The structure of power bounded elements in Fourier–Stieltjes algebras of locally compact groups

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Abstract

Let $G$ be an arbitrary locally compact group and $B(G)$ its Fourier–Stieltjes algebra. An element $u$ of $B(G)$ is called power bounded if $\sup_{n \in \mathbb{N}} \|u^n\| < \infty$. We present a detailed analysis of the structure of power bounded elements of $B(G)$ and characterize them in terms of sets in the coset ring of $G$ and $w^*$-convergence of sequences $(v^n)_{n \in \mathbb{N}}, v \in B(G)$.

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1. Introduction

An element $a$ of a Banach algebra $A$ is said to be power bounded if $\sup_{n \in \mathbb{N}} \|a^n\| < \infty$. During the last decades, power bounded elements of Banach algebras in general and, more intensively, those of the Banach algebras of bounded linear operators on Banach spaces in particular, have been investigated by many authors. As samples, we only mention [1,6,17,20,25] and [26]. Much of the work in this area concerns the relation of power boundedness of bounded linear operators

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to properties of their spectra and resolvent sets. In contrast, much less studied so far is the power boundedness of elements in Banach algebras of harmonic analysis, such as $L^1$- and measure algebras and, more generally, Fourier and Fourier–Stieltjes algebras. The purpose of this paper is to continue our investigation, begun in [18] and [19], of power bounded elements in the Fourier–Stieltjes algebra of a general locally compact group $G$.

The Fourier and the Fourier–Stieltjes algebras, $A(G)$ and $B(G)$, of a locally compact group $G$ have been introduced and studied by Eymard in his seminal article [7]. Both are semisimple commutative Banach algebras and, when $G$ is abelian with dual group $\hat{G}$, they are isometrically isomorphic, by means of the Fourier transform, to the group algebra $L^1(\hat{G})$ and the measure algebra $M(\hat{G})$, respectively. Since the appearance of [7], $A(G)$ and $B(G)$ have become central objects of research in harmonic analysis. The wide range of topics, which have been investigated, includes Banach space and Banach algebra properties of $A(G)$ and $B(G)$ induced by properties of the group $G$, various notions of amenability, ideal theory and spectral synthesis, among many other issues.

As far as we are aware of, the history of the power boundedness problem of measures commences with [3], where Beurling and Helson proved that if $\mu$ is an invertible complex Borel measure on $\mathbb{R}$, then $\sup_{n \in \mathbb{Z}} \| \mu^n \| < \infty$ if and only if $\mu = c\delta_s$, where $c \in \mathbb{T}$, $s \in \mathbb{R}$ and $\delta_s$ denotes the Dirac measure at $s$ (see also Section 1.11 of [12]). Related questions have been studied by several other authors (see, for example, [2,4] and [30]).

For general locally compact abelian groups $G$, the most comprehensive work on power bounded elements in the measure algebra $M(G)$ is due to Schreiber [28]. One of his many interesting results gives, for power bounded $\mu \in M(G)$, a precise description of the restriction of $\hat{\mu}$ to the set $E_{\mu} = \{ \gamma \in \hat{G}: |\hat{\mu}(\gamma)| = 1 \}$. Much of the present work is motivated by [28], which contains a wealth of ideas and has been a source of constant inspiration.

We now briefly describe the main content of this paper. To any $u \in B(G)$, associate the sets

$$E_u = \{ x \in G: |u(x)| = 1 \} \quad \text{and} \quad F_u = \{ x \in G: u(x) = 1 \}.$$  

In Section 3 we prove that for any power bounded $u \in B(G)$, $\theta = w^\ast\lim_{n \to \infty} (\frac{1 + u}{2})^n$ exists, $\theta$ is an idempotent and it is the characteristic function of $F_u$, the interior of $F_u$. In particular, $F_u^0$ is closed (Theorem 3.3 and Corollary 3.4). This is used to obtain a very satisfactory structure theorem for power bounded elements (Theorem 4.5): $u \in B(G)$ is power bounded if and only if $u$ can be written as an orthogonal sum $u = u_1 + u_2$, where $w^\ast\lim_{n \to \infty} u_1^n = 0$, $u_2$ vanishes on $G \setminus E_u^0$ and $u_2|_{E_u^0}$ can be described explicitly in terms of sets in the coset ring of $G$ and affine maps. This description and the fact that $u$ is power bounded whenever $w^\ast\lim_{n \to \infty} u_1^n$ exists, naturally raise the question of characterizing those functions in $B(G)$ for which this $w^\ast$-limit exists. In Theorem 5.3 we show that for any $u \in B(G)$, $w^\ast\lim_{n \to \infty} u_1^n = 0$ precisely when $u$ is power bounded and the set $F_{\lambda u} = \{ x \in G: \lambda u(x) = 1 \}$ has empty interior for every $\lambda \in \mathbb{T}$. As a counterpart, the condition that $w^\ast\lim_{n \to \infty} u_1^n$ exists and is nonzero turns out to be equivalent to that $u$ is power bounded and the interior of $F_{\lambda u}$ is nonempty exactly for $\lambda = 1$ (Theorem 5.7).

