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Ample vector bundles of small Δ-genera

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ABSTRACT

A natural notion of "delta-genus" Δ for a generalized polarized manifold $(X, \mathcal{E})$, strictly related to its associated scroll, is introduced and pairs $(X, \mathcal{E})$ with low Δ are classified. The stronger are the properties enjoyed by the vector bundle $\mathcal{E}$, the larger are the values of Δ attained by the results.

0. Introduction

Let X be a smooth complex projective variety of dimension n and let $\mathcal{L}$ be an ample line bundle on X. In order to study polarized manifolds $(X, \mathcal{L})$ Fujita introduced the Δ-*genus* of $(X, \mathcal{L})$, which is a nonnegative integer defined by the formula

$$\Delta(X, \mathcal{L}) := n + \mathcal{L}^n - h^0(X, \mathcal{L}).$$

The theory developed around this invariant has been a powerful tool in characterizing polarized varieties with Δ small enough [11]. As noticed in [11, p. 176] t here is not a good vector bundle version of the theory of Δ-genus. This sentence motivated our interest in the subject.

Let $\mathcal{E}$ be an ample vector bundle of rank $r \geqslant 2$ on X. There are two obvious polarized varieties naturally associated with $(X, \mathcal{E})$, namely $(X, \det \mathcal{E})$ and the scroll (P, H), where $P = \mathbb{P}_X(\mathcal{E})$ and H is the tautological line bundle. One could be tempted to use their Δ-genera to study $(X, \mathcal{E})$. The natural expectation, however, is to have a new invariant capturing the vector bundle aspects in a better way, e.g. involving the rank r and all Chern classes of $\mathcal{E}$.

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In principle there could be many ways to define a Δ-genus for $(X, \mathcal{E})$, which could be conceived either as a single integer or as an r-tuple of integers (e.g. see [1]). Looking for one integer, a natural definition would be

$$\Delta(X, \mathcal{E}) = nr + f(c_1, \dots, c_r) - h^0(X, \mathcal{E}),$$

where f is a suitable polynomial in the Chern classes of $\mathcal{E}$.

In this paper we define $\Delta(X, \mathcal{E})$ in this way, choosing f to be the polynomial computing the degree of the tautological line bundle of the scroll (P, H). This has the effect of producing the relation $\Delta(X, \mathcal{E}) = (n-1)(r-1) + \Delta(P, H)$, which makes $\Delta(P, H)$ playing a prominent role in our study. As a consequence we immediately get the nonnegative lower bound $(n-1)(r-1)$ for $\Delta(X, \mathcal{E})$.

Our aim is to investigate pairs $(X, \mathcal{E})$ with low Δ. We can do that relying on the classification results available for polarized manifolds of low Δ-genus to be applied to (P, H). Combining them with the special structure of (P, H) leads to remarkable simplifications which enable us to obtain classification results much cleaner than those holding for line bundles. For instance, for $\Delta(X, \mathcal{E}) \leqslant (n-1)(r-1) + 1$, our result is complete, unlike that holding for polarized manifolds with Δ-genus $\leqslant 1$ (e.g. see [11, Problem 6.24]). Actually a still unsolved case in the setting of ample line bundles does not fit our context (Lemma 1.4).

As we said, another obvious polarized manifold associated with $(X, \mathcal{E})$ is $(X, \det \mathcal{E})$. However, the inequality $\Delta(X, \det \mathcal{E}) \leqslant \Delta(X, \mathcal{E})$ holding for $n = 1$ (Remark 3.5) seems to suggest that our $\Delta(X, \mathcal{E})$ is a more relevant character than $\Delta(X, \det \mathcal{E})$. More generally, for any dimension n, even if we assume that $\mathcal{E}$ is spanned and $\det \mathcal{E}$ is very ample in order to have more geometric evidence, the character $\Delta(X, \det \mathcal{E})$ alone seems unable to reflect the possible degeneracy of the image of X via the morphism to an appropriate Grassmannian defined by $\mathcal{E}$. On the contrary, at the end, $\Delta(P, H)$ reveals more meaningful than expected.

The precise formulation of our classification results of pairs $(X, \mathcal{E})$ with small Δ is given in Theorems 3.6 and 3.7 for $\mathcal{E}$ ample, 4.3 for $\mathcal{E}$ ample and spanned by global sections, 5.6 and 6.3 for $\mathcal{E}$ very ample.

Actually, the better are the properties enjoyed by $\mathcal{E}$, the larger are the values of Δ we can include in our investigation. For instance, Theorem 6.3 deals with pairs $(X, \mathcal{E})$ with $n \geqslant 2$ and $\Delta = (n-1)(r-1) + 4$. More generally, when $\mathcal{E}$ is very ample we also provide the list of pairs $(X, \mathcal{E})$ with $n \geqslant 2$ such that either $\Delta \leqslant (n-1)(r-1) + \frac{d}{2}$, where $d = d(P, H)$ (Theorem 5.1), or $\Delta \leqslant nr - 1$ (Theorem 5.4). As a consequence of the results above, we obtain the list of pairs with $\Delta = 2$ for $\mathcal{E}$ ample and spanned (Proposition 4.5) and, when $\mathcal{E}$ is very ample, of those: (a) with $\Delta = 3$ (Proposition 6.1), (b) with $\Delta = 4$ or 5 (Propositions 6.2 and 6.5) under the assumption $n \geqslant 2$.

In all proofs the fact that our definition of $\Delta(X, \mathcal{E})$ relies on the Δ-genus of the polarized manifold (P, H) turns out to be a concrete advantage. Actually, $\Delta(X, \mathcal{E})$ small implies $\Delta(P, H)$ small; according to the theory (see [11] for H ample, and [15,17,18] for H very ample), polarized manifolds with low Δ-genus are rather special and include several special varieties arising from adjunction theory. Since we already know that (P, H) is a scroll, the investigation of scrolls admitting another relevant structure for adjunction theory (e.g. non-trivial reductions, quadric fibrations, del Pezzo and Mukai manifolds, etc.) plays a key role in our analysis. This investigation takes Section 2. Some results we prove to this end are of interest in themselves, e.g. see Propositions 2.8 and 2.12. Another point deserves to be stressed. The map associating (P, H) to $(X, \mathcal{E})$ is not injective. Hence, in the reconstruction process of $(X, \mathcal{E})$ from (P, H), one can meet admissible pairs (P, H) carrying distinct scroll structures. This happens in several instances, some of which are nontrivial, e.g. see Remarks 2.5, 2.10, 5.5 and 5.7. Finally, while the value of the Δ-genus increases, new possible varieties arise as candidates for (P, H) (e.g. see the proof of Theorem 6.3); of course this makes it harder to analyze the compatibility of different structures on (P, H).

The paper is organized as follows: in Section 1 we collect some background material; scrolls carrying a further structure are analysed in Section 2; the Δ-genus of $(X, \mathcal{E})$ is discussed in Section 3 for ample vector bundles; in Section 4 we consider ample vector bundles spanned by global sections, while Sections 5 and 6 are devoted to very ample vector bundles.

1. Background material

We work over the field of complex numbers and we use the standard notation from algebraic geometry. By a little abuse we make no distinction between a line bundle and the corresponding invertible sheaf. Moreover, the tensor products of line bundles are denoted additively. The pullback $i^*\mathcal{E}$ of a vector bundle $\mathcal{E}$ on X by an embedding of projective varieties $i : Y \hookrightarrow X$ is denoted by $\mathcal{E}_Y$. We denote by K_X the canonical bundle of a smooth variety X. The blow-up of a variety X along a smooth subvariety Y is denoted by $\mathrm{Bl}_Y(X)$.

A smooth projective variety X is called a *Fano manifold* if its anticanonical bundle $-K_X$ is ample. For a Fano manifold X, the largest integer, r_X, which divides $-K_X$ in the Picard group $\mathrm{Pic}(X)$ is called the *index* of X while the integer $i_X := \min\{-K_X \cdot C\colon\ C \text{ is a rational curve on } X\}$ is called the *pseudoindex* of X.

Let S be a smooth projective surface. By saying that S is *ruled* we mean that S is birationally equivalent to $B \times \mathbb{P}^1$, where B is a smooth projective curve. We say that S is *geometrically ruled* to mean that it is a $\mathbb{P}^1$-bundle over B.

We set $\mathbb{F}_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ to denote the Segre–Hirzebruch surface of invariant e ($e \geqslant 0$). Then, as in [14, p. 372], C_0 stands for a section of minimal self-intersection and f for a fiber.

A *polarized manifold* is a pair $(X, \mathcal{L})$ consisting of a smooth projective variety X and an ample line bundle $\mathcal{L}$ on X. The degree, the sectional genus and the Δ-genus of a polarized manifold $(X, \mathcal{L})$ of dimension n are defined as $d(X, \mathcal{L}) = \mathcal{L}^n$, $g(X, \mathcal{L}) = 1 + \frac{1}{2}(K_X + (n-1)\mathcal{L}) \cdot \mathcal{L}^{n-1}$ and $\Delta(X, \mathcal{L}) = n + d(X, \mathcal{L}) - h^0(X, \mathcal{L})$, respectively. A polarized manifold $(X, \mathcal{L})$ is said to be a *scroll* over a smooth variety W if there exists a surjective morphism $f : X \longrightarrow W$ such that $(F, \mathcal{L}_F) \cong (\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1))$ with $m = \dim X - \dim W$ for any fiber F of f. This condition is equivalent to saying that $(X, \mathcal{L}) \cong (\mathbb{P}_W(\mathcal{F}), H(\mathcal{F}))$ for some ample vector bundle $\mathcal{F}$ on W, where $H(\mathcal{F})$ is the tautological line bundle on the projective space bundle $\mathbb{P}_W(\mathcal{F})$ associated to $\mathcal{F}$. A polarized manifold $(X, \mathcal{L})$ is said to be a *quadric fibration* over a smooth curve W if there exists a surjective morphism $f : X \longrightarrow W$ and any general fiber F of f is a smooth quadric hypersurface $\mathbb{Q}^{n-1}$ in $\mathbb{P}^n$ with $n = \dim X$ such that $\mathcal{L}_F \cong \mathcal{O}_{\mathbb{Q}^{n-1}}(1)$. A polarized manifold $(X, \mathcal{L})$ is said to be a *Veronese bundle* over a smooth curve W if there exists a $\mathbb{P}^2$-bundle $p : X \longrightarrow W$ such that $\mathcal{L}_F \cong \mathcal{O}_{\mathbb{P}^2}(2)$ for any fiber F of p. A polarized manifold $(X, \mathcal{L})$ is said to be a *del Pezzo manifold* (resp. a *Mukai manifold*) if $K_X + (\dim X - 1)\mathcal{L} = \mathcal{O}_X$ (resp. if $K_X + (\dim X - 2)\mathcal{L} = \mathcal{O}_X$).

Let $(X, \mathcal{L})$ be a polarized manifold. An effective divisor $E \subset X$ is called a (-1)-*hyperplane* if $E \cong \mathbb{P}^{n-1}$, $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ and $\mathcal{L}_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. Sometimes we write E_E instead of $\mathcal{O}_E(E)$ for shortness. Note that the set of (-1)-hyperplanes contained in X is finite and, in case $\dim X \geqslant 3$, any two (-1)-hyperplanes are disjoint. Let $\dim X \geqslant 3$. We will call a pair (Y, L) the (*adjunction theoretic*) *reduction* of $(X, \mathcal{L})$ if there exists a birational morphism $\sigma : X \longrightarrow Y$ which is the contraction of all (-1)-hyperplanes $E_1, \ldots, E_s$ contained in X and L is the (unique) ample line bundle on Y such that $\mathcal{L} = \sigma^* L - E_1 - \cdots - E_s$. We use the expression *simple reduction* to mean that $s = 1$.

Let X be a smooth projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Consider the projective bundle $P := \mathbb{P}_X(\mathcal{E})$ and denote by $H = H(\mathcal{E})$ the tautological line bundle on P. Then (P, H) is a polarized manifold of dimension $n + r - 1$ and degree $d(P, H) = H^{n+r-1}$. Let $\pi : P \to X$ be the bundle projection and, with a little abuse, let us use the symbol c_i to denote $c_i(\mathcal{E})$ as well as $\pi^* c_i(\mathcal{E})$, according to the context. The Chern–Wu relation allows us to write

$$H^r = f(H, c_1, \ldots, c_r) = H^{r-1}c_1 - H^{r-2}c_2 + \cdots + (-1)^{r-1}c_r. \tag{1.1}$$

By (1.1) we can express $d(P, H)$ in the following way

$$H^{n+r-1} = H^{n-1} \cdot f(H, c_1, \ldots, c_r) = H^{n-2} \cdot \big(H \cdot f(H, c_1, \ldots, c_r)\big),$$

and proceed inductively. Then we can use (1.1) again to replace each power $\geqslant r$ of H in terms of smaller powers. This reduces the expression above to a polynomial of degree $r - 1$. On the other hand, coefficients of H^i for $i < r-1$ are cup products of Chern classes, which are obviously zero since $\dim X = n < n+r-1-i$. This leads to a final expression of the form

$$d(P,H) = H^{r-1} \cdot P(c_1, \dots, c_r),$$

where P is a polynomial in the Chern classes expressing a 0-cycle on X, or a finite number of fibers F of π, according to the convention about the meaning of c_i. Recall that $H^{r-1} \cdot F = (H_F)^{r-1} = 1$, hence $d(P,H)$ is the degree of the 0-cycle on X expressed by P. In order to provide an explicit expression of this degree, set

$$\varphi_j = 0 \quad \text{for } j < 0, \qquad \varphi_0 = 1, \quad \text{and} \quad \varphi_1 = c_1,$$

and inductively define

$$\varphi_k = c_1\varphi_{k-1} - c_2\varphi_{k-2} + c_3\varphi_{k-3} - \dots + (-1)^{r-1}c_r\varphi_{k-r} \quad \text{for } k = 2, \dots, n.$$

For instance, for any $r \geqslant 2$ we have $\varphi_2 = c_1^2 - c_2$, $\varphi_3 = c_1\varphi_2 - c_2c_1 + c_3 = c_1^3 - 2c_1c_2 + c_3$, and $\varphi_4 = c_1\varphi_3 - c_2\varphi_2 + c_3c_1 - c_4 = c_1^4 - 3c_1^2c_2 + 2c_1c_3 + c_2^2 - c_4$. Then the iterated procedure described above leads to a recursive expression for $P(c_1, \dots, c_r)$. The result is the following

Lemma 1.1. *The degree of* (P,H) *is given by* $d(P,H) = \varphi_n$.

Here are some examples:

Examples 1.2. (i) Let $r = 2$. Here we list the expressions of φ_k for $3 \leqslant k \leqslant 8$: $\varphi_3 = c_1(c_1^2 - 2c_2)$, $\varphi_4 = c_1^4 - 3c_1^2c_2 + c_2^2$, $\varphi_5 = c_1(c_1^4 - 4c_1^2c_2 + 3c_2^2)$, $\varphi_6 = c_1^6 - 5c_1^4c_2 + 6c_1^2c_2^2 - c_2^3$, $\varphi_7 = c_1(c_1^6 - 6c_1^4c_2 + 10c_1^2c_2^2 - 4c_2^3)$, $\varphi_8 = c_1^8 - 7c_1^6c_2 + 15c_1^4c_2^2 - 10c_1^2c_2^3 + c_2^4$.

(ii) Let $r = 3$. The expressions of φ_k for $4 \leqslant k \leqslant 6$ are the following: $\varphi_4 = c_1^4 - 3c_1^2c_2 + 2c_1c_3 + c_2^2$, $\varphi_5 = c_1^5 - 4c_1^3c_2 + 3c_1^2c_3 + 3c_1c_2^2 - 2c_2c_3$, $\varphi_6 = c_1^6 - 5c_1^4c_2 + 4c_1^3c_3 + 6c_1^2c_2^2 - 6c_1c_2c_3 - c_2^3 + c_3^2$.

