d}_X(u_1)\mathrm{d}_X(u_2 \cdots u_k). \tag{11}$$

Further define $\mathrm{d}_X(1) = 0$ and then extend d_X to $\mathbf{k}\{X\}$ by linearity. Then $(\mathbf{k}\{X\}, \mathrm{d}_X)$ is the free commutative differential algebra of weight λ on the set X.

(b) *Let $\mathbf{k}^{NC}\{X\}$ be the free noncommutative algebra $\mathbf{k}\langle \Delta X\rangle$ on the set ΔX. Define $\mathrm{d}_X^{NC} : \mathbf{k}^{NC}\{X\} \to \mathbf{k}^{NC}\{X\}$ on the noncommutative words from the alphabet set ΔX in the same way as d_X is defined in (a). Then $(\mathbf{k}^{NC}\{X\}, \mathrm{d}_X^{NC})$ is the free noncommutative differential algebra of weight λ on the set X.*

Remark 2.6. Our use of $\mathbf{k}\{X\}$ for free commutative differential algebras of weight λ is consistent with the notation of the usual free commutative differential algebra (when $\lambda = 0$). We do not know a standard notation for free noncommutative differential algebras.

Proof. We just give a proof of (a). The proof of (b) is the same. In either case, it is similar to the proof of the $\lambda = 0$ case [21, p. 70].

Let (R, d) be a commutative λ-differential algebra and let $f : X \to R$ be a set map. We extend f to a λ-differential algebra homomorphism $\bar{f} : \mathbf{k}\{X\} \to R$ as follows.

Let $w = u_1 \cdots u_k, u_i \in \Delta X, 1 \leq i \leq k$, be a commutative word from the alphabet set ΔX. If $k = 1$, then $w = x^{(n)} \in \Delta X$. Define

$$\bar{f}(w) = \mathrm{d}^n(f(x)). \tag{12}$$

Note that this is the only possible definition in order for $\bar{f}$ to be a λ-differential algebra homomorphism. If $k > 1$, recursively define

$$\bar{f}(w) = \bar{f}(u_1)\bar{f}(u_2 \cdots u_k).$$

Further define $\bar{f}(1) = 1$ and then extend $\bar{f}$ to $\mathbf{k}\{X\}$ by linearity. This is the only possible definition in order for $\bar{f}$ to be an algebra homomorphism.

Since $\mathbf{k}\{X\}$ is the free commutative algebra on ΔX, $\bar{f}$ is an algebra homomorphism. So it remains to verify that, for all commutative words $w = u_1 \cdots u_k$ from the alphabet set ΔX,

$$\bar{f}(\mathrm{d}_X(w)) = \mathrm{d}(\bar{f}(w)), \tag{13}$$

for which we use induction on k. The case when $k = 1$ follows immediately from Eq. (12). For the inductive step, by Eq. (11):

$$\begin{aligned}\bar{f}(\mathrm{d}_X(w)) &= \bar{f}(\mathrm{d}_X(u_1)u_2\cdots u_k) + \bar{f}(u_1\mathrm{d}_X(u_2\cdots u_k)) + \lambda\bar{f}(\mathrm{d}_X(u_1)\mathrm{d}_X(u_2\cdots u_k))\\ &= \bar{f}(\mathrm{d}_X(u_1))\bar{f}(u_2\cdots u_k) + \bar{f}(u_1)\bar{f}(\mathrm{d}_X(u_2\cdots u_k)) + \lambda\bar{f}(\mathrm{d}_X(u_1))\bar{f}(\mathrm{d}_X(u_2\cdots u_k)).\end{aligned}$$

Then by Eq. (12), the induction hypothesis on k and the λ-differential algebra relation for d, the last sum equals to $\mathrm{d}(\bar{f}(w))$. □

2.3. *Cofree differential algebras of weight λ*

For any **k**-algebra A, let $A^{\mathbb{N}}$ denote the **k**-module of all functions $f : \mathbb{N} \to A$. We define a product on $A^{\mathbb{N}}$ by defining, for any $f, g \in A^{\mathbb{N}}$, $fg \in A^{\mathbb{N}}$ by

$$(fg)(n) = \sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}\lambda^k f(n-j)g(k+j).$$

Note that this definition is motivated by Proposition 2.1.(b). It is easily checked that this product is commutative, associative, distributive over addition, and has an identity $\mathbf{1}_{A^{\mathbb{N}}}$ defined by $\mathbf{1}_{A^{\mathbb{N}}}(n) = 0$ if $n \neq 0$ and $\mathbf{1}_{A^{\mathbb{N}}}(0) = \mathbf{1}_A$. We call this product the λ-**Hurwitz product** on $A^{\mathbb{N}}$, since if we take $\lambda = 0$, the product reduces to

$$(fg)(n) = \sum_{k=0}^{n}\binom{n}{k} f(n-k)g(k),$$

which is the usual Hurwitz product defined in [20]. We denote the **k**-algebra $A^{\mathbb{N}}$ with this product by DA, and call it the **k**-algebra of λ-**Hurwitz series over A**. Also, there is, for any **k**-algebra A, a homomorphism $\kappa_A : A \to DA$ of **k**-algebras defined by $\kappa_A(a) = a\mathbf{1}_{A^{\mathbb{N}}}$. This makes DA into an A-algebra, where for any $a \in A$ and any $f \in DA$, $af \in DA$ is given by $(af)(n) = a(f(n))$.

The **k**-algebra DA behaves much like the ring of Hurwitz series. The following proposition is one instance of this. We first define a map

$$\partial_A : DA \to DA, \qquad (\partial_A(f))(n) = f(n+1), \quad n \in \mathbb{N},\ f \in DA. \tag{14}$$

Proposition 2.7. *The map ∂_A is a λ-derivation on DA.*

Proof. It is clear that ∂_A is a mapping of **k**-modules, so all that remains is to show that for any $f, g \in DA$,

$$\partial_A(fg) = \partial_A(f)g + f\partial_A(g) + \lambda\partial_A(f)\partial_A(g).$$

But because of the definition of the λ-Hurwitz product, the proof of this equation is virtually identical to the proof of Proposition 2.1(b) and is left to the reader. ■

It follows from Proposition 2.7 that (DA, ∂_A) is a λ-differential **k**-algebra. If $h : A \to B$ is a **k**-algebra homomorphism, one checks that $Dh : DA \to DB$ defined by $((Dh)(f))(n) = h(f(n))$ is a morphism of **k**-algebras, and that $\partial_B \circ Dh = Dh \circ \partial_A$. Recalling that $\mathbf{Dif} = \mathbf{Dif}_\lambda$ denotes the category of λ-differential **k**-algebras, we see that we have a functor $G : \mathbf{Alg_k} \to \mathbf{Dif}$ given on objects $A \in \mathbf{Alg}$ by $G(A) = (DA, \partial_A)$ and on morphisms $h : A \to B$ in $\mathbf{Alg}$ by $G(h) = Dh$ as defined above. Letting $V : \mathbf{Dif} \to \mathbf{Alg}$ denote the forgetful functor defined on objects $(R, \mathrm{d}) \in \mathbf{Dif}$ by $V(R, \mathrm{d}) = R$ and on morphisms $f : (R, \mathrm{d}) \to (S, e)$ in $\mathbf{Dif}$ by $V(f) = f$, we have the following characterization of $G(A)$.

Proposition 2.8. *The functor $G : \mathbf{Alg} \to \mathbf{Dif}$ defined above is the right adjoint of the forgetful functor $V : \mathbf{Dif} \to \mathbf{Alg}$.*

Proof. By [22], it is equivalent to show that there are two natural transformations $\eta : \mathrm{id}_{\mathbf{Dif}} \to GV$ and $\varepsilon : VG \to \mathrm{id}_{\mathbf{Alg}}$ satisfying the equations $G\varepsilon \circ \eta G = G$ and $\varepsilon V \circ V\eta = V$. Here $\mathrm{id}_{\mathbf{Dif}}$ denotes the identity functor on $\mathbf{Dif}$, and similarly for $\mathrm{id}_{\mathbf{Alg}}$.

For any $A \in \mathbf{Alg}$, define $\varepsilon_A : DA \to A$ for any $f \in DA$ by $\varepsilon_A(f) = f(0)$. One checks that ε_A is a morphism of $\mathbf{k}$-algebras, and that if $h : A \to B$ is any morphism of $\mathbf{k}$-algebras, then $\varepsilon_B \circ Dh = h \circ \varepsilon_A$, i.e., ε is a natural transformation as desired.

For any $(R, \mathrm{d}) \in \mathbf{Dif}$, $x \in R$ and $n \in \mathbb{N}$, define $\eta_{(R,\mathrm{d})} : (R, \mathrm{d}) \to (DR, \partial_R)$ by $(\eta_{(R,\mathrm{d})}(x))(n) = \mathrm{d}^{(n)}(x)$. It is not difficult to see that $\eta_{(R,\mathrm{d})}$ is $\mathbf{k}$-linear, and it is immediate from Proposition 2.1(b) that for any $x, y \in R$, $(\eta_{(R,\mathrm{d})}(x))(\eta_{(R,\mathrm{d})}(y)) = (\eta_{(R,\mathrm{d})}(xy))$. Also, it is clear that $\partial_R \circ \eta_{(R,\mathrm{d})} = \eta_{(R,\mathrm{d})} \circ \mathrm{d}$, so that $\eta_{(R,\mathrm{d})}$ is a morphism in $\mathbf{Dif}$. Further, if $f : (R, \mathrm{d}) \to (S, e)$ is a morphism in $\mathbf{Dif}$, then one sees that $\eta_{(S,e)} \circ f = Df \circ \eta_{(R,\mathrm{d})}$. Hence η is a natural transformation.

