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On the subalgebra of $H_*((\mathbb{R}P^\infty)^n;\ \mathbb{F}_2)$ annihilated by Steenrod operations[1]

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Abstract

We define a homomorphism θ on $H^*((\mathbb{R}P^\infty)^n;\ \mathbb{F}_2)$ having the property that it is zero on elements hit by the positive degree elements of the Steenrod algebra. We describe the subalgebra $(\operatorname{Im}\theta)_*$ of Steenrod-annihilated elements of $H_*((\mathbb{R}P^\infty)^n;\ \mathbb{F}_2)$ and in particular we show that it is nilpotent of order $n+1$. We make some conjectures as to properties of $H_*((\mathbb{R}P^\infty)^n;\ \mathbb{F}_2)$ including a nilpotency conjecture that is a strengthening of the conjecture of Peterson, proved by Wood, concerning the degrees containing elements not hit by positive degree Steenrod operations.

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0. Introduction

The classifying space of the group $\mathbb{Z}/2\mathbb{Z}$ is $\mathbb{R}P^\infty$. The multiplication and diagonal group homomorphisms $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\to\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}\to\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ induce maps $\mathbb{R}P^\infty\times\mathbb{R}P^\infty\to\mathbb{R}P^\infty$ and $\mathbb{R}P^\infty\to\mathbb{R}P^\infty\times\mathbb{R}P^\infty$ which turn $H_*((\mathbb{R}P^\infty)^n;\ \mathbb{F}_2)$ into a Hopf algebra. Let $P(n)=\mathbb{F}_2[x_1,x_2,\ldots,x_n]\cong H^*((\mathbb{R}P^\infty)^n;\ \mathbb{F}_2)$. In the Hopf algebra structure on $P(n)$ all of the generators are primitive and the action of the Steenrod algebra on $P(n)$ is determined by $Sq^1(x_j)=x_j^2$ and $Sq^k(xy)=\sum_{i+j=k}Sq^i(x)Sq^j(y)$. The dual to $P(n)$ is given by $P(n)_*=\Gamma[x_1,x_2,\ldots,x_n]$, where $\Gamma[S]$ denotes the divided polynomial algebra on S. Explicitly, $\Gamma[x_1,x_2,\ldots,x_n]=\bigotimes_{i=1}^n\Gamma[x_i]$ where $\Gamma[x]$ has a basis $\{\gamma_k(x)\}_{k\ge 0}$ with multiplication given by $\gamma_i(x)\gamma_j(x)=\binom{i+j}{i}\gamma_{i+j}(x)$ and comultiplication given by $\psi(\gamma_k(x))=\sum_{i+j=k}\gamma_i(x)\otimes\gamma_j(x)$. We will often write simply P for $P(n)$ when there is no possibility of confusion.

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Let $\mathscr{A}$ denote the mod 2 Steenrod algebra and let $I\mathscr{A}$ be the augmentation ideal in $\mathscr{A}$. The opposite algebra of the Steenrod algebra acts on P_* by means of $\langle Sq^i_* a, x\rangle = \langle a, Sq^i x\rangle$. In particular, $Sq^q_*(\gamma_k(x)) = \binom{k-q}{q}\gamma_{k-q}(x)$.

Peterson's problem is to find a basis for $P/((I\mathscr{A})P)$. The equivalent problem after dualizing is to find a basis for $(P/((I\mathscr{A})P))_*$, which is the same as $\operatorname{Ann} P_*$, the elements annihilated by all Steenrod operations. One advantage of working with the dual is that it has additional structure; $\operatorname{Ann} P_*$ forms a subalgebra of P_*.

In the case $n=1$, Peterson's problem is trivial, and complete solutions have been given for $n=2$ [4] and $n=3$ [3]. When $n=1$, a basis for $\operatorname{Ann} P(1)_*$ is $\{\gamma_{2^t-1}(x) \mid t \geq 0\}$. The images of the generators of $\operatorname{Ann} P(1)_*$ under the various compositions of iterated diagonal maps and inclusions $\mathbb{R}P^\infty \to (\mathbb{R}P^\infty)^n$ generate a subalgebra of $\operatorname{Ann} P_*$ which we denote by $\mathscr{S}(n)$ or simply $\mathscr{S}$. The generators of $\mathscr{S}$, denoted $\Lambda^S_t \in \mathscr{S}_{2^t-1}$, are in one-to-one correspondence with pairs consisting of integers t and subsets $S \subset \{1,\ldots,n\}$. We shall describe a homomorphism θ on $P/((I\mathscr{A})P)$ having the property that $(\operatorname{Im}\theta)_* \cong \mathscr{S}$. This separates out the relatively easy to compute part of $P/((I\mathscr{A})P)$, $\operatorname{Im}\theta$, from the unknown and possibly unknowable portion, $\ker\theta$, much as the J-homomorphism separates the homotopy groups of spheres into the known and the unknown. The subalgebra $\mathscr{S}$ of $\operatorname{Ann} P_*$ has some nice properties which we shall describe. Although in this paper we compute it completely only "stably", its computation seems to be quite tractable. In contrast, $\ker\theta$ seems to be very unwieldy in general.

Examining the known cases, the dimensions of $(P(n)/((I\mathscr{A})P(n)))^k$ form a fairly easy-to-understand pattern for $n=1$ and $n=2$, while for $n=3$ the pattern seems to be disrupted by an irregularity in dimensions $8, 19, 41, \ldots, 2^{t+2} + 3(2^{t-1} - 1), \ldots$ This reflects the fact that $\ker\theta = 0$ when $n < 3$ and that when $n=3$, $\ker\theta$ has dimension 1 in the degrees listed above and is 0 in other degrees. It appears to us however that the number of such irregularities (i.e., the dimensions of $\ker\theta$) increases dramatically with n.

The homomorphism θ will actually be defined on P and shown to have the property that $\theta((I\mathscr{A})P) = 0$, thus inducing the homomorphism referred to above as θ. Therefore, for $x \in P$, $\theta(x) = 0$ forms a necessary (but not sufficient) condition for x to be a "hit" element of P; that is, one in the image of positive degree Steenrod operations.

For $k \in \mathbb{N}$, define $\alpha(k)$ and $\beta(k)$ as follows. Let $\alpha(k)$ = number of 1's in the dyadic expansion of k = least r such that k can be written as a sum of r numbers of the form 2^t. Let $\beta(k)$ = least r such that k can be written as a sum of r numbers of the form $2^t - 1$. Clearly $\beta(k+m) \leq \beta(k) + \beta(m)$. Also from the definitions one gets $\alpha(n+k) \leq n$ if and only if $\beta(k) \leq n$ (see Lemma 1.1).

Peterson's conjecture, proved by Wood [5], states that in degree k

$$(P(n)/((I\mathscr{A})P(n)))^k = 0 \quad \text{if } \alpha(n+k) > n$$

or equivalently,

$$(P(n)/((I\mathscr{A})P(n)))^k = 0 \quad \text{if } \beta(k) > n.$$

The algebra $\mathscr{S}$ satisfies the stronger statement that it is nilpotent of order $n+1$ (see Theorem 2.3). Since all of its generators are in degrees of the form $2^t - 1$, it is clear that this implies that $\mathscr{S}$ satisfies the Peterson conjecture. We propose the following strengthening of the Peterson conjecture for $P(n)/((I\mathscr{A})P(n))$.

Conjecture 0.1. $\operatorname{Ann}(P(n))_*$ *is weighted nilpotent of order* $n+1$ *where algebra generators in degree* k *are assigned weight* $\beta(k)$.

By *weighted nilpotent of order* $n+1$ we mean that if x can be written as $x = x_1 x_2 \cdots x_r$ where $x_i \in \operatorname{Ann}(P(n))_*$ for all i and $wt(x_1) + wt(x_2) + \cdots + wt(x_r) \geq n+1$ then $x = 0$. Thus, for example, when $n=3$ the generator in degree 8 counts 2 against the nilpotency limit of 3 and so according to the conjecture not only is any 4-fold product in $\operatorname{Ann} P(3)_*$ equal to 0, but so is any 3-fold product containing the exceptional generator in degree 8. It is clear from the definitions that a positive solution to this conjecture implies Wood's theorem (the Peterson conjecture). The known calculations (cf. [3, 1]) show that the conjecture holds for $n \leq 3$.

