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Classification of pinched positive scalar curvature manifolds

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Abstract

Let (M^n, g), $n \geqslant 3$, be a smooth closed Riemannian manifold with positive scalar curvature R_g. There exists a positive constant $C = C(M, g)$ defined by mean curvature of Euclidean isometric immersions, which is a geometric invariant, such that $R_g \leqslant n(n-1)C$. In this paper we prove that $R_g = n(n-1)C$ if and only if (M^n, g) is isometric to the Euclidean sphere $\mathbb{S}^n(C)$ with constant sectional curvature C. Also, there exists a Riemannian metric g on M^n such that the scalar curvature satisfies the pinched condition

$$\frac{n^2(n-2)}{n-1} C < R_g \leqslant n(n-1)C$$

if and only if M^n is diffeomorphic to the standard sphere $\mathbb{S}^n$.
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Résumé

Soit (M^n, g), $n \geqslant 3$, une variété riemannienne compacte C^∞ avec courbure scalaire R_g positive. Il existe une constante positive $C = C(M, g)$ définie par la courbure moyenne de immersions isométriques euclidiennes, qui est un invariant géométrique, telle que $R_g \leqslant n(n-1)C$. Dans cet article, on démontre que $R_g = n(n-1)C$ si et seulement si (M^n, g) est isométrique à la sphère euclidienne $\mathbb{S}^n(C)$ à courbure sectionnelle C constante. De plus, il existe une metrique riemannienne g sur M^n telle que l'inégalité suivante soit vérifiée

$$\frac{n^2(n-2)}{n-1} C < R_g \leqslant n(n-1)C$$

si et seulement si M^n est difféomorphe à la sphère $\mathbb{S}^n$.
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1. Introduction

In 1951, H.E. Rauch [14] began the study on the relationship between geometrical and topological properties of a manifold considering pinched conditions on its metric. The basic question is the following: is a compact, simply connected manifold M^n, whose sectional curvatures belong to the interval $(1,4]$, necessarily homeomorphic to the sphere $\mathbb{S}^n$? This question was answered positively by M. Berger [1] and W. Klingenberg [10] in the early seventies. These results are known as Sphere theorems. A more general result, known as Differentiable Sphere theorem, where a diffeomorphism is obtained instead of a homeomorphism, was obtained recently by Brendle and Schoen [2]. They proved the following:

Theorem 1.1. *Let (M^n, g) be a compact Riemannian manifold with $1/4$-pinched curvature (that is (M^n, g) has positive sectional curvature and the ratio of the minimum and the maximum of the sectional curvatures is always strictly bigger than a quarter). Then M^n admits a Riemannian metric of constant positive sectional curvature, therefore is diffeomorphic to a spherical space form.*

On the other hand, pinched conditions on the curvature operators as the Ricci tensor Ric_g, Riemann tensor Rm_g and the Weyl tensor W_g have used in order to find Sphere like theorems. In 1986, C. Margerin [12] proved the following Sphere like theorem in the case of 4-dimensional manifolds.

Theorem 1.2. *Let (M^4, g) be a closed 4-dimensional Riemannian manifold with positive scalar curvature. If the pinching condition*

$$WP_g < \frac{1}{6}$$

is satisfied, then M^4 is diffeomorphic to a spherical space form, i.e. M^4 admits a metric with constant positive sectional curvature. Moreover, M^4 is diffeomorphic to $\mathbb{S}^4$ or $\mathbb{R}P^4$, where $WP_g = \frac{|W_g|_g^2 + 2|E_g|_g^2}{R_g^2}$, E_g is the trace-free Ricci tensor and $|\cdot|_g$ is the usual norm of a tensor with respect to the metric g.

Recently, G. Catino and Z. Djadli [3] proved an integral pinching theorem in 3 dimensions. More precisely they showed the following result.

Theorem 1.3. *Let (M^3, g) be a closed 3-dimensional Riemannian manifold with positive scalar curvature. If*

$$\int_{M^3} |Ric_g|_g^2 \, dv_g \leqslant \frac{3}{8} \int_{M^3} R_g^2 \, dv_g,$$

then M^3 is diffeomorphic to a spherical space form.

Now, let us define $\sigma_k(g^{-1}A_g^t)$ the k-th elementary function of the eigenvalues of $g^{-1}A_g^t$, where

$$A_g^t = \frac{1}{n-2}\left(Ric_g - \frac{t}{2(n-1)}R_g g\right).$$

Namely, if we denote by $\lambda_1, \dots, \lambda_n$ the eigenvalues of $g^{-1}A_g^t$,

$$\sigma_k\left(g^{-1}A_g^t\right) = \sum_{1 \leqslant \lambda_{i_1} < \cdots < \lambda_{i_k} \leqslant n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

We denote $A_g^t \in \Gamma_k^+$, if

$$\sigma_j\left(g^{-1}A_g^t\right) > 0$$

for all $j \in \{1, \dots, k\}$. The next result was obtained recently by G. Catino, Z. Djadli and C.B. Ndiaye [4].

Theorem 1.4. *Let (M^n, g) be a closed, locally conformally flat, n-dimensional Riemannian manifold, $n \geqslant 8$ even, with positive scalar curvature and with positive Euler–Poincaré characteristic. There exists a constant*

$$t_0 = t_0\left(n, \operatorname{diam}\left(M^n, g\right), \left|\nabla^2 Rm_g\right|_g\right) < 1$$

such that if

$$A_g^t \in \Gamma_{\frac{n}{2}}^+,$$

for some $t \in [t_0, 1]$, then M^n is diffeomorphic to either $\mathbb{S}^n$ or $\mathbb{R}P^n$.