To see that $G\varepsilon \circ \eta G = G$, let $A \in \mathbf{Alg}$, $f \in DA$ and $n \in \mathbb{N}$. Then

$$((D\varepsilon_A)(\eta_{(DA,\partial_A)}(f)))(n) = \varepsilon_A(\eta_{(DA,\partial_A)}(f)(n)) = \varepsilon_A(\partial_A^{(n)}(f)) = (\partial_A^{(n)}(f))(0) = f(n).$$

Similarly, to see that $\varepsilon V \circ V\eta = V$, let $(R, \mathrm{d}) \in \mathbf{Dif}$, and $x \in R$. Then $\varepsilon_R(\eta_{(R,\mathrm{d})}(x)) = (\eta_{(R,\mathrm{d})}(x))(0) = \mathrm{d}^{(0)}(x) = x$. ■

The following corollary gives a "universal mapping property" characterization of the λ-differential $\mathbf{k}$-algebra of λ-Hurwitz series as the cofree λ-differential $\mathbf{k}$-algebra on any $\mathbf{k}$-algebra A.

Corollary 2.9. *Let (R, d) be any λ-differential $\mathbf{k}$-algebra, and let A be any $\mathbf{k}$-algebra. For any $\mathbf{k}$-algebra homomorphism $f : R \to A$, there is a unique morphism of λ-differential $\mathbf{k}$-algebras $\tilde{f} : (R, \mathrm{d}) \to (DA, \partial_A)$ such that $\varepsilon_A \circ V\tilde{f} = f$.*

Proof. This follows from page 81, Theorem 2 in [22]. □

We next show that DA provides another example of differential Rota–Baxter algebras. Define

$$\pi_A : DA \to DA, \qquad (\pi_A(f))(n) = f(n-1), \quad n \geq 1, \qquad (\pi_A(f))(0) = 0, \quad f \in DA. \tag{15}$$

Proposition 2.10. *The triple (DA, ∂_A, π_A) is a differential Rota–Baxter algebra of weight λ.*

Proof. Since

$$(\pi_A(\partial_A(f)))(n) = (\pi_A(f))(n+1) = f(n)$$

for $f \in DA$, we have $\pi_A \circ \partial_A = \mathrm{id}_{DA}$. Thus we only need to verify that π_A is a Rota–Baxter operator of weight λ. Let $H \in DA$ be defined by

$$H = \pi_A(f)\pi_A(g) - \pi_A(\pi_A(f)g) - \pi_A(f\pi_A(g)) - \lambda\pi_A(fg). \tag{16}$$

By Proposition 2.7, we have $\partial_A(H) = 0$. Thus H is of the form $H : \mathbb{N} \to A$ with $H(n) = 0, n > 0$ and $H(0) = k$ for some $k \in \mathbf{k}$. But by definition, $\pi_A(0) = 0$. Thus $H(0) = 0$ and so $H = 0$. This shows that π_A is a Rota–Baxter operator of weight λ. □

3. Free commutative differential Rota–Baxter algebras

We briefly recall the construction of free commutative Rota–Baxter algebras in terms of mixable shuffles [16,17]. The mixable shuffle product is shown to be the same as the quasi-shuffle product of Hoffman [9,18,19]. Let A be a commutative $\mathbf{k}$-algebra. Define

$$Ш(A) = \bigoplus_{k\in\mathbb{N}} A^{\otimes(k+1)} = A \oplus A^{\otimes 2} \oplus \cdots.$$

Let $\mathfrak{a} = a_0 \otimes \cdots \otimes a_m \in A^{\otimes(m+1)}$ and $\mathfrak{b} = b_0 \otimes \cdots \otimes b_n \in A^{\otimes(n+1)}$. If $m = 0$ or $n = 0$, define

$$\mathfrak{a} \diamond \mathfrak{b} = \begin{cases} (a_0b_0) \otimes b_1 \otimes \cdots \otimes b_n, & m = 0, n > 0, \\ (a_0b_0) \otimes a_1 \otimes \cdots \otimes a_m, & m > 0, n = 0, \\ a_0b_0, & m = n = 0. \end{cases} \tag{17}$$

If $m > 0$ and $n > 0$, inductively (on $m + n$) define

$$\mathfrak{a} \diamond \mathfrak{b} = (a_0 b_0) \otimes ((a_1 \otimes a_2 \otimes \cdots \otimes a_m) \diamond (1 \otimes b_1 \otimes \cdots \otimes b_n) + (1 \otimes a_1 \otimes \cdots \otimes a_m) \diamond (b_1 \otimes \cdots \otimes b_n) + \lambda (a_1 \otimes \cdots \otimes a_m) \diamond (b_1 \otimes \cdots \otimes b_n)) . \tag{18}$$

Extending by additivity, we obtain a $\mathbf{k}$-bilinear map

$$\diamond : Ш(A) \times Ш(A) \to Ш(A),$$

called the mixable shuffle product on $Ш(A)$. Define a $\mathbf{k}$-linear endomorphism P_A on $Ш(A)$ by assigning

$$P_A(x_0 \otimes x_1 \otimes \cdots \otimes x_n) = \mathbf{1}_A \otimes x_0 \otimes x_1 \otimes \cdots \otimes x_n,$$

for all $x_0 \otimes x_1 \otimes \cdots \otimes x_n \in A^{\otimes(n+1)}$ and extending by additivity. Let $j_A : A \to Ш(A)$ be the canonical inclusion map.

Theorem 3.1 ([16,17]). *The pair $(Ш(A), P_A)$, together with the natural embedding $j_A : A \to Ш(A)$, is a free commutative Rota–Baxter $\mathbf{k}$-algebra on A of weight λ. In other words, for any Rota–Baxter $\mathbf{k}$-algebra (R, P) and any $\mathbf{k}$-algebra map $\varphi : A \to R$, there exists a unique Rota–Baxter $\mathbf{k}$-algebra homomorphism $\tilde{\varphi} : (Ш(A), P_A) \to (R, P)$ such that $\varphi = \tilde{\varphi} \circ j_A$ as $\mathbf{k}$-algebra homomorphisms.*

Since $\diamond$ is compatible with the multiplication in A, we will often suppress the symbol $\diamond$ and simply denote xy for $x \diamond y$ in $Ш(A)$, unless there is a danger of confusion.

Let (A, d_0) be a commutative differential $\mathbf{k}$-algebra of weight λ. Define an operator d_A on $Ш(A)$ by assigning

$$\mathrm{d}_A(x_0 \otimes x_1 \otimes \cdots \otimes x_n) = \mathrm{d}_0(x_0) \otimes x_1 \otimes \cdots \otimes x_n + x_0 x_1 \otimes x_2 \otimes \cdots \otimes x_n + \lambda \mathrm{d}_0(x_0) x_1 \otimes x_2 \otimes \cdots \otimes x_n$$

for $x_0 \otimes \cdots \otimes x_n \in A^{\otimes(n+1)}$ and then extending by $\mathbf{k}$-linearity. Here we use the convention that when $n = 0$, $\mathrm{d}_A(x_0) = \mathrm{d}_0(x_0)$.

Theorem 3.2. *Let (A, d_0) be a commutative differential $\mathbf{k}$-algebra of weight λ.*

(a) *$(Ш(A), \mathrm{d}_A, P_A)$ is a commutative differential Rota–Baxter $\mathbf{k}$-algebra of weight λ. The $\mathbf{k}$-algebra embedding*

$$j_A : A \to Ш(A)$$

is a morphism of differential $\mathbf{k}$-algebras of weight λ.

(b) *The quadruple $(Ш(A), \mathrm{d}_A, P_A, j_A)$ is a free commutative differential Rota–Baxter $\mathbf{k}$-algebra of weight λ on (A, d_0), as described by the following universal property: For any commutative differential Rota–Baxter $\mathbf{k}$-algebra (R, d, P) of weight λ and any λ-differential $\mathbf{k}$-algebra map $\varphi : (A, \mathrm{d}_0) \to (R, \mathrm{d})$, there exists a unique λ-differential Rota–Baxter $\mathbf{k}$-algebra homomorphism $\tilde{\varphi} : (Ш(A), \mathrm{d}_A, P_A) \to (R, \mathrm{d}, P)$ such that the diagram*

$$\begin{tikzcd} (A, \mathrm{d}_0) \arrow[r, "j_A"] \arrow[dr, "\varphi"'] & (Ш(A), \mathrm{d}_A) \arrow[d, "\tilde{\varphi}"] \\ & (R, \mathrm{d}) \end{tikzcd}$$

commutes in the category of commutative differential $\mathbf{k}$-algebras of weight λ.

