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A Maximum Principle for Non-linear Elliptic Systems: Boundary Fundamental Estimates

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1. INTRODUCTION

Let Ω be a bounded open set of R^n, $n \geqslant 2$, with points $x = (x, ..., x_n)$. N is an integer > 1, $(\ |\ )_k$ and $\|\cdot\|_k$ are the scalar product and the norm in R^k. We will drop the subscript k when there is no fear of confusion.

If $u: \Omega \to R^N$, we set $Du = (D_1 u, ..., D_n u)$, where, as usual, $D_i = \partial/\partial x_i$. Clearly, $Du \in R^{nN}$ and we denote by $p = (p^i, ..., p^n)$, $p^i \in R^N$, a typical vector of R^{nN}.

$H^1 = H^{1,2}$ and $H_0^1 = H_0^{1,2}$ are the usual Sobolev spaces.

Let us consider the non-linear differential operator of second order

$$Eu = \sum_i D_i a^i(Du), \tag{1.1}$$

where $a^i(p)$ are vectors of R^N. Suppose that

$$a^i \in C^1(R^{nN}) \tag{1.2}$$

$$a^i(0) = 0 \tag{1.3}$$

$$\left\{ \sum_{ij=1}^{n} \sum_{hk=1}^{N} \left| \frac{\partial a_h^i(p)}{\partial p_k^j} \right|^2 \right\}^{1/2} \leqslant M, \qquad \forall p \in R^{nN} \tag{1.4}$$

$$\sum_{ij} \sum_{hk} \frac{\partial a_h^i(p)}{\partial p_k^j} \xi_h^i \xi_k^j \geqslant \nu \|\xi\|^2, \qquad \forall p, \xi \in R^{nN}, \tag{1.5}$$

where M and ν are suitable positive constants.

Then, we say that operator (1.1) has "2-non-linearity" [6].

From (1.3), (1.4) it easily follows that

$$\|a^i(p)\| \leqslant M \|p\|, \qquad \forall p \in R^{nN}. \tag{1.6}$$

A solution of system $Eu = 0$ in Ω is a vector $u \in H^1(\Omega)$, such that

$$\int_\Omega \sum_i (a^i(Du) \mid D_i \varphi)\, dx = 0, \qquad \forall \varphi \in H_0^1(\Omega). \tag{1.7}$$

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For the sake of simplicity, we will confine ourselves to considering second order operators even if, as will be proved, they could be extended to systems of order $2m$.

Now, consider the theory of the regularity, in the $\mathscr{L}^{2,\lambda}$-spaces[1] (in particular the theory of $C^{0,\alpha}$-regularity), for the solutions to non-linear elliptic systems

$$\sum_i D_i a^i(x, u, Du) = a^0(x, u, Du); \tag{1.8}$$

it is known that systems of type

$$\sum_i D_i a^i(Du) = 0 \tag{1.9}$$

play, in that theory, a role quite analogous to the one played, in linear theory, by systems with constant coefficients and reduced to the principal part, i.e.,

$$a^i(p) = \sum_j A_{ij} p^j, \tag{1.10}$$

where A_{ij} are $N \times N$ constant matrices.

This is the reason why it is important to obtain, for the solutions of system (1.9), the $\mathscr{L}^{2,\lambda}$-regularity and above all the so-called "fundamental estimates" for both the vectors u and Du.

In [4, 5] such a problem has been studied in the interior. In Section 3, we will recall and even improve some results obtained in [4, 5]:

Define

$$B(x^0, \sigma) = \{x : \|x - x^0\| < \sigma\}.$$

If $u \in H^1(\Omega)$ is a solution of system (1.9), then there exists an $\varepsilon(v, M, n) \in (0, 1)$ such that, for every ball $B(\sigma) = B(x^0, \sigma) \subset \Omega$ and $\forall t \in (0, 1)$, we have the following interior fundamental estimates,

$$\int_{B(t\sigma)} \|Du\|^2 \, dx \leqslant ct^\lambda \int_{B(\sigma)} \|Du\|^2 \, dx, \tag{1.11}$$

where

$$\lambda = \min(2 + \varepsilon, n). \tag{1.12}$$

Furthermore, we have

$$\int_{B(t\sigma)} \|u - u_{B(t\sigma)}\|^2 \, dx \leqslant ct^{\lambda+2} \int_{B(\sigma)} \|u - u_{B(\sigma)}\|^2 \, dx, \tag{1.13}$$

[1] See [2] and [Q, p. 13].

where

$$u_B = \fint_B u(x)\, dx.$$

And hence, if $2 \leqslant n \leqslant 4$, we get

$$\int_{B(t\sigma)} \|u\|^2\, dx \leqslant ct^n \int_{B(\sigma)} \|u\|^2\, dx. \tag{1.14}$$

The constants c, which appear in (1.11), (1.13), (1.14), depend neither on t, σ nor on x^0.

In particular, from (1.13) it follows that $u \in \mathscr{L}^{2,\lambda+2}_{\text{loc}}(\Omega)$, so that, if $2 \leqslant n \leqslant 4$,

$$u \in C^{0,\alpha}(\Omega) \qquad \text{with} \quad \alpha = 1 - \frac{n-\lambda}{2}. \tag{1.15}$$

Note that this regularity result for the vector u is the best possible. Indeed, if $n > 4$, the vector u is only partially Hölder continuous in Ω (see [4]).

Finally, inequality (1.14) is a cornerstone in proving the maximum principle of Section 8.

In Section 5 we will prove that a fundamental estimate, quite analogous to (1.11), also holds for the solutions of system $Eu = 0$, in the hemisphere

$$B^+(1) = \{x \in B(0, 1) : x_n > 0\};$$

such solutions vanish on the flat part Γ of the boundary

$$\Gamma = \{x \in B(0, 1) : x_n = 0\}.$$

Indeed, under the hypotheses (1.2)–(1.5), we will prove again that $u \in H^{2,2}(B^+(\sigma))$ for every $\sigma < 1$ (Section 4). Then, there again exists an $\varepsilon(\nu, M, n) \in (0, 1)$ such that, $\forall \sigma \in (0, 1)$ and $\forall t \in (0, 1)$, we have

$$\int_{B^+(t\sigma)} \|Du\|^2\, dx \leqslant ct^\lambda \int_{B^+(\sigma)} \|Du\|^2\, dx, \tag{1.16}$$

where $\lambda = \min(2 + \varepsilon, n)$. The proof of this inequality is not elementary.

The interior fundamental estimate (1.11) allows one to obtain the following regularity result (Section 3).

Let $a^i(x, p)$, $i = 1, ..., n$, be vectors of R^N, defined in $\Omega \times R^{nN}$, of class C^1 in p, which satisfy conditions (1.4) and (1.5) for all $(x, p) \in \Omega \times R^{nN}$ and $a^i(x, 0) = 0$ for $x \in \Omega$. Moreover, for every x, $y \in \Omega$ and $p \in R^{nN}$, we suppose that

$$\sum \|a^i(x, p) - a^i(y, p)\| \leqslant \omega(\|x - y\|) \cdot \|p\|, \tag{1.17}$$

where $\omega(t)$, with $t>0$, is a bounded, non-decreasing function which converges to zero when $t \to 0$. If $u \in H^1(\Omega)$ is a solution of the system

$$\sum_i D_i a^i(x, Du+Dg)=0 \qquad \text{in } \Omega \tag{1.18}$$

and $g \in H^{1,(\mu)}(\Omega)$ with $0 \leqslant \mu < \lambda$, then $Du \in L^{2,\mu}_{\text{loc}}(\Omega)$.

Likewise, from the fundamental estimate (1.16) it follows that, if the vectors $a^i(x, p)$ verify, in $B^+(1) \times R^{nN}$, all the conditions listed above, and if $u \in H^1(B^+(1))$ is a solution of the problem

$$\begin{aligned} u&=0 \qquad \text{on } \Gamma \\ \sum_i D_i a^i(x, Du+Dg)&=0 \qquad \text{in } B^+(1), \end{aligned} \tag{1.19}$$

where $g \in H^{1,(\mu)}(B^+(1))$, $0 \leqslant \mu < \lambda$, then $Du \in L^{2,\mu}(B^+(\sigma))$ for every $\sigma \in (0, 1)$ (see Section 6).

By a usual covering argument, from the previous interior, or near the boundary, regularity results, we deduce the following (see Section 7):

If Ω is of class C^2 and $u \in H^1(\Omega)$ is the solution of the Dirichlet problem

$$\begin{aligned} u-g &\in H^1_0(\Omega) \\ \sum_i D_i a^i(Du)&=0 \qquad \text{in } \Omega, \end{aligned} \tag{1.20}$$

where $g \in H^{1,(\mu)}(\Omega)$, $0 \leqslant \mu < \lambda$, then $Du \in L^{2,\mu}(\Omega)$; moreover, the estimate

$$\|Du\|_{L^{2,\mu}(\Omega)} \leqslant C \|Dg\|_{L^{2,\mu}(\Omega)} \tag{1.21}$$

holds. In particular we get $u \in \mathscr{L}^{2,2+\mu}(\Omega)$. As a consequence, if

$$2 \leqslant n \leqslant 4 \qquad \text{and} \qquad n-2 < \mu < \lambda$$

then

$$u \in C^{0,\alpha}(\bar{\Omega}) \qquad \text{with} \quad \alpha = 1 - \frac{n-\mu}{2}. \tag{1.22}$$

Sections 4, 5, and 6 are necessary as, to date, the boundary $\mathscr{L}^{2,\lambda}$-regularity for solutions to non-linear systems with Dirichlet boundary datum has not been stated. By the same procedure, it is certainly possible to study this boundary $\mathscr{L}^{2,\lambda}$-regularity for systems of general type, such as (1.8) also. But we will not deal with this topic in this paper.