It is natural to look for pinched conditions involving only the positive scalar curvature R_g to obtain new Sphere theorems. Conditions such as 1/4-pinched on the scalar curvature is not sufficient because, as we know the Yamabe problem, we can find a conformal deformation of the metric such that 1/4-pinched condition on the scalar curvature is satisfied. So, our hopes of finding pinched conditions involving only the scalar curvature should be based on conditions that involve some kind of geometric invariant. The results cited above involving the scalar curvature and also other measures are in that direction. In this paper we show a geometric invariant and a pinched condition on the scalar curvature that is sufficient to obtain a Sphere like theorem.

2. Preliminaries and notations

In order to state our result, we need to recall some notions. We start by recalling the mean curvature of the manifold (M^n, g). We will consider only closed connected manifolds with positive scalar curvature. Since M^n is a compact Riemannian manifold, from Nash's immersion theorem, we have that M^n can be isometrically immersed into a Euclidean space $\mathbb{R}^{n+p}$, where $p = \frac{(n+2)(n+3)-2n}{2}$. That is, there exists an isometric immersion $\Theta : M^n \to \mathbb{R}^{n+p}$. Thus, M^n can be seen as a compact submanifold isometrically immersed into $\mathbb{R}^{n+p}$. We choose a local field of orthonormal frames $\{e_1, \dots, e_{n+p}\}$ adapted to the Riemannian metric of $\mathbb{R}^{n+p}$ and the dual coframes $\{\omega_1, \dots, \omega_{n+p}\}$ in such a way that, restricted to the submanifold M^n, $\{e_1, \dots, e_n\}$ are tangent to M^n. We have that $\{e_1, \dots, e_n\}$ is a local field of orthonormal frames adapted to the induced Riemannian metric on M^n and $\{\omega_1, \dots, \omega_n\}$ is a local field of its dual coframes on M^n. From Cartan's lemma,

$$\omega_{\alpha i} = \sum_j^n h_{ij}^\alpha \omega_j \quad \text{and} \quad h_{ij}^\alpha = h_{ji}^\alpha.$$

The second fundamental form σ_Θ and the mean curvature vector h_Θ of the immersion Θ are defined by

$$\sigma_\Theta = \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^n h_{ij}^\alpha \omega_i \omega_j e_\alpha$$

and

$$h_\Theta = \frac{1}{n} \sum_{\alpha=n+1}^{n+p} \left(\sum_{i=1}^n h_{ii}^\alpha \right) e_\alpha.$$

We will denote h instead h_Θ. The mean curvature H_Θ and the squared norm of the second fundamental form S_Θ of the immersion Θ are defined by

$$H_\Theta = \frac{1}{n} \sqrt{\sum_{\alpha=n+1}^{n+p} \left(\sum_{i=1}^n h_{ii}^\alpha \right)^2}$$

and

$$S_\Theta = \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^n \left(h_{ij}^\alpha\right)^2.$$

The connection form of Θ is characterized by the structure equations

$$d\omega_i = -\sum_{j=1}^n \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = -\sum_{k=1}^n \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l=1}^n R_{ijkl} \omega_k \wedge \omega_l$$

and

$$R_{ijkl} = \sum_{\alpha=n+1}^{n+p} \left(h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha \right),$$

where R_{ijkl} are the components of the curvature tensor of M^n. Denote by R_{ij} and R_g the components of the Ricci curvature and the scalar curvature of M^n, respectively. So, we have, from the last identity,

$$R_{jk} = \sum_{\alpha=n+1}^{n+p} \left(\sum_{i=1}^n h_{ii}^\alpha h_{jk}^\alpha - \sum_{i=1}^n h_{ik}^\alpha h_{ji}^\alpha \right)$$

and

$$R_g = n^2 H_\Theta^2 - S_\Theta. \tag{1}$$

A simple computation shows that

$$S_\Theta \geqslant n H_\Theta^2. \tag{2}$$

Hence, from (1) and (2),

$$R_g \leqslant n(n-1) H_\Theta^2. \tag{3}$$

Since the scalar curvature R_g is invariant of isometries, we have that the inequality (3) holds for any isometric immersion from M^n into a Euclidean space $\mathbb{R}^{n+p}$. We define

$$\Gamma = \{\Theta;\ \Theta \text{ is an isometric immersion from } M^n \text{ into a Euclidean space } \mathbb{R}^{n+p}\}.$$

Thus, the positive constant

$$C(M,g) = \inf_{\Theta\in\Gamma} \max_{M^n} H_\Theta^2 \tag{4}$$

is a geometric invariant and, from the inequality (3),

$$R_g \leqslant n(n-1)C(M,g). \tag{5}$$

We call $C(M,g)$ the mean curvature of the Riemannian manifold (M,g). Note that, for all $n \geqslant 2$,

$$\frac{n^2(n-2)}{n-1} C(M,g) < n(n-1)C(M,g).$$

If $\mathbb{S}^n(C)$ is a Euclidean sphere with constant sectional curvature C, then $C(\mathbb{S}^n(C),\delta) = C$ and $R_\delta = n(n-1)C$, where δ is the Euclidean metric on $\mathbb{S}^n(C)$. A question that arise here is the following: are Euclidean spheres the only Riemannian manifolds that satisfies $R_g = n(n-1)C(M,g)$?

Now we are ready to state our result on classification of pinched positive scalar curvature manifolds.

Theorem 2.1. *Let (M^n,g), $n \geqslant 3$, be a smooth closed Riemannian manifold with positive scalar curvature R_g and mean curvature $C = C(M,g)$. Then, $R_g = n(n-1)C$ if and only if (M^n,g) is isometric to the Euclidean sphere $\mathbb{S}^n(C)$ with constant sectional curvature C. Also, there exists a Riemannian metric g on M^n such that the scalar curvature satisfies the pinched condition*

$$\frac{n^2(n-2)}{n-1} C < R_g \leqslant n(n-1)C$$

if and only if M^n is diffeomorphic to the standard sphere $\mathbb{S}^n$.