(iii) We compute $d(P,H)$ in case $n = 5$ and any $r \geqslant 2$. We already know the expression of φ_k for $k \leqslant 4$. Then $d(P,H) = \varphi_5 = c_1\varphi_4 - c_2\varphi_3 + c_3\varphi_2 - c_4c_1 + c_5 = c_1^5 - 4c_1^3c_2 + 3c_1^2c_3 + 3c_1c_2^2 - 2c_1c_4 - 2c_2c_3 + c_5$.

In this paper we will work in the following set-up:

1.3. X is a smooth projective variety of dimension n and $\mathcal{E}$ is an ample vector bundle of rank r on X. We will denote by (P,H) the polarized manifold consisting of $P := \mathbb{P}_X(\mathcal{E})$ and $H := H(\mathcal{E})$, the taulogical line bundle of $\mathcal{E}$ on P. Moreover, we will denote by $\pi : P \to X$ the bundle projection and by $d := d(P,H)$ the degree of (P,H).

The following result will be used in Section 3.

Lemma 1.4. *Let* X, $\mathcal{E}$, (P,H) *and* d *be as in* 1.3. *Assume that* $\mathcal{E}$ *has rank* $r \geqslant 2$. *Then* $\Delta(P,H) = 1 = d$ *cannot happen.*

Proof. Assume by contradiction that $\Delta(P,H) = 1 = d$. Using the description in [7, Theorem 13.6] we know that the base locus $\mathrm{Bs}|H|$ of $|H|$ is a single point $x \in P$. Let $F_0 = \mathbb{P}^{r-1}$ be the fiber of $\pi : P \to X$ containing x. As $\dim|H| = n+r-2$, by imposing to contain $r-1$ linearly independent tangent directions to F_0 at x we obtain a linear subsystem $\mathcal{S}$ of $|H|$ of dimension $n+r-2-(r-1) = n-1$, all of whose elements D contain F_0, the general one being a scroll over X. Choose general elements $D_0, \dots, D_{n-1}$ generating $\mathcal{S}$. Then $D_0 \cap \dots \cap D_{n-1} = F_0 + T$, where T cuts every fiber F of π along a linear subspace of codimension $\leqslant n$. Note that if $r-1 \geqslant n$, then T contains a $\mathbb{P}^{r-1-n}$-bundle over X

as an irreducible component. On the other hand, if $2 \leqslant r \leqslant n$, then one of the irreducible components of T maps birationally via π to a subvariety of X of dimension $r-1$. In fact, $\pi(T)$ parameterizes the fibers of P that $D_0, \dots, D_{n-1}$ meet at a same point, and a dimension count shows that $\pi(T)$ has codimension $n-r+1$ in X. In both cases we thus get

$$1 = H^{n+r-1} = H^{r-1} D_0 \cdots D_{n-1} = H^{r-1} F_0 + H^{r-1} T = 1 + H^{r-1} T.$$

Hence $H^{r-1}T = 0$, but this contradicts the ampleness of H. $\square$

Now we prove a result on the cubic surface that we will use in Section 6.

Lemma 1.5. *Let $Y \subset \mathbb{P}^3$ be a smooth cubic surface and let L be an ample line bundle on Y with $g(L) = 4$. Then $L = -2K_Y$.*

Proof. Let $\sigma : Y \to \mathbb{P}^2$ be the birational morphism exhibiting Y as the plane blown-up at general points $p_1, \dots, p_6$; let $\ell = \sigma^* \mathcal{O}_{\mathbb{P}^2}(1)$ and $e_i = \sigma^{-1}(p_i)$. Recall that $-K_Y = 3\ell - \sum_{i=1}^6 e_i$. We can write $L = a\ell - \sum_{i=1}^6 b_i e_i$ for suitable integers $a, b_1, \dots, b_6$. Letting $d = L^2$ the condition $g(L) = 4$ is equivalent to $L \cdot K_Y = 6 - d$. Thus the existence of a line bundle L such that $g(L) = 4$ can be rephrased by saying that in the 6-dimensional euclidean space $\langle e_1, \dots, e_6 \rangle \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^6$ the sphere Σ and the hyperplane Π defined by

$$\sum_{i=1}^{6} b_i^2 = a^2 - d \quad \text{and} \quad \sum_{i=1}^{6} b_i = 3a + 6 - d,$$

respectively, do intersect. To this end it is necessary that the distance of the origin of $\mathbb{R}^6$ from Π does not exceed the radius of Σ. This gives

$$d\bigl(d - 6(a+1)\bigr) + 3\bigl(a^2 + 12a + 12\bigr) \leqslant 0. \tag{1.5.1}$$

On the other hand, by the Hodge index theorem we have $(6-d)^2 = (L \cdot K_Y)^2 \geqslant L^2 \cdot K_Y^2 = 3d$, i.e.

$$d \geqslant 12. \tag{1.5.2}$$

Combining this with (1.5.1) gives

$$3(a-6)^2 \leqslant 0.$$

Therefore $a = 6$ and equality must hold in both (1.5.1) and (1.5.2). Hence $\sum_{i=1}^6 b_i = 3a + 6 - d = 12$ and $\sum_{i=1}^6 b_i^2 = a^2 - d = 24$. Thus $\sum_{i=1}^6 (b_i - 2)^2 = \sum_{i=1}^6 b_i^2 - 4\sum_{i=1}^6 b_i + 24 = 0$ and we conclude that $b_1 = \cdots = b_6 = 2$. In other words, $L = 6\ell - 2\sum_{i=1}^6 e_i = -2K_Y$. $\square$

We conclude this section summarizing the properties of the Δ-genus for a "polarized curve" in the following

Proposition 1.6. *Let $\mathcal{L}$ be an ample line bundle on a smooth projective curve C of genus g, and let $\Delta := \Delta(C, \mathcal{L})$. Then:*

(1) *$\Delta \leqslant g$, with equality if and only if $\mathcal{L}$ is nonspecial (i.e. $h^1(\mathcal{L}) = 0$).*
(2) *If $\mathcal{L}$ is special, then $\deg \mathcal{L} \leqslant 2\Delta$ and $\mathcal{L}$ imposes Δ linearly independent linear conditions on the canonical series $|K_C|$.*
(3) *$\deg \mathcal{L} = 2\Delta$ if and only if $g \geqslant 2$ and either $\mathcal{L} = K_C$, or C is hyperelliptic and $|\mathcal{L}| = \Delta g_2^1$.*

Proof. By definition, $\Delta = 1 + \deg \mathcal{L} - h^0(\mathcal{L})$. Combining this with the Riemann–Roch theorem gives (1). If $\mathcal{L}$ is special, Clifford's theorem gives

$$1 + \deg \mathcal{L} - \Delta = h^0(\mathcal{L}) \leqslant \frac{1}{2} \deg \mathcal{L} + 1,$$

hence $\deg \mathcal{L} \leqslant 2\Delta$. On the other hand, by Serre duality and the Riemann–Roch theorem we get

$$0 < h^0(K_C - \mathcal{L}) = h^1(\mathcal{L}) = g - \Delta = h^0(K_C) - \Delta.$$

This proves (2). Finally, (3) follows from the characterization of the equality in the Clifford theorem. $\square$

Remarks 1.7. Here are some immediate implications of Proposition 1.6.

(i) $\Delta = 0$ if and only if $g = 0$.

(ii) If $\Delta = 1$, then either $g = 1$, or $\mathcal{L}$ is special and $g \geqslant 2$ by (1). Moreover, in the latter case $\deg \mathcal{L} \leqslant 2\Delta = 2$, equality implying that C is hyperelliptic and $|\mathcal{L}|$ is the g^1_2, by (2) and (3) (note that, if $\mathcal{L} = K_C$, then $g = 2$). On the other hand, if $\deg \mathcal{L} = 1$ then $h^0(\mathcal{L}) = 1 + \deg \mathcal{L} - \Delta = 1$, hence $\mathcal{L} = \mathcal{O}_C(p)$ for some point $p \in C$.

(iii) If $\Delta = 2$, then (1) and (2) imply that either $g = 2$, or $\mathcal{L}$ is special, $g \geqslant 3$, and $\deg \mathcal{L} \leqslant 4$. In the latter case, equality $\deg \mathcal{L} = 4$ implies by (3) that either C is a non-hyperelliptic curve with $g = 3$ and $\mathcal{L} = K_C$, or C is hyperelliptic and $|\mathcal{L}| = 2g^1_2$. Let $\mathcal{L}$ be special. If $\deg \mathcal{L} = 1$ then $h^0(\mathcal{L}) = 1 + \deg \mathcal{L} - \Delta = 0$, so $\mathcal{L} = \mathcal{O}_C(p_1 + \cdots + p_m + p - q_1 - \cdots - q_m)$ for some points $p_i, p, q_i \in C$, $m \geqslant 1$, and the points $p_1, \ldots, p_m, p$ impose $(m+1)$ linearly independent linear conditions on the linear series $|K_C + q_1 + \cdots + q_m|$, by (2). If $\deg \mathcal{L} = 2$, then $h^0(\mathcal{L}) = 1$, hence $\mathcal{L} = \mathcal{O}_C(p + p')$ for some points $p, p' \in C$, which impose two linearly independent linear conditions on $|K_C|$ (in particular, if C is hyperelliptic then $p + p' \notin g^1_2$). Finally, if $\deg \mathcal{L} = 3$, then $h^0(\mathcal{L}) = 2$, so that C is trigonal and $|\mathcal{L}|$ is a g^1_3. We have $\mathcal{L} = \mathcal{O}_C(p + p' + p'')$ for some $p, p', p'' \in C$. Thus $p + p' + p''$ is the divisor cut out on the canonical curve by a trisecant line (take into account that C cannot be hyperelliptic, being a trigonal curve of genus $g \geqslant 3$, due to the Castelnuovo–Severi inequality).

2. Scrolls admitting further relevant structures

Let X, $\mathcal{E}$, (P, H) and π be as in 1.3. In this section we study whether the scroll structure of (P, H) given by π is compatible with (P, H) being another relevant variety for adjunction theory.

First we point out the following fact.

Lemma 2.1. *Let (X, L) be a del Pezzo manifold. If X admits a $\mathbb{P}$-bundle structure, then one of the following holds*:

(1) $d(X, L) = 6$ *and X is either* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, *or* $\mathbb{P}^2 \times \mathbb{P}^2$, *or* $\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$;
(2) $d(X, L) = 7$ *and X is* $\mathrm{Bl}_p \mathbb{P}^3$;
(3) $d(X, L) = 8$ *and X is either* $\mathbb{F}_0$, *or* $\mathbb{F}_1$.

Proof. The assertion follows from [11, Theorem 8.11] and the classification of del Pezzo surfaces. $\square$

We deduce the following

Corollary 2.2. *Let X, $\mathcal{E}$ and (P, H) be as in* 1.3, *where $\mathcal{E}$ has rank $r \geqslant 2$. If (P, H) is a del Pezzo manifold, then $(X, \mathcal{E})$ is one of the following*:

(1) $(\mathbb{P}^2, \mathcal{E})$, *where $\mathcal{E}$ is one of the following vector bundles*: $\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3}$, *the tangent bundle* $T_{\mathbb{P}^2}$, $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$;
(2) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)^{\oplus 2})$.

Proof. If $n \geqslant 2$, the assertion follows from Lemma 2.1, cases (1) and (2). If $n = 1$, Lemma 2.1 again (case (3)) says that $P = \mathbb{F}_e$, $e = 0, 1$. However, on $\mathbb{F}_0$ we have $H = -K_P = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)$, while on $\mathbb{F}_1$ it is $H = -K_P = 2C_0 + 3f$. It follows that $H \cdot f = 2$ in both cases, so we get a contradiction, since H is the tautological line bundle on P. $\square$

Lemma 2.3. *Let X be a Fano manifold which is a $\mathbb{P}$-bundle over a smooth curve C. Then X has pseudoindex $\leqslant 2$.*

Proof. We know that $C = \mathbb{P}^1$ (e.g. by [35, Theorem 1.6]). Then $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{F})$ where $\mathcal{F} = \bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^1}(a_i)$, $a_m \geqslant \cdots \geqslant a_2 \geqslant a_1 = 0$. Let ξ and f be the tautological line bundle and a fiber, respectively. Then $K_X = -m\xi + (\sum_{i=1}^m a_i - 2)f$. Let $\gamma \subset X$ be the section corresponding to the surjection of $\mathcal{F}$ onto the trivial summand $\mathcal{O}_{\mathbb{P}^1}(a_1) = \mathcal{O}_{\mathbb{P}^1}$. Then $\xi \cdot \gamma = 0$, hence

$$-K_X \cdot \gamma = \left(m\xi - \left(\sum_{i=1}^m a_i - 2\right)f\right) \cdot \gamma = 2 - \sum_{i=1}^m a_i.$$

By contradiction, suppose that X has pseudoindex $\geqslant 3$. Then from $-K_X \cdot \gamma \geqslant 3$ we get $\sum_{i=1}^m a_i \leqslant -1$, a contradiction. $\square$

Proposition 2.4. *Let X, $\mathcal{E}$ and (P, H) be as in* 1.3. *Assume that X has dimension $n \geqslant 2$ and that $\mathcal{E}$ has rank $r \geqslant 2$. If (P, H) is a Mukai manifold, then $(X, \mathcal{E})$ is one of the following*:

(1) *$(\mathbb{P}^3, \mathcal{E})$, where $\mathcal{E}$ is one of the following vector bundles: $\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4}$, $\mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$, the tangent bundle $T_{\mathbb{P}^3}$, the twist $\mathcal{N}(2)$ of a null-correlation bundle $\mathcal{N}$ on $\mathbb{P}^3$, $\mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 2}$, $\mathcal{O}_{\mathbb{P}^3}(3) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$;*
(2) *$(\mathbb{Q}^3, \mathcal{E})$, where $\mathcal{E}$ is one of the following vector bundles: $\mathcal{O}_{\mathbb{Q}^3}(1)^{\oplus 3}$, the twist $\mathcal{S}(2)$ of a spinor bundle $\mathcal{S}$ on $\mathbb{Q}^3$ (see* [31, *Definition* 1.3]), *$\mathcal{O}_{\mathbb{Q}^3}(2) \oplus \mathcal{O}_{\mathbb{Q}^3}(1)$;*
(3) *(X, h) is a del Pezzo threefold and $\mathcal{E} = h^{\oplus 2}$;*
(4) *$(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{E})$, where $\mathcal{E}$ is either $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(2, 1) \oplus \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(1, 1)$, or $\pi_1^* T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(0, 1)$, π_1 denoting the first projection.*

Proof. As (P, H) is a Mukai manifold, we know that $-K_P = (\dim P - 2)H$. On the other hand, by the canonical bundle formula we have

$$K_P = -rH + \pi^*(K_X + \det \mathcal{E}),$$

from which we derive that $K_X + \det \mathcal{E} = \mathcal{O}_X$ and $r = \dim P - 2 = n + r - 3$. Therefore $n = 3$. Clearly, X is a Fano manifold; moreover, its pseudoindex is $i_X \geqslant i_P$ by [2, Lemme 2.5]. Therefore, as P has index $\dim P - 2 = r$,

$$4 = n + 1 \geqslant i_X \geqslant i_P \geqslant r \geqslant 2, \tag{2.4.1}$$

the first inequality coming from Mori theory, e.g. see [3, Theorem 1.8].

If $r = 3$ or 4, we are in the assumptions of [10, Main Theorem], so we get the following possibilities for $(X, \mathcal{E})$:

(i) $(\mathbb{P}^3, \mathcal{V})$, where $\mathcal{V}$ is either $\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4}$, or $\mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$, or the tangent bundle $T_{\mathbb{P}^3}$;
(ii) $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(1)^{\oplus 3})$;
(iii) X is a $\mathbb{P}^2$-bundle over a smooth curve C and $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3}$ for any fiber $F = \mathbb{P}^2$ of the bundle projection.

Taking into account that X is a Fano 3-fold of pseudoindex $\geqslant 3$ by (2.4.1), the last case cannot occur in view of Lemma 2.3.