(c) *Let X be a set and let $\mathbf{k}\{X\}$ be the free commutative differential algebra of weight λ on X. The quadruple $(Ш(\mathbf{k}\{X\}), \mathrm{d}_{\mathbf{k}\{X\}}, P_{\mathbf{k}\{X\}}, j_X)$ is a free commutative differential Baxter $\mathbf{k}$-algebra of weight λ on X, as described by the following universal property: For any commutative differential Rota–Baxter $\mathbf{k}$-algebra (R, d, P) of weight λ and any set map $\varphi : X \to R$, there exists a unique λ-differential Rota–Baxter $\mathbf{k}$-algebra homomorphism $\tilde{\varphi} : (Ш(\mathbf{k}\{X\}), \mathrm{d}_{\mathbf{k}\{X\}}, P_{\mathbf{k}\{X\}}) \to (R, \mathrm{d}, P)$ such that $\tilde{\varphi} \circ j_X = \varphi$.*

Proof. (a). For any $x = x_0 \otimes \cdots \otimes x_m \in A^{\otimes(m+1)}$, by definition we have

$$\mathrm{d}_A(P_A(x)) = \mathrm{d}_A(1 \otimes x_0 \otimes \cdots \otimes x_m) = x_0 \otimes \cdots \otimes x_m.$$

Thus $\mathrm{d}_A \circ P_A$ is the identity map on $Ш(A)$. So it remains to prove that for any $m, n \in \mathbb{N}_+$ and $x = x_0 \otimes \cdots \otimes x_m \in A^{\otimes(m+1)}$ and $y = y_0 \otimes \cdots \otimes y_n \in A^{\otimes(n+1)}$, we have

$$\mathrm{d}_A(x \diamond y) = \mathrm{d}_A(x) \diamond y + x \diamond \mathrm{d}_A(y) + \lambda \mathrm{d}_A(x) \diamond \mathrm{d}_A(y). \tag{19}$$

If $m = 0$ or $n = 0$, then the equation follows from the definition of d_A. Now consider the case when $m, n \in \mathbb{N}_+$. Denoting $x^+ = x_1 \otimes \cdots \otimes x_m$ and $y^+ = y_1 \otimes \ldots \otimes y_m$, we have $x = x_0 \diamond P_A(x^+)$, $y = y_0 \diamond P_A(y^+)$ and Eq. (18) can be rewritten as

$$\begin{aligned} x \diamond y &= (x_0 y_0) \diamond (P_A(x^+) \diamond P_A(y^+)) \\ &= (x_0 y_0) \diamond \big(P_A(x^+ \diamond P_A(y^+)) + P_A(y^+ \diamond P_A(x^+)) + \lambda P_A(x^+ \diamond y^+)\big) \\ &= (x_0 y_0) \diamond P_A(x^+ \diamond P_A(y^+) + y^+ \diamond P_A(x^+) + \lambda(x^+ \diamond y^+)). \end{aligned}$$

It follows from the definition of d_A that, for any $z_0 \in A$ and $z \in Ш(A)$,

$$\begin{aligned} \mathrm{d}_A(z_0 \diamond P_A(z)) &= \mathrm{d}_0(z_0) \diamond P_A(z) + z_0 \diamond \mathrm{d}_A(P_A(z)) + \lambda \mathrm{d}_0(z_0) \diamond \mathrm{d}_A(P_A(z)) \\ &= \mathrm{d}_0(z_0) \diamond P_A(z) + z_0 \diamond z + \lambda \mathrm{d}_0(z_0) \diamond z. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{d}_A(x \diamond y) &= \mathrm{d}_A((x_0 y_0) \diamond P_A(x^+ \diamond P_A(y^+) + y^+ \diamond P_A(x^+) + \lambda(x^+ \diamond y^+))) \\ &= \mathrm{d}_0(x_0 y_0) \diamond P_A(x^+ \diamond P_A(y^+) + y^+ \diamond P_A(x^+) + \lambda(x^+ \diamond y^+)) \\ &\quad + (x_0 y_0) \diamond (x^+ \diamond P_A(y^+) + y^+ \diamond P_A(x^+) + \lambda(x^+ \diamond y^+)) \\ &\quad + \lambda \mathrm{d}_0(x_0 y_0) \diamond (x^+ \diamond P_A(y^+) + y^+ \diamond P_A(x^+) + \lambda(x^+ \diamond y^+)) \\ &= (\mathrm{d}_0(x_0) y_0 + x_0 \mathrm{d}_0(y_0) + \lambda \mathrm{d}_0(x_0) \mathrm{d}_0(y_0)) \diamond (P_A(x^+) \diamond P_A(y^+)) \\ &\quad + (x_0 y_0) \diamond (x^+ \diamond P_A(y^+) + y^+ \diamond P_A(x^+) + x^+ \diamond y^+) \\ &\quad + \lambda(\mathrm{d}_0(x_0) y_0 + x_0 \mathrm{d}_0(y_0) + \lambda \mathrm{d}_0(x_0) \mathrm{d}_0(y_0)) \diamond (x^+ \diamond P_A(y^+) + y^+ \diamond P_A(x^+) + x^+ \diamond y^+). \end{aligned}$$

Also

$$\begin{aligned} & x \diamond \mathrm{d}_A(y) + y \diamond \mathrm{d}_A(x) + \lambda \mathrm{d}_A(x) \diamond \mathrm{d}_A(y) \\ &\quad = (x_0 \diamond P_A(x^+)) \diamond \mathrm{d}_A(y_0 \diamond P_A(y^+)) + (y_0 \diamond P_A(y^+)) \diamond \mathrm{d}_A(x_0 \diamond P_A(x^+)) \\ &\qquad + \lambda \mathrm{d}_A(x_0 \diamond P_A(x^+)) \diamond \mathrm{d}_A(y_0 \diamond P_A(y^+)) \\ &\quad = (x_0 \diamond P_A(x^+)) \diamond \big(\mathrm{d}_0(y_0) \diamond P_A(y^+) + y_0 \diamond y^+ + \lambda \mathrm{d}_0(y_0) \diamond y^+\big) \\ &\qquad + (y_0 \diamond P_A(y^+)) \diamond \big(\mathrm{d}_0(x_0) \diamond P_A(x^+) + x_0 \diamond x^+ + \lambda \mathrm{d}_0(x_0) \diamond x^+\big) \\ &\qquad + \lambda \left(\mathrm{d}_0(x_0) \diamond P_A(x^+) x_0 \diamond x^+ + \lambda \mathrm{d}_0(x_0) \diamond x^+\right) \diamond \left(\mathrm{d}_0(y_0) \diamond P_A(y^+) y_0 \diamond y^+ + \lambda \mathrm{d}_0(y_0) \diamond y^+\right). \end{aligned}$$

Comparing the last terms of the above two equations, we see that Eq. (19) holds.

The second statement follows directly from the definition of d_A.

(b). Now let (R, d, P) be a commutative differential Rota–Baxter **k**-algebra of weight λ and let $\varphi : (A, \mathrm{d}_0) \to (R, \mathrm{d})$ be a λ-differential **k**-algebra map. Then in particular φ is a **k**-algebra map. So by Theorem 3.1, there is a unique Rota–Baxter **k**-algebra map $\tilde{\varphi} : (Ш(A), P_A) \to (R, P)$ such that

$$\varphi = \tilde{\varphi} \circ j_A \tag{20}$$

in the category of **k**-algebras. We next show that $\tilde{\varphi}$ is a differential **k**-algebra map.

For any $x_0 \otimes x_1 \otimes \cdots \otimes x_n \in A^{\otimes(n+1)}$, we have

$$
\begin{aligned}
\mathrm{d}(\tilde{\varphi}(x_0 \otimes x_1 \otimes \cdots \otimes x_n)) &= \mathrm{d}(\tilde{\varphi}(x_0 \diamond P_A(x_1 \otimes \cdots \otimes x_n))) \\
&= \mathrm{d}(\tilde{\varphi}(x_0)\tilde{\varphi}(P_A(x_1 \otimes \cdots \otimes x_n))) \\
&= \mathrm{d}(\tilde{\varphi}(x_0) P(\tilde{\varphi}(x_1 \otimes \cdots \otimes x_n))) \\
&= \mathrm{d}(\tilde{\varphi}(x_0)) P(\tilde{\varphi}(x_1 \otimes \cdots \otimes x_n)) + \tilde{\varphi}(x_0)\mathrm{d}(P(\tilde{\varphi}(x_1 \otimes \cdots \otimes x_n))) \\
&\quad + \lambda \mathrm{d}(\tilde{\varphi}(x_0))\mathrm{d}(P(\tilde{\varphi}(x_1 \otimes \cdots \otimes x_n))) \\
&= \mathrm{d}(\tilde{\varphi}(x_0))\tilde{\varphi}(P_A(x_1 \otimes \cdots \otimes x_n)) + \tilde{\varphi}(x_0)\tilde{\varphi}(x_1 \otimes \cdots \otimes x_n) \\
&\quad + \lambda \mathrm{d}(\tilde{\varphi}(x_0))\tilde{\varphi}(x_1 \otimes \cdots \otimes x_n).
\end{aligned}
$$