As a direct consequence of Theorem 2.1 we have the following corollary.

Corollary 2.1. *Let (M^n,g), $n \geqslant 3$, be a smooth closed Riemannian manifold with positive scalar curvature R_g. There exists a positive constant*

$$C = C(M,g)$$

such that if

$$T_g(X,X) > \frac{n^2(5n-9)}{4(n-1)} C(M,g),$$

for all unit vectors X, then M^n is diffeomorphic to the standard sphere $\mathbb{S}^n$, where

$$T_g = Ric_g + R_g g.$$

The proof of the above results is based on techniques of isometric immersions. First we find an isometric immersion of codimension p of the manifold M^n in a Euclidean space. So, the pinched conditions allow us to make a reduction of codimension and also to get the sectional curvature is nonnegative. Then, using a classification result obtained by Cheng and Yau in [7] and the mean curvature flow theory, we classify the manifold. I want to clarify here that all the ideas of reduction of codimension is inspired by the work of Q.M. Cheng, especially the work [6].

3. Proof of Theorem 2.1

Assuming that

$$\frac{n^2(n-2)}{n-1}C(M,g) < R_g \leqslant n(n-1)C(M,g),$$

we find an isometric immersion $\Theta : M^n \to \mathbb{R}^{n+p}$, such that

$$\frac{n^2(n-2)}{n-1}H_\Theta^2 < R_g \leqslant n(n-1)C(M,g). \tag{6}$$

In the sequel, we will use the notations H and S instead H_Θ and S_Θ, respectively. Independently, we have

$$d\omega_{\alpha\beta} = -\sum_{\gamma=1}^{n+p}\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta} + \frac{1}{2}\sum_{i,j=1}^{n} R_{\alpha\beta ij}\omega_i\wedge\omega_j$$

and

$$R_{\alpha\beta ij} = \sum_{l=1}^{n}\left(h_{il}^\alpha h_{lj}^\beta - h_{jl}^\alpha h_{li}^\beta\right).$$

Using the exterior differentiation and defining h_{ijk}^α by

$$\sum_{k=1}^{n} h_{ijk}^\alpha\omega_k = dh_{ij}^\alpha - \sum_{k=1}^{n} h_{ik}^\alpha\omega_{kj} - \sum_{k=1}^{n} h_{jk}^\alpha\omega_{ki} - \sum_{\beta=n+1}^{n+p} h_{ij}^\beta\omega_{\beta\alpha},$$

we obtain the Codazzi's equation,

$$h_{ijk}^\alpha = h_{ikj}^\alpha = h_{jik}^\alpha.$$

Again using the exterior differentiation and defining h_{ijkl}^α by

$$\sum_{l=1}^{n} h_{ijkl}^\alpha\omega_l = dh_{ijk}^\alpha - \sum_{l=1}^{n} h_{ljk}^\alpha\omega_{li} - \sum_{l=1}^{n} h_{ilk}^\alpha\omega_{lj} - \sum_{l=1}^{n} h_{ijl}^\alpha\omega_{lk} - \sum_{\beta=n+1}^{n+p} h_{ijk}^\beta\omega_{\beta\alpha},$$

we obtain Ricci's formula for the second fundamental form σ_Θ,

$$h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_{m=1}^{n} h_{mj}^\alpha R_{mikl} + \sum_{m=1}^{n} h_{im}^\alpha R_{mjkl} + \sum_{\beta=n+1}^{n+p} h_{ij}^\beta R_{\beta\alpha kl}.$$

The Laplacian Δh_{ij}^{α} of h_{ij}^{α} is defined by

$$\Delta h_{ij}^{\alpha} = \sum_{k=1}^{n} h_{ijkk}^{\alpha}.$$

From the Codazzi's equation and the Ricci's formula, we obtain, for any α, $n+1 \leqslant \alpha \leqslant n+p$,

$$\begin{aligned}\Delta h_{ij}^{\alpha} &= \sum_{k=1}^{n} h_{kijk}^{\alpha} \\ &= \sum_{k=1}^{n} h_{kikj}^{\alpha} + \sum_{k,m=1}^{n} h_{km}^{\alpha} R_{mijk} + \sum_{k,m=1}^{n} h_{mi}^{\alpha} R_{mkjk} + \sum_{k=1}^{n} \sum_{\beta=n+1}^{n+p} h_{ki}^{\beta} R_{\beta\alpha jk} \\ &= \sum_{k=1}^{n} h_{kkij}^{\alpha} + \sum_{k,m=1}^{n} h_{km}^{\alpha} R_{mijk} + \sum_{k,m=1}^{n} h_{mi}^{\alpha} R_{mkjk} + \sum_{k=1}^{n} \sum_{\beta=n+1}^{n+p} h_{ki}^{\beta} R_{\beta\alpha jk}.\end{aligned}$$