It is worth mentioning that all of our results admit immediate applications to the measure algebra of a locally compact abelian group, leading mostly to new results even in this special case.

2. Preliminaries

For a commutative Banach algebra $A$, we shall always denote by $\Delta(A)$ the Gelfand spectrum of $A$, equipped with the $w^\ast$-topology, and by $a \to \widehat{a}$, where $\widehat{a}(\gamma) = \gamma(a)$ for $\gamma \in \Delta(A)$, the
Gelfand homomorphism of $A$ into $C_0(\Delta(A))$. Recall that $A$ is said to be regular if given a closed subset $F$ of $\Delta(A)$ and $\gamma \in \Delta(A) \setminus F$, there exists $a \in A$ such that $\hat{a}(\gamma) \neq 0$ and $\hat{a}|_F = 0$. For $a \in A$, let $r(a) = \|\hat{a}\|_\infty = \lim_{n \to \infty} \|a^n\|^{1/n}$ denote the spectral radius of $a$. Note that every power bounded element $a \in A$ satisfies $r(a) \leq 1$, and conversely, if $r(a) < 1$, then $a$ is power bounded [28, Theorem 1.2].

We now introduce our main objects of study, the Fourier–Stieltjes algebra $B(G)$ and the Fourier algebra $A(G)$ of a locally compact group $G$. The space $B(G)$ is the linear span of the set $P(G)$ of all continuous positive definite functions on $G$ and can be identified with the dual space of the group $C^*$-algebra $C^*(G)$. In fact, for every $u \in B(G)$, there exist a unitary representation $\pi$ of $G$ and vectors $\xi$ and $\eta$ in the Hilbert space of $\pi$ such that $u(x) = (\pi(x)\xi, \eta)$ for all $x \in G$. Equipped with pointwise multiplication and the norm $\|u\| = \inf\{\|\xi\| \cdot \|\eta\|\}$, where the infimum is taken over all pairs $(\xi, \eta)$ of such representations of $u$, $B(G)$ is a commutative Banach algebra. The Fourier algebra $A(G)$ is the closed ideal of $B(G)$ generated by all compactly supported functions in $B(G)$. Both algebras, $A(G)$ and $B(G)$, are semisimple. The spectrum of $A(G)$ can be canonically identified with $G$. More precisely, the map $x \to \varphi_x$, where $\varphi_x(u) = u(x)$ for $u \in A(G)$, is a homeomorphism from $G$ onto $\Delta(A(G))$. In this sense, $G$ will always be identified with $\Delta(A(G))$ and also with the corresponding open subset of $\Delta(B(G))$. Note that if $u \in B(G)$ is power bounded, then $|u(x)| \leq 1$ for all $x \in G$, since $r(u) \leq 1$.

The algebra $A(G)$ is regular. Actually, given any closed subset $E$ of $G$ and some $x \in G \setminus E$, there exists $u \in A(G)$ such that $u(x) = 1$, $0 \leq u \leq 1$ and $u = 0$ on $E$. We shall frequently use the fact that multiplication in $B(G)$ is separately $w^*$-continuous, that is, for each $v \in B(G)$ the map $u \to vu$ from $B(G)$ into itself is $w^*$-$w^*$-continuous.

Recall that when $G$ is abelian and $\hat{G}$ denotes the dual group of $G$, then the Fourier–Stieltjes transform furnishes isometric isomorphisms between the measure algebra $M(G)$ and $B(\hat{G})$ and the group algebra $L^1(G)$ and $A(\hat{G})$, respectively. Consequently, every result about $A(G)$ or $B(G)$ for general locally compact groups $G$ entails a corresponding statement for the $L^1$- or measure algebra of a locally compact abelian group, respectively. For all this and more information on $B(G)$ and $A(G)$ we refer the reader to [7] and [24]. Further facts will be mentioned whenever required.