Finally, (1.21) and the fundamental estimate (1.14) allow one to obtain the maximum principle contained in Section 8.

This maximum principle is analogous to the one proved in [9] for linear systems with constant coefficients and it is the aim of the present paper. In fact, this maximum principle is an important step in studying the partial regularity of the $H^1 \cap L^\infty(\Omega)$ solutions to system (1.8) when $a^0(x, u, p)$ has quadratic growth (see [10] for the quasi-linear systems).

2. Preliminaries and Notations

We define

$$B(x^0, \sigma) = \{x : \|x - x^0\| < \sigma\};$$

moreover, if $x_n^0 = 0$,

$$B^+(x^0, \sigma) = \{x \in B(x^0, \sigma) : x_n > 0\}$$

$$\Gamma(x^0, \sigma) = \{x \in B(x^0, \sigma) : x_n = 0\}.$$

We will simply write $B^+(\sigma)$, $\Gamma(\sigma)$, and Γ instead of $B^+(0, \sigma)$, $\Gamma(0, \sigma)$, and $\Gamma(0, 1)$, respectively.

Throughout the present paper, Ω will denote a bounded open set of R^n with diameter d_Ω.

If $u \in L^1(\mathscr{B})$, $\mathscr{B}$ is an open non-empty set of Ω, then

$$u_{\mathscr{B}} = \fint_{\mathscr{B}} u(x)\, dx = \frac{1}{\text{meas } \mathscr{B}} \int_{\mathscr{B}} u\, dx. \tag{2.1}$$

If $u \in L^\infty(\Omega)$, we define

$$\|u\|_{\infty, \Omega} = \operatorname*{ess\,sup}_{\Omega} \|u(x)\|. \tag{2.2}$$

If $u \in C^{0,\alpha}(\bar{\Omega})$, $0 < \alpha \leqslant 1$, we set

$$[u]_{\alpha, \Omega} = \sup_{x, y \in \Omega} \frac{\|u(x) - u(y)\|}{\|x - y\|^\alpha} \tag{2.3}$$

and we will say that $u \in C^{0,\alpha}(\Omega)$ if $u \in C^{0,\alpha}(\mathbb{K})$ for every compact subset $\mathbb{K} \subset \Omega$.

If $u \in L^{2,\lambda}(\Omega)$, $0 \leqslant \lambda \leqslant n$, or $u \in \mathscr{L}^{2,\lambda}(\Omega)$ with $0 \leqslant \lambda \leqslant n + 2$, we define, as usual (see [Q]),

$$\|u\|^2_{L^{2,\lambda}(\Omega)} = \sup_{\substack{x^0 \in \Omega \\ 0 < \sigma \leqslant d_\Omega}} \sigma^{-\lambda} \int_{\Omega(x^0, \sigma)} \|u(x)\|^2\, dx \tag{2.4}$$

$$[u]_{\mathscr{L}^{2,\lambda}(\Omega)} = \sup_{\substack{x^0 \in \Omega \\ 0 < \sigma \leqslant d_\Omega}} \sigma^{-\lambda} \int_{\Omega(x^0, \sigma)} \|u(x) - u_{\Omega(x^0, \sigma)}\|^2\, dx, \tag{2.5}$$

where $\Omega(x^0, \sigma) = \Omega \cap B(x^0, \sigma)$.

We say that $u \in H^{1,(\lambda)}(\Omega)$, $0 \leqslant \lambda \leqslant n$, if

$$u \in H^1(\Omega) \qquad \text{and} \qquad Du \in L^{2,\lambda}(\Omega). \tag{2.6}$$

When $u \in H^1(\Omega)$, we define

$$|u|^2_{0,\Omega} = \int_\Omega \|u\|^2\, dx \tag{2.7}$$

$$|u|_{1,\Omega} = |Du|_{0,\Omega}. \tag{2.8}$$

The lemma which follows is a very particular case of Theorem 2.I, p. 15, of [Q]; clearly, it holds even if the ball $B(x^0, \sigma)$ is replaced by the hemisphere $B^+(x^0, \sigma)$.

LEMMA 2.I. *Fix a ball* $B(\sigma) = B(x^0, \sigma) \subset R^n$. *If* $u \in \mathscr{L}^{2,\lambda}(B(\sigma))$ *with* $n < \lambda \leqslant n+2$, *then* $u \in C^{0,\alpha}(B(\sigma))$, *with* $\alpha = (\lambda - n)/2$, *and*

$$[u]_{\alpha,\overline{B(\sigma)}} \leqslant c(n)[u]_{\mathscr{L}^{2,\lambda}(B(\sigma))}. \tag{2.9}$$

The following Caccioppoli-type inequality is well known.

LEMMA 2.II. *If* $u \in H^1(\Omega)$ *is a solution, in* Ω, *of system* (1.9) *under the hypotheses* (1.2)–(1.5), *then for every ball* $B(2\sigma) = B(x^0, 2\sigma) \subset \Omega$

$$|Du|_{0,B(\sigma)} \leqslant c(\nu, M)\, \sigma^{-1}\, \|u - u_{B(2\sigma)}\|_{0,B(2\sigma)}. \tag{2.10}$$

In fact, after the $(N \times N)$-matrices

$$B_{ij} = \{B^{hk}_{ij}\}, \qquad \text{where} \quad B^{hk}_{ij} = \int_0^1 \frac{\partial a^i_h(tp)}{\partial p^j_k}\, dt, \tag{2.11}$$

have been introduced, system (1.9) can be written

$$\sum_{ij} D_i(B_{ij}(Du)\, D_j u) = 0 \qquad \text{in } \Omega \tag{2.12}$$

so that $u \in H^1(\Omega)$ is a solution, in Ω, of a linear elliptic system, whose coefficients $B_{ij}(Du(x))$ belong to $L^\infty(\Omega)$.

Then, (2.10) is a very particular case, for instance, of (1.40), p. 46 of [Q].

LEMMA 2.III. *If* u *belongs to* $L^2(B^+(R))$ *together with its derivatives* $D_i u$, $i = 1, ..., n-1$, *and if*

$$\left| \int_{B^+(R)} (u \mid D_n \varphi)\, dx \right| \leqslant \mathfrak{M}\, |\varphi|_{0,B^+(R)}, \qquad \forall \varphi \in C^\infty_0(B^+(R)) \tag{2.13}$$

then, for every $\sigma < R$, $u \in H^1(B^+(\sigma))$ *and*

$$\|D_n u\|_{0,B^+(\sigma)} \leqslant c\left\{\mathfrak{M} + |u|_{0,B^+(R)} + \sum_{i=1}^{n-1} |D_i u|_{0,B^+(R)}\right\}, \tag{2.14}$$

where the constant c depends on $(R-\sigma)$.

For this lemma see, for instance, [1, Lemma 9.3, p. 112].

LEMMA 2.IV. *If* $u \in H^1(B^+(\sigma))$ *is a solution of the system* $\Delta u = 0$, *and* $u = 0$ *on* $\Gamma(\sigma)$, *then*, $\forall t \in (0, 1)$,

$$|Du|_{0,B^+(t\sigma)} \leqslant ct^n |Du|_{0,B^+(\sigma)}. \tag{2.15}$$

The constant c depends neither on t nor on σ (one can prove that $c = 1$). This lemma is proved in [2, p. 352].

Let $A_{ij}(x)$, $ij = 1, ..., n$, be $N \times N$ matrices, defined in $B^+(\sigma)$, which belong to $L^\infty(B^+(\sigma))$, and suppose that

$$\begin{gathered} M = \sup_{B^+(\sigma)} \left\{\sum_{ij} \|A_{ij}\|^2\right\}^{1/2} \\ \sum_{ij} (A_{ij}\xi^j \mid \xi^i) \geqslant \nu \|\xi\|^2, \qquad \nu > 0, \forall x \in B^+(\sigma) \text{ and } \forall \xi \in R^{nN}. \end{gathered} \tag{2.16}$$

LEMMA 2.V. *If* $u \in H^1(B^+(\sigma))$ *is a solution of the system* $\sum_{ij} D_i[A_{ij} D_j u] = 0$, *and* $u = 0$ *on* $\Gamma(\sigma)$, *then* $\forall t \in (0, 1)$

$$|Du|_{0,B^+(t\sigma)} \leqslant \frac{c(\nu, M)}{\sigma(1-t)} |u|_{0,B^+(\sigma)}. \tag{2.17}$$

See, for instance, [2, Lemma 5.III, p. 329] for the case of only one equation ($N = 1$). The proof of this lemma remains unchanged in the case of several equations ($N \geqslant 1$).

Denote by A_{ij}^* the adjoint of the matrix A_{ij} and set

$$M_- = \sup_{B^+(\sigma)} \left\{\sum_{ij} \left\|\frac{1}{2}(A_{ij} - A_{ij}^*)\right\|^2\right\}^{1/2}. \tag{2.18}$$

LEMMA 2.VI. *For every* $\mu \geqslant 0$ *and* $\xi \in R^{nN}$

$$\sup_{B^+(\sigma)} \left\{\sum_i \left\|(M+\mu)\,\xi^i - \sum_j A_{ij}\xi^j\right\|^2\right\}^{1/2} \leqslant \{M - \nu + \sqrt{\mu^2 + M_-^2}\} \cdot \|\xi\|. \tag{2.19}$$

Moreover, if $\mu > (M_-^2 - \nu^2)/2\nu$, *then*

$$K(\mu) = \frac{M - \nu + \sqrt{\mu^2 + M_-^2}}{M + \mu} < 1. \tag{2.20}$$

Because it concerns inequality (2.19), see [Q], Lemma 8.III, p. 88]. To verify (2.20), an elementary calculation is enough.