Since $R_g > 0$, we have from (5) that the mean curvature vector $h \neq 0$ on M^n. Hence, we have that $e_{n+1} = H^{-1}h$ is a normal vector field defined globally on M^n. We define φ and ψ by

$$\varphi = \sum_{i,j=1}^{n} \left(h_{ij}^{n+1} - H\delta_{ij}\right)^2$$

and

$$\psi = \sum_{\alpha=n+2}^{n+p} \sum_{i,j=1}^{n} \left(h_{ij}^{\alpha}\right)^2,$$

respectively. Then φ and ψ are functions defined on M^n globally, which do not depend on the choice of the orthonormal frame $\{e_1, \ldots, e_n\}$. From the definition of the mean curvature vector h, we obtain that $nH = \sum_{i=1}^{n} h_{ii}^{n+1}$ and $\sum_{i=1}^{n} h_{ii}^{\alpha} = 0$ for $n+2 \leqslant \alpha \leqslant n+p$ on M^n. Setting $H_\alpha = (h_{ij}^{\alpha})$ and defining $N(A) = tr({}^tAA)$ for $n \times n$-matrix A, where $tr(B)$ denotes the trace of the matrix B, by making use of a direct computation we have

$$\begin{aligned}\sum_{\alpha=n+2}^{n+p} \sum_{l,i,j,k=1}^{n} h_{ij}^{\alpha} h_{kl}^{\alpha} R_{lijk} &= \sum_{\alpha=n+2}^{n+p} tr(H_{n+1}H_\alpha)^2 - \sum_{\alpha=n+2}^{n+p} \big(tr(H_{n+1}H_\alpha)\big)^2 \\ &\quad + \sum_{\alpha,\beta=n+2}^{n+p} tr(H_\alpha H_\beta)^2 - \sum_{\alpha,\beta=n+2}^{n+p} \big(tr(H_\alpha H_\beta)\big)^2, \\ \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k,l=1}^{n} h_{ij}^{\alpha} h_{kl}^{\alpha} R_{lkjk} &= nH \sum_{\alpha=n+2}^{n+p} tr\big(H_{n+1}H_\alpha^2\big) - \sum_{\alpha=n+2}^{n+p} tr\big(H_{n+1}^2 H_\alpha^2\big) \\ &\quad - \sum_{\alpha,\beta=n+2}^{n+p} tr(H_\alpha H_\beta H_\beta H_\alpha)\end{aligned}$$

and

$$\sum_{\alpha,\beta=n+1}^{n+p} \sum_{i,j,k=1}^{n} h_{ij}^{\alpha} h_{ki}^{\beta} R_{\beta\alpha jk} = \sum_{\alpha,\beta=n+1}^{n+p} tr(H_\alpha H_\beta)^2 - \sum_{\alpha,\beta=n+1}^{n+p} tr(H_\alpha H_\beta H_\beta H_\alpha).$$

Hence,

$$\begin{aligned}\frac{1}{2}\Delta\psi &= \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}\left(h_{ijk}^{\alpha}\right)^2+\sum_{\alpha=n+2}^{n+p}\sum_{i,j=1}^{n}h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}\\ &= \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}\left(h_{ijk}^{\alpha}\right)^2+nH\sum_{\alpha=n+2}^{n+p}tr\left(H_{n+1}H_{\alpha}^2\right)-\sum_{\alpha=n+2}^{n+p}\left(tr(H_{n+1}H_{\alpha})\right)^2\\ &\quad-\sum_{\alpha,\beta=n+2}^{n+p}N(H_\alpha H_\beta-H_\beta H_\alpha)-\sum_{\alpha,\beta=n+2}^{n+p}\left(tr(H_\alpha H_\beta)\right)^2\\ &\quad+\sum_{\alpha=n+2}^{n+p}tr(H_{n+1}H_\alpha)^2-\sum_{\alpha=n+2}^{n+p}tr\left(H_{n+1}^2H_\alpha^2\right).\end{aligned}\tag{7}$$

According to Lemma A.1 (see Appendix A), and the definition of ψ, we obtain

$$-\sum_{\alpha,\beta=n+2}^{n+p}N(H_\alpha H_\beta-H_\beta H_\alpha)-\sum_{\alpha,\beta=n+2}^{n+p}\left(tr(H_\alpha H_\beta)\right)^2\geqslant-\frac{3}{2}\psi^2.\tag{8}$$

Since $e_{n+1}=\frac{h}{H}$ we have $tr(H_\alpha)=0$ for $\alpha=n+2,\dots,n+p$ and $tr(H_{n+1})=nH$. Note that

$$\begin{aligned}&-\sum_{\alpha=n+2}^{n+p}\left(tr(H_{n+1}H_\alpha)\right)^2+\sum_{\alpha=n+2}^{n+p}tr(H_{n+1}H_\alpha)^2-\sum_{\alpha=n+2}^{n+p}tr\left(H_{n+1}^2H_\alpha^2\right)\\ &\quad=\sum_{\alpha=n+2}^{n+p}\left\{-\left(tr(H_{n+1}H_\alpha)\right)^2+tr(H_{n+1}H_\alpha)^2-tr\left(H_{n+1}^2H_\alpha^2\right)\right\}\\ &\quad=\sum_{\alpha=n+2}^{n+p}\left\{-\left(tr\left((H_{n+1}-HI)H_\alpha\right)\right)^2+tr\left((H_{n+1}-HI)H_\alpha\right)^2-tr\left((H_{n+1}-HI)^2H_\alpha^2\right)\right\},\end{aligned}$$

where I denotes the unit matrix. For a fixed α, $n+2\leqslant\alpha\leqslant n+p$, we can take a local orthonormal frame field $\{e_1,\dots,e_n\}$ such that $h_{ji}^{\alpha}=\lambda_i^{\alpha}\delta_{ij}$. Thus, we have