Letting $S \triangle T$ denote the symmetric difference of the sets S and T we will show that the generators of $\mathscr{S}$ satisfy

$$\Lambda_j^S \Lambda_k^T = \Lambda_j^{S \triangle T} \Lambda_k^T \tag{1}$$

when $j \leq k$, and that generically this is the only relation in $\mathscr{S}$ (see Theorem 3.2 and Corollary 3.11). Kameko has conjectured

Conjecture 0.2 (Kameko [3]). *For all* k *and* n,

$$\operatorname{Dim}(P(n)/((I\mathscr{A})P(n)))^k \leq 1 \cdot 3 \cdot 7 \cdots (2^{n-1} - 1) \cdot (2^n - 1).$$

Carlisle and Wood [2] have shown that for each n there exists a uniform bound $\delta(n)$ for $\operatorname{Dim}(P(n)/((I\mathscr{A})P(n)))^k$. It is a consequence of (1) that $\mathscr{S}$ satisfies Kameko's conjecture. In fact, it is easy to see that the number of elements $\{\Lambda^{S_1} \Lambda^{S_2} \cdots \Lambda^{S_n} \mid S_i \subset \{1, \ldots, n\}\}$ one can obtain as products of n symbols satisfying relation (1) is precisely $1 \cdot 3 \cdot 7 \cdots (2^{n-1} - 1) \cdot (2^n - 1)$. Recalling that $\mathscr{S}(n)$ is nilpotent of order $n+1$ shows that this is an upper bound on $\operatorname{Dim} \mathscr{S}(n)_k$ for all k. It also follows from the discussion above that $1 \cdot 3 \cdot 7 \cdots (2^{n-1} - 1) \cdot (2^n - 1)$ is the best possible uniform bound for $\operatorname{Dim} \mathscr{S}(n)_k$ and thus best possible for $\operatorname{Dim}(P(n)/((I\mathscr{A})P(n)))^k$ (see Corollary 3.12).

The outline for this paper is as follows. Section 1 contains some elementary number-theoretic facts. The definitions of θ and $\mathscr{S}$ appear in Section 2. Section 3 is devoted to a description of $\mathscr{S}$. The paper concludes with a few examples in Section 4.

1. Preliminaries

We begin with some preliminaries on mod 2 binomial coefficients and properties of the functions α and β.

Lemma 1.1. *For $n,\ k \in \mathbb{N}$, $\alpha(n+k) \le n$ if and only if $\beta(k) \le n$.*

Proof. Suppose $\alpha(n+k)=r$ where $r \le n$. Then the binary expansion of $n+k$ is $n+k=2^{t_1}+2^{t_2}+\cdots+2^{t_r}$ where $t_1 < t_2 < \cdots < t_r$. Since $n+k > n$ one can keep subdividing the powers of 2 appearing in this expression until one obtains an expression $n+k=2^{t'_1}+2^{t'_2}+\cdots+2^{t'_n}$ containing n terms (with possible repetition). But then $k=2^{t'_1}-1+2^{t'_2}-1+\cdots+2^{t'_n}-1$ and so $\beta(k) \le n$. Conversely, if $\beta(k)=r$ where $r \le n$ then we can write $k=2^{t_1}-1+2^{t_2}-1+\cdots+2^{t_r}-1+2^{t_{r+1}}-1+2^{t_{r+2}}-1+\cdots+2^{t_n}-1$ where $t_i=0$ for $i > r$. Then $n+k=2^{t_1}+2^{t_2}+\cdots+2^{t_n}$ and so $\alpha(n+k) \le n$.

Lemma 1.2. (a) *If $(2^{t_1}-1)+(2^{t_2}-1)+\cdots+(2^{t_r}-1)=(2^{t'_1}-1)+(2^{t'_2}-1)+\cdots+(2^{t'_r}-1)$ with $0 < t_1 < t_2 < \cdots < t_r$ and $0 < t'_1 \le t'_2 \le \cdots \le t'_{r'}$, then $t_i=t'_i$ for all $i=1,\ldots,r$.*

(b) *If $(2^{t_1}-1)+(2^{t_2}-1)+\cdots+(2^{t_r}-1)=(2^{t'_1}-1)+(2^{t'_2}-1)+\cdots+(2^{t'_{r'}}-1)$ with $0 < t_1 < t_2 < \cdots < t_r$ and $0 < t'_1 < t'_2 < \cdots < t'_{r'}$, then $r=r'$.*

Proof. (a) Let $x=(2^{t_1}-1)+(2^{t_2}-1)+\cdots+(2^{t_r}-1)$ and let $m=2^{t_1}+2^{t_2}+\cdots+2^{t_r}=x+r=2^{t'_1}+2^{t'_2}+\cdots+2^{t'_r}$. Since the t_i's are distinct, $2^{t_1}+2^{t_2}+\cdots+2^{t_r}$ is the binary expansion of m and so is the only way of writing m as a sum of powers of 2 using only r terms. Therefore, $t_i=t'_i$ for all $i=1,\ldots,r$.

(b) Suppose $t_r > t'_{r'}$. Then we get the contradiction $(2^{t_1}-1)+(2^{t_2}-1)+\cdots+(2^{t_r}-1) > 2^{t_r}-1 > 2^{t_r}-1-t_r=1+3+7+\cdots+(2^{t_r-1}-1) \ge (2^{t'_1}-1)+(2^{t'_2}-1)+\cdots+(2^{t'_{r'}}-1)$. Similarly, $t'_{r'} > t_r$ yields a contradiction and therefore $t_r=t'_{r'}$. After cancelling the $2^{t_r}-1$ term, proceeding by induction gives $r=r'$. □

The following lemma is easily verified.

Lemma 1.3. *If $0 < q < 2^t-1$, then*

$$\binom{2^t-1-q}{q} \equiv 0 \pmod 2.$$

From this it follows that a basis for $\operatorname{Ann} P(1)_*$ is $\{\gamma_{2^t-1}(x)\}_{t\ge 0}$.

Lemma 1.4. *If $a+b+c=2^j-1+2^k-1$ for some j and k with $j \le k$ and $a,b,c \ge 0$, then*

$$\binom{b}{2^j-1-a} \equiv \binom{c}{2^j-1-a} \pmod 2.$$

Proof. Let b' and c' be the reductions of b and c modulo 2^j. Then

$$\binom{b'}{2^j-1-a} \equiv \binom{b}{2^j-1-a} \pmod 2$$

and

$$\binom{c'}{2^j-1-a} \equiv \binom{c}{2^j-1-a} \pmod 2.$$

We may assume $a + b' + c' \geq 2^j - 1$ since otherwise b' and c' are less than $2^j - 1 - a$ so that both sides are zero. Therefore, $2^j - 2 < a + b' + c' \leq 3(2^j - 1)$. Since $a + b' + c' \equiv -2 \pmod{2^j}$, this implies $a + b' + c' = 2^{j+1} - 2$. In other words, without loss of generality, we may assume $j = k$. We proceed by induction on b. We begin the induction with $b = 0$ in which case the relation $a + c = 2^{j+1} - 2$ together with the inequalities $a \leq 2^j - 1$, $c \leq 2^j - 1$ forces $a = 2^j - 1$, $c = 2^j - 1$ and so both sides are 1. Set $d = 2^j - 1 - a$. If $d = 0$, then both sides are 1 so assume $d > 0$. Assuming now that when b is replaced by $b - 1$ the formula is known for all d we have

$$\begin{aligned}\binom{b}{d} + \binom{c}{d} &= \binom{b-1}{d} + \binom{b-1}{d-1} + \binom{c+1}{d} - \binom{c}{d-1} \\ &= \binom{b-1}{d} + \binom{b-1}{d-1} + \binom{c+1}{d} + \binom{c}{d-1} \pmod 2 \\ &\equiv 0 \pmod 2\end{aligned}$$

since the first and third terms are congruent and the second and fourth terms are congruent by the induction hypothesis. □

Lemma 1.5. *If* $a + b + c = 2^j - 1 + 2^{j+1} - 1$ *and* $a, b, c \geq 0$, *then*

$$\binom{b}{2^j - 1 - a} + \binom{c}{2^j - 1 - b} + \binom{a}{2^j - 1 - c} \equiv 0 \pmod 2.$$

Proof. Applying Lemma 1.4 and using $b - (2^j - 1 - a) = a + b - (2^j - 1) = 2^{j+1} - 1 - c$ gives that the following expressions are equal modulo 2:

$$\binom{b}{2^j - 1 - a}, \quad \binom{c}{2^j - 1 - a}, \quad \binom{b}{2^{j+1} - 1 - c}, \quad \binom{c}{2^{j+1} - 1 - b}.$$