We can thus assume that $r = 2$. Then we are in the assumptions of [33, Theorem 0.4], which gives all the remaining cases of our statement. □

Remark 2.5. Note that the pairs $(\mathbb{P}^3, \mathcal{N}(2))$ and $(\mathbb{Q}^3, \mathcal{S}(2))$ in cases (1) and (2) give rise to the same polarized manifold (P, H), according to [35, Propositions 2.6 and 3.4]. Looking at P as the incidence variety $\{(x,\ell) \in \mathbb{P}^3 \times \mathcal{C} \mid x \in \ell\}$, where $\mathcal{C} \subset \mathrm{Grass}(1,3)$ is a general linear complex of lines of $\mathbb{P}^3$, and recalling that $\mathcal{C} \cong \mathbb{Q}^3$, the two distinct $\mathbb{P}^1$-bundle structures of P are induced by the projections of $\mathbb{P}^3 \times \mathcal{C}$.

Remark 2.6. Arguing similarly to the previous proof, it is possible to derive directly the classification of Corollary 2.2.

We will use the following generalization of [24, Lemma 1.8]:

Lemma 2.7. *Let X be a smooth projective variety of dimension $n \geqslant 3$ and let L be an ample line bundle on X such that (X, L) is a scroll over a smooth variety Y of dimension $2 \leqslant \dim Y \leqslant n-1$. Suppose that X contains a (-1)-hyperplane with respect to L. Then $X = \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1))$, L being the tautological line bundle. Moreover, (X, L) has $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$ as its simple reduction.*

Proof. Let $\pi : X \to Y$ be the scroll projection and let $E = \mathbb{P}^{n-1}$ be a (-1)-hyperplane contained in X. Since every fiber f of π is a projective space of dimension $n - \dim Y \leqslant n-2$, $\pi(E)$ cannot be a point. Hence $\pi|_E : E \to Y$ is surjective, which implies that $\dim Y = n-1$. We continue by induction on n. For $n = 3$ the assertion is proved in [24, Lemma 1.8]. Therefore assume $n \geqslant 4$. Notice that $L_E = \mathcal{O}_E(-E) = \mathcal{O}_E(1)$. Since Y is smooth, this implies that $Y = \mathbb{P}^{n-1}$ by [27, Theorem 4.1]. Set $\mathcal{W} := \pi_* L$ and let $M := \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. Then L is the tautological line bundle of $\mathcal{W}$ on X and $\mathrm{Pic}(X) \cong \mathbb{Z}^2$ is generated by L, M. Since E is a divisor inside X, its class can be written as $E = aL - bM$ for some integers a, b. Taking into account that $M^n = 0$ and that $\pi|_E$ is surjective, we have

$$a = aL \cdot M^{n-1} = aL \cdot M^{n-1} - bM^n = E \cdot M^{n-1} = E \cdot f > 0$$

and

$$(-1)^{n-1} = (E_E)^{n-1} = E^n = a\bigl(a^{n-1}L^n - na^{n-2}bL^{n-1} \cdot M + \cdots\bigr).$$

This implies $a = 1$, so that $E \cdot f = 1$, i.e. E is a section of π. In particular, $M_E \cong \mathcal{O}_E(1)$, due to the isomorphism $\pi|_E : E \to \mathbb{P}^{n-1}$. Moreover, $L = E + bM$. On the other hand, since

$$\mathcal{O}_E(1) = L_E = E_E + bM_E = \mathcal{O}_E(-1+b),$$

we conclude that $b = 2$, i.e. $L = E + 2M$. Set $\mathcal{U} := \pi_* \mathcal{O}_X(E)$. Then $X = \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{U})$, E being the tautological section. Moreover, $\mathcal{U} = \mathcal{W}(-2)$. For any hyperplane $h = \mathbb{P}^{n-2} \subset \mathbb{P}^{n-1}$ consider the divisor $\pi^* h \in |M|$ and for simplicity denote it by M again. Then we have that $M = \mathbb{P}_h(\mathcal{U}_h)$ is a $\mathbb{P}^1$-bundle on h with a scroll structure given by L_M. Set $e := E \cap M$. Note that

$$e_e = (M \cdot E)_e = (M_E \cdot E_E)_e = (E_E)_e = \bigl(\mathcal{O}_E(-1)\bigr)_e = \mathcal{O}_e(-1),$$

and

$$(L_M)_e = (L_E)_e = \bigl(\mathcal{O}_E(1)\bigr)_e = \mathcal{O}_e(1).$$

Thus e is a (-1)-hyperplane of (M, L_M), which is a scroll over $h = \mathbb{P}^{n-2}$ via $\pi|_M$. It thus follows by induction that $M = \mathbb{P}_h(\mathcal{W}_h)$ where $\mathcal{W}_h = \mathcal{O}_{\mathbb{P}^{n-2}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(1)$ for every hyperplane h of $\mathbb{P}^{n-1}$. This implies that $\mathcal{W} = \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ by [30, Chapter I, Theorem 2.3.2].

To prove the final assertion in the statement note that $M + E$ is spanned. Clearly it is spanned outside E, because M is spanned. The exact cohomology sequence of

$$0 \to M \to M + E \to (M + E)_E = \mathcal{O}_E \to 0,$$

taking into account that $h^1(M) = h^1(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) = 0$, shows that it is spanned also on E. Moreover, $h^0(M + E) = h^0(M) + 1 = n + 1$. So, $|M + E|$ defines a morphism $\sigma : P \to \mathbb{P}^n$. Since $(M + E)_E$ is trivial and $M^n = 0$, we get

$$(M + E)^n = (M + E)\bigl(M^{n-1} + E(\ldots)\bigr) = (M + E)M^{n-1} = (M_E)^{n-1} = 1.$$

Therefore σ is birational and contracts E. Finally, note that $L = 2M + E = \sigma^*\mathcal{O}_{\mathbb{P}^n}(2) - E$. $\square$

We will also use the following results on the compatibility of further scroll structures on (P, H):

Proposition 2.8. *Let X, $\mathcal{E}$, (P, H) and π be as in* 1.3, *where $\mathcal{E}$ has rank $r \geqslant 2$. Suppose that (P, H) admits another scroll structure $p : P \to C$ over a smooth curve C (i.e. $p \neq \pi$). Then $(X, \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2})$.*

Proof. Let $F = \mathbb{P}^{n+r-2}$ be any fiber of p. First assume that $n \geqslant 2$. Then the restriction of π to F is surjective. So $r = 2$ and $X = \mathbb{P}^n$ by a [27, Theorem 4.1]. Now, denote by $G = \mathbb{P}^1$ any fiber of π. Since the restriction of p to G is surjective, it follows that $C = \mathbb{P}^1$. Then P has two $\mathbb{P}$-bundle structures over projective spaces. Therefore $P = \mathbb{P}^n \times \mathbb{P}^1$ by [34, Theorem A]. It follows that $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(a)^{\oplus 2}$ for some $a \geqslant 1$, hence $H = \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(a, 1)$. On the other hand, $H = \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(1, b)$ for some $b \geqslant 1$ due to the scroll structure of (P, H) over $\mathbb{P}^1$. Hence $a = b = 1$, so $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2}$. Now, let $n = 1$ and let $x \in X$ be any point. Since $\pi^{-1}(x) = \mathbb{P}^{r-1}$ is not a fiber of p, it follows that $p(\pi^{-1}(x)) = C$, hence $r = 2$ and $C = \mathbb{P}^1$. So $\dim P = 2$ and $F = \mathbb{P}^1$. Then the restriction of π to F is surjective, so $X = \mathbb{P}^1$. Moreover, $P = \mathbb{P}^1 \times \mathbb{P}^1$, hence $\mathcal{E}$ has the form $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. $\square$

Proposition 2.9. *Let X, $\mathcal{E}$, (P, H) and π be as in* 1.3. *Assume that X has dimension $n \geqslant 2$ and that $\mathcal{E}$ has rank $r \geqslant 2$. Suppose that (P, H) admits another scroll structure $p : P \to S$ over a smooth surface S (i.e. $p \neq \pi$). Then $(X, \mathcal{E})$ is one of the following*:

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 3})$;
(2) $(\mathbb{P}^2, T_{\mathbb{P}^2})$, *where $T_{\mathbb{P}^2}$ is the tangent bundle*;
(3) *$n = r = 2$ and both X and S are $\mathbb{P}^1$-bundles over the same smooth curve.*

Moreover, if $\mathcal{E}$ is very ample, then the only pairs $(X, \mathcal{E})$ as in case (3) *are the following*:

(3a) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)^{\oplus 2})$;
(3b) *X is a $\mathbb{P}^1$-bundle over a smooth curve B and $\mathcal{E}_f = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ for every fiber of the projection $\varphi : X \to B$ and $\varphi \circ \pi$ makes (P, H) a quadric fibration over B.*

Proof. Since $\pi \neq p$ there exists a fiber $F = \mathbb{P}^{n+r-3}$ of p such that the restriction of π to F is not constant.

Assume that $\dim P \geqslant 4$. In this case, the restriction of π to F is surjective, so $r = 3$ and $X = \mathbb{P}^n$ by [27, Theorem 4.1]. Moreover, the restriction of p to a fiber of π, which is a $\mathbb{P}^2$, is surjective, hence $S = \mathbb{P}^2$ for the same reason. Then P has two $\mathbb{P}$-bundle structures over projective spaces. Therefore $P = \mathbb{P}^n \times \mathbb{P}^2$ by [34, Theorem A]. It follows that $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(a)^{\oplus 3}$ for some $a \geqslant 1$, hence $H = \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^2}(a, 1)$. On the other hand, $H = \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^2}(1, b)$ for some $b \geqslant 1$ due to the scroll structure of (P, H) over $\mathbb{P}^2$. Hence $a = b = 1$, so $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 3}$. This gives case (1) in the statement.

Assume now that $\dim P = 3$. Then we are in the assumption of [29, Theorem 2], so we get cases (2) and (3) of our statement. For the last part of the statement we refer to the discussion of Case (C) in the proof of [24, Theorem 2.1]. $\square$

Remark 2.10. In case (3a) the three obvious structures of $(P, H) = (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(1,1,1))$ as a scroll over $\mathbb{P}^1 \times \mathbb{P}^1$ are given by the morphisms $p_i \times p_j$, $i < j$, where p_i is the projection of P onto the i-th factor.

A typical example as in (3b) is given by the two pairs $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,1) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1))$ and $(\mathbb{F}_1, [C_0 + 2f]^{\oplus 2})$, which give rise to the same (P, H).

Scrolls having an additional structure of a quadric fibration over a smooth curve occur very often. First of all we specialize a result of [21] as follows.

Proposition 2.11. *Let X, $\mathcal{E}$, (P, H) and π be as in 1.3. Assume that X has dimension $n = 2$ and that $\mathcal{E}$ has rank $r \geqslant 2$. Suppose that (P, H) admits a quadric fibration $\varphi : P \to C$ over a smooth curve C. Then*

I. $\dim P = 3$ *and either*

(a) $(X, \mathcal{E}) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)^{\oplus 2})$, *or*
(b) *$X = \mathbb{P}_C(\mathcal{V})$, where $\mathcal{V}$ is a vector bundle of rank 2 on C, $\varphi = p \circ \pi$, where $p : X \to C$ is the projection and $\mathcal{E} = \xi \otimes p^*\mathcal{G}$, with $\xi = H(\mathcal{V})$ the tautological line bundle of $\mathcal{V}$ on X and $\mathcal{G}$ a vector bundle of rank 2 on C.*

In particular, φ has no singular fibers.

II. *Moreover, if $C = \mathbb{P}^1$, then $(X, \mathcal{E}) = (\mathbb{F}_e, [C_0 + af] \oplus [C_0 + bf])$, for some integers $a, b > e$.*

Proof. Part I follows from [21, Theorem] recalling that, in our assumption, we have the same polarization for the structures given by π and φ. Now let $C = \mathbb{P}^1$. Then $\mathcal{G} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ for some integers a and b. It follows that $\mathcal{E}$ is decomposable. Then the ampleness of $\mathcal{E}$ implies that $a, b > e$. $\square$

Note the analogy between the situation described by Proposition 2.11 with that arising in case (3) of Proposition 2.9. In the result above note that P has dimension 3. In higher dimension the situation is easier.

Proposition 2.12. *Let X, $\mathcal{E}$ and (P, H) be as in 1.3. Assume that X has dimension $n \geqslant 2$ and that $\mathcal{E}$ has rank $r \geqslant 2$. Suppose that (P, H) admits a quadric fibration $\varphi : P \to C$ over a smooth curve C. If $\dim P \geqslant 4$, then $(X, \mathcal{E}) = (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus 2})$.*

Proof. Denote by $F = \mathbb{P}^{r-1}$ any fiber of π, and by $G = \mathbb{Q}^{n+r-2}$ a general fiber of φ. Note that $G \nsubseteq F$ because $\dim G = n + r - 2 \geqslant r > r - 1 = \dim F$; so the restriction of π to G is not constant.

Since $\dim G = n + r - 2 \geqslant 3$, then $\pi(G)$ must have dimension $\dim G$; hence $n \leqslant n + r - 2 = \dim G = \dim \pi(G) \leqslant \dim X = n$. It follows that $r = 2$, hence $n \geqslant 3$ and $\pi(G) = X$, so $\pi(G)$ is smooth. Then X is either $\mathbb{P}^n$ or $\mathbb{Q}^n$ by [32, Proposition 8]. Due to the quadric fibration structure of (P, H) we have $K_P + nH = \varphi^*\mathcal{L}$, for some line bundle $\mathcal{L}$ on C.

Let $\ell \subset X$ be any line and set $P_\ell := \pi^{-1}(\ell)$. Since $P_\ell = \mathbb{P}_\ell(\mathcal{E}_\ell)$ we have that $P_\ell = \mathbb{F}_e$, a Segre–Hirzebruch surface of invariant e, for some integer $e \geqslant 0$, the ruling being given by $\pi|_{P_\ell} : P_\ell \to \ell$. We want to show that $e = 0$. A fiber $F = \pi|_{P_\ell}^{-1}(x)$, for $x \in \ell$ is obviously a fiber of the scroll (P, H), hence $F \cong \mathbb{P}^1$ and $H \cdot F = 1$. In particular $K_P \cdot F = -2$. Hence, recalling the expression of K_P via φ, we get

$$-2 = K_P \cdot F = -nH \cdot F + \varphi^*\mathcal{L} \cdot F.$$

Suppose that $\varphi|_F$ is constant. Then the above equality gives $n = 2$, a contradiction. It follows that $\varphi(F) = C$, which implies that $C = \mathbb{P}^1$. Moreover, $\varphi|_{P_\ell} : P_\ell \to \mathbb{P}^1$ is a morphism on $\mathbb{P}^1$, distinct from

$\pi|_{P_\ell}$. But the only Segre–Hirzebruch surface admitting a second morphism onto a curve, distinct from the ruling projection is $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. This means that $\mathcal{E}_\ell = \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus 2}$ for some positive integer a. Since $2a = \deg \mathcal{E}_\ell$, this happens for any line $\ell \subset X$ and then we conclude that $\mathcal{E}$ is a uniform vector bundle of splitting type (a, a), where $\det \mathcal{E} = \mathcal{O}_X(2a)$. Now, if $X = \mathbb{P}^n$ we conclude that $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(a)^{\oplus 2}$ by [30, Theorem 3.2.3, p. 55]. Suppose that $X = \mathbb{Q}^n$; then $\mathcal{E}(-a)$ restricts trivially to any line of $\mathbb{Q}^n$ and so it is the trivial vector bundle of rank 2, by [37, Lemma 3.6.1]. This means that $\mathcal{E} = \mathcal{O}_{\mathbb{Q}^n}(a)^{\oplus 2}$. Therefore, $\mathcal{E} = \mathcal{O}_X(a)^{\oplus 2}$ in both cases, where $\deg c_1(\mathcal{E}) = 2a$. In particular we see that $P = X \times \mathbb{P}^1$, with $H = \mathcal{O}_{X\times\mathbb{P}^1}(a, 1)$, π being the first projection. If $X = \mathbb{Q}^n$, then φ is the second projection and the fact that φ is a quadric fibration implies that $\mathcal{O}_{\mathbb{Q}^n}(a) = (\mathcal{O}_{\mathbb{Q}^n\times\mathbb{P}^1}(a, 1))_G = H_G = \mathcal{O}_{\mathbb{Q}^n}(1)$. So $(X, \mathcal{E}) = (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus 2})$.