$$\sum_{i=1}^{n}\lambda_i^{\alpha}=0$$

and

$$tr\left(H_\alpha^2\right)=\sum_{i=1}^{n}\left(\lambda_i^{\alpha}\right)^2.$$

Let $B=H_{n+1}-HI=(b_{ij})$. We have $b_{ij}=b_{ji}$ for all $i,j=1,\dots,n$,

$$\sum_{i=1}^{n}b_{ii}=0$$

and

$$\sum_{i,j=1}^{n}b_{ij}^2=\varphi.$$

Hence, we find

$$\begin{aligned}
&-\big(tr\big((H_{n+1}-HI)H_\alpha\big)\big)^2+tr\big((H_{n+1}-HI)H_\alpha\big)^2-tr\big((H_{n+1}-HI)^2H_\alpha^2\big)\\
&\quad=-\big(tr(BH_\alpha)\big)^2+tr(BH_\alpha)^2-tr\big(B^2H_\alpha^2\big)\\
&\quad=-\left(\sum_{i=1}^n b_{ii}\lambda_i^\alpha\right)^2+\sum_{i=1}^n b_{ij}^2\lambda_i^\alpha\lambda_j^\alpha-\sum_{i=1}^n b_{ij}^2\big(\lambda_i^\alpha\big)^2.
\end{aligned}$$

Since λ_i^α and b_{ij} for $i,j=1,\dots,n$ satisfy the conditions in Lemma A.2 (see Appendix A), we obtain

$$\begin{aligned}
&-\big(tr\big((H_{n+1}-HI)H_\alpha\big)\big)^2+tr\big((H_{n+1}-HI)H_\alpha\big)^2-tr\big((H_{n+1}-HI)^2H_\alpha^2\big)\\
&\quad\geqslant-\varphi\, tr\big(H_\alpha^2\big).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sum_{\alpha=n+2}^{n+p}\big\{-\big(tr\big((H_{n+1}-HI)H_\alpha\big)\big)^2+tr\big((H_{n+1}-HI)H_\alpha\big)^2-tr\big((H_{n+1}-HI)^2H_\alpha^2\big)\big\}\\
&\quad\geqslant-\varphi\sum_{\alpha=n+2}^{n+p}tr\big(H_\alpha^2\big)=-\varphi\psi. \qquad (9)
\end{aligned}$$

Also,

$$\begin{aligned}
nH\sum_{\alpha=n+2}^{n+p}tr\big(H_{n+1}H_\alpha^2\big)&=nH\sum_{\alpha=n+2}^{n+p}tr\big((H_{n+1}-HI)H_\alpha^2\big)+nH^2\sum_{\alpha=n+2}^{n+p}tr\big(H_\alpha^2\big)\\
&=nH\sum_{\alpha=n+2}^{n+p}tr\big((H_{n+1}-HI)H_\alpha^2\big)+nH^2\psi. \qquad (10)
\end{aligned}$$

Using the same assertion as above, we have, for fixed α, $n+2\leqslant\alpha\leqslant n+p$,

$$tr\big((H_{n+1}-HI)H_\alpha^2\big)=\sum_{i=1}^n b_{ii}\big(\lambda_i^\alpha\big)^2.$$

From Lemmas A.3 and A.4 (see Appendix A), we obtain

$$tr\big((H_{n+1}-HI)H_\alpha^2\big)\geqslant-\frac{n-2}{\sqrt{n(n-1)}}\sqrt{\varphi}\, tr\big(H_\alpha^2\big).$$

Thus, from (10), we conclude

$$nH\sum_{\alpha=n+2}^{n+p}tr\big(H_{n+1}H_\alpha^2\big)\geqslant nH^2\psi-\sqrt{\frac{n}{n-1}}(n-2)H\sqrt{\varphi}\,\psi. \qquad (11)$$

From (7), (8), (9) and (11), we find the following inequality:

$$\frac{1}{2}\Delta\psi\geqslant\sum_{\alpha=n+2}^{n+p}\sum_{i,j=1}^n\big(h_{ijk}^\alpha\big)^2+\left(nH^2-\sqrt{\frac{n}{n-1}}(n-2)H\sqrt{\varphi}-\varphi-\frac{3}{2}\psi\right)\psi. \qquad (12)$$

Since

$$\left(\sqrt{\frac{n(n-2)}{n-1}}H-\sqrt{(n-2)\varphi}\right)^2\geqslant 0,$$

we obtain

$$-\sqrt{\frac{n}{n-1}}(n-2)H\sqrt{\varphi}\geqslant -\frac{n}{2}\left(\frac{n-2}{n-1}\right)H^2-\frac{(n-2)}{2}\varphi. \tag{13}$$

Therefore, from (12) and (13), we have the inequality

$$\frac{1}{2}\Delta\psi\geqslant\sum_{\alpha=n+2}^{n+p}\sum_{i,j=1}^{n}\left(h_{ijk}^{\alpha}\right)^2+\left(nH^2-\frac{n}{2}\left(\frac{n-2}{n-1}\right)H^2-\frac{(n-2)}{2}\varphi-\varphi-\frac{3}{2}\psi\right)\psi. \tag{14}$$

Observe that

$$-\frac{(n-2)}{2}\varphi-\varphi=\frac{n^2}{2}H^2-\frac{n}{2}S+\frac{n}{2}\psi. \tag{15}$$