Also

$$
\begin{aligned}
\tilde{\varphi}(\mathrm{d}_A(x_0 \otimes x_1 \otimes \cdots \otimes x_n)) &= \tilde{\varphi}(\mathrm{d}_A(x_0 \diamond P_A(x_1 \otimes \cdots \otimes x_n))) \\
&= \tilde{\varphi}(\mathrm{d}_A(x_0) \diamond P_A(x_1 \otimes \cdots \otimes x_n) + x_0 \diamond \mathrm{d}_A(P_A(x_1 \otimes \cdots \otimes x_n)) \\
&\quad + \lambda \mathrm{d}_A(x_0) \diamond \mathrm{d}_A(P_A(x_1 \otimes \cdots \otimes x_n))) \\
&= \tilde{\varphi}(\mathrm{d}_A(x_0) \diamond P_A(x_1 \otimes \cdots \otimes x_n) + x_0 \diamond (x_1 \otimes \cdots \otimes x_n) \\
&\quad + \lambda \mathrm{d}_A(x_0) \diamond (x_1 \otimes \cdots \otimes x_n)) \\
&= \tilde{\varphi}(\mathrm{d}_A(x_0))\tilde{\varphi}(P_A(x_1 \otimes \cdots \otimes x_n)) + \tilde{\varphi}(x_0)\tilde{\varphi}(x_1 \otimes \cdots \otimes x_n) \\
&\quad + \lambda \tilde{\varphi}(\mathrm{d}_A(x_0))\tilde{\varphi}(x_1 \otimes \cdots \otimes x_n).
\end{aligned}
$$

Since

$$
\begin{aligned}
\mathrm{d}(\tilde{\varphi}(x_0)) &= \mathrm{d}(\tilde{\varphi}(j_A(x_0))) = \mathrm{d}(\varphi(x_0)) = \varphi(\mathrm{d}_0(x_0)) \\
&= \tilde{\varphi}(j_A(\mathrm{d}_0(x_0))) = \tilde{\varphi}(\mathrm{d}_A(j_A(x_0))) = \tilde{\varphi}(\mathrm{d}_A(x_0)),
\end{aligned}
$$

we have proved that

$$
\mathrm{d}(\tilde{\varphi}(x_0 \otimes x_1 \otimes \cdots \otimes x_n)) = \tilde{\varphi}(\mathrm{d}_A(x_0 \otimes x_1 \otimes \cdots \otimes x_n)).
$$

This shows that $\tilde{\varphi}$ is a differential **k**-algebra homomorphism. Since φ and j_A are differential **k**-algebra homomorphisms, we see that Eq. (2) holds in the category of differential **k**-algebras.
(c). The forgetful functor from the category $\mathbf{DRB}_\lambda$ to the category **Set** of sets is the composition of the forgetful functors from $\mathbf{DRB}_\lambda$ to **Alg** and from **Alg** to **Set**. By Theorem 1 in page 101 of [22], the adjoint functor of a composed functor is the composition of the adjoint functors. This proves (c). □

4. Free noncommutative differential Rota–Baxter algebras

We now consider the noncommutative analog of Section 3.

4.1. Free noncommutative Rota–Baxter algebras

We first summarize the construction of free noncommutative Rota–Baxter algebras on a set X in terms of angularly decorated planar rooted trees. See [11] (as well as [2]) for further details.

4.1.1. Rota–Baxter algebra on rooted trees

We follow the notations and terminologies in [8,28]. A free tree is an undirected graph that is connected and contains no cycles. A **rooted tree** is a free tree in which a particular vertex has been distinguished as the **root**. A **planar rooted tree** (also called an **ordered rooted tree**) is a rooted tree with a fixed embedding into the plane. For example,

The depth $d(T)$ of a rooted tree T is the length of the longest path from its root to its leaves.

Let $\mathcal{T}$ be the set of planar rooted trees. A **planar rooted forest** is a noncommutative concatenation of planar rooted trees, denoted by $T_1 \sqcup \cdots \sqcup T_b$ with $T_1, \ldots, T_b \in \mathcal{T}$. $b = b(F)$ is called the breadth of F. The depth $d(F)$ of F is the maximal depth of the trees T_i, $1 \le i \le b$. Let $\mathcal{F}$ be the set of planar rooted forests. Then $\mathcal{F}$ is the free semigroup generated by $\mathcal{T}$ with the product $\sqcup$, and $\mathbf{k}\,\mathcal{F}$ with the product $\sqcup$ is the free noncommutative nonunitary **k**-algebra on the alphabet set $\mathcal{T}$. We are going to define, for each fixed $\lambda \in \mathbf{k}$, another product $\diamond = \diamond_\lambda$ on $\mathbf{k}\,\mathcal{F}$, making it into a unitary Rota–Baxter algebra (of weight λ). We will suppress λ to ease notation.

For the rest of this paper, a tree or forest means a planar rooted tree or a planar rooted forest unless otherwise specified. Let $\lfloor T_1 \sqcup \cdots \sqcup T_b \rfloor$ denote the usual **grafting** of the trees $T_1, \ldots, T_b$ by adding a new root together with an edge from the new root to the root of each of the trees $T_1, \ldots, T_b$. If a tree F is the grafting of a forest, we denote that forest by $\overline{F}$. This is not the algebraic closure of a field F. Our notation should not cause any confusion since neither fields nor algebraic closure will appear in this paper.

Let $\mathcal{F}_n$, $n \ge 0$, be the set of planar rooted forests with depth less than or equal to n. Then we have the depth filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$ such that $\mathcal{F} = \cup_{n \in \mathbb{N}} \mathcal{F}_n$. By using the grafting and the filtration $\mathcal{F}_n$, we recursively defined in [11] a map

$$\diamond : \mathcal{F} \times \mathcal{F} \to \mathbf{k}\mathcal{F}$$

with the following properties

(a) For trees F and F',

$$F \diamond F' = \begin{cases} F, & \text{if } F' = \bullet, \\ F', & \text{if } F = \bullet, \\ \lfloor \overline{F} \diamond \lfloor \overline{F}' \rfloor \rfloor + \lfloor \lfloor \overline{F} \rfloor \diamond \overline{F}' \rfloor + \lambda \lfloor \overline{F} \diamond \overline{F}' \rfloor, & \text{if } F = \lfloor \overline{F} \rfloor, F' = \lfloor \overline{F}' \rfloor. \end{cases} \tag{21}$$

(b) For forests $F = T_1 \sqcup \cdots \sqcup T_b$ and $F' = T'_1 \sqcup \cdots \sqcup T'_{b'}$,

$$F \diamond F' = T_1 \sqcup \cdots \sqcup T_{b-1} \sqcup (T_b \diamond T'_1) \sqcup T'_2 \cdots \sqcup T_{b'}. \tag{22}$$

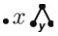
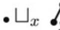
Then $\diamond$ extends to a binary operation $\diamond$ on $\mathbf{k}\mathcal{F}$ by bilinearity. As an example, we have


(23)

It was shown in [11] that $(\mathbf{k}\mathcal{F}, \diamond)$ is a Rota–Baxter **k**-algebra.

4.1.2. Free Rota–Baxter algebra on a set X

Let X be a non-empty set. Let $F \in \mathcal{F}$ with $\ell = \ell(F)$ leaves. Let X^F denote the set of pairs $(F; \vec{x})$ where $\vec{x}$ is in $X^{(\ell(F)-1)}$. Such a pair $(F; \vec{x})$ is called an **angularly decorated rooted forest** since $(F; \vec{x})$ can be identified with the forest F together with an ordered decoration of $\vec{x}$ on the angles of F. We use the convention that $X^\bullet = \{(\bullet; 1)\}$. For example, we have

 is denoted by  in [11].