The following existence lemma is well known. To obtain its proof it suffices, for instance, to argue as in Lemma 2.XI of [8].

LEMMA 2.VII. *Under the hypotheses* (1.2)–(1.5), *for every* $g \in H^1(\Omega)$ *and* $f^i \in L^2(\Omega)$, $i = 1, ..., n$, *there exists a unique vector* $u \in H_0^1(\Omega)$, *which is the solution of the Dirichlet problem*

$$\begin{gathered} u \in H_0^1(\Omega) \\ \sum_i D_i a^i(Du + Dg) = \sum_i D_i f^i \qquad \text{in } \Omega. \end{gathered} \tag{2.21}$$

Moreover, the inequality

$$|Du|_{0,\Omega}^2 \leqslant c(\nu, M) \sum_i |f^i - a^i(Dg)|_{0,\Omega}^2 \tag{2.22}$$

holds.

3. INTERIOR FUNDAMENTAL ESTIMATES AND AN INTERIOR REGULARITY RESULT

Let $u \in H^1(\Omega)$ be a solution of the system

$$\sum_i D_i a^i(Du) = 0 \qquad \text{in } \Omega \tag{3.1}$$

in the sense that (1.7) holds. The vectors $a^i(p)$ satisfy the conditions (1.2)–(1.5). Then, it is known (see, for instance, [5, Theorem 1.I]) that

$$u \in H_{\text{loc}}^2(\Omega);$$

moreover, for every ball $B(2\sigma) = B(x^0, 2\sigma) \subset\subset \Omega$

$$|Du|_{1,B(\sigma)} \leqslant c(\nu, M)\, \sigma^{-1}\, |Du - (Du)_{B(2\sigma)}|_{0,B(2\sigma)}. \tag{3.2}$$

If we consider (1.7) at $\varphi = D_s\psi$, $s = 1, ..., n$, with $\psi \in C_0^\infty(B(\sigma))$, then we obtain

$$\int_{B(\sigma)} \sum_i (D_s a^i(Du) \mid D_i\psi)\, dx = 0;$$

in addition, for $i, j = 1, \ldots, n$ and $h, k = 1, \ldots, N$, we set

$$A_{ij} = \{A_{ij}^{hk}\} \qquad \text{with} \quad A_{ij}^{hk}(p) = \frac{\partial a_h^i(p)}{\partial p_k^j}. \tag{3.3}$$

Then, we have

$$\int_{B(\sigma)} \sum_{ij} (A_{ij}(Du)\, D_j D_s u \mid D_i \psi)\, dx = 0, \qquad s = 1, \ldots, n, \forall \psi \in C_0^\infty(B(\sigma)). \tag{3.4}$$

Furthermore, we define

$$U = Du$$

$$\mathscr{A}_{ij}(p) = \left\{ \begin{array}{c|c|c} A_{ij} & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & A_{ij} \end{array} \right\}, \qquad n^2\ (N \times N)\text{-blocks}. \tag{3.5}$$

Then, from (3.4) it follows that $U \in H^1(B(\sigma))$ is a solution of the system

$$\int_{B(\sigma)} \sum_{ij} (\mathscr{A}_{ij}(U)\, D_j U \mid D_i \varphi)\, dx = 0, \qquad \forall \varphi \in H_0^1(B(\sigma)). \tag{3.6}$$

Taking into account (1.2), (1.5), the $(nN \times N)$-matrices $\mathscr{A}_{ij}(p)$ turn out to be continuous and elliptic, i.e.,

$$\sum_{ij} (\mathscr{A}_{ij}(p)\, \xi^j \mid \xi^i) \geqslant v \|\xi\|^2$$

$\forall p \in R^{nN}$ and for every $\xi = (\xi^1, \ldots, \xi^n)$ with $\xi^i \in R^{Nn}$. Moreover, by (1.4), it results that $\mathscr{A}_{ij}(U(x)) \in L^\infty(B(\sigma))$.

Hence, a known theorem (see [Q, p. 90, Theorem 8.I]) enables one to conclude that there exists an $\varepsilon(v, M, n) \in (0, 1)$ such that, for every $t \in (0, 1)$,

$$|Du|^2_{1, B(t\sigma)} \leqslant ct^\varepsilon\, |Du|^2_{1, B(\sigma)} \tag{3.7}$$

where the constant c depends neither on t, σ nor on x^0.

Now, we can prove the following theorem, which improves, in case $n = 2$, the result of [4, 5].

THEOREM 3.I. *If $u \in H^1(\Omega)$ is a solution of system* (3.1) *then, for every ball $B(\sigma) = B(x^0, \sigma) \subset \Omega$ and $\forall t \in (0, 1)$, we have*

$$|Du|^2_{0, B(t\sigma)} \leqslant ct^\lambda\, |Du|^2_{0, B(\sigma)} \tag{3.8}$$

where

$$\lambda = \min(2+\varepsilon, n) \tag{3.9}$$

and the constant c does not depend on t, σ, x^0.

Proof. The cases $n>2$ and $n=2$ will be considered separately.

If $n>2$, let us suppose $0<t<\tau<\frac{1}{2}$. Then

$$|Du|^2_{0,B(t\sigma)} \leqslant c(n)\left(\frac{t}{\tau}\right)^n |Du|^2_{0,B(\tau\sigma)} + 2\,|Du-(Du)_{B(\tau\sigma)}|^2_{0,B(\tau\sigma)}. \tag{3.10}$$

On the other hand, by Poincaré's inequality and (3.7)

$$\begin{aligned}|Du-(Du)_{B(\tau\sigma)}|^2_{0,B(\tau\sigma)} &\leqslant c(n)(\tau\sigma)^2\,|Du|^2_{1,B(\tau\sigma)}\\ &\leqslant c\tau^{2+\varepsilon}\sigma^2\,|Du|^2_{1,B(\sigma/2)}.\end{aligned} \tag{3.11}$$

Taking into account inequality (3.2), from (3.10), (3.11) we get

$$|Du|^2_{0,B(t\sigma)} \leqslant c(n)\left(\frac{t}{\tau}\right)^n |Du|^2_{0,B(\tau\sigma)} + c\tau^{2+\varepsilon}\,|Du|^2_{0,B(\sigma)}.$$

Then, since $n>2+\varepsilon$, by Lemma 1.I, p. 7, of [Q]

$$|Du|^2_{0,B(t\sigma)} \leqslant c\left(\frac{t}{\tau}\right)^{2+\varepsilon} |Du|^2_{0,B(\tau\sigma)} + ct^{2+\varepsilon}\,|Du|^2_{0,B(\sigma)}.$$

Taking the limit for $\tau\to\frac{1}{2}$, we obtain inequality (3.8) $\forall t\in(0,\frac{1}{2})$. However, (3.8) is clearly true for $\frac{1}{2}\leqslant t<1$ too.

If $n=2$ the proof is slightly more complicated. First of all, taking into account Poincaré's inequality and (3.2), from (3.7) we deduce that for every ball $B(\sigma)=B(x^0,\sigma)\subset\Omega$ and $\forall t\in(0,1)$

$$|Du-(Du)_{B(t\sigma)}|^2_{0,B(t\sigma)} \leqslant ct^{2+\varepsilon}\,|Du-(Du)_{B(\sigma)}|^2_{0,B(\sigma)}. \tag{3.12}$$

Then, $Du\in\mathscr{L}^{2,2+\varepsilon}_{\text{loc}}(\Omega)$ or, due to the properties of the $\mathscr{L}^{2,\lambda}$-spaces (see Lemma 2.I), $Du\in C^{0,\varepsilon/2}(\Omega)$. Furthermore, from (3.12), we get that, for every ball $B(\sigma)=B(x^0,\sigma)\subset\Omega$

$$\sigma^{2+\varepsilon}[Du]^2_{\varepsilon/2,\overline{B(\sigma/2)}} \leqslant c\,|Du|^2_{0,B(\sigma)}, \tag{3.13}$$

where the constant c depends neither on σ nor on x^0.

Then, for every ball $B(\sigma)=B(x^0,\sigma)\subset\Omega$ and $\forall t\in(0,\frac{1}{2})$

$$|Du|^2_{0,B(t\sigma)} \leqslant c(n)(t\sigma)^n\,\|Du\|^2_{\infty,B(\sigma/2)}. \tag{3.14}$$

On the other hand,

$$\sigma^n \|Du\|^2_{\infty, B(\sigma/2)} \leqslant 2\sigma^{2+\varepsilon}[Du]^2_{\varepsilon/2, \overline{B(\sigma/2)}} + c(n)\, |Du|^2_{0, B(\sigma)}. \tag{3.15}$$

When $0 < t < \frac{1}{2}$, taking into account (3.13), from (3.14) and (3.15) the thesis (3.8) still follows. Obviously, (3.8) is true also for $\frac{1}{2} \leqslant t < 1$.

We now give the interior fundamental estimate for the vector u.