For any group $H$, the coset ring $R(H)$ is the Boolean ring generated by all cosets of subgroups of $H$. If $H$ is a topological group, then the closed coset ring $\mathcal{R}_c(H)$ is defined to be

$$\mathcal{R}_c(H) = \{ E \in R(H) : E \text{ is closed in } H \}.$$ 

For a locally compact abelian group $G$, the sets in $\mathcal{R}_c(G)$ have been described explicitly and independently by Gilbert [10] and Schreiber [29]. Forrest [8] verified that the analogous description is valid for arbitrary locally compact groups $G$. Compare [27] for $R(\mathbb{Z})$.

For any power bounded element $u$ of $B(G)$, the sets $E_u$ and $F_u$ are in the closed coset ring $\mathcal{R}_c(G)$ of $G$ [18]. Actually, we shall see that each set $E_u$ coincides with $F_u$, for another power bounded element $v$ of $B(G)$, namely $v = |u|$.

We have to fix some notation. The locally compact group $G$ will always be equipped with a left invariant Haar measure. If $M$ is any subset of $G$, $1_M$ denotes the characteristic function of $M$ and $|M|$ the measure of $M$ whenever $M$ is measurable. If $A$ is a linear space and $\ell$ a linear functional of $A$, the image of $a \in A$ under $\ell$ is written as $\langle \ell, a \rangle$. Note that, if $f \in L^1(G) \subseteq C^*(G)$ and $u \in B(G) = C^*(G)^*$, then $\langle u, f \rangle = \int_G f(x)u(x) \, dx$. Moreover, for any function $f$ on $G$ and $x \in G$, the left translation $L_x f$ is defined by $L_x f(y) = f(x^{-1}y)$, $y \in G$. 
3. The sequence \((u^n)_{n \in \mathbb{N}}\) and the set \(F_u\)

Let \(G\) be a locally compact group and \(u\) a power bounded element of \(B(G)\). In this section we prove that \(w^*-\lim_{n \to \infty} (1+u)^n\) exists and we determine its limit. We start with two lemmas which are of independent interest.

**Lemma 3.1.** Let \(u \in B(G)\) and suppose that \(\theta = w^*-\lim_{n \to \infty} u^n\) exists. Then

(i) \(\theta\) is an idempotent and satisfies \(\theta u = \theta\). More precisely, \(\theta = 1_{F_u}\).

(ii) The set \(F_u^o\) is closed in \(G\) and the boundary \(\partial F_u = F_u \setminus F_u^o\) is a set of measure zero.

**Proof.** (i) Since multiplication in \(B(G)\) is \(w^*\)-continuous in each variable, we have

\[
\theta = w^*\lim_{n \to \infty} u^{n+1} = u \left( w^*\lim_{n \to \infty} u^n \right) = u\theta.
\]

This equation implies that \(\theta = u^n\theta \) for all \(n \in \mathbb{N}\) and hence \(\theta\) is an idempotent. By Host’s idempotent theorem [15], \(\theta = 1_E\) for some open and closed set in \(R(G)\). Since \(\theta u = \theta\), \(E \subseteq F_u\) and so \(E \subseteq F_u^o\). However, the reverse inclusion also holds. To see this, let \(V \subseteq F_u^o\) be any relatively compact open subset of \(G\). Then

\[
|V| = \langle u^n, 1_V \rangle \to \langle \theta, 1_V \rangle = \int_V \theta(x) \, dx.
\]

Thus \(|V| = \int_V \theta(x) \, dx\) and hence \(V \setminus E\) must have measure zero. This is impossible unless \(V \subseteq E\), since \(E\) is closed in \(G\) and therefore \(V \setminus E\) is open in \(G\). It follows that \(F_u^o \subseteq E\), and this completes the proof of (i).

(ii) By (i), \(F_u^o\) is closed in \(G\). Let \(M\) be a measurable subset of \(G\) of finite positive measure and assume that \(u(x) = 1\) for all \(x \in M\). Consider \(1_M\) as an element of \(L^1(G)\). Since \(u = 1\) on \(M\), we have

\[
|M| = \int_G u^n(x) \, dx \to \int_M \theta(x) \, dx = \int_M \theta(x) \, dx.
\]

As in the proof of (i), it follows that \(M \setminus F_u^o\) is a set of measure zero. Since Haar measure is a regular measure, this in turn implies that \(\partial F_u = F_u \setminus F_u^o\) cannot have positive measure. □

**Lemma 3.2.** Let \(u\) be a power bounded element of \(B(G)\) and suppose that \(\|u^{n+1} - u^n\| \to 0\) as \(n \to \infty\). Then \(\theta = w^*-\lim_{n \to \infty} u^n\) exists.