The assertion is proved once we show that X cannot be $\mathbb{P}^n$. Were it so, then it would be $(P, H) = (\mathbb{P}^n \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^1}(a, 1))$. Note that P has to contain a smooth divisor G such that $(G, H_G) = (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$, which is a fiber of φ. Then

$$-nH_G = K_G = (K_P + G)_G = (K_P)_G. \tag{2.12.1}$$

Set $M = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$. Since $\mathrm{Pic}(G) \cong \mathbb{Z}$ generated by H_G, we can write $M_G = \lambda H_G$ for some integer λ, and $\lambda \geqslant 0$ as M is nef. Recalling the canonical bundle formula and the fact that $\deg c_1(\mathcal{E}) = 2a$, we thus get $(K_P)_G = (-2H + (2a - n - 1)M)_G = (-2 + \lambda(2a - n - 1))H_G$. Hence (2.12.1) gives

$$2 - n = \lambda(2a - n - 1). \tag{2.12.2}$$

Since $n \geqslant 3$, we see that $\lambda \neq 0$, hence $\lambda \geqslant 1$. It follows that $2a = n + 1 + \frac{2-n}{\lambda} \leqslant n + 1 + 2 - n = 3$, which implies $a = 1$. But putting this value in (2.12.2), we get $\lambda = \frac{n-2}{n-1} < 1$, a contradiction. □

2.13. In the sequel we will need to face the same situation as in Proposition 2.12 also with $\dim P = 3$. In this case, set $\mathcal{V} = \varphi_* H$. Then $\mathcal{V}$ is a vector bundle of rank $\dim P + 1 = 4$ on C. Consider the projective bundle $\psi : \mathbb{P}_C(\mathcal{V}) \to C$ and let $\xi = H(\mathcal{V})$ be tautological line bundle. Then P embeds fiberwise into $\mathbb{P}_C(\mathcal{V})$ as a divisor $P \in |2\xi - \psi^* B|$, for some $B \in \mathrm{Pic}(C)$, and $H = \xi_P$.

Lemma 2.14. *Let X, $\mathcal{E}$, (P, H) and d be as in* 1.3. *Assume that X has dimension $n \geqslant 2$ and that $\mathcal{E}$ has rank $r \geqslant 2$. Suppose that (P, H) admits a quadric fibration $\varphi : P \to C$ over a smooth curve C and let $\mathcal{V} = \varphi_* H$. If* $\dim P = 3$, *then (P, H) has degree $d = \frac{3}{2} \deg \mathcal{V}$.*

Proof. The number δ of singular fibers of φ is given by $\delta = 2\deg \mathcal{V} - 4b$, where $b = \deg B$ [8, (3.3)]. Then $\psi^* B$ is numerically equivalent to bD, where $D \cong \mathbb{P}^3$ is a fiber of ψ. So, taking into account the Chern–Wu relation, we get

$$d = H^3 = (\xi_P)^3 = \xi^3(2\xi - bD) = 2\xi^4 - b\xi^3 D = 2\deg \mathcal{V} - b.$$

Therefore,

$$\delta = -6\deg \mathcal{V} + 4d.$$

Now note that $\dim P = 3$ implies $n = 2$. So, taking into account also the scroll structure of (P, H) over a surface, it turns out from part I of Proposition 2.11 that $\delta = 0$ and this proves the assertion. □

3. Ample vector bundles of very small Δ-genus

In order to extend the notion of Δ-genus to the vector bundle setting, we give the following definition.

Definition 3.1. Let $\mathcal{E}$ be an ample vector bundle of rank r on a smooth projective variety X of dimension n. We define the Δ-*genus* of the pair $(X,\mathcal{E})$ as the integer

$$\Delta(X,\mathcal{E}) := nr + d - h^0(X,\mathcal{E}), \tag{3.1.1}$$

$d := d(P,H)$ being given by Lemma 1.1.

Remark 3.2. Let $\mathcal{E}$ be a rank-r ample vector bundle on a smooth curve C and assume that $\mathcal{E}$ is decomposable as $\mathcal{E} = \oplus_{i=1}^r \mathcal{L}_i$, where all the $\mathcal{L}_i$'s are ample. Then

$$\Delta(C,\mathcal{E}) = \Delta\left(C, \bigoplus_{i=1}^r \mathcal{L}_i\right) = r + \sum_{i=1}^r \deg \mathcal{L}_i - \sum_{i=1}^r h^0(C,\mathcal{L}_i) = \sum_{i=1}^r \Delta(C,\mathcal{L}_i). \tag{3.2.1}$$

Notice that for the Δ-genus of a decomposable ample vector bundle on a smooth variety of dimension $\geqslant 2$ we have only superadditivity; for instance, consider the following

Example 3.3. Let X be a smooth projective variety of dimension $n = 3$ and let $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are two ample line bundles on X. According to Example 1.2(i), $d = c_1(c_1^2 - 2c_2)$. Hence

$$\begin{aligned}\Delta(X,\mathcal{E}) &= 6 + (\mathcal{L}_1 + \mathcal{L}_2)\cdot\left(\mathcal{L}_1^2 + \mathcal{L}_2^2\right) - \left(h^0(\mathcal{L}_1) + h^0(\mathcal{L}_2)\right) \\ &> \sum_{i=1}^2 \left(3 + \mathcal{L}_i^3 - h^0(\mathcal{L}_i)\right) = \sum_{i=1}^2 \Delta(X,\mathcal{L}_i).\end{aligned}$$

Note that for a decomposable ample vector bundle $\mathcal{E}$ on a smooth curve C, we have $\Delta(C,\mathcal{E}) \geqslant 0$ as a consequence of Remark 3.2. Moreover, the same holds for the pair $(X,\mathcal{E})$ in the example. On the other hand, looking at Definition 3.1, it is not immediate to see that the Δ-genus of any ample vector bundle is a nonnegative integer, as it is for line bundles. However, this is the case, as the following proposition shows.

Proposition 3.4. *Let X, $\mathcal{E}$ and (P,H) be as in* 1.3. *Then*

$$\Delta(X,\mathcal{E}) = (n-1)(r-1) + \Delta(P,H).$$

In particular, $\Delta(X,\mathcal{E}) \geqslant (n-1)(r-1)$.

Proof. Let F be any fiber of the scroll projection $\pi : P \longrightarrow X$. Since H is ample, due to the ampleness of $\mathcal{E}$, we can compute the Δ-genus of the pair (P,H). We have

$$\Delta(X,\mathcal{E}) = nr + H^{n+r-1} - h^0(\mathcal{E}) = (n-1)(r-1) + \Delta(P,H).$$

Now, the final assertion follows from the non-negativity of Δ-genus for ample line bundles. $\square$

Remark 3.5. According to Proposition 3.4, one could observe that our definition of $\Delta(X,\mathcal{E})$ for ample vector bundles of rank $r \geqslant 2$ still relies on the Δ-genus of a polarized manifold. However, in studying pairs with low $\Delta(X,\mathcal{E})$, this will be an advantage, since the scroll structure of (P,H) prevents this pair from entering in a range for which Fujita's classification is not complete. On the other hand, one could consider another obvious polarized manifold associated with $(X,\mathcal{E})$, instead of (P,H), namely $(X,\det\mathcal{E})$. For instance, let $n = 1$, so that $\Delta(X,\mathcal{E}) = \Delta(P,H)$, and let g be the genus of X. If $g = 0$, then $\Delta(X,\mathcal{E}) = 0 = \Delta(X,\det\mathcal{E})$, according to Proposition 1.6. If $g = 1$, then $\Delta(X,\mathcal{E}) =$

$r + \deg \mathcal{E} - h^0(\mathcal{E}) = r$, while $\Delta(X, \det \mathcal{E}) = 1$ by Proposition 1.6. Now, let $g \geqslant 2$ and suppose that $\mathcal{E}$ is very ample. Then $h^0(\det \mathcal{E}) \geqslant h^0(\mathcal{E}) + r - 2$, by [19, Theorem]. This says that

$$\Delta(X, \mathcal{E}) \geqslant \Delta(X, \det \mathcal{E}) + 2r - 3 > \Delta(X, \det \mathcal{E}).$$

This seems to suggest that $\Delta(X, \mathcal{E})$ is a more relevant character than $\Delta(X, \det \mathcal{E})$: actually the list of pairs $(X, \mathcal{E})$ satisfying $\Delta(X, \mathcal{E}) \leqslant (n-1)(r-1) + \delta$ is expected to include that of pairs such that $\Delta(X, \det \mathcal{E}) \leqslant \delta$. Apart from case $n = 1$ discussed before, if $n \geqslant 2$ this is certainly true for $\delta = 0$, as one can see comparing [11, Theorem 5.10] with Theorem 3.6 below, taking into account that $\deg \mathcal{E}_C \geqslant r$ for any rational curve $C \subset X$. Moreover, this is true also for $\delta = 1$ at least for pairs $(X, \mathcal{E})$ such that $c_1(\mathcal{E})^2 \geqslant 3$ (compare [11, Chapters 8 and 9] with Theorems 3.6 and 3.7).

Now we want to classify pairs $(X, \mathcal{E})$ whose Δ-genus is small. We start with pairs whose Δ-genus is minimum, i.e. equal to $(n-1)(r-1)$. Since the case of line bundles on manifolds of dimension $\geqslant 2$ has already been settled by Fujita (see [11, Theorem 5.10]), and in view of Proposition 1.6, we confine to ample vector bundles of rank at least two.

Theorem 3.6. *Let X and $\mathcal{E}$ be as in* 1.3. *Assume that $\mathcal{E}$ has rank $r \geqslant 2$. Then $\Delta(X, \mathcal{E}) = (n-1)(r-1)$ if and only if $(X, \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2})$, or $(X, \mathcal{E}) = (\mathbb{P}^1, \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(a_i))$, with $a_r \geqslant \cdots \geqslant a_1 \geqslant 1$. In particular, $\Delta(X, \mathcal{E}) = 0$ if and only if $(X, \mathcal{E})$ is as in the latter case.*

Proof. Let (P, H) be as in 1.3. By Proposition 3.4, the assertion on the Δ-genus is equivalent to $\Delta(P, H) = 0$. Therefore the polarized manifold (P, H) satisfies the assumption of [11, Theorem 5.10]. Note that, since $\mathrm{rk}(\mathrm{Pic}(P)) \geqslant 2$, the only possibility is case (3) of [11, Theorem 5.10], namely (P, H) is the scroll of an ample vector bundle $\mathcal{V}$ on $\mathbb{P}^1$ via a morphism $p : P \to \mathbb{P}^1$. If $\pi = p$, then $\mathcal{E} = \mathcal{V}$ is a direct sum of line bundles of positive degrees and we are done. If $\pi \neq p$, then the assertion follows by Proposition 2.8. Viceversa, for the pairs $(X, \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2})$ and $(\mathbb{P}^1, \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(a_i))$, a direct check shows that $\Delta(P, H) = 0$. In the former case, it is useful to note that $P = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(n+1)})$, with H being the tautological line bundle. The final assertion in the statement is obvious, recalling Proposition 3.4. $\square$

Theorem 3.7. *Let X and $\mathcal{E}$ be as in* 1.3. *Assume that $\mathcal{E}$ has rank $r \geqslant 2$. Then $\Delta(X, \mathcal{E}) = (n-1)(r-1) + 1$ if and only if $(X, \mathcal{E})$ is one of the following*:

(1) *$(\mathbb{P}^2, \mathcal{E})$, where $\mathcal{E}$ is one of the following vector bundles*: *$\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3}$, the tangent bundle $T_{\mathbb{P}^2}$, $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$*;
(2) *$(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)^{\oplus 2})$.*

Moreover, $\Delta(X, \mathcal{E}) = 1$ if and only if $(X, \mathcal{E}) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2})$.

Proof. Let (P, H) be as in 1.3. By Proposition 3.4, the assertion on the Δ-genus is equivalent to $\Delta(P, H) = 1$. Therefore the polarized manifold (P, H) satisfies the assumptions of [11, Chapters 8 and 9], which gives the following possibilities: (a) $H^{n+r-1} = 1$, (b) $H^{n+r-1} = 2$ and there is a finite morphism $p : P \to \mathbb{P}^{n+r-1}$ of degree 2 such that $H = p^*\mathcal{O}_{\mathbb{P}^{n+r-1}}(1)$, or (c) $H^{n+r-1} \geqslant 3$ and (P, H) is a del Pezzo manifold.

Case (a) is ruled out by Lemma 1.4. In case (b) we get a contradiction with the Picard number if $\dim P \geqslant 3$ by a result of Lazarsfeld (see [26, Proposition 3.1]). If $\dim P = 2$, let $B \in |\mathcal{O}_{\mathbb{P}^2}(2b)|$ be the branch locus of the double cover $p : P \to \mathbb{P}^2$. Comparing the expression of $K_P = -2H + \pi^*(K_X + \det \mathcal{E})$ with that given by the ramification formula $K_P = p^*\mathcal{O}_{\mathbb{P}^2}(b - 3)$ we conclude that $b = 1$ and $K_X + \det \mathcal{E} = \mathcal{O}_X$. But this gives $(P, H) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1))$, which contradicts the condition $\Delta(P, H) = 1$. Hence we are left with case (c), and the assertion follows from Corollary 2.2.

Conversely, for all pairs as in (1) and (2) it is clear that $\Delta(P, H) = 1$.

As to the final assertion, recall Proposition 3.4 and note that $(n-1)(r-1)=0$ cannot occur. This would imply $\Delta(P,H)=1$ and the previous part of the proof shows that this is not compatible with $n=1$. Therefore, $n=r=2$ and $\Delta(X,\mathcal{E})=(n-1)(r-1)$, so $(X,\mathcal{E})=(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2})$ by Theorem 3.6. $\square$

4. Ample and spanned vector bundles

When the ample vector bundle $\mathcal{E}$ is spanned, we can do a further step in the analysis of the low values of $\Delta(X,\mathcal{E})$. We need some preliminary lemmata.

Lemma 4.1. *Let X, $\mathcal{E}$ and d be as in* 1.3. *Assume that $\mathcal{E}$ is spanned and has rank $r \geqslant 2$. Then $d \geqslant 3$ unless $(X,\mathcal{E})=(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})$.*

Proof. Let (P,H) be as in 1.3 and assume that $d \leqslant 2$. Then (P,H) is one of the pairs listed in [25, Lemma 0.6.1], namely:

(i) $(\mathbb{P}^{n+r-1},\mathcal{O}_{\mathbb{P}^{n+r-1}}(1))$;
(ii) $(\mathbb{Q}^{n+r-1},\mathcal{O}_{\mathbb{Q}^{n+r-1}}(1))$;
(iii) $p: P \to \mathbb{P}^{n+r-1}$ is a double cover and $H=p^*\mathcal{O}_{\mathbb{P}^{n+r-1}}(1)$.

Clearly case (ii) for $n+r-1 \geqslant 3$ and case (i) are ruled out, as in our assumptions P has Picard number at least two. Similarly, in view of [26, Proposition 3.1], we rule out case (iii) for $n+r-1 \geqslant 3$. Therefore $n+r-1=2$, and (P,H) is a surface scroll over a smooth curve.

Let $B \in |\mathcal{O}_{\mathbb{P}^2}(2b)|$, for a positive integer b, be the branch locus of the double cover p in (iii). We have $p_*\mathcal{O}_P=\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(-b)$, hence $h^1(\mathcal{O}_P)=h^1(p_*\mathcal{O}_P)=0$. Then, in both cases (ii) and (iii), we are left with (P,H) a rational scroll. It follows that $\Delta(P,H)=0$, which is equivalent to $\Delta(X,\mathcal{E})=(n-1)(r-1)$ by Proposition 3.4. Therefore, noting that $\deg\mathcal{E}=d=2$, we deduce that $(X,\mathcal{E})=(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})$ by Theorem 3.6. $\square$

Lemma 4.2. *Let X, $\mathcal{E}$ and (P,H) be as in* 1.3. *Assume that $\mathcal{E}$ is spanned and has rank $r \geqslant 2$. Then $\Delta(X,\mathcal{E})=(n-1)(r-1)+2$ if and only if one of the following holds*:

(1) *X is a smooth curve of genus* 1 *and $r=2$*;
(2) $g(P,H) \geqslant 2$.