Similarly, there are equivalent expressions for each of the other two terms. Since $a + b + c > 2^{j+1} + 2^j - 3$ it is not possible to have all of $a < 2^j$, $b < 2^j$, and $c < 2^j$. So assume that at least one of a, b, c is greater than or equal to 2^j; say $a \geq 2^j$. If $a \geq 2^{j+1}$, then

$$\binom{b}{2^j - 1 - a} = 0 \quad \text{and} \quad \binom{a}{2^j - 1 - c} \equiv \binom{a - 2^{j+1}}{2^j - 1 - c} \equiv 0 \pmod 2$$

since $a + c \leq a + b + c = 2^{j+1} + 2^j - 2 < 2^{j+1} + 2^j - 1$ and so $2^j - 1 - c > a - 2^{j+1}$. Similarly,

$$\binom{c}{2^j - 1 - b} \equiv \binom{a}{2^j - 1 - b} \equiv 0 \pmod 2$$

so again all three terms are congruent to 0. Therefore, suppose $2^j \leq a < 2^{j+1}$. Then

$$\binom{b}{2^j - 1 - a} = 0$$

and

$$\binom{a}{2^j-1-c} \equiv \binom{a}{2^{j+1}-1-c} \equiv \binom{a}{2^j-1-b} \equiv \binom{c}{2^j-1-b} \pmod 2,$$

using that

$$\binom{x}{y} \equiv \binom{x}{y+2^j} \pmod 2$$

when $0 \le y < 2^j \le x < 2^{j+1}$. □

2. The homomorphism θ and its dual

Let B be the free Boolean ring on $\{x_1, x_2, \ldots, x_n\}$. That is $B = P/J$ where J is the ideal generated by $\{x - x^2\}_{x \in P}$ where the grading on P is ignored. Let $\pi : P \to B$ be the canonical projection. For a monomial $a \in P$, we can regard $\pi(a)$ as a subset of $\{1, \ldots, n\}$ in the obvious way. We remark in passing that $\pi(Sq^i(x)) = \binom{k}{i}\pi(x)$ if $x \in P^k$.

Consider a partition $\{k_1, k_2, \ldots, k_r\}$ of a positive integer k into positive integers of the form $2^t - 1$; in degree k define $\theta_{(k_1, k_2, \ldots, k_r)} : P^k \to B^{\otimes r}$ to be the composite

$$P^k \hookrightarrow P \xrightarrow{\psi^{r-1}} P^{\otimes r} \xrightarrow{\text{projection}} P^{k_1} \otimes P^{k_2} \otimes \cdots \otimes P^{k_r} \xrightarrow{\pi \otimes \pi \otimes \cdots \otimes \pi} B^{\otimes r}.$$

We use the maps $\theta_{(k_1, k_2, \ldots, k_r)}$ to define a map

$$\theta_k : P^k \to \bigoplus_{\substack{k_1, k_2, \ldots, k_r \\ k_i = 2^{t_i} - 1 \\ t_i > 0}} B^{\otimes r}$$

where the sum is indexed over all arbitrary length partitions of k into strictly positive integers of the form $2^t - 1$. Define θ_k to be the map whose projection to the factor indexed by $\{k_1, k_2, \ldots, k_r\}$ is $\theta_{(k_1, k_2, \ldots, k_r)}$. Extend $\theta_{(k_1, k_2, \ldots, k_r)}$ and θ_k to all of P by defining them to be 0 outside of gradation k. Define θ to be the map which in gradation k is θ_k.

Theorem 2.1. *$\theta_{(k_1, k_2, \ldots, k_r)}(\Phi(x)) = 0$ for $k_1, k_2, \ldots, k_r$ of the form $2^t - 1$, $\Phi \in I\mathscr{A}$, and $x \in P$.*

Remark. The theorem has content only when $k = |\Phi(x)|$.

Corollary 2.2. $\theta((IA)P) = 0$.

Proof of Theorem 2.1. Assume $k = |\Phi(x)|$. It suffices to consider the case when $\Phi = Sq^q$ and x is a monomial. The projection of $\psi^{r-1}(Sq^q x)$ onto $P^{k_1} \otimes P^{k_2} \otimes \cdots \otimes P^{k_r}$

is a sum of terms each of which lies in $Sq^{q_1}P^{k_1-q_1} \otimes Sq^{q_2}P^{k_2-q_2} \otimes \cdots \otimes Sq^{q_r}P^{k_r-q_r}$ for some $q_1, q_2, \ldots, q_r$ with $q_1 + q_2 + \cdots + q_r = q$ and $q_i \geq 0$ for all i. It suffices to show that the sum of the terms from each such partition contributes 0, so fix a partition $q_1, q_2, \ldots, q_r$. Let $Sq^{q_1}y_1 \otimes Sq^{q_2}y_2 \otimes \cdots \otimes Sq^{q_r}y_r$, where $y_j \in P^{k_j-q_j}$, be a term in $\psi^{r-1}(Sq^q x)$ coming from the partition $q_1, q_2, \ldots, q_r$. Since π keeps track only of which variables appear and Sq^m does not change this, the contribution from this partition is $e_1\pi(y_1) \otimes e_2\pi(y_2) \otimes \cdots \otimes e_r\pi(y_r)$, where e_i is the number of terms in $Sq^{q_i}(y_i)$. Therefore,

$$e_i = \binom{k_i - q_i}{q_i} = \binom{2^{t_i} - 1 - q_i}{q_i},$$

where $k_i = 2^{t_i} - 1$. Therefore, by Lemma 1.3, $e_i = 0$ unless $q_i = 0$. However, the q_i's cannot all be 0 since they add to q. Thus, e_i equals 0 for at least one i and therefore $e_1\pi(y_1) \otimes e_2\pi(y_2) \otimes \cdots \otimes e_r\pi(y_r) = 0$. $\square$

For each nonempty subset S of $\{1, \ldots, n\}$ we have an inclusion $i_S : (\mathbb{R}P^\infty)^{|S|} \to (\mathbb{R}P^\infty)^n$. Thus, each such subset gives rise to a "diagonal map" ψ^S defined as the composite $\mathbb{R}P^\infty \xrightarrow{\psi^{|S|-1}} (\mathbb{R}P^\infty)^{|S|} \xrightarrow{i_S} (\mathbb{R}P^\infty)^n$. We extend the notation by setting $\psi^\emptyset$ to be the zero map. For each subset S and for each integer t define an element $\Lambda_t^S \in (P_*)_{2^t-1}$ by $\Lambda_t^S = \psi^S(\gamma_{2^t-1}(x))$. Explicitly, $\Lambda_t^S = \sum \gamma_{i_1}(x_1)\gamma_{i_2}(x_2)\cdots\gamma_{i_r}(x_r)$ where the sum runs over all sequences $i_1, i_2, \ldots, i_r$ of positive integers such that $\sum_{m=1}^r i_m = 2^t - 1$ and $i_m = 0$ for $m \notin S$. Since $\gamma_{2^t-1}(x)$ belongs to $\operatorname{Ann} P(1)_*$ it follows that Λ_t^S lies in $\operatorname{Ann} P_*$. Let $\mathscr{S}$ be the subalgebra of $\operatorname{Ann} P_*$ generated by the elements $\{\Lambda_t^S \mid t > 0,\ S \subset \{1, \ldots, n\}\}$.

Theorem 2.3. *$\mathscr{S}$ is nilpotent of order $n+1$.*

Proof. All of the generators of $\mathscr{S}$ have odd degree and the product in P_* of any $n+1$ elements of odd degree is 0. $\square$

Corollary 2.4. *$\mathscr{S}_m = 0$ whenever $\beta(m) > n$.*

This is immediate from Theorem 2.3 and the fact that the generators of $\mathscr{S}$ are in degrees of the form $2^t - 1$.