THEOREM 3.II. *If $u \in H^1(\Omega)$ is a solution of system* (3.1) *and*

$$2 \leqslant n \leqslant 4 \tag{3.16}$$

then $\forall B(\sigma) = B(x^0, \sigma) \subset \Omega$ and $\forall t \in (0, 1)$

$$|u|^2_{0, B(t\sigma)} \leqslant ct^n\, |u|^2_{0, B(\sigma)}, \tag{3.17}$$

where c depends neither on σ, t nor on x^0.

Proof. By Poincaré's inequality and Lemma 2.II, from (3.8) it follows that for every ball $B(\sigma) = B(X^0, \sigma) \subset \Omega$ and $\forall t \in (0, 1)$

$$|u - u_{B(t\sigma)}|^2_{0, B(t\sigma)} \leqslant ct^{2+\lambda}\, |u - u_{B(\sigma)}|^2_{0, B(\sigma)}, \tag{3.18}$$

where λ is defined as in (3.8). Then, it is sufficient to repeat the proof we have given in the previous theorem for the case $n = 2$.

Inequality (3.18) implies that $u \in \mathscr{L}^{2, \lambda+2}_{\text{loc}}(\Omega)$ and for every ball $B(x^0, \sigma) \subset \Omega$

$$\sigma^{2+\lambda}[u]^2_{\mathscr{L}^{2,2+\lambda}(B(\sigma/2))} \leqslant c\, |u|^2_{0, B(\sigma)}. \tag{3.19}$$

Then, because $n < \lambda + 2$, having set $\alpha = 1 - (n - \lambda)/2$, by Lemma 2.I we have that

$$\sigma^{2+\lambda}[u]_{\alpha, \overline{B(\sigma/2)}} \leqslant c\, |u|^2_{0, B(\sigma)}, \tag{3.20}$$

where the constant c depends neither on σ nor on x^0.

From this we obtain that, for every ball $B(\sigma)$ and $\forall t \in (0, \frac{1}{2})$,

$$\begin{aligned} |u|^2_{0, B(t\sigma)} &\leqslant c(n)(t\sigma)^n\, \|u\|^2_{\infty, B(\sigma/2)} \\ &\leqslant c(n)\, t^n \{\sigma^{n+2\alpha}[u]^2_{\alpha, \overline{B(\sigma/2)}} + |u|^2_{0, B(\sigma)}\} \\ &\leqslant ct^n\, |u|^2_{0, B(\sigma)}. \end{aligned}$$

Finally, inequality (3.17) trivially holds if $\frac{1}{2} \leqslant t < 1$. Remark that condition (3.16) cannot be weakened unless system (3.1) has a particular structure. For instance, for the linear systems (1.10), with constant coefficients, estimate (3.17) holds without any condition on n.

We now consider the operator

$$\sum_i D_i a^i(x, Du),$$

where the $a^i(x, p)$ are vectors of R^N, defined in $\Lambda = \Omega \times R^{nN}$, continuous in x and of class C^1 in p; such vectors satisfy assumptions (1.3)–(1.5), i.e.,

$$a^i(x, 0) = 0, \qquad \forall x \in \Omega \tag{3.21}$$

$$\left\{ \sum_{ij=1}^{n} \sum_{hk=1}^{N} \left| \frac{\partial a_h^i(x, p)}{\partial p_k^j} \right|^2 \right\}^{1/2} \leqslant M, \qquad \forall (x, p) \in \Lambda \tag{3.22}$$

$$\sum_{ij} \sum_{hk} \frac{\partial a_h^i(x, p)}{\partial p_k^j} \xi_h^i \xi_k^j \geqslant \nu \|\xi\|^2, \qquad \nu > 0, \tag{3.23}$$

for every $(x, p) \in \Lambda$ and $\forall \xi \in R^{nN}$.

From (3.21) and (3.22) it again follows that

$$\|a^i(x, p)\| \leqslant M \|p\|, \qquad \forall (x, p) \in \Lambda. \tag{3.24}$$

Furthermore, we supose that there exists a bounded non-negative function $\omega(t)$, on $t > 0$, which is non-decreasing and converges to zero when $t \to 0$, such that $\forall x, y \in \Omega$ and $p \in R^{nN}$

$$\left\{ \sum_i \|a^i(x, p) - a^i(y, p)\|^2 \right\}^{1/2} \leqslant \omega(\|x - y\|) \cdot \|p\|. \tag{3.25}$$

The fundamental estimate (3.8) enables us to obtain the following interior regularity result.

THEOREM 3.III. *Let $u \in H^1(\Omega)$ be a solution of system*

$$\sum_i D_i a^i(x, Du + Dg) = 0 \qquad \text{in } \Omega \tag{3.26}$$

under the assumptions (3.21)–(3.25), *and suppose that $g \in H^{1,(\mu)}(\Omega)$ with $0 < \mu < \lambda$. Then, for every open set $\Omega^* \subset\subset \Omega$, we have that $Du \in L^{2,\mu}(\Omega^*)$ and inequality*

$$\|Du\|_{L^{2,\mu}(\Omega^*)} \leqslant c\{|Du|_{0,\Omega} + \|Dg\|_{L^{2,\mu}(\Omega)}\} \tag{3.27}$$

holds, where the constant c depends also on $d = \operatorname{dist}(\bar{\Omega}^*, \partial\Omega)$.

Proof. Fix $B(\sigma) = B(x^0, \sigma)$ with $x^0 \in \Omega^*$ and $\sigma \leqslant d$. In $B(\sigma)$ we decompose u as $v-w$; w is the solution of the Dirichlet problem (recall Lemma 2.VII)

$$\begin{gathered} w \in H_0^1(B(\sigma)) \\ \sum_i D_i a^i(x^0, Dw+Du+Dg) = \sum_i D_i a^i(x, Du+Dg) \end{gathered} \tag{3.28}$$

while $v \in H^1(B(\sigma))$ is a solution of system

$$\sum_i D_i a^i(x^0, Dv+Dg) = 0. \tag{3.29}$$

From (2.22) we get

$$|Dw|^2_{0,B(\sigma)} \leqslant c(v, M) \sum_i |a^i(x, Du+Dg) - a^i(x^0, Du+Dg)|^2_{0,B(\sigma)}.$$

Then, taking into account hypothesis (3.25), we have

$$|Dw|^2_{0,B(\sigma)} \leqslant c(v, M)\, \omega^2(\sigma)\{|Du|^2_{0,B(\sigma)} + |Dg|^2_{0,B(\sigma)}\}. \tag{3.30}$$

As far as $(v+g)$ is concerned, the hypotheses of Theorem 3.I are fulfilled; then $\forall t \in (0, 1)$

$$|Dv+Dg|^2_{0,B(t\sigma)} \leqslant ct^\lambda |Dv+Dg|^2_{0,B(\sigma)}$$

and so, $\forall t \in (0, 1)$,

$$|Dv|^2_{0,B(t\sigma)} \leqslant ct^\lambda |Dv|^2_{0,B(\sigma)} + c|Dg|^2_{0,B(\sigma)}. \tag{3.31}$$

From (3.30) and (3.31), $\forall t \in (0, 1)$ it easily follows that

$$|Du|^2_{0,B(t\sigma)} \leqslant c\{t^\lambda + \omega^2(\sigma)\}\, |Du|^2_{0,B(\sigma)} + c\sigma^\mu \|Dg\|^2_{L^{2,\mu}(\Omega)}. \tag{3.32}$$

Hence, by Lemma 2.VII of [5], it follows that $\forall \tau \in (0, \lambda-\mu)$ there exists a positive $\sigma_\tau \leqslant d$ such that, if $\sigma \leqslant \sigma_\tau$ and $t \in (0, 1)$,

$$\begin{aligned} |Du|^2_{0,B(t\sigma)} \leqslant{} & (1+c)\, t^{\lambda-\tau}\, |Du|^2_{0,B(\sigma)} \\ & + K(c, \tau, \lambda, \mu)(t\sigma)^\mu\, \|Dg\|^2_{L^{2,\mu}(\Omega)}. \end{aligned}$$

This implies that for every $\sigma \leqslant \sigma_\tau$

$$|Du|_{0,B(\sigma)\cap\Omega^*} \leqslant c\sigma^\mu\{\sigma_\tau^{-\mu}|Du|^2_{0,\Omega} + \|Dg\|^2_{L^{2,\mu}(\Omega)}\}. \tag{3.33}$$

Therefore, recalling (2.4), Theorem 3.III is proved.

Note that, if $\omega = 0$, in particular if $a^i = a^i(p)$, then $\sigma_\tau = d$ (see [Q, Lemma 1.I, p. 7] or [5, Lemma 2.VI]).

4. Differentiability near the Boundary

In the hemisphere $B^+(1)$ let us consider the problem

$$\begin{aligned} & u \in H^1(B^+(1)) \\ & u = 0 \qquad \text{on } \Gamma \\ & \sum_i D_i a^i(Du) = 0 \qquad \text{in } B^+(1). \end{aligned} \tag{4.1}$$

The last equality means that

$$\int_{B^+(1)} \sum_i (a^i(Du) \mid D_i \varphi)\, dx = 0, \qquad \forall \varphi \in H_0^1(B^+(1)). \tag{4.2}$$

Let us suppose that the vector mappings $p \to a^i(p)$ satisfy the conditions (1.2)–(1.5). Then, we want to prove the following differentiability theorem:

Theorem 4.I. *If $u \in H^1(B^+(1))$ is a solution of problem* (4.1), *under the conditions* (1.2)–(1.5), *for every $\sigma < 1$*

$$u \in H^2(B^+(\sigma)) \tag{4.3}$$

and

$$|Du|_{1,B^+(\sigma)} \leqslant \frac{c(\nu, M)}{(1-\sigma)} |Du|_{0,B^+(1)}. \tag{4.4}$$

Proof. Define

$$\tau_{r,\rho} u(x) = u(x + \rho e^r) - u(x),$$

where $\{e^r\}_{r=1,\dots,n}$ is the standard base of R^n.