Proof. By Proposition 3.4 the assumption on the Δ-genus of $(X,\mathcal{E})$ is equivalent to $\Delta(P,H)=2$. Therefore by [11, Theorem 10.2] one of the following holds:

(i) $P=\mathbb{P}_C(\mathcal{F})$, with $\mathcal{F}$ an ample vector bundle over an elliptic curve C and $H=H(\mathcal{F})$ (which implies that we can assume $\mathcal{F}$ spanned);
(ii) $g(P,H) \geqslant 2$.

If (i) holds, then $h^1(\mathcal{O}_P)=h^1(\mathcal{O}_C)=1$, as C is an elliptic curve; hence $h^1(\mathcal{O}_X)=1$, too. Combining with Proposition 2.8, we derive that the bundle structures of P on X and on C have to coincide; in particular X is an elliptic curve. On the other hand, for such a curve and for any ample (and spanned) vector bundle $\mathcal{E}$, we know that $\Delta(P,H)=r$, since $h^0(H)=h^0(\mathcal{E})=\deg\mathcal{E}+r(1-g(C))$ by the Riemann–Roch theorem, as $h^1(\mathcal{E})=h^1(\mathcal{E}\otimes K_C)=0$. Therefore $r=2$ and we get case (1) in the statement. Case (ii) corresponds to (2). $\square$

As we have seen, case (1) in Lemma 4.2 is effective. So we continue assuming in the following that $g(P,H) \geqslant 2$.

Theorem 4.3. *Let X, $\mathcal{E}$, (P,H) and d be as in* 1.3. *Assume that $d \geqslant 4$, $\mathcal{E}$ is spanned of rank $r \geqslant 2$ and $g(P,H) \geqslant 2$. Then $\Delta(X,\mathcal{E})=(n-1)(r-1)+2$ if and only if $(X,\mathcal{E})$ is one of the following*:

(1) $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(1)^{\oplus 2})$;
(2) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,1))$;
(3) $(\mathbb{F}_1, [C_0 + 2f]^{\oplus 2})$;
(4) *X is a smooth hyperelliptic curve and $\mathcal{E} = \mathcal{L}^{\oplus 2}$, where $\mathcal{L} \in \mathrm{Pic}(X)$ is the line bundle giving the g_2^1 of X.*

Proof. By Proposition 3.4, the assumption on the Δ-genus of $(X, \mathcal{E})$ is equivalent to $\Delta(P, H) = 2$.

Assume first that $d \geqslant 5$. As shown in [11, (10.7)], the adjoint bundle $K_P + (n + r - 2)H$ defines a morphism $q : P \to \mathbb{P}^1$ giving to (P, H) the structure of a quadric fibration over $\mathbb{P}^1$.

We claim that $\dim P \geqslant 3$. If this is not the case, then $\dim P = 2$, hence $n = 1$ and $r = 2$. Then by [11, (10.7.2)], (P, H) is one of the following:

(a1) P is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $(12 - d)$ points and $H = \sigma^*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,3)) - \sum E_i$, where $\sigma : P \to \mathbb{P}^1 \times \mathbb{P}^1$ is the blow-up and E_i are the exceptional divisors;
(a2) $P = \mathbb{F}_1$ or a blow-up of $\mathbb{F}_1$ at a point on the (-1)-curve;
(a3) $P = \mathbb{F}_2$.

In case (a1) $d = 12$, as P is a $\mathbb{P}$-bundle, and, of course, $P = \mathbb{F}_1$ in case (a2). Therefore in all cases $P = \mathbb{F}_e$ for some $e \leqslant 2$. Since these surfaces are rational, (P, H) can only be a scroll over $\mathbb{P}^1$, but then $\Delta(P, H) = 0$, a contradiction. This proves the claim.

Next we claim that $n \geqslant 2$. Indeed, if this is not the case, then the general fiber of π, which is a $\mathbb{P}^{r-1}$ with $r - 1 = \dim P - 1 \geqslant 2$, cannot map surjectively onto $\mathbb{P}^1$ via q; hence it has to be a fiber of q, which is impossible.

First let $\dim P = 3$. Taking into account Lemma 2.14, a closed check of the list in [11, (10.7.3)] shows that the only possibility is $P = \mathbb{P}^1 \times \mathbb{F}_1$ with $d = 9$; thus we immediately see that H corresponds to the Segre embedding. According to Remark 2.10, this (P, H) leads to the pairs (2) and (3) in the statement. Conversely, for these two pairs it is immediate to check that $\Delta(P, H) = 2$, hence $\Delta(X, \mathcal{E}) = 3$.

Next let $\dim P \geqslant 4$. Then $(X, \mathcal{E}) = (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus 2})$ by Proposition 2.12. By computing the Δ-genus we find that

$$2n - 2 = \Delta\big(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus 2}\big) = (n-1)(r-1) + 2 = n + 1$$

if and only if $n = 3$. This gives case (1) in the statement.

Now let $d = 4$. There are two possibilities according to whether the morphism Φ_H associated to $|H|$ is birational or not [11, 10.8]. When Φ_H is birational, [11, Theorem 10.8.1], recalling that $g(P, H) \geqslant 2$, gives the following possibilities:

(b1) P is isomorphic to a smooth quartic hypersurface of $\mathbb{P}^{\dim P + 1}$;
(b2) P is $\mathbb{P}^1 \times \mathbb{P}^1$ blown-up at 8 points;
(b3) $\dim P = 3$, (P, H) is a quadric fibration over $\mathbb{P}^1$, and P embeds fiberwise as a divisor in $\mathbb{P}_{\mathbb{P}^1}(\mathcal{V})$ as in 2.13 with $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}$.

Under our assumptions all these possibilities cannot happen. Indeed, the fact that our P has a $\mathbb{P}$-bundle structure rules out both case (b2) and case (b1) when $\dim P \geqslant 3$. On the other hand, for $\dim P = 2$, P is a $\mathbb{P}^1$-bundle over a smooth curve, hence it cannot be a $K3$ surface as in (b1). Finally, case (b3) gives a contradiction with Lemma 2.14.

Now assume that Φ_H is not birational. Then according to [11, (10.8.2)], $\Phi_H : P \to W$ is a double cover where either

(c1) $W = \mathbb{Q}^{n+r-1}$ is a smooth quadric, or
(c2) $n + r - 1 = 2$ and W is a quadric cone.

Case (c2) cannot occur. Otherwise the vertex v of W would be in the branch locus of Φ_H, P being smooth. By taking the desingularization $\nu : \mathbb{F}_2 \to W$ and blowing-up $\mu : \widetilde{P} \to P$ of P at the point $\Phi_H^{-1}(v)$, we would get a commutative diagram:

$$\begin{tikzcd} \widetilde{P} \arrow[r, "\mu"] \arrow[d, "\varphi"'] & P \arrow[d, "\Phi_H"] \\ \mathbb{F}_2 \arrow[r, "\nu"'] & W, \end{tikzcd}$$

where φ is the double cover induced by Φ_H. Note that K_P^2 is even, since P is a $\mathbb{P}^1$-bundle over a smooth curve, and then $K_{\widetilde{P}}^2 = K_P^2 - 1$ is odd. On the other hand, by the ramification formula, $K_{\widetilde{P}} = \varphi^*(K_{\mathbb{F}_2} + \mathcal{B})$, where $|2\mathcal{B}|$ is the linear system containing the branch divisor of φ. Then $K_{\widetilde{P}}^2 = 2(K_{\mathbb{F}_2} + \mathcal{B})^2$ is even, a contradiction.

Now, consider case (c1). If $n + r - 1 \geqslant 3$, then P has Picard number one by [4], but this is impossible since (P, H) is a scroll. Thus $W = \mathbb{P}^1 \times \mathbb{P}^1$ and the branch divisor of Φ_H is a smooth element $B \in |2\mathcal{B}|$, where $\mathcal{B} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)$ for some integers $a, b \geqslant 0$ and we can suppose $a \geqslant b$ in view of the symmetry. First assume that $b = 0$; thus B is a union of fibers of the first projection p_1 of $\mathbb{P}^1 \times \mathbb{P}^1$. Then Φ_H is induced by a double cover of smooth curves $\rho : \Gamma \to \mathbb{P}^1$ branched at the points corresponding to the connected components of B and we have a commutative diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\Phi_H"'] & \Gamma \arrow[d, "\rho"] \\ \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "p_1"'] & \mathbb{P}^1. \end{tikzcd}$$

This shows that $P = \Gamma \times \mathbb{P}^1$, so $X = \Gamma$ is a smooth curve admitting a g_2^1, hence a hyperelliptic curve, since $g(X) = g(P, H) \geqslant 2$, and $\mathcal{E} = \mathcal{L}^{\oplus 2}$ for some ample line bundle $\mathcal{L} \in \operatorname{Pic}(X)$. Writing the numerical class of H as $\xi + (\deg \mathcal{L}) f$, where ξ is the tautological line bundle on P of the trivial vector bundle $\mathcal{O}_X^{\oplus 2}$ and f is a fiber, we get $4 = d = H^2 = \xi^2 + (2 \deg \mathcal{L}) \xi \cdot f = 2 \deg \mathcal{L}$. Then $\deg \mathcal{L} = 2$ and $\mathcal{L}$ has to be spanned, $\mathcal{E}$ being so. Therefore, $|\mathcal{L}|$ is the g_2^1 of X. This gives case (4) in the statement.

Now suppose that $b > 0$. Then $\mathcal{B}$ is ample and

$$h^1(\mathcal{O}_P) = h^1(\Phi_{H*}\mathcal{O}_P) = h^1\big(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-a, -b)\big) = 0$$

by the Kodaira vanishing theorem. This says that the base curve of the $\mathbb{P}^1$-bundle P is $\mathbb{P}^1$, hence $P = \mathbb{F}_e$ for some e. Ramification formula gives

$$K_P = \Phi_H^*(K_{\mathbb{P}^1 \times \mathbb{P}^1} + \mathcal{B}) = \Phi_H^*\big(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a - 2, b - 2)\big),$$

and $8 = K_{\mathbb{F}_e}^2 = K_P^2$ implies $(a, b) = (4, 3)$. But then K_P would be ample, according to the above formula, which is clearly impossible. $\square$

We finally consider $d = 3$. In this case, since H is ample and spanned, $\phi_H : P \to \mathbb{P} := \mathbb{P}^{n+r-1}$ is a triple cover.

Proposition 4.4. *Let X, $\mathcal{E}$, (P,H) and d be as in* 1.3. *Assume that $d=3$, $\mathcal{E}$ is spanned of rank $r \geqslant 2$, $g(P,H) \geqslant 2$ and $\Delta(P,H)=2$. Then the triple cover $\phi_H : P \to \mathbb{P}$ defined by $|H|$ is not of triple section type* (*in the sense of* [9]). *Moreover, $n=r=2$.*

Proof. Recall that P has Picard number $\rho(P) \geqslant 2$. It thus follows from [26, Proposition 3.1] that $3=\deg \phi_H \geqslant \dim P = n+r-1$. Therefore we know that

$$n+r-1 \leqslant 3. \tag{4.4.1}$$

By contradiction, suppose ϕ_H is of triple section type. Then [26, Proposition 3.2] (see also [9, Theorem 2.1]) says that the relative dualizing sheaf of ϕ_H is isomorphic to $\phi_H^* \mathcal{O}_{\mathbb{P}}(\lambda)$ for some $\lambda \in \mathbb{Z}$. We know that $K_P = -rH + \pi^*(K_X + \det \mathcal{E})$ where $\pi : P \to X$ is the bundle projection; on the other hand, $\phi_H^* \mathcal{O}_{\mathbb{P}}(1) = H$. Hence we get

$$\lambda H = K_P - \phi_H^* K_{\mathbb{P}} = (-r+n+r)H + \pi^*(K_X + \det \mathcal{E}).$$

Therefore $\lambda = n$ and $K_X + \det \mathcal{E} = \mathcal{O}_X$, due to the injectivity of the homomorphism $\pi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(P)$, whence $K_P = -rH$. Then the genus formula gives

$$2g(P,H) - 2 = \big(K_P + (n+r-2)H\big)H^{n+r-2} = (n-2)H^{n+r-1} = 3(n-2).$$

This shows that n has to be even. Moreover, $n \geqslant 4$ since $g(P,H) \geqslant 2$. But this contradicts (4.4.1).

Furthermore, since $r \geqslant 2$, according to (4.4.1) we have two possibilities: either

(a) $n=1$ with $r=2, 3$, or
(b) $n=r=2$.

So the proof is complete when we show that case (a) cannot occur.

Assume by contradiction that case (a) holds. Let S denote a smooth element of $|H|$ when $r=3$ and P itself when $r=2$. In both cases we know that $|H_S|$ induces a triple cover $\phi : S \to \mathbb{P}^2$. Then we can write $\phi_* \mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{T}$, where $\mathcal{T}$ is a rank-2 vector bundle on $\mathbb{P}^2$, and the branch locus of ϕ is an element of $|2 \det \mathcal{T}^*|$. Set $b_i := c_i(\mathcal{T})$. By applying the Riemann–Hurwitz formula to the curve $\phi^* \ell \in |H_S|$, where $\ell \subset \mathbb{P}^2$ is a general line, we thus get $2g(H_S) - 2 = 3(-2) + (-2b_1)$. Since $g(H_S) = g(P,H)$ this gives

$$-b_1 = g(P,H) + 2. \tag{4.4.2}$$

But (P,H) is a scroll over the smooth curve X, hence $g(P,H) = q$, the genus of X, so that (4.4.2) reads as $-b_1 = q+2$. Moreover, (S, H_S) itself is a surface scroll over X. So we have $K_S^2 = 8(1-q)$ and the topological Euler–Poincaré characteristic is $e(S) = 4(1-q)$. Thus, eliminating b_2 from Miranda's formulas [28, Proposition 10.3]

$$K_S^2 = 27 + 12b_1 + 2b_1^2 - 3b_2 \quad \text{and} \quad e(S) = 9 + 6b_1 + 4b_1^2 - 9b_2,$$

we get $q(q-1) = 0$. But this is a contradiction in view of our assumption $g(P,H) \geqslant 2$. □

To conclude the discussion for $d=3$, according to Proposition 4.4 it remains to analyze the very restricted case $n=r=2$. However, this requires more work than expected and will be done in a separate paper.

We conclude this section classifying pairs $(X, \mathcal{E})$ where $\mathcal{E}$ is an ample and spanned vector bundle of rank $r \geqslant 2$ such that $\Delta(X, \mathcal{E}) = 2$.