Hence,

$$\begin{aligned}\frac{1}{2}\Delta\psi&\geqslant\sum_{\alpha=n+2}^{n+p}\sum_{i,j=1}^{n}\left(h_{ijk}^{\alpha}\right)^2+\left(\frac{n}{2}\left(\frac{n^2H^2}{n-1}-S\right)+\left(\frac{n-3}{2}\right)\psi\right)\psi\\&\geqslant\sum_{\alpha=n+2}^{n+p}\sum_{i,j=1}^{n}\left(h_{ijk}^{\alpha}\right)^2+\left(\frac{n-3}{2}\right)\psi^2,\end{aligned}$$

since (14) and (15) hold and

$$\frac{n}{2}\left(\frac{n^2H^2}{n-1}-S\right)\geqslant 0.$$

This last inequality is true because from (6) we have

$$\frac{n^2(n-2)}{n-1}H^2<R_g=n^2H^2-S$$

and consequently

$$\frac{n^2H^2}{n-1}\geqslant S.$$

Again from (6) we obtain that

$$R_g\geqslant(n-2)S.$$

So, it follows from this last inequality and Theorem 4.1 in [5] that the sectional curvature of M^n is nonnegative. Consequently, the Ricci curvature of M^n has a lower bound. By applying the generalized maximum principle due to Omori [13] and Yau [15] to the function ψ, we have that there exists a sequence (p_k) in M^n such that

$$\lim_{k\to\infty}\psi(p_k)=\sup_M\psi\quad\text{and}\quad\limsup_{k\to\infty}\Delta\psi(p_k)\leqslant 0.$$

Since (6) holds and H is bounded, we have that $(h_{ij}^{\alpha}(p_k))$, for any $i,j=1,2,\ldots,n$ and any $\alpha=n+1,\ldots,n+p$, is a bounded sequence. Hence we can assume that $\lim_{k\to\infty}h_{ij}^{\alpha}(p_k)=\bar{h}_{ij}^{\alpha}$, if necessary, we can take a subsequence. So, we have that $\psi = 0$ for $n \geqslant 4$. Now suppose that $n = 3$. So, if $\sup \psi \neq 0$, we find

$$\lim_{k\to+\infty}\left(S - \frac{n^2H^2}{n-1}\right)(p_k) = 0 \quad \text{and} \quad \lim_{k\to+\infty}\sqrt{\frac{n}{n-1}}H(p_k) = \sqrt{\varphi}.$$

Let

$$\lim_{k\to+\infty} H(p_k) = \overline{H}, \qquad \lim_{k\to+\infty} S(p_k) = \overline{S} \quad \text{and} \quad \lim_{k\to+\infty} \varphi(p_k) = \overline{\varphi}.$$

We have

$$\overline{S} = \frac{n\overline{H}^2}{n-1}, \qquad \overline{\varphi} = \frac{n\overline{H}^2}{n-1} \quad \text{and} \quad \overline{S} = \sup\psi + \overline{\varphi} + n\overline{H}^2 = \frac{n^2\overline{H}^2}{n-1} + \sup\psi.$$

Hence, $\sup \psi = 0$. This is impossible. Then, we obtain

$$\psi = 0 \quad \text{and} \quad \sum_{\alpha=n+2}^{n+p}\sum_{i,j=1}^{n}\left(h^{\alpha}_{ijk}\right)^2 = 0.$$

We conclude from Erbacher's theorem [8] (or Theorem 1 in [16]) that M^n lies in a totally geodesic submanifold $\mathbb{R}^{n+1}$ of the Euclidean space $\mathbb{R}^{n+p}$.

If $R_g = n(n-1)C$, it follows from the classification result obtained by Cheng and Yau [7] that (M^n, g) is isometric to the Euclidean sphere $\mathbb{S}^n(C)$ with constant sectional curvature C, since in this case R_g is constant and the sectional curvature is nonnegative. In the general case, it follows from the theory of mean curvature flow (see [9], for example) that M^n is diffeomorphic to the standard sphere $\mathbb{S}^n$, since M^n is a closed hypersurface immersed in $\mathbb{R}^{n+1}$ and the sectional curvature is nonnegative.

Finally, if $f : M^n \to \mathbb{S}^n$ is a diffeomorphism, then the metric $g = f^*\delta$, where δ is the standard metric on $\mathbb{S}^n$, is a metric on M^n such that

$$\frac{n^2(n-2)}{n-1}C(M,g) < R_g \leqslant n(n-1)C(M,g).$$

4. Proof of Corollary 2.1

Consider an isometric immersion $\Theta : M^n \to \mathbb{R}^{n+p}$. Choose a local field of orthonormal frames $\{e_1, \ldots, e_{n+p}\}$ adapted to the Riemannian metric of $\mathbb{R}^{n+p}$ in such a way that, restricted to the submanifold M^n, $\{e_1, \ldots, e_n\}$ are tangent to M^n. We have that $\{e_1, \ldots, e_n\}$ is a local field of orthonormal frames adapted to the induced Riemannian metric on M^n. We have that

$$R_g = n^2H^2 - S.$$

Hence,

$$\begin{aligned} n^2H^2 = R_g &+ \sum_{\alpha=n+1}^{n+p}\left(\left(h^{\alpha}_{11}\right)^2 + \left(h^{\alpha}_{22} + \cdots + h^{\alpha}_{nn}\right)^2 + 2\sum_{i<j}\left(h^{\alpha}_{ij}\right)^2\right) \\ &- 2\sum_{\alpha=n+1}^{n+p}\sum_{2\leqslant i<j\leqslant n} h^{\alpha}_{ii}h^{\alpha}_{jj} \end{aligned}$$

$$= R_g + \frac{1}{2}\sum_{\alpha=n+1}^{n+p}\left(\left(h_{11}^{\alpha}+h_{22}^{\alpha}+\cdots+h_{nn}^{\alpha}\right)^2+\left(h_{11}^{\alpha}-h_{22}^{\alpha}-\cdots-h_{nn}^{\alpha}\right)^2\right)$$

$$+2\sum_{\alpha=n+1}^{n+p}\left(\sum_{i<j}\left(h_{ij}^{\alpha}\right)^2\right)-2\sum_{\alpha=n+1}^{n+p}\sum_{2\leqslant i<j\leqslant n}h_{ii}^{\alpha}h_{jj}^{\alpha}$$

$$= R_g + \frac{n^2H^2}{2} - 2\sum_{2\leqslant i<j\leqslant n}K_{ij} + 2\sum_{\alpha=n+1}^{n+p}\left(\sum_{j=2}\left(h_{1j}^{\alpha}\right)^2\right).$$

