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A MATROID GENERALIZATION OF A THEOREM OF MENDELSOHN AND DULMAGE*

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Received 12 November 1971

Abstract. A matroid generalization is given to a theorem of Mendelsohn and Dulmage concerning assignments in bipartite graphs. The generalized theorem has applications in optimization theory and provides a simple proof of a theorem of Nash-Williams.

1. A theorem of Mendelsohn and Dulmage

A theorem of Mendelsohn and Dulmage [3] was originally proved for (0, 1) matrices. However, it can be regarded as a theorem on bipartite matchings and hence it is a special case (finite) of Banach's mapping theorem [1]. We first give a simple proof of this theorem using the concept of a matching.

Theorem 1.1 (Mendelsohn–Dulmage). *Let $G(X, Y)$ be a bipartite graph, the two parts being X and Y. Let S, T be subsets of X, Y, respectively, such that each set has an assignment into the other part. Then there is an assignment between two subsets of X, Y which contains both S and T.*

Proof. Let f: $S \to Y$ and g: $T \to X$ be the assignments. Write

$$F = \{(x, f(x)) : x \in S\}, \qquad H = \{(g(y), y) : y \in T\}.$$

* Research sponsored by the Air Force Office of Scientific Research Grant AFOSR-71-2076.
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F and H are matchings in G and they cover respectively S and T. Form the symmetric difference $F \triangle H$. It consists of the five types of cycles and chains shown in Fig. 1. (Some of these chains could be infinite in one or both directions if G is an infinite graph). In each case above we can select a matching set of edges $M \subseteq F \triangle H$ such that it covers all the vertices of $X \cup Y$ covered by $F \triangle H$. Then $M \cup (F \cap H)$ is a matching that covers $S \cup T$.

Remark. Banach's mapping theorem is the same as Theorem 1.1 for arbitrary $|X \cup Y|$, possibly infinite. For the other variations of Banach's theorem, see [4].

We proceed to prove our main theorem. We shall write a matroid as $M = (E, I)$, where E is the set of elements and I is the family of independent subsets of E.

Theorem 1.2. *Let M_1, M_2 be two matroids on E and I_1, I_2 each be independent in both matroids. Then there exists a set $I \subseteq I_1 \cup I_2$ independent in M_1, M_2 such that*

$$\mathrm{sp}^i(I) \supseteq \mathrm{sp}^i(I_i), \quad i = 1, 2 ,$$

where $\mathrm{sp}^i(\ )$ *stands for span in matroid M_i.*

To see how these two theorems are related, we take E = set of edges of the graph $G(X, Y)$ and consider the natural partitions of E defined

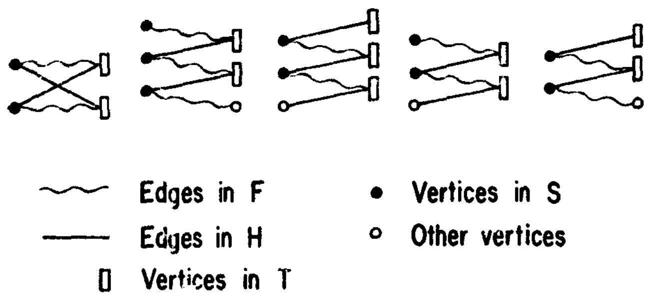

Fig. 1. Components of $F \triangle H$.

as follows: $P_1 = \{E_x : x \in X\}$, $P_2 = \{E_y : y \in Y\}$, where E_v = set of edges incident with vertex v. Let M_i, $i = 1, 2$, be the partition matroids corresponding to P_i, i.e., a set I is independent in M_i if and only if $|I \cap E_x| \leq 1$ for all x (resp. $|I \cap E_y| \leq 1$ for all y). Then

I_1 = the edges of the assignment of S,
I_2 = the edges of the assignment of T,
I = the edges of the assignment given by the theorem.

Proof of Theorem 1.2. If $\mathrm{sp}^2(I_1) \supseteq I_2$, there is nothing to prove. Let $e \in I_2 \backslash \mathrm{sp}^2(I_1)$; $I_1 + e$ is in $\mathcal{I}_2$. If $I_1 + e$ belongs to $\mathcal{I}_1$, let $I'_1 = I_1 + e$. Otherwise, there exists an M_1-circuit C such that $e \in C \subseteq I_1 + e$. Now $C - e \subseteq I_1 \cap I_2$, since I_2 is in $\mathcal{I}_1$. Choose $e' \in C \cap (I_1 \backslash I_2)$ and define $I'_1 = I_1 - e' + e$. We have $I'_1 \in \mathcal{I}_1$ and $\mathrm{sp}^1(I'_1) = \mathrm{sp}^1(I_1)$ and also I'_1 is trivially independent in M_2. However, $|I'_1 \cap I_2| > |I_1 \cap I_2|$. Thus we can apply the same procedure to define $I_1^{(k)}$, $k = 1, 2, \ldots$ such that $\mathrm{sp}^1(I_1^{(k)}) = \mathrm{sp}^1(I_1)$, $I_1^{(k)} \in \mathcal{I}_1 \cap \mathcal{I}_2$ until $\mathrm{sp}^2(I_1^{(k)}) \supseteq I_2$. Then $I = I^{(k)}$ proves the theorem.