**Proof.** Since the sequence \((u^n)_{n \in \mathbb{N}}\) is bounded, it has \(w^*\)-convergent subnets. We have to show that there is only one \(w^*\)-cluster point. Suppose that

\[
\theta_1 = w^*-\lim_{\alpha} u^{n_\alpha} \quad \text{and} \quad \theta_2 = w^*-\lim_{\beta} u^{m_\beta}
\]

for subnets \((u^{n_\alpha})_\alpha\) and \((u^{m_\beta})_\beta\) of \((u^n)_n\). The hypothesis that \(\|u^{n+1} - u^n\| \to 0\) combined with the two facts that the norm of \(B(G)\) is lower semi-continuous with respect to the \(w^*\)-topology and that multiplication in \(B(G)\) is \(w^*\)-continuous in each variable when the other is fixed, implies that \(\theta_1 u = \theta_1\) and \(\theta_2 u = \theta_2\). Thus
\[ \theta_1 u^{m_1} = \theta_1 \quad \text{and} \quad \theta_2 u^{n_2} = \theta_2 \]

for all \( \alpha \) and \( \beta \). Passing to \( w^\ast \)-limits, we get that

\[ \theta_1 \theta_2 = \theta_1 \quad \text{and} \quad \theta_2 \theta_1 = \theta_2, \]

and hence \( \theta_1 = \theta_2 \). Consequently, \( \theta = w^\ast - \lim_{n \to \infty} u^n \) exists. \( \square \)

There is some explanation in order on why the condition \( \| u^{n+1} - u^n \| \to 0 \) enters the scene. To \( u \) associate the multiplication operator \( M_u : B(G) \to B(G) \) defined by \( M_u v = uv, v \in B(G) \).

Then \( M_u \) is also power bounded and satisfies \( \| M_u^{n+1} - M_u^n \| = \| u^{n+1} - u^n \| \) for all \( n \in \mathbb{N} \). Now, for power bounded operators \( T \) in Banach spaces, the condition that \( \lim_{n \to \infty} \| T^{n+1} - T^n \| = 0 \) turned out to be very important because, by a result due to Katznelson and Tzafriri [20], it is equivalent to \( \sigma(T) \cap \mathbb{T} \subseteq \{1\} \). In the case \( T = M_u \) for some power bounded element \( u \) of \( B(G) \), it is easy to see that the condition \( \| u^{n+1} - u^n \| \to 0 \) implies that \( \sigma(M_u) \cap \mathbb{T} \subseteq \{1\} \). Indeed, for any \( \gamma \in \Delta(B(G)) \) with \( |\hat{u}(\gamma)| = 1 \), we have

\[ |\hat{u}(\gamma) - 1| = |\hat{u}(\gamma)^{n+1} - \hat{u}(\gamma)^n| \leq \| u^{n+1} - u^n \| \to 0, \]

so that \( \hat{u}(\gamma) = 1 \). In particular, \( E_u = F_u \). Note that, for every power bounded \( u \in B(G) \), the set \( \sigma(M_u) \cap \mathbb{T} \), although not equal to \( E_u \) and \( F_u \), is very closely related to the sets \( E_u \) and \( F_u \). We also remark that the spectrum \( \sigma(M_u) = \hat{u}(\Delta(B(G))) \) of \( M_u \) is in general much bigger than the closure of the set \( u \).  

The preceding two lemmas lead to the following result.

**Theorem 3.3.** Let \( G \) be an arbitrary locally compact group and let \( u \in B(G) \) be such that \( \| u^{n+1} - u^n \| \to 0 \) as \( n \to \infty \). Then the following are equivalent:

(i) \( u \) is power bounded.

(ii) The set \( F_u^0 \) is closed in \( G \), belongs to the coset ring \( \mathcal{R}(G) \) and satisfies \( w^\ast - \lim_{n \to \infty} 1_{G \setminus F_u^0} u^n = 0 \).