Lemma 2.5. *For a monomial $Y \in P_n$ having degree $2^t - 1$, $\langle \Lambda_t^S, Y \rangle = 1$ if and only if $\pi(Y) \subset S$.*

Proof. $\langle \Lambda_t^S, Y \rangle = \langle \psi_*^S(\gamma_{2^t-1}(x)), Y \rangle = \langle \gamma_{2^t-1}(x), \psi^{S^*} Y \rangle$. Since ψ^{S^*} sends variables in S to x and those outside of S to 0, the lemma follows. $\square$

Corollary 2.6. *For a monomial $Y \in P_n$, $\langle \Lambda_{j_1}^{R_1}\Lambda_{j_2}^{R_2}\cdots\Lambda_{j_r}^{R_r}, Y\rangle$ = number of terms $y_1 \otimes y_2 \otimes \cdots \otimes y_r$ in $\psi^{r-1}Y$ such that $|y_i| = 2^{j_i} - 1$ and $\pi(y_i) \subset R_i$ for all $i = 1,\ldots r$.*

Fix a positive integer r and positive integers $j_1, j_2, \ldots, j_r$; given $x \in P$ and a collection $\mathfrak{S} = \{S_1, S_2, \ldots, S_r\}$ of non-empty subsets of $\{1, 2, \ldots, n\}$, let $c_{\mathfrak{S},x}$ be the coefficient of $S_1 \otimes S_2 \otimes \cdots \otimes S_r$ in $\theta_{2^{j_1}-1, 2^{j_2}-1, \ldots, 2^{j_r}-1}(x)$.

Lemma 2.7. *If $\Lambda = \Lambda_{j_1}^{S_1}\Lambda_{j_2}^{S_2}\cdots\Lambda_{j_r}^{S_r} \in \mathscr{S}$, then $\langle \Lambda, x\rangle = \sum c_{\mathfrak{S}',x}$ where the sum runs over all $\mathfrak{S}' = \{S_1', S_2', \ldots, S_r'\}$ such that $S_i' \subset S_i$ for all $i = i, \ldots, r$.*

Proof. By Corollary 2.6, in computing $\langle \Lambda, x\rangle$ there is a contribution of 1 for each term in $\psi^{r-1}x$ of the form $a_1 \otimes a_2 \otimes \cdots \otimes a_r$ where $|a_i| = 2^{j_i} - 1$ and $\pi(a_i) \subset S_i$. Such a term also contributes 1 to the sum on the right-hand side through $c_{\mathfrak{S}',x}$ for $\mathfrak{S}' = \{\pi(a_1), \pi(a_2), \ldots, \pi(a_r)\}$. Conversely, by the definition of θ, every contribution to the sum on the right comes from such a term in $\psi^{r-1}x$. □

Theorem 2.8. *$\langle \mathscr{S}, x\rangle = 0$ if and only if $x \in \ker\theta$.*

Proof. If $x \in \ker\theta$, then $c_{\mathfrak{S},x} = 0$ for all $\mathfrak{S}$ and so it follows from Lemma 2.7 that $\langle \mathscr{S}, x\rangle = 0$. Conversely, suppose that $\langle \mathscr{S}, x\rangle = 0$. Then for every product of generators $\Lambda = \Lambda_{j_1}^{S_1}\Lambda_{j_2}^{S_2}\cdots\Lambda_{j_r}^{S_r} \in \mathscr{S}$, $0 = \langle \Lambda, x\rangle = \sum c_{\mathfrak{S}',x}$ where the sum is as in Lemma 2.7. We wish to show that $\theta(x) = 0$ or equivalently that $c_{\mathfrak{S},x} = 0$ for all $\mathfrak{S}$. It suffices to consider the case where x is a monomial. For $\mathfrak{S} = \{S_1, S_2, \ldots, S_r\}$, let $N(\mathfrak{S}) = \sum_{i=1}^{r} |S_i|$. To show that $c_{\mathfrak{S},x} = 0$ for all $\mathfrak{S}$ containing r sets we proceed by induction on $N(\mathfrak{S})$. We begin the induction with $N(\mathfrak{S}) = r$. In this case, each set S_i is a singleton and so has no proper subsets. Thus, $c_{\mathfrak{S},x}$ is the unique term on the right-hand side of the sum in Lemma 2.7 and so $c_{\mathfrak{S},x} = 0$. Now suppose by induction that $c_{\mathfrak{S}',x} = 0$ holds for all $\mathfrak{S}'$ such that $N(\mathfrak{S}') < N(\mathfrak{S})$. Then $c_{\mathfrak{S}',x} = 0$ for all $\mathfrak{S}'$ such that $\mathfrak{S}' = \{S_1', S_2', \ldots, S_r'\}$ where S_i' is a proper subset of S_i for some $i = 1, \ldots, r$. Therefore Lemma 2.7 yields $0 = c_{\mathfrak{S},x} + 0$ and so $c_{\mathfrak{S},x} = 0$ to complete the induction. □

Corollary 2.9. $\mathscr{S} \cong (\operatorname{Im}\theta)_*$.

3. Relations in $\mathscr{S}$

Theorem 3.1. *Let S and T be subsets of $\{1, 2, \ldots, n\}$ such that $S \cup T = \{1, 2, \ldots, n\}$, and let m be a monomial in $P^{2^j-1+2^k-1}$. Write $m = m_1 m_2 m_3$ where m_1 is the product of the factors of m which contain variables from $S - T$, m_2 contains those from $S \cap T$, and m_3 contains those from $T - S$. Then*

$$\langle \Lambda_j^S \Lambda_k^T, m\rangle = \binom{|m_2|}{2^j - 1 - |m_1|}.$$

Proof.

$$\langle \Lambda_j^S \Lambda_k^T, m \rangle = \langle \Lambda_j^S \otimes \Lambda_k^T, \psi(m_1)\psi(m_2)\psi(m_3) \rangle.$$

$$= \left\langle \Lambda_j^S \otimes \Lambda_k^T, \sum_p \sum_q \sum_r (m'_{1_p} \otimes m''_{1_p})(m'_{2_q} \otimes m''_{2_q})(m'_{3_r} \otimes m''_{3_r}) \right\rangle$$

$$= \left\langle \gamma_{2^j-1}(x) \otimes \gamma_{2^k-1}(x), \sum_{p,q,r} \psi^{S^*}(m'_{1_p} m'_{2_q} m'_{3_r}) \otimes \psi^{T^*}(m''_{1_p} m''_{2_q} m''_{3_r}) \right\rangle.$$

Since i_A^* projects onto the factors corresponding to the subset A, terms with $m'_3 \neq 1$ or $m''_1 \neq 1$ give 0. Therefore,

$$\langle \Lambda_j^S \Lambda_k^T, m \rangle = \left\langle \gamma_{2^j-1}(x) \otimes \gamma_{2^k-1}(x), \sum_q \psi^{S^*}(m_1 m'_{2_q}) \otimes \psi^{T^*}(m''_{2_q} m_3) \right\rangle$$

$$= \sum_q \left\langle \gamma_{2^j-1}(x), \psi^{S^*}(m_1 m'_{2_q}) \right\rangle \left\langle \gamma_{2^k-1}(x), \psi^{T^*}(m''_{2_q} m_3) \right\rangle$$

$$= \sum_q \left\langle \gamma_{2^j-1}(x), x^{|m_1 m'_{2_q}|} \right\rangle \left\langle \gamma_{2^k-1}(x), x^{|m''_{2_q} m_3|} \right\rangle$$

$$= \sum_q \delta_{2^j-1}^{|m_1 m'_{2_q}|} \delta_{2^k-1}^{|m''_{2_q} m_3|}.$$

Since $|m_1| + |m'_{2_q}| + |m''_{2_q}| + |m_3| = 2^j - 1 + 2^k - 1$ for each q, $|m_1| + |m'_{2_q}| = 2^j - 1$ if and only if $|m''_{2_q}| + |m_3| = 2^k - 1$. Therefore,

$$\delta_{2^j-1}^{|m_1 m'_{2_q}|} \delta_{2^k-1}^{|m'_{2_q} m_3|} = \left(\delta_{2^j-1}^{|m_1 m'_{2_q}|} \right)^2 = \delta_{2^j-1}^{|m_1 m'_{2_q}|}$$

for each q. Thus,

$$\langle \Lambda_j^S \Lambda_k^T, m \rangle = \sum_q \delta_{2^j-1}^{|m_1 m'_{2_q}|} = \sum_q \delta_{2^j-1-|m_1|}^{|m'_{2_q}|} = \binom{|m_2|}{2^j - 1 - |m_1|},$$

since $\sum_q \delta_{2^j-1-|m_1|}^{|m'_{2_q}|}$ counts the number of terms of the form $a \otimes b$ with $|a| = 2^j - 1 - |m_1|$ in the coproduct of m_2. □

Let $S \triangle T$ denote the symmetric difference of sets S and T.