Proposition 4.5. *Let X and $\mathcal{E}$ be as in 1.3. Assume that $\mathcal{E}$ is spanned of rank $r \geqslant 2$. Then $\Delta(X,\mathcal{E}) = 2$ if and only if $(X,\mathcal{E})$ is one of the following*:

(1) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2})$;
(2) $(\mathbb{P}^2, \mathcal{E})$, *where $\mathcal{E}$ is either the tangent bundle $T_{\mathbb{P}^2}$, or $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$*;
(3) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)^{\oplus 2})$;
(4) *X is a smooth curve of genus 1 and $r = 2$*;
(5) *X is a smooth hyperelliptic curve and $\mathcal{E} = \mathcal{L}^{\oplus 2}$, where $\mathcal{L} \in \mathrm{Pic}(X)$ is the line bundle giving the g^1_2 of X.*

Proof. Let (P,H) be as in 1.3. In view of our assumptions and of Proposition 3.4, we have $0 \leqslant \Delta(P,H) \leqslant 2$. We can therefore split the proof according to this value. If $\Delta(P,H) = 0$, then $2 = \Delta(X,\mathcal{E}) = (n-1)(r-1)$; hence we get case (1) of the statement by Theorem 3.6. If $\Delta(P,H) = 1$, then $2 = \Delta(X,\mathcal{E}) = (n-1)(r-1)+1$; hence we get cases (2) and (3) of the statement by Theorem 3.7. If $\Delta(P,H) = 2$, then $2 = \Delta(X,\mathcal{E}) = (n-1)(r-1)+2$; hence $n = 1$, and we get cases (4) and (5) of the statement by combining Lemmata 4.1 and 4.2, Theorem 4.3 and Proposition 4.4. $\square$

5. Very ample vector bundles

As we have seen in Section 4, the analysis of low values of $\Delta(X,\mathcal{E})$ can be pushed further as long as the vector bundle $\mathcal{E}$ enjoys better properties than the bare ampleness. In this section we consider very ample vector bundles $\mathcal{E}$ and we obtain, among other results, a complete descriptions of pairs $(X,\mathcal{E})$ up to $\Delta(X,\mathcal{E}) = (n-1)(r-1)+3$. First we prove some results relying on the classification of projective manifolds of low degree due to Ionescu.

Theorem 5.1. *Let X, $\mathcal{E}$, (P,H) and d be as in 1.3. Assume that X has dimension $n \geqslant 2$ and that $\mathcal{E}$ is very ample of rank $r \geqslant 2$. Then $\Delta(X,\mathcal{E}) > (n-1)(r-1) + \frac{d}{2}$ unless $(X,\mathcal{E})$ is one of the following*:

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2})$;
(2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus 2})$;
(3) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 3})$, $n \leqslant 6$;
(4) $(\mathbb{P}^3, \mathcal{E})$, *where $\mathcal{E}$ is one of the following vector bundles: $\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4}$, $\mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$, the tangent bundle $T_{\mathbb{P}^3}$, the twist $\mathcal{N}(2)$ of a null-correlation bundle $\mathcal{N}$ on $\mathbb{P}^3$, $\mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 2}$, $\mathcal{O}_{\mathbb{P}^3}(3) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$, $\mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$*;
(5) $(\mathbb{Q}^3, \mathcal{E})$, *where $\mathcal{E}$ is one of the following vector bundles*: $\mathcal{O}_{\mathbb{Q}^3}(1)^{\oplus 3}$, *the twist $\mathcal{S}(2)$ of a spinor bundle $\mathcal{S}$ on $\mathbb{Q}^3$, $\mathcal{O}_{\mathbb{Q}^3}(2) \oplus \mathcal{O}_{\mathbb{Q}^3}(1)$*;
(6) *(X,h) is a del Pezzo threefold and $\mathcal{E} = h^{\oplus 2}$*;
(7) $(\mathbb{P}^2, \mathcal{E})$, *where $\mathcal{E}$ is either the tangent bundle $T_{\mathbb{P}^2}$, or $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$*;
(8) $(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{E})$, *where $\mathcal{E}$ is either $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(2,1) \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(1,1)$, or $\pi_1^* T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(0,1)$, $T_{\mathbb{P}^2}$ and π_1 denoting the tangent bundle on $\mathbb{P}^2$ and the first projection respectively*;
(9) *$n = r = 2$, $X = \mathbb{P}_C(\mathcal{V})$, where $\mathcal{V}$ is a vector bundle of rank 2 on a smooth curve C, and $\mathcal{E} = \xi \otimes p^*\mathcal{G}$, where $\xi = H(\mathcal{V})$ is the tautological line bundle of $\mathcal{V}$ on X, $\mathcal{G}$ is a vector bundle of rank 2 on C, $p : X \to C$ is the projection and $2h^0(\mathcal{V} \otimes \mathcal{G}) \geqslant 3(\deg \mathcal{V} + \deg \mathcal{G} + 2)$*;
(10) *$n = 2$, X is ruled and (P,H) has a unique scroll structure over a surface.*

Proof. Assume that $\Delta(X,\mathcal{E}) \leqslant (n-1)(r-1) + \frac{d}{2}$. We note first that, H being very ample, there is an embedding of P in $\mathbb{P}^N$ with $N = h^0(\mathcal{E}) - 1$; moreover, the assumption on the Δ-genus is equivalent to the condition $d \leqslant 2\,\mathrm{codim}_{\mathbb{P}^N} P + 2$.

This allows us to apply [16, Theorem I] to (P,H), obtaining the following possibilities:

(i) (P,H) is a scroll over a smooth curve C;
(ii) (P,H) is a scroll over a (birationally) ruled surface S;
(iii) (P,H) is a quadric fibration over a smooth curve C;
(iv) (P,H) is a del Pezzo manifold;

(v) (P, H) is a Mukai manifold;
(vi) (P, H) admits a reduction (Y, L) which is one of the following pairs:
  (vi-a) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$;
  (vi-b) $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$;
  (vi-c) $(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$;
  (vi-d) a Veronese bundle over a smooth curve.

We proceed with a case-by-case analysis.

Case (i). We can apply Proposition 2.8, hence we have $(X, \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2})$. Notice that this pair satisfies $\Delta(X, \mathcal{E}) = (n-1)(r-1)$ in view of Theorem 3.6, so we get case (1) in the statement.

Case (ii). If (P, H) has a unique scroll structure over a surface, then we get case (10) in the statement (see Remark 5.2). Otherwise we can apply Proposition 2.9, which gives three possibilities. If $(X, \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 3})$, then the condition $\Delta(X, \mathcal{E}) \leqslant (n-1)(r-1) + \frac{d}{2}$ gives $n \leqslant 6$, and we get case (3) in the statement. Pair $(X, \mathcal{E}) = (\mathbb{P}^2, T_{\mathbb{P}^2})$ fits into case (7). In this case $\Delta(X, \mathcal{E}) = 2 < (n-1)(r-1) + \frac{d}{2}$. The last possibility gives (9), as shown in the discussion of *Case* (iii) below.

Case (iii). In this case P is endowed with two morphisms $\pi : P \to X$ and $\varphi : P \to C$. Denote by $F = \mathbb{P}^{r-1}$ any fiber of π, and by $G = \mathbb{Q}^{n+r-2}$ a general fiber of φ. Note that $G \not\subseteq F$ because $\dim G = n + r - 2 \geqslant r > r - 1 = \dim F$; so the restriction of π to G is not constant.

If $\dim G = n + r - 2 \geqslant 3$, then we get case (2) of the statement with $n \geqslant 3$ by Proposition 2.12. Note that this case is effective, since $d = 2(n+1)$, hence $\Delta(X, \mathcal{E}) = 2n - 2 \leqslant 2n = (n-1)(r-1) + \frac{d}{2}$.

If $\dim G = n + r - 2 = 2$, then $n = r = 2$. So, (P, H) is both a scroll over a smooth surface and a quadric fibration over a smooth curve (with respect to the same polarization). Hence we are in the assumptions of part I of Proposition 2.11, which gives two possibilities, namely (a) and (b). Of course, (a) corresponds to case (2) of the statement for $n = 2$. In case (b) we have $c_1(\mathcal{E}) = 2\xi + p^* c_1(\mathcal{G})$ and $c_2(\mathcal{E}) = \xi^2 + \xi \cdot p^* c_1(\mathcal{G})$. Moreover, $\xi^2 = \deg \mathcal{V}$ by the Chern–Wu relation, and $d = H^3 = c_1(\mathcal{E})^2 - c_2(\mathcal{E})$. Finally, $h^0(\mathcal{E}) = h^0(p_* \xi \otimes \mathcal{G}) = h^0(\mathcal{V} \otimes \mathcal{G})$, by projection formula. It thus follows that condition $\Delta(X, \mathcal{E}) \leqslant (n-1)(r-1) + \frac{d}{2}$ is equivalent to

$$h^0(\mathcal{V} \otimes \mathcal{G}) \geqslant 3 + \frac{3(\deg \mathcal{V} + \deg \mathcal{G})}{2}.$$

Then we get case (9) in the statement. Note that also the pair in (a) satisfies this condition.

Case (iv). We are in the assumptions of Corollary 2.2. As already observed, the first case of this corollary fits into case (9) of the statement. Therefore we get cases (7), (3) with $n = 2$ and (9) of the statement. Indeed, in these cases, $\Delta(X, \mathcal{E}) = (n-1)(r-1) + 1 \leqslant (n-1)(r-1) + \frac{d}{2}$, in view of Theorem 3.7 and Lemma 1.4.

Case (v). We are in the assumptions of Proposition 2.4, hence we get cases (4), except for the last vector bundle, (5), (6) and (8) of the statement. In fact, a direct computation shows that $\Delta(X, \mathcal{E}) \leqslant (n-1)(r-1) + \frac{d}{2}$ in all these cases. The very ampleness of the indecomposable $\mathcal{E}$ in case (8) follows taking into account the Euler sequence on $\mathbb{P}^2$ pulled-back to X via π_1 and twisted by $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(0, 1)$. This also shows that $h^0(\mathcal{E}) = 3h^0(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(1, 1)) - h^0(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(0, 1)) = 16$. On the other hand, a straightforward computation gives $d = 24$. Therefore $\Delta(X, \mathcal{E}) = 14 = (n-1)(r-1) + \frac{d}{2}$.

As to pairs $(\mathbb{P}^3, \mathcal{N}(2))$ and $(\mathbb{Q}^3, \mathcal{S}(2))$, we already pointed out in Remark (2.5) that they give rise to the same (P, H). So, to prove their effectiveness, it is enough to deal with $(\mathbb{Q}^3, \mathcal{S}(2))$, in view of Proposition 3.4. Thus consider the exact sequence (e.g. see [31, Theorem 2.8])

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_{\mathbb{Q}^3}^{\oplus 4} \longrightarrow \mathcal{S}(1) \longrightarrow 0$$

and twist it by $\mathcal{O}_{\mathbb{Q}^3}(1)$. This shows the very ampleness of $\mathcal{S}(2)$. Since $h^1(\mathcal{S}(1)) = 0$ and $h^0(\mathcal{S}(1)) = 4$ by [31, Theorem 2.3], we derive $h^0(\mathcal{S}(2)) = 4h^0(\mathcal{O}_{\mathbb{Q}^3}(1)) - h^0(\mathcal{S}(1)) = 16$. Moreover, $c_1(\mathcal{S}(2)) =$

$\mathcal{O}_{\mathbb{Q}^3}(3)$, while $c_2(\mathcal{S}(2)) = c_2(\mathcal{S} \otimes \mathcal{O}_{\mathbb{Q}^3}(2)) = \mathcal{O}_{\mathbb{Q}^3}(2)^2 + c_1(\mathcal{S}) \cdot \mathcal{O}_{\mathbb{Q}^3}(2) + c_2(\mathcal{S}) = \frac{5}{2}\mathcal{O}_{\mathbb{Q}^3}(1)^2$. It thus follows that $d = c_1(c_1^2 - 2c_2) = 24$, hence $\Delta(\mathbb{Q}^3, \mathcal{S}(2)) = 14 = (n-1)(r-1) + \frac{d}{2}$.

Case (vi). By dimensional reasons the pair (P, H) is a scroll over either a smooth surface, or a smooth threefold (only in case (vi-c)), so $n = 2$ or 3 (only in case (vi-c)). It follows by Lemma 2.7 that $(X, \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1))$, $n = 2$ or 3. Moreover, the only pair occurring as a nontrivial reduction (in fact simple reduction) of (P, H) is that in (vi-c). Thus, either $(X, \mathcal{E})$ is the last pair in (4), or the reduction morphism has to be an isomorphism, i.e. $(P, H) = (Y, L)$. But since $\rho(P) = 2$, this can happen only in case (vi-d). However, since (P, H) is also a scroll over a smooth surface, this situation gives a contradiction with [5, Theorem 2].

Note that for the last pair in (4) $\Delta(X, \mathcal{E}) = 7 < (n-1)(r-1) + \frac{d}{2}$. □

Remark 5.2. Of course not all pairs as in case (9) are true exceptions for the inequality in the statement. For instance, let $X = \mathbb{P}^2$ and consider $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b)$, with $a, b \geqslant 1$. Then $(X, \mathcal{E})$ is a true exception if and only if a and b satisfy the inequality $ab \leqslant 3(a+b) - 2$. E.g. this condition holds for $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, for any $a \geqslant 1$; but it does not hold for $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(a)^{\oplus 2}$, as soon as $a \geqslant 6$.

Remark 5.3. Suppose that the pair $(X, \mathcal{E})$, with $\mathcal{E}$ a very ample vector bundle, satisfies the condition $\Delta(X, \mathcal{E}) > (n-1)(r-1) + \frac{d}{2}$. If $d \geqslant 2 \dim P - 2$, then clearly $\Delta(X, \mathcal{E}) > (n-1)(r-1) + \dim P - 1 = nr - 1$. So, all pairs $(X, \mathcal{E})$ making exception to the inequality

$$\Delta(X, \mathcal{E}) > nr - 1 \tag{5.3.1}$$

and satisfying $d \geqslant 2 \dim P - 2$ also appear in the list of exceptions provided by Theorem 5.1. On the other hand, if a pair $(X, \mathcal{E})$ appearing in the list of exceptions in Theorem 5.1 is such that $d \leqslant 2 \dim P - 3$ then certainly it cannot satisfy the inequality (5.3.1). On the other hand, it is easy to check that there are pairs $(X, \mathcal{E})$ in the list of exceptions in Theorem 5.1 not satisfying the condition $d \leqslant 2 \dim P - 3$, e.g. $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus 2})$. So, it deserves to study inequality (5.3.1) by its own in order to find pairs making exception. The result is the following.

Theorem 5.4. *Let X and $\mathcal{E}$ be as in* 1.3. *Assume that X has dimension $n \geqslant 2$ and that $\mathcal{E}$ is very ample of rank $r \geqslant 2$. Then $\Delta(X, \mathcal{E}) > nr - 1$ unless $(X, \mathcal{E})$ is one of the following*:

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2})$;
(2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus 2})$;
(3) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 3})$;
(4) $(\mathbb{P}^2, \mathcal{E})$, *where $\mathcal{E}$ is one of the following vector bundles*: $\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus r}$ *with $r = 3$ or 4*, $T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus(r-2)}$ *where $T_{\mathbb{P}^2}$ is the tangent bundle and $r = 2$ or* 3, $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus(r-1)}$ *with $r = 2$ or* 3;
(5) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,2))$;
(6) $(\mathbb{F}_1, [C_0 + 2f]^{\oplus 2})$.

Proof. Assume that $\Delta(X, \mathcal{E}) \leqslant nr - 1$. Let (P, H), π and d be as in 1.3. We note first that, H being very ample, there is an embedding of P in $\mathbb{P}^N$ with $N = h^0(\mathcal{E}) - 1$; moreover, the assumption on the Δ-genus is equivalent to the condition $d \leqslant N$.