The proof will be divided into two steps. First let us suppose that

$$r = 1, \dots, n-1. \tag{4.5}$$

In this case one argues exactly as in the interior differentiability case. Let us choose

$$\sigma < 1, \qquad \sigma_0 = \frac{1+\sigma}{2}, \qquad |\rho| < 1 - \sigma_0,$$

and the function $\theta \in C_0^\infty(R^n)$ fulfilling these properties,

$$0 \leqslant \theta \leqslant 1, \qquad \theta = 1 \text{ in } B(\sigma), \qquad \theta = 0 \text{ in } R^n \backslash B(\sigma_0); \tag{4.6}$$

then, taking into account the fact that $u=0$ on Γ, in (4.2) we can assume

$$\varphi=\tau_{r,-\rho}(\theta^2\tau_{r,\rho}u), \qquad r=1, \ldots, n-1,$$

and we obtain

$$\int_{B^+(1)} \sum_i (\tau_{r,\rho} a^i(Du) \mid D_i(\theta^2\tau_{r,\rho}u))\, dx=0. \tag{4.7}$$

Setting

$$B_{ij}^{hk}(x)=\int_0^1 \frac{\partial a_h^i(Du+t\tau_{r,\rho}\, Du)}{\partial p_k^j}\, dt, \qquad B_{ij}=\{B_{ij}^{hk}\}, h, k=1, \ldots, N \tag{4.8}$$

we have that

$$\tau_{r,\rho}a^i(Du)=\sum_{j=1}^{n} B_{ij}\tau_{r,\rho}\, D_j u.$$

Then, from (4.7), we obtain

$$\int_{B^+(1)} \theta^2 \sum_{ij} (B_{ij}\tau_{r,\rho}\, D_j u \mid \tau_{r,\rho}\, D_i u)\, dx$$

$$=-\int_{B^+(1)} \sum_{ij} (B_{ij}\tau_{r,\rho}\, D_j u \mid \tau_{r,\rho} u\cdot D_i\theta^2)\, dx. \tag{4.9}$$

By keeping in mind (1.5) and (1.4), from (4.9) we easily obtain

$$\int_{B^+(\sigma)} \|\tau_{r,\rho}\, Du\|^2\, dx \leqslant \frac{c(v, M)}{(1-\sigma)^2}\int_{B^+(\sigma_0)} \|\tau_{r,\rho}u\|^2\, dx$$

$$\leqslant \frac{c(v, M)}{(1-\sigma)^2}|\rho|^2 \int_{B^+(1)} \|Du\|^2\, dx.$$

From this, because of Nirenberg's well-known lemma, we conclude that there exists $D_r\, Du\in L^2(B^+(\sigma))$, $r=1, \ldots, n-1$, and

$$\sum_{r=1}^{n-1} \int_{B^+(\sigma)} \|D_r\, Du\|^2\, dx \leqslant \frac{c(v, M)}{(1-\sigma)^2}\int_{B^+(1)} \|Du\|^2\, dx. \tag{4.10}$$

In case $r=n$, we argue as follows:
Fix $0<\sigma<R<1$ and $0<\rho<(1-R)/2$. We want to estimate the integral

$$\int_{B^+(R)} (D_n u \mid D_n\varphi)\, dx, \qquad \varphi\in C_0^\infty(B^+(R)). \tag{4.11}$$

For this purpose, we observe that $\forall x \in B^+(R) + \rho e^n$

$$\tau_{n,-\rho} a^n(Du) = B_{nn}[\tau_{n,-\rho} D_n u] + \sum_{j=1}^{n-1} B_{nj}[\tau_{n,-\rho} D_j u], \tag{4.12}$$

where the B_{ij} are defined as in (4.8), ρ being replaced by $-\rho$.

Now, the B_{nn} is a non-singular matrix; in fact, assuming $\xi = (0, ..., 0, \xi^n)$, from the ellipticity condition (1.5) we deduce

$$(B_{nn}(x)\, \xi^n \mid \xi^n) \geq \nu \|\xi^n\|^2, \qquad \forall \xi^n \in R^N \text{ and } \forall x \in B^+(1),$$

so that

$$\det B_{nn} \neq 0 \qquad \text{and} \qquad \|B_{nn}^{-1}(x)\| \leq \frac{\sqrt{N}}{\nu}, \qquad \forall x \in B^+(1).^2$$

In conclusion, from (4.12) we get

$$\tau_{n,-\rho} D_n u = B_{nn}^{-1}[\tau_{n,-\rho} a^n(Du) + G(Du)], \qquad \forall x \in B^+(R) + \rho e^n, \tag{4.13}$$

where $G(Du) = -\sum_{j=1}^{n-1} B_{nj}[\tau_{n,-\rho} D_j u]$.

On the other hand, taking into account (1.4) and (4.10), from (4.2) it follows that $D_n a^n(Du)$ exists and belongs to $L^2(B^+(R))$, $\forall R < 1$. Moreover,

$$\begin{aligned}\int_{B^+(R)} \|D_n a^n(Du)\|^2\, dx &= \int_{B^+(R)} \left\| \sum_{i=1}^{n-1} \sum_{j=1}^{n} \frac{\partial a^i(Du)}{\partial p_k^j} D_{ij} u \right\|^2 dx \\ &\leq \frac{c(\nu, M)}{(1-R)^2} \int_{B^+(1)} \|Du\|^2\, dx. \end{aligned} \tag{4.14}$$

Finally, integral (4.11) can be estimated as follows:

Set $\mathscr{B}^+(R, \rho) = B^+(R) \cap [B^+(R) + \rho e^n]$. For every $\varphi \in C_0^\infty(B^+(R))$ we have

$$\begin{aligned}&\int_{B^+(R)} (D_n u \mid \tau_{n,\rho} \varphi)\, dx \\ &\quad = \int_{\mathscr{B}^+(R,\rho)} (D_n u(x - \rho e^n) \mid \varphi(x))\, dx - \int_{B^+(R)} (D_n u \mid \rho)\, dx \\ &\quad = \int_{\mathscr{B}^+(R,\rho)} (\tau_{n,-\rho} D_n u \mid \varphi)\, dx - \int_{B^+(R) \setminus \mathscr{B}^+(R,\rho)} (D_n u \mid \varphi)\, dx.\end{aligned}$$

If ρ is small enough, the last integral vanishes because φ has a compact

[2] Recall that, if $C = \{C^{hk}\}$, then $\|C\| = \{\sum_{hk} |C^{hk}|^2\}^{1/2}$.

support in $B^+(R)$. Then, taking into account (4.13), (4.14), and (4.10), if ρ is small enough we get[3]

$$\left|\int_{B^+(R)} (D_n u \mid \tau_{n,\rho}\varphi)\, dx\right|$$

$$= \left|\int_{\mathscr{B}^+(R,\rho)} (\tau_{n,-\rho} a^n(Du) + G(Du) \mid (B_{nn}^{-1})^* \varphi)\, dx\right|$$

$$\leqslant c(\nu, M) \cdot |\varphi|_{0,B^+(R)} \cdot \left\{\int_{\mathscr{B}^+(R,\rho)} \|\tau_{n,-\rho} a^n(Du)\|^2 + \sum_{j=1}^{n-1} \|\tau_{n,-\rho} D_j u\|^2\, dx\right\}^{1/2}$$

$$\leqslant c(\nu, M)\, \rho\, |\varphi|_{0,B^+(R)} \left\{\int_{B^+(R+\rho)} \|D_n a^n(Du)\|^2 + \sum_{j=1}^{n-1} \|D_{nj} u\|^2\, dx\right\}^{1/2}$$

$$\leqslant \frac{c(\nu, M)}{(1-R)} \rho \cdot |\varphi|_{0,B^+(R)} \cdot |Du|_{0,B^+(1)}.$$

From this, by dividing all sides by ρ and taking the limit for $\rho \to 0$, we obtain that for every $\varphi \in C_0^\infty(B^+(R))$

$$\left|\int_{B^+(R)} (D_n u \mid D_n \varphi)\, dx\right| \leqslant \frac{c(\nu, M)}{(1-R)} |Du|_{0,B^+(1)} \cdot |\varphi|_{0,B^+(R)}. \tag{4.15}$$

Now, we only need to apply Lemma 2.III to obtain that $D_n u \in H^1(B^+(\sigma))$, $\forall \sigma < R < 1$, and

$$\int_{B^+(\sigma)} \|D_{nn} u\|^2\, dx \leqslant \frac{c(\nu, M)}{(1-\sigma)^2} \int_{B^+(1)} \|Du\|^2\, dx. \tag{4.16}$$

From (4.10) and (4.16), Theorem 4.I follows.

5. The Boundary Fundamental Estimates

Let $u \in H^1(B^+(1))$ be a solution of the problem (4.1). Having stated in Section 4 that $u \in H^2(B^+(\sigma))$ for every $\sigma < 1$, we can argue as in Section 3.