Let $(F; \vec{x}) \in X^F$. Let $F = T_1 \sqcup \cdots \sqcup T_b$ be the decomposition of F into trees. We consider the corresponding decomposition of decorated forests. If $b = 1$, then F is a tree and $(F; \vec{x})$ has no further decompositions. If $b > 1$, denote $\ell_i = \ell(T_i)$, $1 \le i \le b$, to be the number of leaves of T_i. Then

$$(T_1; (x_1, \ldots, x_{\ell_1 - 1})), (T_2; (x_{\ell_1+1}, \ldots, x_{\ell_1+\ell_2-1})), \ldots, (T_b; (x_{\ell_1+\cdots+\ell_{b-1}+1}, \ldots, x_{\ell_1+\cdots+\ell_b}))$$

are well-defined angularly decorated trees when $\ell(T_i) > 1$. If $\ell(T_i) = 1$, then $x_{\ell_{i-1}+\ell_i-1} = x_{\ell_{i-1}}$ and we use the convention $(T_i; x_{\ell_{i-1}+\ell_i-1}) = (T_i; \mathbf{1})$. With this convention, we have,

$$\begin{aligned}(F; (x_1, \ldots, x_{\ell-1})) = {} & (T_1; (x_1, \ldots, x_{\ell_1-1}))x_{\ell_1}(T_2; (x_{\ell_1+1}, \ldots, x_{\ell_1+\ell_2-1}))x_{\ell_1+\ell_2} \\ & \cdots x_{\ell_1+\cdots+\ell_{b-1}}(T_b; (x_{\ell_1+\cdots+\ell_{b-1}+1}, \ldots, x_{\ell_1+\cdots+\ell_b})).\end{aligned}$$

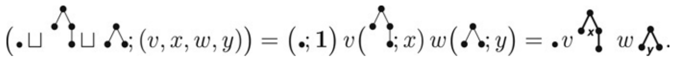
We call this the **standard decomposition** of $(F; \vec{x})$ and abbreviate it as

$$(F; \vec{x}) = (T_1; \vec{x}_1)x_{i_1}(T_2; \vec{x}_2)x_{i_2} \cdots x_{i_{b-1}}(T_b; \vec{x}_b) = D_1 x_{i_1} D_2 x_{i_2} \cdots x_{i_{b-1}} D_b \tag{24}$$

where $D_i = (T_i; \vec{x}_i)$, $1 \le i \le b$. So $i_j = \ell_1 + \cdots + \ell_j$ and

$$\vec{x}_j = \begin{cases} (x_{\ell_1+\cdots+\ell_{j-1}+1}, \ldots, x_{\ell_1+\cdots+\ell_j-1}), & \ell_j > 1, \\ \mathbf{1}, & \ell_j = 1. \end{cases}$$

For example,



Let $\mathbf{k}\langle X\rangle = \bigoplus_{n\ge 0} \mathbf{k}\, X^n$ be the noncommutative polynomial algebra on X. Denote its basis elements by vectors and its product by vector concatenation: for $\vec{x} = (x_1, \ldots, x_m)$, $\vec{x}\,' = (x_1', \ldots, x_n')$, define

$$(\vec{x}, \vec{x}\,') = (x_1, \ldots, x_m, x_1', \ldots, x_n').$$

Define the $\mathbf{k}$-module

$$Ш^{NC}(X) = \bigoplus_{F\in\mathcal{F}} X^F.$$

For $D = (F; \vec{x}) \in X^F$ and $D' = (F'; \vec{x}\,') \in X^{F'}$, define

$$D \bar{\diamond} D' = (F \diamond F'; (\vec{x}, \vec{x}\,')), \tag{25}$$

where $\diamond$ is defined in Eqs. (21) and (22). For example, from Eq. (23) we have

(26)

Extending the product $\bar{\diamond}$ bi-additively, we obtain a binary operation

$$\bar{\diamond} : Ш^{NC}(X) \otimes Ш^{NC}(X) \to Ш^{NC}(X).$$

For $(F; \vec{x}) \in X^F$, define

$$P_X(F; \vec{x}) = \lfloor (F; \vec{x}) \rfloor = (\lfloor F \rfloor; \vec{x}) \in X^{\lfloor F \rfloor}, \tag{27}$$

extending to a linear operator on $Ш^{NC}(X)$. Let

$$j_X : X \to Ш^{NC}(X) \tag{28}$$

be the map sending $a \in X$ to $(\bullet \sqcup \bullet; a)$. The following theorem is proved in [11].

Theorem 4.1. *The quadruple $(Ш^{NC}(X), \bar{\diamond}, P_X, j_X)$ is the free unitary Rota–Baxter algebra of weight λ on the set X. More precisely, for any unitary Rota–Baxter algebra (R, P) and map $f : X \to R$, there is a unique unitary Rota–Baxter algebra morphism $\bar{f} : Ш^{NC}(X) \to R$ such that $f = \bar{f} \circ j_X$.*

4.2. Free noncommutative differential Rota–Baxter algebras

The following is the noncommutative analog of Theorem 3.2.

Theorem 4.2. *Let $(\mathbf{k}^{NC}\{X\}, \mathrm{d}_X^{NC}) = (\mathbf{k}\langle \Delta X\rangle, \mathrm{d}_X^{NC})$ be the free noncommutative differential algebra of weight λ on a set X, constructed in Theorem 2.5. Let $Ш^{NC}(\Delta X)$ be the free noncommutative Rota–Baxter algebra of weight λ on ΔX, constructed in Theorem 4.1.*

(a) *There is a unique extension $\bar{\mathrm{d}}_X^{NC}$ of d_X^{NC} to $Ш^{NC}(\Delta X)$ so that $(Ш^{NC}(\Delta X), \bar{\mathrm{d}}_X^{NC}, P_{\Delta X})$ is a differential Rota–Baxter algebra of weight λ.*
(b) *The differential Rota–Baxter algebra $Ш^{NC}(\Delta X)$ thus obtained is the free differential Rota–Baxter algebra of weight λ over X.*

Proof. (a). We define a λ-derivation $\bar{\mathrm{d}}_X^{NC}$ on $Ш^{NC}(\Delta X)$ as follows. Let $F \in \mathcal{F}$ and let $D \in (\Delta X)^F$ be the forest F with angular decoration by $\vec{y} \in (\Delta X)^{\ell(F)-1}$. Let

$$D = (F; \vec{y}) = (T_1; \vec{y}_1) y_{i_1} (T_2; \vec{y}_2) y_{i_2} \cdots y_{i_{b-1}} (T_b; \vec{y}_b)$$

be the standard decomposition of D in Eq. (24). We define $\bar{\mathrm{d}}_X^{NC}$ by induction on the breadth $b = b(F)$ of F. If $b = 1$, then F is a tree so either $F = \bullet$ or $F = \lfloor \overline{F} \rfloor$ for a forest $\overline{F}$. Accordingly we define

$$\bar{\mathrm{d}}_X^{NC}(F; \vec{y}) = \begin{cases} 0, & \text{if } F = \bullet, \\ (\overline{F}; \vec{y}), & \text{if } F = \lfloor \overline{F} \rfloor. \end{cases} \tag{29}$$

We note that this is the only way to define $\bar{\mathrm{d}}_X^{NC}$ in order to obtain a differential Rota–Baxter algebra since $\bullet$ is the identity and $(F; \vec{y}) = \lfloor (\overline{F}; \vec{y}) \rfloor$.

If $b > 1$, then $F = T_1 \sqcup F_t$ for another forest $F_t = T_2 \sqcup \cdots \sqcup F_b$ (t in F_t stands for the tail). So

$$D = (F; \vec{y}) = (T_1; \vec{y}_1) y_{i_1} (F_t; \vec{y}_t) = D_1 y_{i_1} D_t$$

where $D_1 = (T_1; \vec{y}_1)$ and $D_t = (T_2; \vec{y}_2) y_{i_2} \cdots y_{i_{b-1}} (T_b; \vec{y}_b)$. We then define

$$\begin{aligned} \bar{\mathrm{d}}_X^{NC}(D) = {} & \bar{\mathrm{d}}_X^{NC}(T_1; \vec{y}_1) y_{i_1} (F_t; \vec{y}_t) + (T_1; \vec{y}_1) \mathrm{d}(y_{i_1}) (F_t; \vec{y}_t) + (T_1; \vec{y}_1) y_{i_1} \bar{\mathrm{d}}_X^{NC}(F_t; \vec{y}_t) \\ & + \lambda \Big(\bar{\mathrm{d}}_X^{NC}(T_1; \vec{y}_1) d(y_{i_1}) (F_t; \vec{y}_t) + \bar{\mathrm{d}}_X^{NC}(T_1; \vec{y}_1) y_{i_1} \bar{\mathrm{d}}_X^{NC}(F_t; \vec{y}_t) \\ & + (T_1; \vec{y}_1) \mathrm{d}(y_{i_1}) \bar{\mathrm{d}}_X^{NC}(F_t; \vec{y}_t) \Big) + \lambda^2 \bar{\mathrm{d}}_X^{NC}(T_1; \vec{y}_1) \mathrm{d}(y_{i_1}) \bar{\mathrm{d}}_X^{NC}(F_t; \vec{y}_t), \end{aligned} \tag{30}$$

where $\bar{\mathrm{d}}_X^{NC}(T_1; \vec{y}_1)$ is defined in Eq. (29) and $\bar{\mathrm{d}}_X^{NC}(F_t; \vec{y}_t)$ is defined by the induction hypothesis. Note that by Eq. (22),

$$(T_1; \vec{y}_1) y_{i_1} (F_t; \vec{y}_t) = (T_1; \vec{y}_1) \diamond (\bullet y_{i_1} \bullet) \diamond (F_t; \vec{y}_t).$$