This allows us to apply the main Theorem of [18] to (P, H), and we obtain the following possibilities recalling that $\rho(P) \geqslant 2$:

(i) (P, H) is a del Pezzo manifold with $3 \leqslant \dim P \leqslant 4$ and $d = 6, 7$;
(ii) P is the Segre embedding of $\mathbb{P}^1 \times \mathbb{F}_1$, where $\mathbb{F}_1$ is embedded in $\mathbb{P}^4$ as a rational scroll of degree 3;
(iii) (P, H) is a scrolls over $\mathbb{P}^2$; more precisely, $P = \mathbb{P}_{\mathbb{P}^2}(\mathcal{F})$ where $\mathcal{F}$ is either $T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, or $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$, or $\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4}$, and H stands for the tautological line bundle;
(iv) (P, H) is a scroll over $\mathbb{P}^1$ with $d \geqslant \dim P$;

(v) there is a vector bundle $\mathcal{G}$ over $\mathbb{P}^1$ of rank $\dim P+1 \geqslant 4$ and of splitting type $\eta = (\eta_0, \dots, \eta_{n+r-1})$ such that, if L is the tautological line bundle on $\mathbb{P}_{\mathbb{P}^1}(\mathcal{G})$ and G denotes a fiber of the projection $\mathbb{P}_{\mathbb{P}^1}(\mathcal{G}) \longrightarrow \mathbb{P}^1$, P embeds fiberwise in $\mathbb{P}_{\mathbb{P}^1}(\mathcal{G})$ as a divisor $P \in |2L + \beta G|$ with $L_P = H$ and one of the following holds:
   (v-a) $N = d = 2\dim P - 1$, $\eta = (1, \dots, 1, 0, 0)$, $\beta = 1$;
   (v-b) $N = d = 2\dim P$, $\eta = (1, \dots, 1, 0)$, $\beta = 0$;
   (v-c) $N = d = 2\dim P + 1$, $\eta = (1, \dots, 1)$, $\beta = -1$;
   (v-d) $\dim P \geqslant 4$, $N = d+1 = 2\dim P + 1$, $\eta = (1, \dots, 1)$, $\beta = -2$ or, equivalently, $P \cong \mathbb{P}^1 \times \mathbb{Q}^{n+r-2}$ Segre embedded;
   (v-e) $N = d = 2\dim P + 2$, $\eta = (2, 1, \dots, 1)$, $\beta = -2$.

We proceed with a case-by-case analysis.

Case (i). We are in the assumptions of Corollary 2.2, hence we get cases (2) with $n = 2$ and (4) with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3}$, the tangent bundle $T_{\mathbb{P}^2}$, $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ of the statement. Indeed, in all these cases $\Delta(X, \mathcal{E}) = (n-1)(r-1) + 1 < nr - 1$, by Theorem 3.7.

Case (ii). In this case $P = \mathbb{P}_{\mathbb{F}_1}(\mathcal{L}^{\oplus 2})$, where $\mathcal{L} = [C_0 + 2f] \in \operatorname{Pic}(\mathbb{F}_1)$ because $H = \pi^*[C_0 + 2f] + q^*\mathcal{O}_{\mathbb{P}^1}(1)$, where q and π denote the projections. A direct computation shows that $\Delta(\mathbb{F}_1, \mathcal{L}^{\oplus 2}) = 3 = nr - 1$, and this gives case (6) in the statement.

Case (iii). Denote by $p : \mathbb{P}_{\mathbb{P}^2}(\mathcal{F}) \to \mathbb{P}^2$ the projection. If $n \geqslant 3$, then $p \neq \pi$. Since $\dim P = 4$ or 5, it follows from Proposition 2.9 that the only possibility is $(X, \mathcal{E}) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 3})$. Notice that in this case $\Delta(X, \mathcal{E}) = 7 < 8 = nr - 1$, so we get case (3) in the statement. Assume now that $n = 2$. Arguing as in case (d) of the proof of [23, Lemma 2.5], we show that $p = \pi$. We reproduce the argument for the convenience of the reader. Assume by contradiction that $p \neq \pi$. Then there is a fiber F of p such that $\pi|_F : F \longrightarrow X$ is not constant. Hence $F = \mathbb{P}^2$, otherwise $\pi|_F$ would give a fibration of $\mathbb{P}^3$ either onto X or onto a curve of X, which is a contradiction. Moreover $\pi|_F : F \longrightarrow X$ is a surjective morphism onto a smooth projective surface, hence $X = \mathbb{P}^2$ by [27, Theorem 4.1]. By [34, Theorem A], we obtain $P = \mathbb{P}^2 \times \mathbb{P}^2$, which is not one of our cases. We have thus proved that $p = \pi$. Therefore we obtain case (4) of the statement with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4}$, $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$, $T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$. Indeed a direct computation shows that in all these cases $\Delta(X, \mathcal{E}) \leqslant nr - 1$, equality occurring for the last two pairs.

Case (iv). As (P, H) is a scroll over $\mathbb{P}^1$, we can apply Proposition 2.8; hence $(X, \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2})$. Then, by Theorem 3.6, $\Delta(X, \mathcal{E}) = (n-1)(r-1) < nr - 1$; so we get case (1) in the statement.

Case (v). We can argue as in *Case* (iii) of Theorem 5.1, so we get the following possibilities:

(α) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus 2})$;
(β) $n = r = 2$, $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{V})$, where $\mathcal{V}$ is a vector bundle of rank 2 on $\mathbb{P}^1$, and $\mathcal{E} = \xi \otimes p^*\mathcal{M}$, where $\xi = H(\mathcal{V})$ is the tautological line bundle of $\mathcal{V}$ on X, $\mathcal{M}$ is a vector bundle of rank 2 on $\mathbb{P}^1$, $p : X \to \mathbb{P}^1$ is the projection.

Recall that $\Delta(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus 2}) = 2n - 2 < nr - 1$. So ($\alpha$) leads to case (2) in the statement. As to case (β), since the base curve is $\mathbb{P}^1$, we can write: $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}$ for a nonnegative integer a and $\mathcal{M} = \mathcal{O}_{\mathbb{P}^1}(\alpha_1) \oplus \mathcal{O}_{\mathbb{P}^1}(\alpha_2)$ for some integers α_1, α_2, with $\alpha_1 \geqslant \alpha_2$. Then $\mathcal{E} = [\xi + \alpha_1 f] \oplus [\xi + \alpha_2 f]$, f being a fiber of p, hence $\alpha_1 \geqslant \alpha_2 > 0$ due to the ampleness. We can compute $d = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = (2\xi + (\alpha_1 + \alpha_2)f)^2 - (\xi + \alpha_1 f)(\xi + \alpha_2 f) = 3(a + \alpha_1 + \alpha_2)$ and, by projection formula, $h^0(\mathcal{E}) = h^0(\mathcal{V} \otimes \mathcal{M}) = h^0(\mathcal{O}_{\mathbb{P}^1}(a + \alpha_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a + \alpha_2) \oplus \mathcal{O}_{\mathbb{P}^1}(\alpha_1) \oplus \mathcal{O}_{\mathbb{P}^1}(\alpha_2)) = 2(a + \alpha_1 + \alpha_2 + 2)$. It follows that $\Delta(X, \mathcal{E}) = a + \alpha_1 + \alpha_2$. Then $\Delta(X, \mathcal{E}) \leqslant nr - 1$ if and only if $a + \alpha_1 + \alpha_2 \leqslant 3$. Therefore we have the following possibilities

(a) $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ and $\mathcal{E} = [\xi + f]^{\oplus 2}$ or $\mathcal{E} = [\xi + 2f] \oplus [\xi + f]$;
(b) $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{E} = [\xi + f]^{\oplus 2}$.

So (a) leads to the cases (2) with $n = 2$ and (5) of the statement, while, noting that $\xi = C_0 + f$, (b) leads to case (6) of the statement. □

Remark 5.5. Note that cases (5) and (6) of Theorem 5.4 give rise to the same pair (P, H), according to Remark 2.10.

Here is the result announced at the beginning of this section.

Theorem 5.6. *Let X and $\mathcal{E}$ be as in* 1.3. *Assume that $\mathcal{E}$ is very ample of rank $r \geqslant 2$. Then $\Delta(X, \mathcal{E}) = (n-1)(r-1) + 3$ if and only if $(X, \mathcal{E})$ is one of the following*:

(1) $(\mathbb{Q}^4, \mathcal{O}_{\mathbb{Q}^4}(1)^{\oplus 2})$;
(2) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 3})$;
(3) $(\mathbb{P}^2, \mathcal{E})$, *where $\mathcal{E}$ is one of the following vector bundles*: $\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4}$, $T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ *where $T_{\mathbb{P}^2}$ is the tangent bundle*, $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$;
(4) $(\mathbb{P}^2, \mathcal{E})$, *where $\mathcal{E}$ is a very ample vector bundle of rank* 2 *with $c_1(\mathcal{E}) = 4$ and $3 \leqslant c_2(\mathcal{E}) \leqslant 10$*;
(5) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E})$, *where $\mathcal{E}$ is either* $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 3) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$, *or* $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 2)^{\oplus 2}$;
(6) $(\mathbb{F}_1, [C_0 + 3f] \oplus [C_0 + 2f])$;
(7) $(\mathbb{F}_2, [C_0 + 3f]^{\oplus 2})$;
(8) *X is a smooth curve of genus* 1 *and $\mathcal{E}$ is any very ample vector bundle of rank* 3.

Proof. If $n + r \geqslant 5$, then $\Delta(X, \mathcal{E}) = (n-1)(r-1) + 3 \leqslant nr - 1$; hence, $n \geqslant 2$, the list of pairs we are looking for is a subset of the list in Theorem 5.4. Checking that list, an easy computation as we did in the course of the proof of Theorem 5.4 leads to the pairs (1)–(3). So we can confine to study the following possibilities: either

$$n = r = 2, \quad \text{or} \quad n = 1, \quad r \geqslant 2. \tag{5.6.1}$$

Let (P, H) be as in 1.3. First of all, according to Ionescu's classification results in [15, Section 4], as rephrased in [23, Theorem 3.7 and Remark 3.9], taking also into account that P has Picard number at least two and negative Kodaira dimension, we see that (P, H) can only be one of the following pairs:

(a) a 3-dimensional scroll over a smooth curve of genus 1;
(b) a quadric fibration over $\mathbb{P}^1$ with $g(P, H) = 3$;
(c) a scroll over $\mathbb{P}^2$ with $g(P, H) = 3$; furthermore, if $\dim P \geqslant 4$, then $P = \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ or $P = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2})$, with H being the tautological line bundle in each case, or P is the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^3$;
(d) a Bordiga surface (i.e. (P, H) has $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$ as adjunction theoretic reduction), with $6 \leqslant d \leqslant 16$;
(e) either P is a del Pezzo surface with $K_P^2 = 2$ and $H_P = -2K_P$, or (P, H) admits such a pair as simple adjunction theoretic reduction.

Pairs in (d) and (e) are not compatible with the scroll structure of (P, H).

In case (c) we have $X = \mathbb{P}^2$. To see this, first note that (P, H) cannot be a scroll over a curve by Proposition 2.8. So $n \geqslant 2$, and then, having $\mathbb{P}^2$ as a base surface of a scroll structure, it follows from Proposition 2.9 that $X = \mathbb{P}^2$. Therefore $r = 2$, according to (5.6.1). It follows that $(X, \mathcal{E})$ is as in case (4) in view of [15, Theorem 4.2 and Proposition 4.7] and [22, Lemma 4].

If (b) holds, then (P, H) cannot be a scroll over a smooth curve; otherwise $0 = h^1(\mathcal{O}_P) = g(P, H) = 3$, a contradiction. Then necessarily $n = r = 2$ in view of (5.6.1). Using Lemma 2.14 we get $d = \frac{3}{2} \deg \mathcal{V}$, where $\mathcal{V} = \varphi_* H$, $\varphi : P \to \mathbb{P}^1$ being the quadric fibration morphism. Comparing the degree d with the degree of the vector bundle appearing in the second column of Table 1 in [12] ($\mathcal{V}$, in our notation), we thus see that $d = 12$ and $\mathcal{V}$ is one of the following: $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(3)^{\oplus 2}$, $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(3)$, or $\mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 4}$.

On the other hand, we are in the assumption of part II of Proposition 2.11, hence $X = \mathbb{F}_e$ and $\mathcal{E} = [C_0 + af] \oplus [C_0 + bf]$, with $a, b \geqslant e + 1$. So we can compute $12 = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = (2C_0 + (a+b)f)^2 - (C_0 + af) \cdot (C_0 + bf) = 3(a + b - e)$, whence $4 + e = a + b \geqslant 2e + 2$. It thus follows that $e \leqslant 2$. We can suppose $a \geqslant b$. So, if $e = 0$, then $(a, b) = (3, 1)$, or $(2, 2)$, which leads to case (5) in the statement. If $e = 1$, then $(a, b) = (3, 2)$; so we get case (6) in the statement. Finally, for $e = 2$, $(a, b) = (3, 3)$, which gives case (7) in the statement.

It only remains to consider case (a). Note that any scroll over a smooth curve of genus 1 has $\Delta(P, H) = \dim P$, so our condition together with Proposition 2.8 implies that P has only one scroll structure and that $r = 3$. Therefore we are in case (8).

On the other hand, a direct computation shows that $\Delta(P, H) = 3$ in all cases of the statement. □

Remark 5.7. Note that case (5) with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,3) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$ and case (7) in Theorem 5.6 lead to the same pair (P, H). To see this, let (P, H) be the scroll defined by the pair in (7). Then $P = \mathbb{F}_2 \times \mathbb{P}^1$. Let π_i be the i-th projection of P, $i = 1, 2$, and let p be the ruling projection of $\mathbb{F}_2$. Then $(p \circ \pi_1, \pi_2) : P \to \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$ exhibits a new scroll structure of (P, H) over $\mathbb{F}_0$. As $\Delta(P, H) = 3$, this other scroll structure must correspond to one of the two pairs $(\mathbb{F}_0, \mathcal{E})$ in (5). On the other hand, if $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2)^{\oplus 2}$, then $P = \mathbb{F}_0 \times \mathbb{P}^1$, but by [6, Theorem 6] this would imply that $\mathbb{F}_0 = \mathbb{F}_2$, a contradiction.

6. Very ample vector bundles of small Δ-genus

In this section we characterize pairs $(X, \mathcal{E})$, with $\mathcal{E}$ a very ample vector bundle of rank at least two, whose Δ-genus is small. The situation is completely settled when $\Delta \leqslant 1$ under the weaker assumption that $\mathcal{E}$ is merely ample by Theorems 3.6 and 3.7, and when $\Delta = 2$ for ample and spanned vector bundles by Proposition 4.5. So here we start our analysis with pairs with $\Delta(X, \mathcal{E}) = 3$.

Proposition 6.1. *Let X and $\mathcal{E}$ be as in* 1.3. *Assume that $\mathcal{E}$ is very ample of rank $r \geqslant 2$. Then $\Delta(X, \mathcal{E}) = 3$ if and only if $(X, \mathcal{E})$ is one of the following*:

(1) $(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2})$;
(2) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3})$;
(3) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2))$;
(4) $(\mathbb{F}_1, [C_0 + 2f]^{\oplus 2})$;
(5) *X is a smooth curve of genus* 1 *and $\mathcal{E}$ is any very ample vector bundle of rank* 3.

Proof. Let (P, H) and d be as in 1.3. In view of our assumptions and of Proposition 3.4, we have $0 \leqslant \Delta(P, H) \leqslant 3$. We proceed according to this value. If $\Delta(P, H) = 0$, then $3 = \Delta(X, \mathcal{E}) = (n-1)(r-1)$; hence we get case (1) of the statement by Theorem 3.6. If $\Delta(P, H) = 1$, then $3 = \Delta(X, \mathcal{E}) = (n-1)(r-1) + 1$; hence we get case (2) of the statement by Theorem 3.7. If $\Delta(P, H) = 2$, then $3 = \Delta(X, \mathcal{E}) = (n-1)(r-1) + 2$. Noting that $\mathcal{E}$ very ample implies $d \geqslant 4$, we get cases (3) and (4) of the statement by combining Lemma 4.2 and Theorem 4.3. If $\Delta(P, H) = 3$, then $3 = \Delta(X, \mathcal{E}) = (n-1)(r-1) + 3$; hence we get cases (5) of the statement by Theorem 5.6. □

Proposition 6.2. *Let X and $\mathcal{E}$ be as in* 1.3. *Assume that X has dimension $n \geqslant 2$ and that $\mathcal{E}$ is very ample of rank $r \geqslant 2$. Then $\Delta(X, \mathcal{E}) = 4$ if and only if $(X, \mathcal{E})$ is one of the following*:

(1) $(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 2})$;
(2) $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(1)^{\oplus 2})$;
(3) $(\mathbb{P}^2, \mathcal{E})$, *where $\mathcal{E}$ is a very ample vector bundle of rank* 2 *with $c_1(\mathcal{E}) = 4$ and $3 \leqslant c_2(\mathcal{E}) \leqslant 10$*;
(4) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E})$, *where $\mathcal{E}$ is either $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,3) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$, or $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2)^{\oplus 2}$*;
(5) $(\mathbb{F}_1, [C_0 + 3f] \oplus [C_0 + 2f])$;
(6) $(\mathbb{F}_2, [C_0 + 3f]^{\oplus 2})$.