We set $U = Du$ (that is, $U^s = D_s u$, $s = 1, \ldots, n$) and we define the matrices A_{ij}, $ij = 1, \ldots, n$, as in (3.3). Fix $\sigma < 1$. Each vector U^s belongs to $H^1(B^+(\sigma))$ and is a solution in $B^+(\sigma)$ of the system (see (3.4))

$$\sum_{ij} D_i(A_{ij}(U)\, D_j U^s) = 0, \qquad s = 1, \ldots, n. \tag{5.1}$$

[3] $(B_{nn}^{-1})^*$ is the adjoint of the matrix B_{nn}^{-1}.

Furthermore

$$U^s = 0 \text{ on } \Gamma \qquad \text{for} \quad s = 1, \ldots, n-1. \tag{5.2}$$

THEOREM 5.I. *If* $u \in H^1(B^+(1)$ *is a solution of the problem* (4.1) *under the assumptions* (1.2)–(1.5), *then there exists an* $\varepsilon(\nu, M, n) \in (0, 1)$ *such that, for every* $t, \sigma \in (0, 1)$

$$|Du|^2_{1,B^+(t\sigma)} \leqslant ct^{\varepsilon} |Du|^2_{1,B^+(\sigma)} \tag{5.3}$$

where c *depends neither on* t *nor on* σ.

Proof. Fix $\sigma \in (0, 1)$ and choose

$$\mu = \frac{M^2 - \nu^2}{\nu}. \tag{5.4}$$

We decompose each vector U^s, $s = 1, \ldots, n-1$, as $V + W$, where W is the solution of the Dirichlet problem

$$\begin{gathered} W \in H^1_0(B^+(\sigma)) \\ (M+\mu)\, \Delta W = \sum_i D_i \left\{ (M+\mu)\, D_i U^s - \sum_j A_{ij}(U)\, D_j U^s \right\}, \qquad \text{in} \quad B^+(\sigma), \end{gathered} \tag{5.5}$$

whereas $V \in H^1(B^+(\sigma))$ is a solution of the problem

$$\begin{aligned} V &= 0 \qquad \text{on } \Gamma(\sigma) \\ \Delta V &= 0 \qquad \text{in } B^+(\sigma). \end{aligned} \tag{5.6}$$

Then, as is well known, W verifies the inequality

$$\int_{B^+(\sigma)} \|DW\|^2\, dx \leqslant \frac{1}{(M+\mu)^2} \int_{B^+(\sigma)} \sum_i \left\| (M+\mu)\, D_i U^s - \sum_j A_{ij}\, D_j U^s \right\|^2 dx;$$

therefore, by Lemma 2.VI,

$$|DW|_{0,B^+(\sigma)} \leqslant K(\mu)\, |DU^s|_{0,B^+(\sigma)}. \tag{5.7}$$

Taking into account Lemma 2.IV, V verifies the inequality

$$|DV|_{0,B^+(t\sigma)} \leqslant ct^{n/2}\, |DV|_{0,B^+(\sigma)}, \qquad \forall t \in (0, 1). \tag{5.8}$$

Because $U^s = V + W$, from (5.7) and (5.8), it follows easily that $\forall t \in (0, 1)$

$$|DU^s|_{0,B^+(t\sigma)} \leqslant \{c(1+K)\, t^{n/2} + K\} \cdot |DU^s|_{0,B^+(\sigma)}.$$

Because of (5.4), the constant K is <1; then by Lemma 1.V, p. 12, of [Q], there exists $\eta \in (0, 1)$ such that for every $t \in (0, 1)$

$$|DU^s|^2_{0,B^+(t\sigma)} \leqslant ct^{\eta n} |DU^s|^2_{0,B^+(\sigma)}, \qquad s = 1, \ldots, n-1.$$

Set $\varepsilon = \eta n$. We can suppose that $\varepsilon \in (0, 1)$, then we conclude that $\forall t \in (0, 1)$

$$\sum_{s=1}^{n-1} |D_s Du|^2_{0,B^+(t\sigma)} \leqslant ct^\varepsilon \sum_{s=1}^{n-1} |D_s Du|^2_{0,B^+(\sigma)}. \tag{5.9}$$

Because $D_j U^i = D_i U^j$, to obtain (5.3) we need only to estimate the integral of the vector $D_{nn} u$.

Remark that $A_{nn}(p)$ is a non-singular matrix; in fact, from the ellipticity condition (1.5), we deduce

$$(A_{nn}(p)\, \eta \,|\, \eta) \geqslant \nu \, \|\eta\|^2, \qquad \forall p R^{nN} \text{ and } \eta \in R^N$$

and so

$$\det A_{nn}(p) \neq 0 \qquad \text{and} \qquad \|A_{nn}^{-1}\| \leqslant \frac{\sqrt{N}}{\nu}, \qquad \forall p \in R^{nN}.$$

On the other hand

$$D_i a^i(U) = \sum_{j=1}^{n} A_{ij}(U)\, D_j U^i, \qquad i = 1, \ldots, n.$$

In particular

$$D_n a^n(U) = A_{nn}(U)\, D_n U^n + \sum_{j=1}^{n-1} A_{nj}(U)\, D_j U^n. \tag{5.10}$$

Moreover, from system (1.9),

$$D_n a^n(U) = -\sum_{i=1}^{n-1} \sum_{j=1}^{n} A_{ij}(U)\, D_j U^i. \tag{5.11}$$

Then, from (5.10), (5.11), we get

$$D_n U^n = -A_{nn}^{-1}(U) \left\{ \sum_{i=1}^{n-1} \sum_{j=1}^{n} A_{ij}(U)\, D_j U^i + \sum_{j=1}^{n-1} A_{nj}(U)\, D_j U^n \right\}$$

so that

$$\|D_n U^n\| \leqslant c(\nu, M) \sum_{i=1}^{n-1} \sum_{j=1}^{n} \|D_j U^i\|. \tag{5.12}$$

From (5.12), taking into account (5.9), it follows that $\forall t \in (0, 1)$

$$|D_{nn} u|^2_{0,B^+(t\sigma)} \leqslant ct^\varepsilon \sum_{s=1}^{n-1} |D_s\, Du|^2_{0,B^+(\sigma)}. \tag{5.13}$$

Clearly, (5.3) follows from (5.9) and (5.13).

Now, we are ready to prove the following boundary fundamental estimate.

THEOREM 5.II. *If* $u \in H^1(B^+(1))$ *is a solution of the problem* (4.1) *then, for every* $\sigma \leqslant 1$ *and* $\forall t \in (0, 1)$

$$|Du|^2_{0,B^+(t\sigma)} \leqslant ct^\lambda\, |Du|^2_{0,B^+(\sigma)} \tag{5.14}$$

where

$$\lambda = \min(2+\varepsilon, n) \tag{5.15}$$

and the constant c depends neither on t nor on σ.

Proof. Inequality (5.14) is clearly true for $\frac{1}{2} \leqslant t < 1$. Then it is enough to consider the case $0 < t < \frac{1}{2}$.

Taking into account Poincaré's inequality and (4.4), from the estimate (5.3) we get that, for every $\sigma \in (0, 1)$ and $t \in (0, \frac{1}{2})$

$$|Du - (Du)_{B^+(t\sigma)}|^2_{0,B^+(t\sigma)} \leqslant ct^{2+\varepsilon}\, |Du|^2_{0,B^+(\sigma)}. \tag{5.16}$$

That being stated, if $n > 2$ the inequality (5.14) follows by arguing as in the analogous case of Theorem 3.I. Conversely, if $n = 2$, from (5.16) and the interior estimate (3.12), it follows that

$$Du \in \mathscr{L}^{2,2+\varepsilon}(B^+(\sigma)), \qquad \forall \sigma < 1$$

and so (cf. Lemma 2.I) for every $\sigma \leqslant 1$

$$\sigma^{2+\varepsilon} [Du]^2_{\varepsilon/2,\overline{B^+(\sigma/2)}} \leqslant c\, |Du|^2_{0,B^+(\sigma)}. \tag{5.17}$$

On the other hand,

$$\sigma^n\, \|Du\|^2_{\infty,B^+(\sigma/2)} \leqslant 2\sigma^{2+\varepsilon} [Du]^2_{\varepsilon/2,\overline{B^+(\sigma/2)}} + c(n)\, |Du|^2_{0,B^+(\sigma)}. \tag{5.18}$$

Now, we conclude as in the analogous case of Theorem 3.I: for every $\sigma \leqslant 1$ and $t \in (0, \frac{1}{2})$

$$\begin{aligned} |Du|^2_{0,B^+(t\sigma)} &\leqslant c(n)(t\sigma)^n\, \|Du\|^2_{\infty,B^+(\sigma/2)} \\ &\leqslant c(n)\, t^n \{\sigma^{2+\varepsilon} [Du]^2_{\varepsilon/2,\overline{B^+(\sigma/2)}} + |Du|^2_{0,B^+(\sigma)}\} \\ &\leqslant ct^n\, |Du|^2_{0,B^+(\sigma)}. \end{aligned}$$

6. A Boundary Regularity Result

We now consider the operator

$$\sum_i D_i a^i(x, Du),$$

where the $a^i(x, p)$ are vectors of R^N, defined in $\Lambda^+ = B^+(1) \times R^{nN}$, continuous in x and of class C^1 in p; such vectors satisfy conditions (3.21)–(3.25), where Λ is replaced by Λ^+.

The fundamental estimate (5.14) enables us to obtain the following boundary regularity result, which is quite analogous to that of Theorem 3.III.