So if $\bar{\mathrm{d}}_X^{NC}$ were to satisfy the λ-Leibniz rule Eq. (2) with respect to the product $\overline{\diamond}$, then we must have

$$\begin{aligned} \bar{\mathrm{d}}_X^{NC}(D) = {} & \bar{\mathrm{d}}_X^{NC}(T_1; \vec{y}_1) \diamond (\bullet y_{i_1} \bullet) \diamond (F_t; \vec{y}_t) + (T_1; \vec{y}_1) \diamond \bar{\mathrm{d}}_X^{NC}(\bullet y_{i_1} \bullet) \diamond (F_t; \vec{y}_t) \\ & + (T_1; \vec{y}_1) \diamond (\bullet y_{i_1} \bullet) \diamond \bar{\mathrm{d}}_X^{NC}(F_t; \vec{y}_t) + \lambda \bar{\mathrm{d}}_X^{NC}(T_1; \vec{y}_1) \diamond (\bullet \mathrm{d}_X^{NC}(y_{i_1}) \bullet) \diamond (F_t; \vec{y}_t) \\ & + \lambda \bar{\mathrm{d}}_X^{NC}(T_1; \vec{y}_1) \diamond (\bullet y_{i_1} \bullet) \diamond \bar{\mathrm{d}}_X^{NC}(F_t; \vec{y}_t) + \lambda (T_1; \vec{y}_1) \diamond (\bullet \mathrm{d}_X^{NC}(y_{i_1}) \bullet) \diamond \bar{\mathrm{d}}_X^{NC}(F_t; \vec{y}_t) \\ & + \lambda^2 \bar{\mathrm{d}}_X^{NC}(T_1; \vec{y}_1) \diamond (\bullet \mathrm{d}_X^{NC}(y_{i_1}) \bullet) \diamond \bar{\mathrm{d}}_X^{NC}(F_t; \vec{y}_t). \end{aligned} \tag{31}$$

Since $\bar{\mathrm{d}}_X^{NC}$ is to extend $\mathrm{d}_X^{NC} : \mathbf{k}^{NC}\{X\} \to \mathbf{k}^{NC}\{X\}$, we have

$$\bar{\mathrm{d}}_X^{NC}(\bullet y_{i_1} \bullet) = \bar{\mathrm{d}}_X^{NC}(j_{\Delta X}(y_{i_1})) = j_{\Delta X}(\mathrm{d}_X^{NC}(y_{i_1})) = \bullet \mathrm{d}_X^{NC}(y_{i_1}) \bullet .$$

So by Eq. (22), Eq. (31) agrees with Eq. (30). Thus $\bar{\mathrm{d}}_X^{NC}(D)$ is the unique map that satisfies the λ-Leibniz rule (2).

We also have the short-hand notation,

$$\bar{\mathrm{d}}_X^{NC}(D) = \bar{\mathrm{d}}_X^{NC}(D_1) y_{i_1} D_t + D_1 \bar{\mathrm{d}}_X^{NC}(y_{i_1} D_t) + \lambda \bar{\mathrm{d}}_X^{NC}(D_1) \bar{\mathrm{d}}_X^{NC}(y_{i_1} D_t), \tag{32}$$

where

$$\bar{\mathrm{d}}_X^{NC}(y_{i_1} D_t) := \mathrm{d}_X^{NC}(y_{i_1}) D_t + y_{i_1} \bar{\mathrm{d}}_X^{NC}(D_t) + \lambda \mathrm{d}_X^{NC}(y_{i_1}) \bar{\mathrm{d}}_X^{NC}(D_t).$$

Similarly, we can also write $D = D_h y_{i_{b-1}} D_b$ where D_h (h stands for the head) is an angularly decorated forest and D_b is an angularly decorated tree. Then

$$\bar{\mathrm{d}}_X^{NC}(D) = \bar{\mathrm{d}}_X^{NC}(D_h y_{i_{b-1}}) D_b + D_h y_{i_{b-1}} \bar{\mathrm{d}}_X^{NC}(D_b) + \lambda \bar{\mathrm{d}}_X^{NC}(D_h y_{i_{b-1}}) \bar{\mathrm{d}}_X^{NC}(D_b). \tag{33}$$

In fact, write

$$D = v_1 v_2 \cdots v_{2b-1},$$

where

$$v_j = \begin{cases} D_{(j-1)/2}, & j \text{ odd}, \\ y_{i_{j/2}}, & j \text{ even}. \end{cases}$$

Then using Eq. (30) and an induction on b, we obtain the "general Leibniz formula" of weight λ with respect to the concatenation product:

$$\bar{\mathrm{d}}_X^{NC}(D) = \sum_{I \subseteq [2b-1]} \lambda^{|I|-1} v_{I,1} v_{I,2} \cdots v_{I,2b-1}, \tag{34}$$

where $[2b-1] = \{1, \ldots, 2b-1\}$ and

$$v_{I,j} = \begin{cases} v_j, & j \notin I, \\ \bar{\mathrm{d}}_X^{NC}(v_j), & j \in I,\ j \text{ odd}, \\ \mathrm{d}_X^{NC}(v_j), & j \in I,\ j \text{ even}. \end{cases}$$

We now prove that $\bar{\mathrm{d}}_X^{NC}$ is a derivation of weight λ with respect to the product $\bar{\diamond}$. Let D and D' be angularly decorated forests and write

$$D = (F; \vec{y}) = (T_1; \vec{y}_1) y_{i_1} (T_2; \vec{y}_2) y_{i_2} \cdots y_{i_{b-1}} (T_b; \vec{y}_b) = D_h y_{i_{b-1}} D_b$$

and

$$D' = (F'; \vec{y}\,') = (T_1'; \vec{y}_1') y_{i_1}' (T_2'; \vec{y}_2') y_{i_2}' \cdots y_{i_{b'-1}}' (T_{b'}'; \vec{y}_{b'}') = D_1' y_{i_1}' D_t'$$

be as above with angularly decorated trees D_b, D_1', angularly decorated forests D_h, D_t' and $y_{i_{b-1}}, y_{i_1}' \in \Delta X$. Then by Eq. (22) (see [11] for further details), $D \bar{\diamond} D'$ has the standard decomposition

$$\begin{aligned} D \bar{\diamond} D' &= (T_1; \vec{y}_1) y_{i_1} \cdots y_{i_{b-1}} \left((T_b; \vec{y}_b) \bar{\diamond} (T_1'; \vec{y}_1') \right) y_{i_1}' \cdots y_{i_{b'-1}}' (T_{b'}'; \vec{y}_{b'}') \\ &= D_h y_{i_{b-1}} (D_b \diamond D_1') y_{i_1}' D_t' \end{aligned} \tag{35}$$

where

$$\begin{aligned} D_b \diamond D_1' &= (T_b; \vec{y}_b) \bar{\diamond} (T_1'; \vec{y}_1') \\ &= \begin{cases} (\bullet; \mathbf{1}), & \text{if } T_b = T_1' = \bullet \text{ (so } \vec{y}_b = \vec{y}_1' = \mathbf{1}), \\ (T_b, \vec{y}_b), & \text{if } T_1' = \bullet,\ T_b \neq \bullet, \\ (T_1', \vec{y}_1'), & \text{if } T_1' \neq \bullet,\ T_b = \bullet, \\ \lfloor (T_b; \vec{y}) \bar{\diamond} (\overline{F}_1'; \vec{y}\,') \rfloor + \lfloor (\overline{F}_b; \vec{y}) \bar{\diamond} (T_1'; \vec{y}\,') \rfloor & \\ \quad + \lambda \lfloor (\overline{F}_b; \vec{y}) \bar{\diamond} (\overline{F}_1'; \vec{y}\,') \rfloor, & \text{if } T_1' = \lfloor \overline{F}_1' \rfloor \neq \bullet,\ T_b = \lfloor \overline{F}_b \rfloor \neq \bullet. \end{cases} \end{aligned} \tag{36}$$

By Eqs. (35) and (34), we have

$$\begin{aligned} \bar{\mathrm{d}}_X^{NC}(D \bar{\diamond} D') &= \bar{\mathrm{d}}_X^{NC} \left((D_h y_{i_{b-1}})(D_b \bar{\diamond} D_1')(y_{i_1}' D_{b'}') \right) \\ &= \bar{\mathrm{d}}_X^{NC}(D_h y_{i_{b-1}})(D_b \bar{\diamond} D_1')(y_{i_1}' D_{b'}') + (D_h y_{i_{b-1}}) \bar{\mathrm{d}}_X^{NC}(D_b \bar{\diamond} D_1')(y_{i_1}' D_{b'}') \\ &\quad + (D_h y_{i_{b-1}})(D_b \bar{\diamond} D_1') \bar{\mathrm{d}}_X^{NC}(y_{i_1}' D_{b'}') + \lambda \bar{\mathrm{d}}_X^{NC}(D_h y_{i_{b-1}}) \bar{\mathrm{d}}_X^{NC}(D_b \bar{\diamond} D_1')(y_{i_1}' D_{b'}') \\ &\quad + \lambda \bar{\mathrm{d}}_X^{NC}(D_h y_{i_{b-1}})(D_b \bar{\diamond} D_1') \bar{\mathrm{d}}_X^{NC}(y_{i_1}' D_{b'}') + \lambda (D_h y_{i_{b-1}}) \bar{\mathrm{d}}_X^{NC}(D_b \bar{\diamond} D_1') \bar{\mathrm{d}}_X^{NC}(y_{i_1}' D_{b'}') \\ &\quad + \lambda^2 \bar{\mathrm{d}}_X^{NC}(D_h y_{i_{b-1}}) \bar{\mathrm{d}}_X^{NC}(D_b \bar{\diamond} D_1') \bar{\mathrm{d}}_X^{NC}(y_{i_1}' D_{b'}'). \end{aligned} \tag{37}$$