Theorem 3.2.

$$\Lambda_j^S \Lambda_k^T = \Lambda_j^{S \triangle T} \Lambda_k^T \tag{1}$$

when $j \leq k$.

Proof. Let m be a monomial in $P^{2^j-1+2^k-1}$. Write $m=m_1m_2m_3m_4$ where m_1 is the product of the factors of m which contain variables from $S-T$, m_2 contains those from $S\cap T$, m_3 contains those from $T-S$, and m_4 contains those from $\{1,\ldots,n\}-(S\cup T)$. The Kronecker product is given by

$$\langle \Lambda_j^S\Lambda_k^T, m\rangle = \langle \Lambda_j^S \otimes \Lambda_k^T,\ \psi(m)\rangle.$$

If $m_4\neq 1$ then this is 0 as is the corresponding expression with $S\triangle T$ replacing S. Assume now that $m_4=1$. Then applying Theorem 3.1 gives

$$\langle \Lambda_j^S\Lambda_k^T, m\rangle = \binom{|m_2|}{2^j-1-|m_1|}.$$

Replacing S by $S\triangle T$ interchanges the roles of m_2 and m_3. However,

$$\binom{|m_2|}{2^j-1-|m_1|} = \binom{|m_3|}{2^j-1-|m_1|}$$

by Lemma 1.4 because $|m|=|m_1|+|m_2|+|m_3|=2^j-1+2^k-1$. Therefore, $\Lambda_j^S\Lambda_k^T$ and $\Lambda_j^{S\triangle T}\Lambda_k^T$ have the same Kronecker product with every monomial and so $\Lambda_j^S\Lambda_k^T = \Lambda_j^{S\triangle T}\Lambda_k^T$. □

Theorem 3.3.

$$\Lambda_j^S\Lambda_{j+1}^T = \Lambda_j^S\Lambda_{j+1}^{S\triangle T} + \Lambda_j^{S\triangle T}\Lambda_{j+1}^S.$$

Proof. Let $m=m_1m_2m_3m_4$ be a monomial in $P^{2^j-1+2^{j-1}-1}$ where m_1, m_2, m_3, and m_4 are as in the previous proof. As before, if $m_4\neq 1$ then all the terms are 0 so assume that $m_4=1$. Applying Theorem 3.1 shows that we are required to prove that

$$\binom{|m_2|}{2^j-1-|m_1|} = \binom{|m_1|}{2^j-1-|m_2|} + \binom{|m_1|}{2^{j+1}-1-|m_2|},$$

which follows from Lemmas 1.4 and 1.5. □

We will show that generically relation (1) is the only relation in $\mathscr{S}$. More precisely, we will show that if $\Lambda_{j_1}^{S_1}\Lambda_{j_2}^{S_2}\cdots\Lambda_{j_r}^{S_r} = \Lambda_{j_1}^{S_1'}\Lambda_{j_2}^{S_2'}\cdots\Lambda_{j_r}^{S_r'}$ and there is sufficient separation $j_r \gg j_{r-1} \gg \cdots \gg j_2 \gg j_1 \gg j_0 = 0$ between the degrees of the monomials, then $\Lambda_{j_1}^{S_1}\Lambda_{j_2}^{S_2}\cdots\Lambda_{j_r}^{S_r}$ is obtained from $\Lambda_{j_1}^{S_1'}\Lambda_{j_2}^{S_2'}\cdots\Lambda_{j_r}^{S_r'}$ by repeated application of (1). We do not determine the minimum separation needed precisely, although it will become clear from the proof that $j_{i+1}-j_i \geq n$ is sufficient. As illustrated by Theorem 3.3, there are more relations when the separation is small.

Let $\mathscr{F}$ be the free commutative algebra on symbols Λ_j^S and let $\mathscr{T}$ be $\mathscr{F}$ modulo relation (1). There is a canonical surjective quotient map $\mathrm{Quot}:\mathscr{T}\to\mathscr{S}$. Define a partial order on the monomials of $\mathscr{T}$ by $\Lambda \prec \Lambda'$ if $\Lambda=\Lambda'$ or if there exist representatives

$$\Lambda = \Lambda_{j_1}^{S_1}\Lambda_{j_2}^{S_2}\cdots\Lambda_{j_r}^{S_r}, \qquad \Lambda' = \Lambda_{j_1}^{S_1'}\Lambda_{j_2}^{S_2'}\cdots\Lambda_{j_r}^{S_r'}$$

in $\mathscr{F}$ such that $(S_1, S_2, \dots, S_r) \leq (S_1', S_2', \dots, S_r')$ in the right lexicographical ordering by containment; that is, if there exists i_0 such that $S_i = S_i'$ for $i > i_0$ and $S_{i_0} \subsetneq S_{i_0}'$.

Suppose $j_r \gg j_{r-1} \gg \cdots \gg j_2 \gg j_1 \gg j_0 = 0$. Because the j_i's are strictly increasing, Lemma 1.2 implies that $n = (2^{j_1} - 1) + (2^{j_2} - 1) + \cdots + (2^{j_r} - 1)$ is the unique expression of n as a sum of r numbers of the form $2^t - 1$. Given $\Lambda = \Lambda_{j_1}^{S_1} \Lambda_{j_2}^{S_2} \cdots \Lambda_{j_r}^{S_r} \in \mathscr{T}$, using the separation we will describe a monomial $X(\Lambda) \in P_n$ having the properties that $\langle \mathrm{Quot}(\Lambda), X(\Lambda) \rangle = 1$ and that $\langle \mathrm{Quot}(\Lambda_{j_1}^{R_1} \Lambda_{j_2}^{R_2} \cdots \Lambda_{j_r}^{R_r}), X(\Lambda) \rangle = 1$ implies $\Lambda \prec \Lambda_{j_1}^{R_1} \Lambda_{j_2}^{R_2} \cdots \Lambda_{j_r}^{R_r}$. This will show that if $\mathrm{Quot}(\Lambda) = \mathrm{Quot}(\Lambda_{j_1}^{R_1} \Lambda_{j_2}^{R_2} \cdots \Lambda_{j_r}^{R_r})$ in $\mathscr{T}$ then $\Lambda = \Lambda_{j_1}^{R_1} \Lambda_{j_2}^{R_2} \cdots \Lambda_{j_r}^{R_r}$ (see Corollary 3.11) which is what we meant when we said that generically relation (1) is the only relation in $\mathscr{S}$.

Let $Q = \{1, 2, 4, \dots, 2^{j_1 - 1}\}$. Using the separation hypothesis we may assume $j_1 \geqslant |S_1|$; choose a surjection $\phi : Q \to S_1$. For $x \in S_1 \cup S_2 \cup \cdots \cup S_r$ define

$$d_i(x) = \begin{cases} 1 & \text{if } x \in S_i, \\ 0 & \text{if } x \notin S_i. \end{cases}$$

For a monomial

$$\Lambda = \Lambda_{j_1}^{S_1} \Lambda_{j_2}^{S_2} \cdots \Lambda_{j_r}^{S_r} \in \mathscr{T},$$

choose a representative $\tau(\Lambda) = \Lambda_{j_1}^{R_1} \Lambda_{j_2}^{R_2} \cdots \Lambda_{j_r}^{R_r} \in \mathscr{F}$ for Λ, elements $z_i(\Lambda) \in R_i - \bigcup_{j<i} R_j$, and a monomial $X(\Lambda) \in P_n$ as follows. Let $\hat{\Lambda} = \Lambda_{j_2}^{S_2} \Lambda_{j_3}^{S_3} \cdots \Lambda_{j_r}^{S_r}$ and assume that $\tau(\hat{\Lambda})$, $\{z_i(\hat{\Lambda})\}_{i=2}^{r}$, and $X(\hat{\Lambda})$ have already been chosen, with $z_i(\hat{\Lambda}) \in R_i - \bigcup_{j<i} R_j$. Write $\tau(\hat{\Lambda}) = \Lambda_{j_2}^{R_2} \Lambda_{j_3}^{R_3} \cdots \Lambda_{j_r}^{R_r}$. By appropriate application of relation (1), we can find R_1 such that

$$\Lambda_{j_1}^{S_1} \Lambda_{j_2}^{R_2} \cdots \Lambda_{j_r}^{R_r} = \Lambda_{j_1}^{R_1} \Lambda_{j_2}^{R_2} \cdots \Lambda_{j_r}^{R_r} \quad \text{in } \mathscr{T}$$