Proof. Let (P, H) and d be as in 1.3. In view of our assumptions and of Proposition 3.4, we have $0 \leqslant \Delta(P, H) \leqslant 3$. We proceed according to this value. If $\Delta(P, H) = 0$, then $4 = \Delta(X, \mathcal{E}) = (n-1)(r-1)$; hence we get case (1) of the statement by Theorem 3.6. If $\Delta(P, H) = 1$, then $4 = \Delta(X, \mathcal{E}) = (n-1)(r-1) + 1$; hence we get a contradiction by Theorem 3.7. If $\Delta(P, H) = 2$, then $4 = \Delta(X, \mathcal{E}) = (n-1)(r-1) + 2$. Noting that $\mathcal{E}$ very ample implies $d \geqslant 4$, we get case (2) of the statement by combining Lemma 4.2 and Theorem 4.3. If $\Delta(P, H) = 3$, then $4 = \Delta(X, \mathcal{E}) = (n-1)(r-1) + 3$; hence we get cases (3)–(6) of the statement by Theorem 5.6. □

In order to classify pairs $(X, \mathcal{E})$ as above with $\Delta(X, \mathcal{E}) = 5$ we first need to study pairs $(X, \mathcal{E})$ such that $\Delta(X, \mathcal{E}) = (n-1)(r-1) + 4$.

Theorem 6.3. *Let X and $\mathcal{E}$ be as in* 1.3. *Assume that X has dimension $n \geqslant 2$ and that $\mathcal{E}$ is very ample of rank $r \geqslant 2$. Then $\Delta(X, \mathcal{E}) = (n-1)(r-1) + 4$ if and only if $(X, \mathcal{E})$ is one of the following*:

(1) $(\mathbb{Q}^5, \mathcal{O}_{\mathbb{Q}^5}(1)^{\oplus 2})$;
(2) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E})$, *where $\mathcal{E}$ is one of the following vector bundles*: $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)^{\oplus 3}$, $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,2) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$, $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,2)$;
(3) $(\mathbb{F}_e, [C_0 + \alpha f] \oplus [C_0 + \beta f])$, *where $e \leqslant 3$, $\alpha, \beta \geqslant e + 1$ and $\alpha + \beta = e + 5$*;
(4) *X is a cubic surface in $\mathbb{P}^3$ and $\mathcal{E}$ is a very ample vector bundle of rank* 2 *such that* $\det \mathcal{E} = -2K_X$.

Proof. Let (P, H), π and d be as in 1.3 and note that $m := \dim P = n + r - 1 \geqslant 3$. According to [17, Theorem 3] (P, H) as a projective manifolds of dimension $m \geqslant 3$ with Δ-genus 4 is one the following:

(a) $P \subset \mathbb{P}^{m+1}$ is a smooth hypersurface of degree 6 and H is the hyperplane bundle;
(b) $m = 3$, $d = 7$, $g(P, H) = 6$, and the adjunction mapping (defined by $K_P + H$) makes P a fibration in cubic surfaces over $\mathbb{P}^1$;
(c) $m = 3$, P is the projection of a smooth complete intersection of type $(2, 2, 2)$ from a point of itself and H is the hyperplane bundle;
(d) $P \subset \mathbb{P}^{m+3}$ is a smooth complete intersection of type $(2, 2, 2)$ and H is the hyperplane bundle;
(e) $m = 3$, (P, H) has sectional genus 4 and it is a scroll over a smooth surface Σ, where either $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$ or $\Sigma \subset \mathbb{P}^3$ is a cubic surface;
(f) $P = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, and H gives the Segre embedding;
(g) (P, H) has sectional genus 4 and is a hyperquadric fibration over $\mathbb{P}^1$;
(h) (P, H) is a scroll over a smooth curve of genus 2;
(i) $m = 4$ and (P, H) is a scroll over a smooth curve of genus 1.

Clearly cases (a) and (d) are ruled out, since P has Picard number $\geqslant 2$. Cases (h) and (i) cannot occur in view of Proposition 2.8.

To handle the cases in which (P, H) is a scroll over a smooth surface, say Σ, it is useful to recall the following fact. Let $\rho : P \to \Sigma$ be the scroll projection and let $S \in |H|$ be a smooth surface. Then the morphism $\rho|_S : S \to \Sigma$ is the reduction morphism of the pair (S, H_S). Let $\mathcal{F} := \rho_* H$. Then $\mathcal{F}$ is a vector bundle of rank $m - 1$ on Σ, $P = \mathbb{P}_\Sigma(\mathcal{F})$ and H is the tautological line bundle of $\mathcal{F}$ on P. In particular, $\mathcal{F}$ is very ample, so being H, and then also $\det \mathcal{F}$ is very ample. From the canonical bundle formula for P, by adjunction, we see that $K_S + H_S = \rho|_S^*(K_\Sigma + \det \mathcal{F})$. This says that the pair $(\Sigma, \det \mathcal{F})$ is the reduction of (S, H_S). In particular,

$$g(\Sigma, \det \mathcal{F}) = g(S, H_S) = g(P, H). \tag{6.3.1}$$

It follows that $(\Sigma, \det \mathcal{F})$ has to be found in the list of surfaces polarized by a very ample line bundle of sectional genus $g(P, H)$.

Now we can prove the following claims.

Claim 1. Case (b) does not occur.

By contradiction, let $\varphi : P \to \mathbb{P}^1$ be the fibration and let G be the general fiber. Then $K_G = -H_G$ and $H_G^2 = 3$ since (G, H_G) is a smooth cubic surface in $\mathbb{P}^3$. Note that $\pi(G) = X$. Actually any fiber F of π is a $\mathbb{P}^1$, hence $\pi(G)$ cannot be a point; moreover it cannot be a curve, otherwise (G, H_G) would be a scroll, which is not the case. Therefore $a := G \cdot F > 0$.

Suppose that some fiber F_0 of π is contained in G. Then

$$0 < a = G \cdot F = G \cdot F_0 = G_G \cdot F_0 = 0,$$

since $\mathcal{O}_G(G)$ is trivial. This is a contradiction. It follows that $\pi|_G : G \to X$ is a finite morphism of degree a. Looking at G as a divisor inside P we can write $G \in |aH + \pi^*D|$ for some $D \in \mathrm{Div}(X)$. For shortness set $A := K_X + \det \mathcal{E}$. From the equalities

$$-H_G = K_G = (K_P + G)_G = (K_P)_G = \big(-2H + \pi^*A\big)_G$$

we get

$$H_G = \pi\big|_G^*(K_X + \det \mathcal{E}). \tag{6.3.2}$$

Then

$$3 = (H_G)^2 = \big(\pi\big|_G^* A\big)^2 = \big(\pi^*A\big)^2 \cdot \big(aH + \pi^*D\big) = aA^2,$$

which shows that $a = 1$ or 3.

Let $a = 1$. Then $\pi|_G : G \to X$ is an isomorphism. It turns out that X itself is isomorphic to a smooth cubic surface of $\mathbb{P}^3$. From (6.3.2) we also get

$$\pi\big|_G^*(K_X + \det \mathcal{E}) = H_G = -K_G = \pi\big|_G^*(-K_X),$$

and taking into account the isomorphism induced by $\pi|_G$ on the Picard groups we obtain $\det \mathcal{E} = -2K_X$. Since $K_X^2 = 3$, this implies that $\det \mathcal{E}$ has genus 4. On the other hand, according to (6.3.1), it must be $g(X, \det \mathcal{E}) = g(P, H) = 6$. This gives a contradiction.

Let $a = 3$. Then $G \in |3H + \pi^*D|$. Recalling (6.3.2), we have

$$\mathcal{O}_G = \mathcal{O}_G(G) = \big(3H + \pi^*D\big)_G = 3H_G + \big(\pi^*D\big)_G = \pi\big|_G^*(3A + D)$$

and the injectivity of the homomorphism induced by $\pi|_G$ on the Picard groups shows that $D = -3A$. Hence $G \in |3(H - \pi^*A)|$ and then

$$3 = H_G^2 = 3H^2 \cdot \big(H - \pi^*A\big) = 3\big(H^3 - H^2 \cdot \pi^*A\big).$$

Recalling that $d = H^3 = 7$ and the Chern–Wu relation, this gives $(\det \mathcal{E}) \cdot A = 6$. By the genus formula this is equivalent to saying that $\det \mathcal{E}$ has genus 4. But this is a contradiction in view of (6.3.1), because $g(P, H) = 6$.

Claim 2. Case (c) does not occur.

We argue by contradiction. Let $V \subset \mathbb{P}^6$ be a smooth complete intersection of type $(2, 2, 2)$, let $\sigma : P \to V$ be the blowing-up at $p \in V$ and let $E = \sigma^{-1}(p) = \mathbb{P}^2$ be the exceptional divisor. We have $H = (\sigma^*\mathcal{O}_{\mathbb{P}^6}(1))_V - E$. Recalling that $K_V = (\mathcal{O}_{\mathbb{P}^6}(-1))_V$ we get $K_P = \sigma^*K_V + 2E = -H + E$. On the other hand, $K_P = -2H + \pi^*(K_X + \det \mathcal{E})$, so that combining the two expressions of K_P we get

$$H + E = \pi^*(K_X + \det \mathcal{E}). \tag{6.3.3}$$

Now note that the projection $\pi : P \to X$ cannot map E to a point, hence $\pi|_E : E \to X$ is a surjective morphism. Hence $E \cdot F > 0$ for every fiber F of π. But then we would get from (6.3.3)

$$0 < (H + E) \cdot F = \pi^*(K_X + \det \mathcal{E}) \cdot F = 0,$$

a contradiction.

Then we deal with the remaining cases.

Let (P, H) be as in (e). If $\Sigma \subset \mathbb{P}^3$ is a smooth cubic surface, then according to Proposition 2.9, the only scroll structure of (P, H) derives from $(X, \mathcal{E})$, where $X = \Sigma$ and $\det \mathcal{E}$ is a very ample line bundle of genus 4. It thus follows from Lemma 1.5 that $\det \mathcal{E} = -2K_\Sigma$. This gives case (4) in the statement.

If $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$, then we have to distinguish according to whether (P, H) admits one or more scroll structures.

If (P, H) has a single scroll structure, it derives from $(X, \mathcal{E})$, and in this case $X = \mathbb{F}_0$ with $\det \mathcal{E} = [2C_0 + 5f]$ or $[3C_0 + 3f]$, because $\det \mathcal{E}$ is a very ample line bundle of genus 4, by (6.3.1). If $\det \mathcal{E} = [2C_0 + 5f]$, then $\mathcal{E}_f = \mathcal{O}_f(1)^{\oplus 2}$ for every fiber of the first projection $p : \mathbb{F}_0 \to \mathbb{P}^1$. Then $\mathcal{E} \otimes [-C_0] = p^*\mathcal{G}$ for some vector bundle $\mathcal{G}$ of rank 2 on $\mathbb{P}^1$. Hence we immediately see that $\mathcal{E} = [C_0 + 3f] \oplus [C_0 + 2f]$ or $[C_0 + 4f] \oplus [C_0 + f]$. This situation fits into case (3) in the statement for $e = 0$.

On the other hand, if $\det \mathcal{E} = [3C_0 + 3f]$, then $\mathcal{E}_f = \mathcal{O}_f(2) \oplus \mathcal{O}_f(1)$, and arguing as in the proof of [13, Lemma 1.2] we get an exact sequence

$$0 \to [2C_0 + sf] \to \mathcal{E} \to [C_0 + tf] \to 0,$$

where $t \geqslant 1$ and $s + t = 3$. But then $\mathcal{E}_{C_0} = \mathcal{O}_{C_0}(s) \oplus \mathcal{O}_{C_0}(t)$ and the ampleness of $\mathcal{E}$ implies that also $s \geqslant 1$. It thus follows that the above exact sequence splits and we get $\mathcal{E} = [2C_0 + 2f] \oplus [C_0 + f]$ or $[2C_0 + f] \oplus [C_0 + 2f]$. This gives case (2) in the statement with $r = 2$.

If the scroll structure of (P, H) is not unique, then $(X, \mathcal{E}) = (\mathbb{F}_e, [C_0 + \alpha f] \oplus [C_0 + \beta f])$, for some integers $\alpha, \beta \geqslant e + 1$, according to Propositions 2.9 and 2.11. In this case, recalling that $\det \mathcal{E}$ has genus 4 by (6.3.1), we see also that $\alpha + \beta = e + 5$ and then $e \leqslant 3$. This gives case (3) in the statement.

In case (g), if $\dim P = 3$ we get the same conclusion, hence case (3) again, due to Proposition 2.11. On the other hand, if $\dim P \geqslant 4$, then Proposition 2.12 tells us that $(X, \mathcal{E}) = (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus 2})$. But then

$$2n - 2 = \Delta\big(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus 2}\big) = (n-1)(r-1) + 4 = n + 3$$

shows that $n = 5$. This gives case (1) in the statement.

Finally if (P, H) is as in (f), then $(X, \mathcal{E})$ is as in case (2) in the statement with $r = 3$.

On the other hand, a direct computation shows that $\Delta(X, \mathcal{E}) = (n-1)(r-1) + 4$ in all cases (1)–(3) and in the only known pair as in (4) (see Remark 6.4). This concludes the proof. □

Remark 6.4. An obvious example of a vector bundle $\mathcal{E}$ as in case (4) is $[-K_X]^{\oplus 2}$. At present we do not know if this is the only possibility [20, Remark 3.7]. We can add that if (P, H) is a Fano bundle, then this is the only possibility according to [36].

Proposition 6.5. *Let X and $\mathcal{E}$ be as in* 1.3. *Assume that X has dimension $n \geqslant 2$ and that $\mathcal{E}$ is very ample of rank $r \geqslant 2$. Then $\Delta(X, \mathcal{E}) = 5$ if and only if $(X, \mathcal{E})$ is one of the following*:

(1) $(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1)^{\oplus 2})$;
(2) $(\mathbb{P}^2, \mathcal{E})$, *where $\mathcal{E}$ is either $T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, with $T_{\mathbb{P}^2}$ the tangent bundle, or $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$*;
(3) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E})$, *where $\mathcal{E}$ is either $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,2) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$, or $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,2)$*;
(4) $(\mathbb{F}_e, [C_0 + \alpha f] \oplus [C_0 + \beta f])$, *where $e \leqslant 3$, $\alpha, \beta \geqslant e + 1$ and $\alpha + \beta = e + 5$*;
(5) *X is a cubic surface in $\mathbb{P}^3$ and $\mathcal{E}$ is a very ample vector bundle of rank* 2 *such that* $\det \mathcal{E} = -2K_X$.

Proof. Let (P, H) and d be as in 1.3. In view of our assumptions and of Proposition 3.4, we have $0 \leqslant \Delta(P, H) \leqslant 4$. We proceed according to this value. If $\Delta(P, H) = 0$, then $5 = \Delta(X, \mathcal{E}) = (n-1)(r-1)$; hence we get case (1) of the statement by Theorem 3.6. If $\Delta(P, H) = 1$, then $5 = \Delta(X, \mathcal{E}) = (n-1)(r-1)+1$; hence we get a contradiction by Theorem 3.7. If $\Delta(P, H) = 2$, then $5 = \Delta(X, \mathcal{E}) = (n-1)(r-1)+2$. Noting that $\mathcal{E}$ very ample implies $d \geqslant 4$, we get a contradiction by combining Lemma 4.2 and Theorem 4.3. If $\Delta(P, H) = 3$, then $5 = \Delta(X, \mathcal{E}) = (n-1)(r-1)+3$; hence we get case (2) of the statement by Theorem 5.6. If $\Delta(P, H) = 4$, then $5 = \Delta(X, \mathcal{E}) = (n-1)(r-1)+4$; hence we get cases (3)–(5) of the statement by Theorem 6.2. $\square$

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