Using Eq. (36), we have

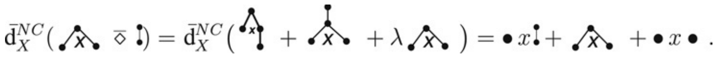
$$\bar{\mathrm{d}}_X^{NC}(D_b \bar{\diamond} D_1') = \bar{\mathrm{d}}_X^{NC}(D_b) \bar{\diamond} D_1' + D_b \bar{\diamond} \bar{\mathrm{d}}_X^{NC}(D_1') + \lambda \bar{\mathrm{d}}_X^{NC}(D_b) \bar{\diamond} \bar{\mathrm{d}}_X^{NC}(D_1'). \tag{38}$$

Applying this to Eq. (37), we find that the resulting expansion for $\bar{\mathrm{d}}_X^{NC}(D \bar{\diamond} D')$ agrees with the expansion of

$$\bar{\mathrm{d}}_X^{NC}(D) \bar{\diamond} D' + D \bar{\diamond} \bar{\mathrm{d}}_X^{NC}(D') + \lambda \bar{\mathrm{d}}_X^{NC}(D) \bar{\diamond} \bar{\mathrm{d}}_X^{NC}(D')$$

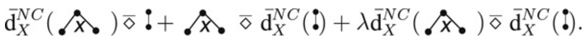
after applying Eq. (32) to $\bar{\mathrm{d}}_X^{NC}(D)$ and applying Eq. (33) to $\bar{\mathrm{d}}_X^{NC}(D')$.

As an example, from Eq. (26), we have



(39)

This agrees with



(b). The proof of the freeness of $Ш^{NC}(D(X))$ as a free differential Rota–Baxter algebra of weight λ is the same as the proof of the freeness of $Ш(D(X))$ in Theorem 3.2. $\square$

5. Structure of a differential algebra on forests

We now give the structure of a differential Rota–Baxter algebra of weight λ to rooted forests without decorations. It should be possible to derive this as a special case from a suitable generalization of the construction in Theorem 4.2. To avoid making the process too complicated, we give a direct construction. See [15] for the work of Grossman and Larson on differential algebra structures on their Hope algebra of trees.

Let $(\mathbf{k}\mathcal{F}, \diamond, \lfloor\rfloor)$ be the Rota–Baxter algebra of planar rooted forests defined in Section 4.1.1. Let $F \in \mathcal{F}$ be a rooted forest. By Eq. (22), the unique decomposition $F = T_1 \sqcup \cdots \sqcup T_b$ into rooted trees $T_1, \ldots, T_b \in \mathcal{T}$ gives the decomposition

$$F = T_1 \diamond (\bullet \sqcup \bullet) \diamond T_2 \diamond (\bullet \sqcup \bullet) \diamond \cdots \diamond (\bullet \sqcup \bullet) \diamond T_b. \tag{40}$$

Denote this by

$$F = V_1 \diamond V_2 \diamond \cdots \diamond V_{2b-1}, \tag{41}$$

where

$$V_i = \begin{cases} T_{(i+1)/2}, & i \text{ odd}, \\ (\bullet \sqcup \bullet), & i \text{ even}. \end{cases}$$

We call Eq. (41) the **$\diamond$-standard decomposition** of F. This decomposition is unique since it is uniquely determined by the unique decomposition of F into rooted trees.

We define a linear operator

$$d_{\mathcal{F}} : \mathbf{k}\mathcal{F} \to \mathbf{k}\mathcal{F}. \tag{42}$$

as follows. First let V be either $\bullet \sqcup \bullet$ or a tree. Since a tree is of the form $\bullet$ or $\lfloor \overline{V} \rfloor$ for a forest $\overline{V}$, it makes sense to define

$$d_{\mathcal{F}}(V) = \begin{cases} 0, & V = \bullet, \\ 1, & V = \bullet \sqcup \bullet, \\ \overline{V}, & V = \lfloor \overline{V} \rfloor. \end{cases} \tag{43}$$

Next let $F \in \mathcal{F}$ have the $\diamond$-standard decomposition in Eq. (41). Define

$$d_{\mathcal{F}}(F) = \sum_{\emptyset \neq I \subseteq [k]} \lambda^{|I|-1} V_{I,1} \diamond \cdots \diamond V_{I,k}, \tag{44}$$

where for $I \subseteq [k]$,

$$V_{I,i} = \begin{cases} V_i, & i \notin I, \\ d_{\mathcal{F}}(V_i), & i \in I \end{cases} \tag{45}$$

with $d_{\mathcal{F}}(V_i)$ as defined in Eq. (43). Finally extend $d_{\mathcal{F}}$ to $\mathbf{k}\mathcal{F}$ by $\mathbf{k}$-linearity.

It is clear that $d_{\mathcal{F}}$ satisfies the recursive relation

$$d_{\mathcal{F}}(F) = d_{\mathcal{F}}(V_1) \diamond (V_2 \diamond \cdots \diamond V_k) + V_1 \diamond d_{\mathcal{F}}(V_2 \diamond \cdots \diamond V_k) + \lambda d_{\mathcal{F}}(V_1) \diamond d_{\mathcal{F}}(V_2 \diamond \cdots \diamond V_k). \tag{46}$$

We give some examples. By the third case in Eq. (43), we have

$$d_{\mathcal{F}}(\lfloor \bullet\sqcup\bullet \rfloor) = \bullet \sqcup \bullet, \quad d_{\mathcal{F}}(\lfloor \bullet \sqcup \lfloor \bullet \rfloor \rfloor) = \bullet \sqcup \lfloor \bullet \rfloor. \tag{47}$$

Further, since $\bullet \sqcup \lfloor \bullet \rfloor = (\bullet \sqcup \bullet) \diamond \lfloor \bullet \rfloor$, we have

$$d_{\mathcal{F}}(\bullet \sqcup \lfloor \bullet \rfloor) = \bullet \diamond \lfloor \bullet \rfloor + (\bullet \sqcup \bullet) \diamond \bullet + \lambda(\bullet \diamond \bullet) = \lfloor \bullet \rfloor + \bullet \sqcup \bullet + \lambda \bullet. \tag{48}$$

Similarly,

$$\begin{aligned} d_{\mathcal{F}}(\lfloor \bullet\sqcup\bullet \rfloor \sqcup \bullet) &= d_{\mathcal{F}}(\lfloor \bullet\sqcup\bullet \rfloor \diamond (\bullet \sqcup \bullet)) \\ &= (\bullet \sqcup \bullet) \diamond (\bullet \sqcup \bullet) + \lfloor \bullet\sqcup\bullet \rfloor \diamond \bullet + \lambda(\bullet \sqcup \bullet) \diamond \bullet \\ &= \bullet \sqcup \bullet \sqcup \bullet + \lfloor \bullet\sqcup\bullet \rfloor + \lambda(\bullet \sqcup \bullet). \end{aligned} \tag{49}$$

As another example, from the $\diamond$-standard decomposition

$$\lfloor \bullet\sqcup\bullet \rfloor \sqcup \lfloor \bullet \rfloor = \lfloor \bullet\sqcup\bullet \rfloor \diamond (\bullet \sqcup \bullet) \diamond \lfloor \bullet \rfloor,$$

by Eqs. (23), (46) and (48) we have

$$\begin{aligned} d_{\mathcal{F}}(\lfloor \bullet\sqcup\bullet \rfloor \sqcup \lfloor \bullet \rfloor) &= d_{\mathcal{F}}(\lfloor \bullet\sqcup\bullet \rfloor) \diamond \big((\bullet \sqcup \bullet) \diamond \lfloor \bullet \rfloor\big) + \lfloor \bullet\sqcup\bullet \rfloor \diamond d_{\mathcal{F}}\big((\bullet \sqcup \bullet) \diamond \lfloor \bullet \rfloor\big) + \lambda d_{\mathcal{F}}(\lfloor \bullet\sqcup\bullet \rfloor) \diamond d_{\mathcal{F}}\big((\bullet \sqcup \bullet) \diamond \lfloor \bullet \rfloor\big) \\ &= (\bullet \sqcup \bullet) \diamond \big((\bullet \sqcup \bullet) \diamond \lfloor \bullet \rfloor\big) + \lfloor \bullet\sqcup\bullet \rfloor \diamond \big(\lfloor \bullet \rfloor + \bullet \sqcup \bullet + \lambda \bullet\big) + \lambda(\bullet \sqcup \bullet) \diamond \big(\lfloor \bullet \rfloor + \bullet \sqcup \bullet + \lambda \bullet\big) \\ &= \bullet \sqcup \bullet \sqcup \bullet \sqcup \lfloor \bullet \rfloor + \lfloor \bullet\sqcup\bullet \rfloor \diamond \lfloor \bullet \rfloor + \lfloor \bullet\sqcup\bullet \rfloor \sqcup \bullet + \lambda \lfloor \bullet\sqcup\bullet \rfloor + \lambda \bullet \sqcup \lfloor \bullet \rfloor + \lambda \bullet \sqcup \bullet \sqcup \bullet + \lambda^2 \bullet \sqcup \bullet \\ &= \lfloor \bullet \sqcup \lfloor \bullet \rfloor \rfloor + \lfloor \lfloor \bullet\sqcup\bullet \rfloor \rfloor + 2\lambda \lfloor \bullet\sqcup\bullet \rfloor + \lfloor \bullet\sqcup\bullet \rfloor \sqcup \bullet + \bullet \sqcup \bullet \sqcup \bullet \sqcup \lfloor \bullet \rfloor + \lambda \bullet \sqcup \lfloor \bullet \rfloor + \lambda \bullet \sqcup \bullet \sqcup \bullet + \lambda^2 \bullet \sqcup \bullet. \end{aligned}$$

Theorem 5.1. *The triple* $(\mathbf{k}\mathcal{F}, d_{\mathcal{F}}, \lfloor\ \rfloor)$ *is a differential Rota–Baxter algebra of weight* λ.