Thus,

$$\frac{n^2}{4}H^2 \geqslant Ric_g(e_1, e_1).$$

Since e_1 is an arbitrary unit vector, we find

$$\frac{n^2}{4}H^2 \geqslant Ric_g(X, X),$$

for all unit vectors X. Consequently,

$$\frac{n^2}{4}C(M, g) \geqslant Ric_g(X, X), \tag{16}$$

for all unit vectors X. Now, if

$$T_g(X, X) > \frac{n^2(5n-9)}{4(n-1)}C(M, g),$$

for all unit vectors X, where

$$T_g = Ric_g + R_g g,$$

we have from (16) that

$$\frac{n^2(n-2)}{n-1}C(M, g) < R_g \leqslant n(n-1)C(M, g).$$

Then, from Theorem 2.1, M^n is diffeomorphic to the standard sphere $\mathbb{S}^n$.

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Appendix A

For the convenience of the reader, we include here the lemmas used in the proof of Theorem 2.1. Lemma A.1 can be found in [11] while the other lemmas can be found in [6]. With the exception of Lemma A.1, we rewrite the proof of them here.

Lemma A.1. *For symmetric matrices $H_1, \dots, H_p$, $p \geqslant 2$, put $S_{\alpha\beta} = tr(H_\alpha H_\beta)$, $S = \sum_{\alpha=1}^{p} S_{\alpha\alpha}$ and $N(H_\alpha) = tr({}^t H_\alpha H_\alpha)$. Then,*

$$\sum_{\alpha,\beta=1}^{p} N(H_\alpha H_\beta - H_\beta H_\alpha) + \sum_{\alpha,\beta=1}^{p} S_{\alpha\beta}^2 \leqslant \frac{3}{2} S^2,$$

and equality holds if and only if one of the following conditions hold:

(1) $H_1 = H_2 = \cdots = H_p = 0$.
(2) *Only two of the matrices $H_1, H_2, \dots, H_p$ are different from zero. Moreover, assuming $H_1 \neq 0$, $H_2 = 0$ and $H_3 = H_4 = \cdots = H_p = 0$ then $S_{11} = S_{22}$ and there exists an orthogonal matrix T such that*

$${}^t T H_1 T = \sqrt{\frac{S_{11}}{2}} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$${}^t T H_2 T = \sqrt{\frac{S_{22}}{2}} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Lemma A.2. *Let $a_1, \dots, a_n$ and b_{ij} for $i, j = 1, \dots, n$ be real numbers satisfying $\sum_{i=1}^{n} a_i = 0$, $\sum_{i=1}^{n} b_{ii} = 0$, $\sum_{i,j=1}^{n} b_{ij}^2 = b$ and $b_{ij} = b_{ji}$ for $i, j = i, \dots, n$. Then*

$$-\left(\sum_{i=1}^{n} b_{ii} a_i\right)^2 + \sum_{i,j=1}^{n} b_{ij}^2 a_i a_j - \sum_{i,j=1}^{n} b_{ij}^2 a_i^2 \geqslant -\sum_{i=1}^{n} a_i^2 b.$$

Proof. We consider the function

$$f(x_{ij}) = -\left(\sum_{i=1}^{n} x_{ii} a_i\right)^2 - \frac{1}{2} \sum_{i,j=1}^{n} x_{ij}^2 (a_i - a_j)^2$$

subjecting to the constraint conditions

$$\sum_{i=1}^{n} x_{ii} = 0 \quad \text{and} \quad \sum_{i,j=1}^{n} x_{ij}^2 = b.$$

By making use of the method of Lagrange multiplier, we shall calculate the minimum of the function $f(x_{ij})$ with these constraint conditions. Let

$$g = f(x_{ij}) + \lambda \sum_{i=1}^{n} x_{ii} + \mu\left(\sum_{i,j=1}^{n} x_{ij}^2 - b\right),$$

where λ and μ are the Lagrange multipliers. We have

$$g = -\left(\sum_{i=1}^{n} x_{ii}a_i\right)^2 - \frac{1}{2}\sum_{i,j=1}^{n} x_{ij}^2(a_i - a_j)^2 + \lambda\sum_{i=1}^{n} x_{ii} + \mu\left(\sum_{i,j=1}^{n} x_{ij}^2 - b\right).$$

If f attains its minimum f_0 at some point (x_{ij}), we have

$$-2\sum_{i=1}^{n} a_i x_{ii} a_j + \lambda + 2\mu x_{jj} = 0 \quad \text{for } j = 1, \dots, n \tag{17}$$

and

$$-x_{ij}(a_j - a_i)^2 + 2\mu x_{ij} = 0 \quad \text{for } i \neq j. \tag{18}$$

Thus, we have

$$-\left(\sum_{i=1}^{n} x_{ii}a_i\right)^2 + \mu\sum_{j=1}^{n} x_{jj}^2 = 0$$

and

$$-\frac{1}{2}\sum_{i,j=1}^{n} x_{ij}^2(a_i - a_j)^2 + \mu\sum_{i,j=1,\ i\neq j}^{n} x_{ij}^2 = 0.$$