THEOREM 6.I. *Let $u \in H^1(B^+(1))$ be a solution of the problem*

$$\begin{aligned} u &= 0 \qquad \text{on } \Gamma \\ \sum_i D_i a^i(x, Du + Dg) &= 0 \qquad \text{in } B^+(1). \end{aligned} \tag{6.1}$$

Let us suppose that $g \in H^{1,(\mu)}(B^+(1))$ with $0 < \mu < \lambda$. Then, for every $R < 1$, $Du \in L^{2,\mu}(B^+(R))$ and the inequality

$$\|Du\|_{L^{2,\mu}(B^+(R))} \leqslant c\{|Du|_{0,B^+(1)} + \|Dg\|_{L^{2,\mu}(B^+(1))}\} \tag{6.2}$$

holds.

Proof. We will reason in the same way as in Theorem 3.III. Fix R, $0 < R < 1$. In any hemisphere $B^+(x^0, \sigma)$, with $\sigma < 1 - R$ and centered in $x^0 \in \Gamma(R)$, we write $u = v - w$, where w is the solution of the Dirichlet problem

$$\begin{aligned} & w \in H_0^1(B^+(x^0, \sigma)) \\ & \sum_i D_i a^i(x^0, Dw + Du + Dg) = \sum_i D_i a^i(x, Du + Dg) \end{aligned} \tag{6.3}$$

whereas $v \in H^1(B^+(x^0, \sigma))$ is a solution of the problem

$$\begin{aligned} v &= 0 \qquad \text{on } \Gamma(x^0, \sigma) \\ \sum_i D_i a^i(x^0, Dv + Dg) &= 0 \qquad \text{in } B^+(x^0, 0). \end{aligned} \tag{6.4}$$

Taking into account (2.22) and hypothesis (3.25), we have (see (3.30))

$$|Dw|^2_{0,B^+(x^0,\sigma)} \leqslant c(\nu, M)\, \omega^2(\sigma)\{|Du|^2_{0,B^+(x^0,\sigma)} + |Dg|^2_{0,B^+(x^0,\sigma)}\}. \tag{6.5}$$

Because of the fundamental estimate (5.14), the inequality

$$|Dv|^2_{0,B^+(x^0,t\sigma)} \leqslant ct^\lambda |Dv|^2_{0,B^+(x^0,\sigma)} + c|Dg|^2_{0,B^+(x^0,\sigma)}, \qquad \forall t \in (0,1) \quad (6.6)$$

holds for the vector v (see (3.31)).

And so, $\forall t \in (0, 1)$

$$|Du|^2_{0,B^+(x^0,t\sigma)} \leqslant c\{t^\lambda + \omega^2(\sigma)\}\,|Du|^2_{0,B^+(x^0,\sigma)} + c\sigma^\mu \|Dg\|^2_{L^{2,\mu}(B^+(1))}. \quad (6.7)$$

Hence, by Lemma 2.VII of [5], it follows that $\forall \tau \in (0, \lambda-\mu)$ there exists a positive $\sigma_\tau \leqslant 1-R$ such that, if $\sigma \leqslant \sigma_\tau$,

$$|Du|^2_{0,B^+(x^0,\sigma)} \leqslant c\sigma^\mu\{\sigma_\tau^{-\mu}|Du|^2_{0,B^+(1)} + \|Dg\|^2_{L^{2,\mu}(B^+(1))}\}. \quad (6.8)$$

We now consider the case of $x^0 \in B^+(R)$ with $x^0_n > 0$. Fix σ, $0<\sigma\leqslant\sigma_\tau/2$. If $x^0_n \leqslant \sigma$, then $B(x^0,\sigma) \cap B^+(R) \subset B^+(\bar{x}^0, 2\sigma)$, where $\bar{x}^0 = (x^0_1, ..., x^0_{n-1}, 0)$, and so, because of (6.7),

$$|Du|^2_{0,B(x^0,\sigma)\cap B^+(R)} \leqslant c\sigma^\mu\{\sigma_\tau^{-\mu}|Du|^2_{0,B^+(1)} + \|Dg\|^2_{L^{2,\mu}(B^+(1))}\}. \quad (6.9)$$

On the contrary, if $x^0_n > \sigma$, then $B(x^0,\sigma)$ is an interior ball of $B^+(1)$; therefore, because of the interior regularity result (3.33), with Ω replaced by $B^+(1)$, estimate (6.9) still holds.

We conclude that, in any case, if $x^0 \in B^+(R)$ and $\sigma \leqslant \sigma_\tau/2$, inequality (6.9) holds. However, (6.9) is trivially true for $\sigma_\tau/2 < \sigma \leqslant \sigma_\tau$ too.

Recalling (2.4), Theorem 6.I follows from (6.9).

7. A Global Regularity Result

Let $u \in H^1(\Omega)$ be the solution of the Dirichlet problem

$$\begin{aligned} &u - g \in H^1_0(\Omega) \\ &\sum_i D_i a^i(Du) = 0 \qquad \text{in } \Omega, \end{aligned} \quad (7.1)$$

where $g \in H^{1,(\mu)}(\Omega)$ with $0 \leqslant \mu < \lambda$; the open set Ω is of class C^2 and the vector mappings $a^i(p)$, $i=1, ..., n$, belong to $C^1(R^{nN})$ and satisfy the conditions (1.2)–(1.5).

Note that, assuming $w = u - g$, problem (7.1) can be written in the equivalent form

$$\begin{aligned} &w \in H^1_0(\Omega) \\ &\sum D_i a^i(Dw + Dg) = 0 \qquad \text{in } \Omega. \end{aligned} \quad (7.2)$$

We premise some notation and remarks. As Ω is of class C^2, if $x^0 \in \partial\Omega$, about x^0 there is an open neighborhood $\mathscr{B}$ such that $\bar{\mathscr{B}}$ is mapped, by a mapping $\mathscr{T}$ of class C^2 together with its inverse, onto the ball $\overline{B(0,1)}$ and, in particular, $\Omega \cap \mathscr{B}$ is sent in $B^+(1)$ and $\partial\Omega \cap \mathscr{B}$ in Γ.

We set

$$\frac{\partial \mathscr{T}(x)}{\partial x} = \left\{\frac{\partial \mathscr{T}_i(x)}{\partial x_j}\right\}$$

$$J(x) = \left|\det \frac{\partial \mathscr{T}(x)}{\partial x}\right|;$$

moreover, for all $y \in B(0, 1)$ and $p \in R^{nN}$, we define

$$\alpha_{ij}(y) = \frac{\partial \mathscr{T}_i}{\partial x_j}(\mathscr{T}^{-1}(y))$$

$$\beta_{ij}(y) = \left(\frac{\partial \mathscr{T}_i}{\partial x_j}\frac{1}{J}\right)(\mathscr{T}^{-1}(y))$$

$$q^j(y, p) = \sum_{r=1}^{n} \alpha_{rj}(y)\, p^r \tag{7.3}$$

$$q(y, p) = (q^1, \ldots, q^n)$$

$$A^s(y, p) = \sum_{i=1}^{n} \beta_{si}(y)\, a^i(q(y, p)).$$

Clearly, q^j and A^s are vectors of R^N defined in $B(0, 1) \times R^{nN}$; moreover α_{ij} and β_{ij} are functions of class $C^1(\overline{B(0, 1)})$. Then, by definition (7.3) and assumptions (1.2)–(1.5), it is not difficult to prove that the vectors $A^s(y, p)$ verify all the conditions (3.21)–(3.25), where ν, M and $\omega(t)$ are replaced by $c(\mathscr{T})\,\nu$, $c(\mathscr{T})\,M$, $c(\mathscr{T})\,t$; $c(\mathscr{T})$ being a suitable positive constant which depends on $\mathscr{T}$.

The following notation will be suitable: if $y \in B^+(1)$ and u is a vector function defined in $\mathscr{B} \cap \Omega$, then

$$U(y) = u(\mathscr{T}^{-1}(y)).$$

That being stated, from (7.2) we get, in particular,

$$\int_{\Omega \cap \mathscr{B}} \sum_i (a^i(Dw + Dg) \,|\, D_i\varphi)\, dx = 0, \qquad \text{for all} \quad \varphi \in H_0^1(\Omega \cap \mathscr{B});$$

then, making use of the transformation of co-ordinates $y = \mathscr{T}(x)$, we obtain that W is a solution of problem

$$\begin{aligned} W &\in H^1(B^+(1)) \\ W &= 0 \qquad \text{on } \Gamma \\ \sum_s D_s A^s(y, DW + DG) &= 0 \qquad \text{in } B^+(1). \end{aligned} \tag{7.4}$$

As $\mathscr{T}$ is of class C^2 and $g \in H^{1,(\mu)}(\Omega \cap \mathscr{B})$, then also G belongs to $H^{1,(\mu)}(B^+(1))$ and

$$\|DG\|_{L^{2,\mu}(B^+(1))} \leqslant c(\mathscr{T}) \|Dg\|_{L^{2,\mu}(\Omega \cap \mathscr{B})} \tag{7.5}$$

(see [2, Theorem V, p. 375]). Then, we may apply Theorem 6.I and we get, for all $R \in (0, 1)$,

$$\|DW\|_{L^{2,\mu}(B^+(R))} \leqslant c\{|DW|_{0,B^+(1)} + \|DG\|_{L^{2,\mu}(B^+(1))}\}. \tag{7.6}$$

Consequently,

$$[U]_{\mathscr{L}^{2,\mu+2}(B^+(R))} \leqslant c\{|DU|_{0,B^+(1)} + \|DG\|_{L^{2,\mu}(B^+(1))}\}. \tag{7.7}$$

Denote by $\mathscr{B}(R)$ the inverse image of $B(0, R)$. Since the mapping $\mathscr{T}$ of class C^2 preserves the desired $\mathscr{L}^{2,\lambda}$-properties [2, Theorem V, p. 375], from (7.6) and (7.7) we derive

$$\begin{aligned} &[u]_{\mathscr{L}^{2,\mu+2}(\Omega \cap \mathscr{B}(R))} + \|Du\|_{L^{2,\mu}(\Omega \cap \mathscr{B}(R))} \\ &\quad \leqslant c\{|Du|_{0,\Omega} + \|Dg\|_{L^{2,\mu}(\Omega)}\}. \end{aligned} \tag{7.8}$$

Using this local regularity result near the boundary together with Theorem 3.III, we can prove, by a usual covering argument, the global regularity result which follows.