Proof. By the third case of Eq. (43), $d_{\mathcal{F}} \circ \lfloor\ \rfloor = \mathrm{id}$. So we only need to show that $d_{\mathcal{F}}$ is a differential operator of weight λ, that is, $d_{\mathcal{F}}$ satisfies the λ-Leibniz rule in Eq. (2):

$$d_{\mathcal{F}}(F \diamond F') = d_{\mathcal{F}}(F) \diamond F' + F \diamond d_{\mathcal{F}}(F') + \lambda d_{\mathcal{F}}(F) \diamond d_{\mathcal{F}}(F'). \tag{50}$$

This is not immediate since the $\diamond$-standard decomposition of $F \diamond F'$ is not the product of the $\diamond$-standard decomposition of F and F'.

First let F and F' be trees. Then F is either $\bullet$ or $\lfloor \overline{F} \rfloor$ for a forest $\overline{F}$. Similarly for F'. Since $\bullet$ is the unit, Eq. (2) trivially holds if $F = \bullet$ or $F' = \bullet$. If $F = \lfloor \overline{F} \rfloor$ and $F' = \lfloor \overline{F}' \rfloor$. Then by the Rota–Baxter equation (4) and Eq. (45), we have

$$d_{\mathcal{F}}(F \diamond F') = \overline{F} \diamond F' + F \diamond \overline{F}' + \lambda \overline{F} \diamond \overline{F}'. \tag{51}$$

This is Eq. (50).

In general, let F and F' be forests and let

$$F = V_1 \diamond \cdots \diamond V_{2b-1}, \qquad F' = V'_1 \diamond \cdots \diamond V'_{2b'-1}$$

be their $\diamond$-standard decompositions from Eq. (40). Then

$$F \diamond F' = V_1 \diamond \cdots \diamond V_{2b-2} \diamond (V_{2b-1} \diamond V'_1) \diamond V'_2 \diamond \cdots \diamond V'_{2b'-1}$$

is the $\diamond$-standard decomposition of $F \diamond F'$. Here $V_{2b-1} \diamond V'_1 = \sum_k Z''_k$ is a tree or a linear combination of trees Z''_k given in Eq. (21). As in Eq. (41), we rewrite it as

$$F \diamond F' = W_1 \diamond \cdots \diamond W_{2(b+b'-1)-1}.$$

In particular, $W_{2b-1} = V_{2b-1} \diamond V'_1 = \sum_k Z''_k$. Then by definition,

$$d_{\mathcal{F}}(F \diamond F') = \sum_{\emptyset \neq J \subseteq [2(b+b'-1)-1]} \lambda^{|J|-1} W_{J,1} \diamond \cdots \diamond W_{J,2(b+b'-1)-1} \tag{52}$$

with $W_{J,j}$ defined in the same way as $T_{I,i}$ in Eq. (43) and $d_{\mathcal{F}}(W_{2b-1}) = \sum_k d_{\mathcal{F}}(Z''_k)$. Depending on whether or not $2b-1 \in J$, we can rewrite Eq. (52) as

$$\begin{aligned} d_{\mathcal{F}}(F \diamond F') &= \sum_{2b-1 \in J \subseteq [2(b+b'-1)-1]} \lambda^{|J|-1} W_{J,1} \diamond \cdots \diamond W_{J,2b-2} \diamond d_{\mathcal{F}}(W_{2b-1}) \diamond \cdots \diamond W_{J,2(b+b'-1)-1} \\ &\quad + \sum_{2b-1 \notin J \subseteq [2(b+b'-1)-1]} \lambda^{|J|-1} W_{J,1} \diamond \cdots \diamond W_{J,2b-2} \diamond W_{J,2b-1} \diamond \cdots \diamond W_{J,2(b+b'-1)-1} \\ &= \left(\sum_{\check{I} \subseteq [2b-2]} \lambda^{|\check{I}|} (V_{\check{I},1} \diamond \cdots \diamond V_{\check{I},2b-2}) \right) \diamond d_{\mathcal{F}}(V_{2b-1} \diamond V'_1) \diamond \left(\sum_{\check{I}' \subseteq \{2,\ldots,2b'-1\}} \lambda^{|\check{I}'|} V'_{\check{I}',2} \diamond \cdots \diamond V'_{\check{I}',2b'-1} \right) \\ &\quad + \sum_{\substack{\check{I} \subseteq \{1,\ldots,2b-2\} \\ \check{I}' \subseteq \{2,\ldots,2b'-1\} \\ \check{I} \neq \emptyset \text{ or } \check{I}' \neq \emptyset}} \lambda^{|\check{I}|+|\check{I}'|-1} (V_{\check{I},1} \diamond \cdots \diamond V_{\check{I},2b-2}) \diamond (V_{2b-1} \diamond V'_1) \diamond (V'_{\check{I}',2} \diamond \cdots \diamond V'_{\check{I}',2b'-1}). \end{aligned} \tag{53}$$

By Eq. (51),

$$d_{\mathcal{F}}(V_{2b-1} \diamond V'_1) = d_{\mathcal{F}}(V_{2b-1}) \diamond V'_1 + V_{2b-1} \diamond d_{\mathcal{F}}(V'_1) + \lambda d_{\mathcal{F}}(V_{2b-1}) \diamond d_{\mathcal{F}}(V'_1).$$

Denote $I \subseteq [2b-1]$ and $I' \subseteq [2b'-1]$. We can write the first sum in Eq. (53) as

$$\begin{aligned} &\left(\sum_{\substack{2b-1 \in I \\ 1 \notin I'}} + \sum_{\substack{2b-1 \notin I \\ 1 \in I'}} + \sum_{\substack{2b-1 \in I \\ 1 \in I'}} \right) \lambda^{|I|+|I'|-1} V_{I,1} \diamond \cdots \diamond (V_{I,2b-1} \diamond V'_{I',1}) \diamond \cdots \diamond V'_{I',2b'-1} \\ &\quad = \left(\sum_{\substack{2b-1 \in I \\ I' = \emptyset}} + \sum_{\substack{2b-1 \in I \\ 1 \notin I' \neq \emptyset}} + \sum_{\substack{I = \emptyset \\ 1 \in I'}} + \sum_{\substack{2b-1 \notin I \neq \emptyset \\ 1 \in I'}} + \sum_{\substack{2b-1 \in I \\ 1 \in I'}} \right) \\ &\quad \lambda^{|I|+|I'|-1} V_{I,1} \diamond \cdots \diamond (V_{I,2b-1} \diamond V'_{I',1}) \diamond \cdots \diamond V'_{I',2b'-1}. \end{aligned} \tag{54}$$

For the second sum in Eq. (53), we have

$$\left(\sum_{\substack{2b-1 \notin I \neq \emptyset \\ I' = \emptyset}} + \sum_{\substack{I = \emptyset \\ 1 \notin I' \neq \emptyset}} + \sum_{\substack{2b-1 \notin I \neq \emptyset \\ 1 \notin I' \neq \emptyset}} \right) \lambda^{|I|+|I'|-1} V_{I,1} \diamond \cdots \diamond (V_{I,2b-1} \diamond V'_{I',1}) \diamond \cdots \diamond V'_{I',2b'-1}. \tag{55}$$

The first sum on the right-hand side of Eq. (54) adding to the first sum in Eq. (55) gives

$$\begin{aligned} &\sum_{I \neq \emptyset, I' = \emptyset} \lambda^{|I|+|I'|-1} V_{I,1} \diamond \cdots \diamond (V_{I,2b-1} \diamond V'_1) \diamond \cdots \diamond V'_{2b'-1} \\ &\quad = \sum_{I \neq \emptyset} \lambda^{|I|-1} V_{I,1} \diamond \cdots \diamond (V_{I,2b-1} \diamond V'_1) \diamond \cdots \diamond V'_{2b'-1} \\ &\quad = d_{\mathcal{F}}(F) \diamond F'. \end{aligned}$$

Similarly, the third sum on the right-hand side of Eq. (54) added to the second term in Eq. (55) gives $F \diamond d_{\mathcal{F}}(F')$. The remaining terms on the right-hand side of Eqs. (54) and (55) add to $\lambda d_{\mathcal{F}}(F) \diamond d_{\mathcal{F}}(F')$. This proves the λ-Leibniz rule (2). □

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