and $z_i(\hat{\Lambda}) \notin R_1$, for $i = 2, \dots, r$. Set $\tau(\Lambda) = \Lambda_{j_1}^{R_1} \Lambda_{j_2}^{R_2} \cdots \Lambda_{j_r}^{R_r}$, $z_i(\Lambda) = z_i(\hat{\Lambda})$, for $i \geq 2$, and choose $z_1(\Lambda)$ to be any element of R_1. (The set R_1 is actually determined by the previous choices of $z_i(\hat{\Lambda})$.) For any sets $S_1, S_2, \dots, S_r$ we can make these choices independently of $j_1, j_2, \dots, j_r$ (subject to $0 = j_0 \ll j_1 \ll j_2 \ll \cdots \ll j_r$). That is, we may assume that if $0 = j_0 \ll j_1 \ll j_2 \ll \cdots \ll j_r$ and $0 = \tilde{j}_0 \ll \tilde{j}_1 \ll \tilde{j}_2 \ll \cdots \ll \tilde{j}_r$ and

$$\tau(\Lambda_{j_1}^{S_1} \Lambda_{j_2}^{S_2} \cdots \Lambda_{j_r}^{S_r}) = \Lambda_{j_1}^{R_1} \Lambda_{j_2}^{R_2} \cdots \Lambda_{j_r}^{R_r},$$

then

$$\tau(\Lambda_{\tilde{j}_1}^{S_1} \Lambda_{\tilde{j}_2}^{S_2} \cdots \Lambda_{\tilde{j}_r}^{S_r}) = \Lambda_{\tilde{j}_1}^{R_1} \Lambda_{\tilde{j}_2}^{R_2} \cdots \Lambda_{\tilde{j}_r}^{R_r}$$

and

$$z_i(\Lambda_{j_1}^{S_1} \Lambda_{j_2}^{S_2} \cdots \Lambda_{j_r}^{S_r}) = z_i(\Lambda_{\tilde{j}_1}^{S_1} \Lambda_{\tilde{j}_2}^{S_2} \cdots \Lambda_{\tilde{j}_r}^{S_r}) \quad \text{for } i = 1, \dots, r.$$

Set

$$X(\Lambda) = X(\hat{\Lambda})^{2^{j_1}} \prod_{q \in Q} (M_q(\Lambda))^q$$

with $M_q(\Lambda) = \phi(q)z_2(\Lambda)z_3(\Lambda)\cdots z_r(\Lambda)$ for $q \in Q$, where in this expression we have identified each variable in the polynomial algebra $P(n)$ with its index. For we write X for $X(\Lambda)$, $\hat{X}$ for $X(\hat{\Lambda})$, z_i for $z_i(\Lambda)$, M_q for $M_q(\Lambda)$, and M for $\prod_{q\in Q}(M_q)^q$.

Lemma 3.4. $\langle \Lambda_{j_1}^{R_1}\Lambda_{j_2}^{R_2}\cdots\Lambda_{j_r}^{R_r}, X\rangle = \langle \Lambda_{j_2-j_1}^{R_2}\Lambda_{j_3-j_1}^{R_3}\cdots\Lambda_{j_r-j_1}^{R_r}, \hat{X}\rangle\langle \Lambda_{j_1}^{R_1}\Lambda_{j_1}^{R_2}\cdots\Lambda_{j_1}^{R_r}, M\rangle$.

Proof. A term $(\hat{x}_1)^{2^{j_1}}m_1\otimes(\hat{x}_2)^{2^{j_1}}m_2\otimes\cdots\otimes(\hat{x}_r)^{2^{j_1}}m_r$ in $\psi^{r-1}(X) = (\psi^{r-1}\hat{X})^{2^{j_1}}\psi^{r-1}(M)$ contributes 1 to $\langle \Lambda_{j_1}^{R_1}\Lambda_{j_2}^{R_2}\cdots\Lambda_{j_r}^{R_r}, X\rangle$ whenever $2^{j_1}|\hat{x}_i|+|m_i| = 2^{j_i}-1$ and $\pi((\hat{x}_i)^{2^{j_1}}m_i)\subset R_i$ for all $i = 1,\ldots,r$. The first condition is equivalent to requiring both $|\hat{x}_i| = 2^{j_i-j_1}-1$ and $|m_i| = 2^{j_1}-1$ (because Q contains no powers of 2 as large as 2^{j_1}) and the second is equivalent to requiring both $\pi(\hat{x}_i)\subset R_i$ and $\pi(m_i)\subset R_i$. Noting that $|\hat{x}_1| = 0$ so the requirement $\pi(\hat{x}_1)\subset R_1$ is superfluous, we see that these are precisely the same conditions under which the pair $\hat{x}_1\otimes\hat{x}_2\otimes\cdots\otimes\hat{x}_r$, $m_1\otimes m_2\otimes\cdots\otimes m_r$ contributes 1 to the right-hand side. □

Essentially the same proof yields

Lemma 3.5. $\langle \Lambda_{j_1}^{R_1}\Lambda_{j_1}^{R_2}\cdots\Lambda_{j_1}^{R_r}, M\rangle = \prod_{q=1}^{r} c_q$ *where* $c_q = \langle \Lambda_1^{R_1}\Lambda_1^{R_2}\cdots\Lambda_1^{R_r}, M_q\rangle$.

Lemma 3.6. *The numbers* c_q *are given by*

$$\begin{aligned} c_q &= \langle \Lambda_1^{R_1}\Lambda_1^{R_2}\cdots\Lambda_1^{R_r}, M_q\rangle \\ &\equiv |\{\sigma\in\Sigma_r \mid \phi(q)\in R_{\sigma(1)} \text{ and } z_i\in R_{\sigma(i)} \text{ for } i>1\}| \pmod 2. \end{aligned}$$

Proof. This is immediate from the definitions. □

As before, consider a monomial $\Lambda = \Lambda_{j_1}^{S_1}\Lambda_{j_2}^{S_2}\cdots\Lambda_{j_r}^{S_r}\in\mathcal{T}$ with $0 = j_0 \leqslant j_1 \leqslant j_2 \leqslant\cdots\leqslant j_r$. By replacing our representative $\Lambda_{j_1}^{S_1}\Lambda_{j_2}^{S_2}\cdots\Lambda_{j_r}^{S_r}$ by $\tau(\Lambda)$, we may assume that $\Lambda_{j_1}^{S_1}\Lambda_{j_2}^{S_2}\cdots\Lambda_{j_r}^{S_r} = \tau(\Lambda)$. Suppose $\Lambda' = \Lambda_{j_1}^{R_1}\Lambda_{j_2}^{R_2}\cdots\Lambda_{j_r}^{R_r}$ is a monomial in $\mathcal{T}$ such that $\langle \mathrm{Quot}(\Lambda'), X\rangle = 1$, where again by replacing $\Lambda_{j_1}^{R_1}\Lambda_{j_2}^{R_2}\cdots\Lambda_{j_r}^{R_r}$ by $\tau(\Lambda')$, we may assume that $\Lambda_{j_1}^{R_1}\Lambda_{j_2}^{R_2}\cdots\Lambda_{j_r}^{R_r} = \tau(\Lambda')$. Then $\langle \mathrm{Quot}(\Lambda_{j_2-j_1}^{R_2}\Lambda_{j_3-j_1}^{R_3}\cdots\Lambda_{j_r-j_1}^{R_r}), \hat{X}\rangle = 1$ and $\langle \mathrm{Quot}(\Lambda_{j_1}^{R_1}\Lambda_{j_1}^{R_2}\cdots\Lambda_{j_r}^{R_r}), M\rangle = 1$. We wish to show that $\Lambda \prec \Lambda'$. By induction

$$\langle \mathrm{Quot}(\Lambda_{j_2-j_1}^{R_2}\Lambda_{j_3-j_1}^{R_3}\cdots\Lambda_{j_r-j_1}^{R_r}), \hat{X}\rangle = 1$$

implies that

$$\Lambda_{j_2-j_1}^{S_2}\Lambda_{j_3-j_1}^{S_3}\cdots\Lambda_{j_r-j_1}^{S_r} \prec \Lambda_{j_2-j_1}^{R_2}\Lambda_{j_3-j_1}^{R_3}\cdots\Lambda_{j_r-j_1}^{R_r} \quad \text{in } \mathcal{T}.$$