THEOREM 7.I. *Let $u \in H^1(\Omega)$ be the solution of Dirichlet problem* (7.1) *and suppose that*

$$\begin{aligned} &\Omega \textit{ is of class } C^2, \\ &g \in H^{1,(\mu)}(\Omega) \textit{ with } 0 \leqslant \mu < \lambda \end{aligned} \tag{7.9}$$

then

$$u \in H^{1,(\mu)}(\Omega) \cap \mathscr{L}^{2,\mu+2}(\Omega) \tag{7.10}$$

and

$$[u]_{\mathscr{L}^{2,\mu+2}(\Omega)} + \|Du\|_{L^{2,\mu}(\Omega)} \leqslant c\,\|Dg\|_{L^{2,\mu}(\Omega)}. \tag{7.11}$$

In particular, if

$$2 \leqslant n \leqslant 4 \qquad and \qquad n-2<\mu<\lambda \tag{7.12}$$

then $u \in C^{0,\alpha}(\bar{\Omega})$, *with* $\alpha = 1-(n-\mu)/2$, *and the inequality*

$$[u]_{\alpha,\Omega} \leqslant c\,\|Dg\|_{L^{2,\mu}(\Omega)} \tag{7.13}$$

holds.

Proof. Around every $x^0 \in \partial\Omega$ there is an open neighborhood $\mathscr{B}$ such that $\bar{\mathscr{B}}$ is mapped, by a mapping $\mathscr{T}$ of class C^2 together with its inverse, onto $\overline{B(0,1)}$ and, in particular, $\mathscr{B} \cap \Omega$ is carried in $B^+(1)$. Since $\partial\Omega$ is a compact, only a finite number of such neighborhoods are needed to cover it, say $\mathscr{B}_1, ..., \mathscr{B}_m$.

For each $\mathscr{B}_j$, we can suppose that R is close enough to 1, such that $\mathscr{B}_1(R), ..., \mathscr{B}_m(R)$ still cover $\partial\Omega$.

Then there exists an open set $\Omega_0 \subset\subset \Omega$ such that $\Omega_0, \mathscr{B}_1(R), ..., \mathscr{B}_m(R)$ cover $\bar{\Omega}$.

Theorem 3.III can be applied to the open set Ω_0; therefore, from (3.27), taking into account that $u = w + g$, we have

$$[u]_{\mathscr{L}^{2,\mu+2}(\Omega_0)} + \|Du\|_{L^{2,\mu}(\Omega_0)} \leqslant c\{|Du|_{0,\Omega} + \|Dg\|_{L^{2,\mu}(\Omega)}\}. \tag{7.14}$$

Inequality (7.8) holds for each of the mapped neighborhoods $\mathscr{B}_j(R)$, $j = 1, ..., m$, so that

$$[u]_{\mathscr{L}^{2,\mu+2}(\Omega\cap\mathscr{B}_j(R))} + \|Du\|_{L^{2,\mu}(\Omega\cap\mathscr{B}_j(R))} \leqslant c\{|Du|_{0,\Omega} + \|Dg\|_{L^{2,\mu}(\Omega)}\}. \tag{7.15}$$

Now, by Lemma 2.VII, we get

$$|Du|_{0,\Omega} \leqslant |Dw|_{0,\Omega} + |Dg|_{0,\Omega} \leqslant c(\nu, M)\,|Dg|_{0,\Omega}. \tag{7.16}$$

Inequality (7.11) follows from estimates (7.14), (7.15), (7.16). Finally, (7.13) is a consequence of (7.11) and Lemma 2.I, where the ball $B(\sigma)$ can be replaced by an open set Ω of class C^2 (see [Q, Theorem 2.I, p. 15]).

Remark 7.I. Theorem 7.I holds also for the solution of the Dirichlet problem

$$u - g \in H_0^1(\Omega)$$

$$\sum_i D_i a^i(x, Du) = 0 \qquad \text{in } \Omega,$$

where the vectors $a^i(x, p)$, $i = 1, ..., n$, verify the assumptions (3.21)–(3.25) instead of (1.2)–(1.5). Even the proof remains unchanged. However, we confine ourselves to considering only this case, which will be useful in next section.

8. A Maximum Principle

In this section we will prove a maximum principle, which is the main purpose of the present paper. The principle concerns the non-linear elliptic operators $Eu = \sum_i D_i a^i(Du)$ which, as mentioned in the Introduction, play a role analogous to that played, in linear theory, by the elliptic operators $\mathscr{E}u = \sum_{ij} D_i A_{ij} D_j u$ with constant coefficients A_{ij}.

For the case of linear operators $\mathscr{E}u$ see [9]. In our case, after the results of Sections 3–7 have been obtained, the proof of the maximum principle can be carried out using a method similar to that in [9].

Of course, we also supose here that vectors $a^i(p)$, $i = 1, ..., n$, fulfill assumptions (1.2)–(1.5).

Theorem 8.I. *Let $u \in H^1(\Omega)$ be the solution of the Dirichlet problem*

$$\begin{gathered} u - g \in H^1_0(\Omega) \\ \sum_i D_i a^i(Du) = 0 \qquad \text{in } \Omega. \end{gathered} \tag{8.1}$$

Suppose that

$$\Omega \text{ is of class } C^2 \text{ and convex,} \tag{8.2}$$

$$g \in H^{1,(n-2)} \cap L^\infty(\Omega) \qquad \text{and} \qquad \|Dg\|_{L^{2,n-2}(\Omega)} \leqslant c\,\|g\|_{\infty,\Omega} \tag{8.3}$$

$$2 \leqslant n \leqslant 4. \tag{8.4}$$

Then, $u \in L^\infty(\Omega)$ and

$$\|u\|_{\infty,\Omega} \leqslant c\,\|g\|_{\infty,\Omega}. \tag{8.5}$$

Proof. We need a reason as in Section 2 of [9]. Let $x^0 \in \Omega$; set $d = \text{dist}(x^0, \partial\Omega)$ and suppose that $y^0 \in \partial\Omega$ is such that $\|x^0 - y^0\| = d$.

As $2 \leqslant n \leqslant 4$, by the fundamental estimate (3.17) it results that

$$|u|^2_{0,B(x^0,td)} \leqslant ct^n |u|^2_{0,B(x^0,d)} \leqslant ct^n |u|^2_{0,\Omega \cap B(y^0,2d)} \tag{8.6}$$

for every $t \in (0, 1)$, where the constant c depends neither on t, d nor on x^0.

On the other hand taking into account that $\Omega \cap B(y^0, 2d)$ is convex and $u - g \in H^1_0(\Omega)$, so that the Poincaré inequality is valid, we get

$$\begin{aligned} |u|^2_{0,\Omega \cap B(y^0,2d)} &\leqslant 2\,|u - g|^2_{0,\Omega \cap B(y^0,2d)} + c(n)\, d^n\, \|g\|^2_{\infty,\Omega} \\ &\leqslant c(n)\{d^2\, |D(u-g)|^2_{0,\Omega \cap B(y^0,2d)} + d^n\, \|g\|^2_{\infty,\Omega}. \end{aligned} \tag{8.7}$$

Moreover, by the regularity Theorem 7.I and the hypothesis (8.3)

$$|D(u-g)|^2_{0,\Omega \cap B(y^0,2d)} \leqslant cd^{n-2}\, \|D(u-g)\|^2_{L^{2,n-2}(\Omega)} \leqslant cd^{n-2}\, \|g\|^2_{\infty,\Omega}. \tag{8.8}$$

From (8.6)–(8.8) we get

$$\fint_{B(x^0, td)} \|u\|^2 \, dx \leqslant c \, \|g\|^2_{\infty,\Omega}, \qquad \forall t \in (0, 1), \tag{8.9}$$

where c depends neither on t nor on x^0. Taking the limit for $t \to 0$, from (8.9) we obtain

$$\|u(x^0)\| \leqslant c \, \|g\|_{\infty,\Omega} \qquad \text{for a.e.} \quad x^0 \in \Omega.$$

Therefore, (8.5) is proved.

Note that condition (8.4) on n cannot be improved. Moreover, the hypothesis that Ω is convex is not crucial.

The previous maximum principle is just what is needed in the proof of the partial Hölder continuity of the $H^1 \cap L^\infty(\Omega)$-solutions of the non-linear elliptic system (1.8) when the vector $a^0(x, u, p)$ has quadratic growth.

See [10] for the quasi-linear case, namely when

$$a^i(x, u, p) = \sum_j A_{ij}(x, u) \, p^j.$$

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