From (17) and $\sum_{i=1}^{n} a_i = 0$, we have $\lambda = 0$ and

$$\left(\mu - \sum_{j=1}^{n} a_j^2\right)\sum_{i=1}^{n} x_{ii}a_i = 0,$$

$$\mu\sum_{j=1}^{n} x_{jj}^2 - \left(\sum_{i=1}^{n} x_{ii}a_i\right) = 0.$$

If $\sum_{i=1}^{n} x_{ii}a_i \neq 0$, we have $\mu = \sum_{j=1}^{n} a_j^2$. Hence,

$$f_0 = -\mu b = -\sum_{j=1}^{n} a_j^2 b.$$

If $\sum_{i=1}^{n} x_{ii}a_i = 0$, we have $\mu\sum_{j=1}^{n} x_{jj}^2 = 0$. Note that $\mu = 0$ yields $f_0 = 0$. If $\mu \neq 0$, we have $\sum_{j=1}^{n} x_{jj}^2 = 0$. Hence, $b = 0$ or there exists $i \neq j$ such that $x_{ij} \neq 0$. From (18), we obtain

$$2\mu = (a_i - a_j)^2 \leqslant 2\sum_{j=1}^{n} a_j^2.$$

Therefore,

$$f_0 \geqslant -2\sum_{j=1}^{n} a_j^2 b.$$

Since $\sum_{i=1}^{n} a_i = 0$, $\sum_{i=1}^{n} b_{ii} = 0$, $\sum_{i,j=1}^{n} b_{ij}^2 = b$ and $b_{ij} = b_{ji}$ for $i, j = i, \dots, n$ hold, we find

$$-\left(\sum_{i=1}^{n} b_{ii}a_i\right)^2 + \sum_{i,j=1}^{n} b_{ij}^2 a_i a_j - \sum_{i,j=1}^{n} b_{ij}^2 a_i^2$$
$$= -\left(\sum_{i=1}^{n} b_{ii}a_i\right)^2 - \frac{1}{2}\sum_{i,j=1}^{n} b_{ij}^2 (a_i - a_j)^2 \geqslant -2\sum_{j=1}^{n} a_j^2 b.$$

Thus, we complete the proof of the lemma. □

Lemma A.3. *Let $b_i, i = 1, \dots, n$, be real numbers such that $\sum_{i=1}^{n} b_i = 0$ and $\sum_{i=1}^{n} b_i^2 = B$, where $B = const. \geqslant 0$. Then*

$$\sum_{i=1}^{n} b_i^4 - \frac{B^2}{n} \leqslant \frac{(n-2)^2}{n(n-1)} B^2.$$

Proof. We consider the function

$$f(y) = \sum_{i=1}^{n} y_i^4 - \frac{B^2}{n}$$

with constraint conditions $\sum_{i=1}^{n} y_i = 0$ and $\sum_{i=1}^{n} y_i^2 = B$. Thus, at least one $y_i^2 \geqslant \frac{B}{n}$ for some i. Assume that $y_n^2 \geqslant \frac{B}{n}$. From $\sum_{i=1}^{n} y_i = 0$, we have

$$y_n^2 = \left(\sum_{i=1}^{n-1} y_i\right)^2 \leqslant (n-1)\sum_{i=1}^{n-1} y_i^2 = (n-1)\big(B - y_n^2\big),$$
$$y_n^2 - \frac{B}{2} = \sum_{1 \leqslant i < j \leqslant n-1} y_i y_j$$

and

$$y_n^2 \leqslant \frac{(n-1)B}{n}.$$

Hence,

$$f(y) = \sum_{i=1}^{n-1} y_i^4 + y_n^4 - \frac{B^2}{n}\left(\sum_{i=1}^{n-1} y_i^2\right)^2 - 2\sum_{1 \leqslant i < j \leqslant n-1} y_i^2 y_j^2 + y_n^4 - \frac{B^2}{n}$$
$$\leqslant \big(B - y_n^2\big)^2 - \frac{4}{(n-1)(n-2)}\left(\sum_{1 \leqslant i < j \leqslant n-1} y_i y_j\right)^2 + y_n^4 - \frac{B^2}{n}$$
$$= \frac{2n(n-3)}{(n-1)(n-2)}\big(y_n^4 - By_n^2\big) + \left(\frac{n-1}{n} - \frac{1}{(n-1)(n-2)}\right)B^2.$$

Since the maximum of the function $t^2 - Bt$ in the interval $[\frac{1}{n}B, \frac{n-1}{n}B]$ is $-\frac{n-1}{n^2}B^2$, we obtain

$$f(y) \leqslant \frac{(n-2)^2}{n(n-1)} B^2.$$

Thus, we complete the proof of the lemma. □

Lemma A.4. *Let a_i and b_i for $i = 1, \dots, n$ be real numbers such that $\sum_{i=1}^n a_i = 0$ and $\sum_{i=1}^n a_i^2 = a$, where $a = const. \geqslant 0$. Then*

$$\sum_{i=1}^n a_i b_i^2 \geqslant \sqrt{\sum_{i=1}^n b_i^4 - \frac{(\sum_{i=1}^n b_i^2)^2}{n}} \sqrt{a}.$$

Proof. By making use of the method of Lagrange multiplier, we calculate the minimum of the function $g(x) = \sum_{i=1}^n x_i b_i^2$ with constraint conditions $\sum_{i=1}^n x_i = 0$ and $\sum_{i=1}^n x_i^2 = a$. If the function g attains its minimum g_0 in some point x, then we have, at point x,

$$b_i^2 + \lambda + 2\mu x_i = 0 \quad \text{for } i = 1, \dots, n,$$

where λ and μ are the Lagrange multipliers. Hence, we have

$$g_0 = -2\mu a, \qquad \lambda = -\frac{\sum_{i=1}^n b_i^2}{n},$$
$$\sum_{i=1}^n b_i^4 - \frac{(\sum_{i=1}^n b_i^2)^2}{n} + 2\mu g_0 = 0.$$

Thus, we complete the proof of the lemma. □

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