If

$$\Lambda_{j_2-j_1}^{S_2}\Lambda_{j_3-j_1}^{S_3}\cdots\Lambda_{j_r-j_1}^{S_r} \neq \Lambda_{j_2-j_1}^{R_2}\Lambda_{j_3-j_1}^{R_3}\cdots\Lambda_{j_r-j_1}^{R_r} \quad \text{in } \mathcal{T},$$

this implies that $\Lambda \prec \Lambda'$ so assume that

$$\Lambda_{j_2-j_1}^{S_2}\Lambda_{j_3-j_1}^{S_3}\cdots\Lambda_{j_r-j_1}^{S_r} = \Lambda_{j_2-j_1}^{R_2}\Lambda_{j_3-j_1}^{R_3}\cdots\Lambda_{j_r-j_1}^{R_r} \quad \text{in } \mathcal{T}.$$

Therefore,

$$\Lambda_{j_1}^{R_1}\Lambda_{j_2}^{R_2}\cdots\Lambda_{j_r}^{R_r}=\Lambda_{j_1}^{R_1}\Lambda_{j_2}^{S_2}\cdots\Lambda_{j_r}^{S_r} \quad\text{and}\quad \Lambda_{j_1}^{R_1}\Lambda_{j_1}^{R_2}\cdots\Lambda_{j_1}^{R_r}=\Lambda_{j_1}^{R_1}\Lambda_{j_1}^{S_2}\cdots\Lambda_{j_1}^{S_r}.$$

Since

$$\langle \mathrm{Quot}(\Lambda_{j_1}^{R_1}\Lambda_{j_1}^{S_2}\cdots\Lambda_{j_1}^{S_r}), M\rangle=\langle \mathrm{Quot}(\Lambda_{j_1}^{R_1}\Lambda_{j_1}^{R_2}\cdots\Lambda_{j_1}^{R_r}), M\rangle=1,$$

applying Lemmas 3.5 and 3.6 gives that $c_q(R_1)=1$ for all $q=1,\dots,r$, where

$$c_q(R)\equiv|\{\sigma\in\Sigma_r\mid\phi(q)\in A_{\sigma(1)} \text{ and } z_i\in A_{\sigma(i)} \text{ for } i>1\}| \pmod 2,$$

with $A_1=R$ and $A_i=S_i$ for $i>1$. Given R, let $W(R)=\{\sigma\in\Sigma_r\mid z_i\in A_{\sigma(i)} \text{ for } i>1\}$, where the A_i's are as above. Since $z_i\notin S_j$ for $j<i$, any $\sigma\in W(R)$ satisfies either $\sigma(i)\geq i$ or $\sigma(i)=1$ for all $i>1$. We will write the identity of Σ_r as the cycle (1). Then if $\sigma\in W(R)$, σ is a single cycle (possibly the identity) beginning with 1 and having monotonically increasing entries. In particular, the only element of $W(R)$ satisfying $\sigma(1)=1$ is (1). For $q\in Q$, let $W^q(R)=\{\sigma\in W(R)\mid\phi(q)\in A_{\sigma(1)}\}$. Let $\tilde{W}(R)=W(R)-\{(1)\}$ and let $\tilde{W}^q(R)=W^q(R)\cap\tilde{W}(R)$. Then $\tilde{W}^q(R)=\{\sigma\in\tilde{W}(R)\mid d_{\sigma(1)}(\phi(q))=1\}$. For $x\in S_1$, let

$$\chi^x(R)=\begin{cases}1 & \text{if } x\in R,\\ 0 & \text{if } x\notin R.\end{cases}$$

The identity is in $W^q(R)$ if and only if $\phi(q)\in R$. Therefore, from the definitions and Lemma 3.6 we get

Lemma 3.7. *For* $x\in S_1, c_q(R)\equiv|W^q(R)|\equiv\chi^x(R)+|\tilde{W}^q(R)| \pmod 2$ *for any* q *such that* $\phi(q)=x$.

For $i=1,\dots,r$ set $W_i(R)=\{\sigma\in W(R)\mid\sigma(1)=i\}$. Define $\tilde{W}_i(R), W_i^q(R)$, and $\tilde{W}_i^q(R)$ to be the intersection of the corresponding unsubscripted set with $W_i(R)$. Note that $\tilde{W}^q(R)=\coprod_{\{i>1\mid d_i(\phi(q))=1\}}\tilde{W}_i(R)$ and observe that for $i>1$, $\tilde{W}_i(R)=W_i(R)$.

Lemma 3.8. (a) *For* $m>1$, $W_m(R)\triangle W_m(R\triangle S_m)=\{(1\ m)\}$.

(b) *For* $m>i>1$, $W_m(R)=W_m(R\triangle S_i)$.

Proof. (a) Let σ belong to $W_m(R)$. For $j>1$, except for $j=\sigma^{-1}(1)$, $z_j\in S_{\sigma(j)}$ while $z_{\sigma^{-1}(1)}\in R$. Since $\sigma\in W_m(R)$, $\sigma^{-1}(1)\geq m$. Unless $\sigma=(1\ m)$, $\sigma^{-1}(1)>m$ and so $z_{\sigma^{-1}(1)}\notin S_m$ and thus $z_{\sigma^{-1}(1)}\in R-S_m\subset R\triangle S_m$, so that $\sigma\in W_m(R\triangle S_m)$. However, if $\sigma=(1\ m)$ is in $W_m(R)$ then $z_m\in R\cap S_m$ and so $z_{\sigma(m)}\notin R\triangle S_m$ and therefore $(1\ m)\notin W_m(R\triangle S_m)$. Conversely, if $\sigma=(1\ m)\notin W_m(R)$ then $z_m\in S_m-R\subset R\triangle S$ and so $(1\ m)$ is in $W_m(R\triangle S_m)$. By symmetry, the same considerations apply to $\sigma\in R\triangle S_m$. Therefore $W_m(R)\triangle W_m(R\triangle S_m)=\{(1\ m)\}$.

(b) Since $m>i$ and σ in $W_m(R)$ or $W_m(R\triangle S_i)$ satisfies $\sigma^{-1}(1)>i$, the argument is the same as in part (a) without its exceptional case of the transposition. □

Theorem 3.9. *Given $\Lambda^{R}_{j_1}\Lambda^{S_2}_{j_2}\cdots\Lambda^{S_r}_{j_r}$, there exists R' such that*

$$\Lambda^{R}_{j_1}\Lambda^{S_2}_{j_2}\cdots\Lambda^{S_r}_{j_r}=\Lambda^{R'}_{j_1}\Lambda^{S_2}_{j_2}\cdots\Lambda^{S_r}_{j_r}\quad \text{in } \mathcal{T}$$

and $|W_m(R')|\equiv 0(2)$ for all $m>1$.

Proof. Assume by induction that R'_i has been chosen such that

$$\Lambda^{R'_i}_{j_1}\Lambda^{S_2}_{j_2}\cdots\Lambda^{S_r}_{j_r}=\Lambda^{R}_{j_1}\Lambda^{S_2}_{j_2}\cdots\Lambda^{S_r}_{j_r}\quad \text{in } \mathcal{T}$$

and $|W_m(R'_i)|\equiv 0(2)$ for all $m>i$.

Set

$$R'_{i-1}=\begin{cases} R'_i & \text{if } |W_i(R'_i)|\equiv 0(2),\\ R'_i\triangle S_i & \text{if } |W_i(R'_i)|\equiv 1(2).\end{cases}$$

By Lemma 3.8, replacing R'_i by R'_{i-1} does not affect $W_m(\,)$ for $m>i$ but changes the parity of $|W_i(\,)|$ appropriately. Therefore, $R'=R'_1$ satisfies the conditions of the theorem. □

Theorem 3.10. *Suppose $j_r\gg j_{r-1}\gg\cdots\gg j_2\gg j_1\gg j_0=0$ and let $\Lambda=\Lambda^{S_1}_{j_1}\Lambda^{S_2}_{j_2}\cdots\Lambda^{S_r}_{j_r}$. Then*

$$\langle \mathrm{Quot}(\Lambda), X(\Lambda)\rangle = 1,$$

and, conversely,

$$\text{if } \langle \mathrm{Quot}(\Lambda^{R_1}_{j_1}\Lambda^{R_2}_{j_2}\cdots\Lambda^{R_r}_{j_r}), X(\Lambda)\rangle = 1 \quad \text{then } \Lambda\prec\Lambda^{R_1}_{j_1}\Lambda^{R_2}_{j_2}\cdots\Lambda^{R_r}_{j_r}.$$