2. Application to optimization

Suppose θ_1 and θ_2 are two different criteria of optimality, such that

$$\mathrm{sp}^i(A) \supseteq \mathrm{sp}^i(B)$$

implies

$$A \geq B \quad (\theta_i), \quad i = 1, 2,$$

i.e., A is to be preferred to B with respect to θ_i. Let I_1, I_2 be sets in the family $\mathcal{I}_1 \cap \mathcal{I}_2$ which are maximal with respect to θ_1, θ_2, respectively. Then by Theorem 1.2, there exists a set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ which is maximal with respect to both θ_1 and θ_2.

Specifically, let X, Y in the bipartite graph $G(X, Y)$ represent men and jobs to be matched, where the edges denote the compatibility relation. Suppose θ_1 is a union-determined criterion of optimality based on seniority of men and θ_2 is a management-determined criterion of optimality based on priority of jobs. Let I_1 be a subset of edges representing a union-optimal assignment of men to jobs, possibly as determined in [2], and let I_2 be a management-optimal subset. Then by Theorem 1.2 (or 1.1), there exists an assignment $I \subseteq I_1 \cup I_2$ which is simultaneously union-optimal and management-optimal.

3. Proof of a theorem of Nash-Williams

Theorem 1.2 provides a simple and direct proof of a theorem of Nash-Williams [5].

Theorem 3.1. *Let $M_1 = (E, \mathcal{I}_1)$ be a matroid and $h: E \to E_0$ be a mapping of E into E_0. Then $M_0 = (E_0, \mathcal{I}_0)$ is a matroid, where*

$$\mathcal{I}_0 = \{I_0 \subseteq E_0 : \textit{for some } I_1 \in \mathcal{I}_1,\ h(I_1) = I_0\} .$$

Proof. It is sufficient to show that if I_p, I_{p+1} are two sets in $\mathcal{I}_0$, respectively with p and $p+1$ elements, there exists a set $h(I) \in \mathcal{I}_0$ with $p+1$ or more elements such that $I_p \subseteq h(I) \subseteq I_p \cup I_{p+1}$. Let $M_2 = (E, \mathcal{I}_2)$ be a partition matroid, where

$$\mathcal{I}_2 = \{I_2 \subseteq E : |I_2 \cap h^{-1}(e)| \leq 1 \text{ for all } e \in E_0\} .$$

Let I'_p, I'_{p+1} be sets in $\mathcal{I}_1$, respectively with p and $p+1$ elements, such that $h(I'_p) = I_p$ and $h(I'_{p+1}) = I_{p+1}$. The sets I'_p, I'_{p+1} are independent in M_2 as well as M_1, and we can apply Theorem 1.2. Thus there is a set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ such that

$$\mathrm{sp}^1(I) \supseteq \mathrm{sp}^1(I'_{p+1}) ,$$

hence $|I| \geq p+1$, and

$$\mathrm{sp}^2(I) \supseteq \mathrm{sp}^2(I'_p) ,$$

from which it follows that

$$h(I) \supseteq h(I'_p) = I_p .$$

Also h is one-one on I and $I \subseteq I'_p \cup I'_{p+1}$, which implies that $|h(I)| \geq p + 1$ and $h(I) \subseteq I_p \cup I_{p+1}$. Thus I_0 defines the independent sets of a matroid.

References

[1] S. Banach, Un théorème sur les transformations biunivoques, Fund. Math. 6 (1924) 236–239.
[2] D. Gale, Optimal assignments in an ordered set; An application of matroid theory, J. Combinatorial Theory 4 (1968) 176–180.
[3] N.S. Mendelsohn and A.L. Dulmage, Some generalizations of the problem of distinct representatives, Canad. J. Math. 10 (1958) 230–241.
[4] L. Mirsky, Transversal theory (Academic Press, New York, 1971).
[5] J.A. Nash-Williams, An application of matroids to graph theory, in: Theorie des Graphes, Journées Internationales d'Études, Rome (Dunod, Paris, 1966) 263–265.