Proof. It follows from the earlier lemmas that $\langle \mathrm{Quot}(\Lambda), X(\Lambda)\rangle = 1$. Suppose now that $\langle \mathrm{Quot}(\Lambda^{R_1}_{j_1}\Lambda^{R_2}_{j_2}\cdots\Lambda^{R_r}_{j_r}), X(\Lambda)\rangle = 1$. As above we may assume that

$$\Lambda^{R_2}_{j_2}\Lambda^{R_3}_{j_3}\cdots\Lambda^{R_r}_{j_r}=\Lambda^{S_2}_{j_2}\Lambda^{S_3}_{j_3}\cdots\Lambda^{S_r}_{j_r}\quad \text{in } \mathcal{T}.$$

Then $c_q(R_1)=1$ for all q and so by Lemma 3.7 we get that for $x\in S_1$, $\chi^x(R_1)\equiv 1+|\tilde W^q(R_1)| \pmod 2$ for any q such that $\phi(q)=x$. By Theorem 3.9 we can find R' such that $\Lambda^{R'}_{j_1}\Lambda^{S_2}_{j_2}\cdots\Lambda^{S_r}_{j_r}=\Lambda^{R_1}_{j_1}\Lambda^{S_2}_{j_2}\cdots\Lambda^{S_r}_{j_r}$ in $\mathcal{T}$ and $W_m(R')\equiv 0(2)$ for all $m>1$. Given $x\in S_1$ choose $q\in Q$ such that $x=\phi(q)$. Then

$$\chi^x(R')\equiv 1+|\tilde W^q(R')|\equiv 1+\sum_{\substack{m>1;\\ d_m(\phi(q))=1}}|\tilde W_m(R')|\equiv 1+\sum_{\substack{m>1;\\ d_m(\phi(q))=1}}0\equiv 1 \pmod 2.$$

Therefore x belongs to R' and so S_1 is contained in R'. Since

$$\Lambda^{R'}_{j_1}\Lambda^{S_2}_{j_2}\cdots\Lambda^{S_r}_{j_r}=\Lambda^{R_1}_{j_1}\Lambda^{S_2}_{j_2}\cdots\Lambda^{S_r}_{j_r}=\Lambda^{R_1}_{j_1}\Lambda^{R_2}_{j_2}\cdots\Lambda^{R_r}_{j_r}\quad \text{in } \mathcal{T},$$

this says that $\Lambda\prec\Lambda^{R_1}_{j_1}\Lambda^{R_2}_{j_2}\cdots\Lambda^{R_r}_{j_r}$. □

Corollary 3.11. *Suppose* $\mathrm{Quot}(\Lambda_{j_1}^{S_1}\Lambda_{j_2}^{S_2}\cdots\Lambda_{j_r}^{S_r}) = \mathrm{Quot}(\Lambda_{j_1}^{R_1}\Lambda_{j_2}^{R_2}\cdots\Lambda_{j_r}^{R_r})$, *where* $j_r \gg j_{r-1} \gg \cdots \gg j_2 \gg j_1 \gg j_0 = 0$. *Then* $\Lambda_{j_1}^{S_1}\Lambda_{j_2}^{S_2}\cdots\Lambda_{j_r}^{S_r} = \Lambda_{j_1}^{R_1}\Lambda_{j_2}^{R_2}\cdots\Lambda_{j_r}^{R_r}$ *in* $\mathscr{T}$.

Proof. Let $j_0, j_1, \ldots, j_r$ be as stated. Let V be the subspace of $\mathscr{T}$ spanned by the set of all monomials $\Lambda_{j_1}^{S_1}\Lambda_{j_2}^{S_2}\cdots\Lambda_{j_r}^{S_r}$. Theorem 3.10 shows that the image under Quot of a basis for V is linearly independent in $\mathscr{S}$. □

Corollary 3.12. *Let* $\delta(n)$ *be the least integer such that* $\mathrm{Dim}(P(n)/((I\mathscr{A})P(n)))^k \le \delta(n)$ *for all* k. *Then* $\delta(n) \ge 1\cdot 3\cdot 7\cdots(2^{n-1}-1)\cdot(2^n-1)$.

Note. As mentioned in the introduction, Carlisle and Wood [2] have shown that such a $\delta(n)$ exists for each n.

Proof. The equivalence classes of monomials in $\{\Lambda_{j_1}^{S_1}\Lambda_{j_2}^{S_2}\cdots\Lambda_{j_r}^{S_r} \mid S_i \subset \{1,\ldots,n\}\}$ are linearly independent in $\mathscr{T}$ and it is easy to see that modulo relation (1) there are $1\cdot 3\cdot 7\cdots(2^{n-1}-1)\cdot(2^n-1)$ such monomials. By Corollary 3.11, these are linearly independent in $\mathscr{S}$ whenever $j_r \gg j_{r-1} \gg \cdots \gg j_2 \gg j_1 \gg j_0 = 0$ and so $1\cdot 3\cdot 7\cdots(2^{n-1}-1)\cdot(2^n-1)$ is a lower bound for $\delta(n)$. □

4. Examples

In this section we give a couple of examples of elements in $\ker\theta$.

In 3 variables the least degree element in $\ker\theta$ lies in degree 8. Let $K = x^3y^3z^2 + x^3y^2z^3 + x^2y^3z^3$:

$$\begin{aligned}\theta_{(1,7)}(K) &= \pi(x)\otimes\pi(x^2y^3z^2) + \pi(y)\otimes\pi(x^3y^2z^2) + \pi(x)\otimes\pi(x^2y^2z^3)\\ &\quad + \pi(z)\otimes\pi(x^3y^2z^2) + \pi(y)\otimes\pi(x^2y^2z^3) + \pi(z)\otimes\pi(x^2y^3z^2)\\ &= x\otimes xyz + y\otimes xyz + x\otimes xyz + z\otimes xyz + y\otimes xyz + z\otimes xyz = 0.\end{aligned}$$

Similarly, $\theta_{(1,1,3,3)}(x)$, $\theta_{(1,1,1,1,1,3)}(x)$, and $\theta_{(1,1,1,1,1,1,1,1)}(x)$ are zero. To show that $K \ne 0$ in $P/((I\mathscr{A})P)$ we exhibit an element of $A \in \mathrm{Ann}\,P_*$ such that $\langle A, K\rangle = 1$. Let $[i, j, k]$ denote the element $\gamma_i(x)\gamma_j(y)\gamma_k(z) \in P_*$. Let $A = [6,1,1] + [5,2,1] + [3,4,1] + [3,3,2]$. Then $\langle A, K\rangle = \langle [3,3,2], x^3y^3z^2\rangle = 1$ and it is easily verified that $A \in \mathrm{Ann}\,P_*$. Notice that the superficially similar-looking element $x^7y^7z^6 + x^7y^6z^7 + x^6y^7z^7$ is 0 in $P/((I\mathscr{A})P)$. In fact, since $\beta(20) = 4 > 3$, $(P(3)/((I\mathscr{A})P(3)))^{20}$ has no nonzero elements.

For a second example, consider $K = w^4x^4y^3z^3 + w^3x^3y^4z^4 \in (P(4)/((I\mathscr{A})P(4)))^{14}$. Using analogous notation, let $A = [1,1,6,6] + [1,2,6,5] + [1,3,5,5] + [1,4,6,3] + [1,5,3,5] + [1,6,1,6] + [1,6,2,5] + [1,6,4,3] + [2,1,5,6] + [2,2,5,5] + [2,3,6,3] + [2,4,5,3] + [2,5,1,6] + [2,5,2,5] + [2,5,4,3] + [2,6,3,3] + [3,1,4,6] + [3,2,4,5] + [3,3,2,6] + [3,4,4,3] + [3,5,1,5] + [4,1,3,6] + [4,2,3,5] + [4,3,1,6] + [4,3,2,5] + [4,3,4,3] + [4,4,3,3] + [5,1,2,6] + [5,2,2,5] + [5,3,3,3] + [5,4,2,3] + [5,5,1,3] + [6,1,1,6] +

$[6,2,1,5]+[6,3,2,3]+[6,4,1,3]$. Clearly $\langle A, K\rangle = 1$. A short calculation similar to that above shows that $\theta(K) = 0$. What our computer regards as a short computation shows that $A \in \operatorname{Ann} P_*$.

The relatively pleasant-looking form of K in these examples suggests that while $\operatorname{Im}\theta$ is best studied by dualizing and looking at $\mathscr{S}$, $\ker\theta$ might be more tractable than its dual.

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