**Proof.** Note first that, since \( \| u^{n+1} - u^n \| \to 0 \), the sets \( E_u \) and \( F_u \) coincide by the above remark.

(i) \( \Rightarrow \) (ii) By Lemma 3.2, since \( u \) is power bounded and \( \| u^{n+1} - u^n \| \to 0 \), \( \theta = w^\ast - \lim_{n \to \infty} u^n \) exists, and by Lemma 3.1, \( \theta = 1_{F_u^0} \).

It remains to show that \( 1_{G \setminus F_u^0} u^n \to 0 \) in the \( w^\ast \)-topology. For that, since \( 1_{G \setminus F_u^0} u^n \) is power bounded, it suffices to show that \( \int_{G \setminus F_u} u^n(x) f(x) \, dx \to 0 \) for every \( f \in L^1(G) \). To see this, note first that \( F_u \setminus F_u^0 \) has Haar measure zero by Lemma 3.1. Since \( F_u = E_u \) and \( u(x)^n \to 0 \) for every \( x \in G \setminus E_u \), for each \( f \in L^1(G) \), the Lebesgue’s dominated convergence theorem yields that

\[ \{ 1_{G \setminus F_u^0} u^n, f \} \int_{G \setminus E_u} u^n(x) f(x) \, dx = \int_{G \setminus E_u} u^n(x) f(x) \, dx \to 0 \]

as \( n \to \infty \).

(ii) \( \Rightarrow \) (i) Since \( w^\ast - \lim_{n \to \infty} 1_{G \setminus F_u^0} u^n = 0 \), the function \( 1_{G \setminus F_u^0} u \) is power bounded. Since \( 1_{F_u^0} u \) is an idempotent and \( 1_{F_u^0} u \) and \( 1_{G \setminus F_u^0} u \) are orthogonal, it follows that \( u \) is power bounded. \( \square \)
For any power bounded element \( u \in B(G) \), the element \( \frac{1+u}{2} \) is also power bounded and, as shown and repeatedly used in [18],

\[
\lim_{n \to \infty} \left\| \left( \frac{1+u}{2} \right)^{n+1} - \left( \frac{1+u}{2} \right)^n \right\| = 0.
\]

So, by Lemma 3.2, \( w^*-\lim_{n \to \infty} \left( \frac{1+u}{2} \right)^n \) exists and equals the characteristic function of \( F^0_{\frac{1+u}{2}} = F^0_u \). We record these important facts for use in subsequent sections.

**Corollary 3.4.** For any power bounded \( u \in B(G) \), \( \theta = w^*-\lim_{n \to \infty} \left( \frac{1+u}{2} \right)^n \) exists and \( \theta \) is the characteristic function of the set \( F_{\frac{1+u}{2}} \).

Let \( u \in B(G) \) be power bounded. Then \(|u|\), the absolute value of \( u \), is also power bounded. In fact, since \( \|\bar{\alpha}^n\| = \|u^n\| \) [7, p. 197],

\[
\|u^{2n}\| = \left\| \bar{\alpha}^n u^n \right\| \leq \|\bar{\alpha}^n\| \cdot \|u^n\| \quad \text{and} \quad \|u^{2n+1}\| \leq \|u^n\| \cdot \|u^{2n}\|
\]

for all \( n \in \mathbb{N} \), it follows that \( \sup_{m \in \mathbb{N}} \|u^m\| < \infty \). As \( E_u = F|u| \), the following is a consequence of Lemma 3.1 and Corollary 3.4 applied to \( |u| \).

**Proposition 3.5.** For any power bounded \( u \in B(G) \), the set \( E^0_u \) is closed in \( G \) and \( \partial E_u \) has measure zero.

Now let \( G \) be a connected locally compact group and let \( u \in B(G) \) be power bounded. Then, since the set \( E^0_u \) is open and closed, either \( E_u = G \) or \( E^0_u = \emptyset \). In the first case, as shown in [19, Corollary 3.6], \( u = \lambda \gamma \) for some \( \lambda \in \mathbb{T} \) and some continuous character \( \gamma \) of \( G \). In the second case, we conclude from Proposition 3.5 the following result.

**Corollary 3.6.** Suppose that \( G \) is connected and \( u \in B(G) \) is power bounded such that \( E_u \neq G \). Then \(|u(x)| < 1 \) for almost all \( x \in G \).

We close this section with an application of Corollary 3.6 to abelian groups. The question of when a bounded continuous function on the dual group is the Fourier–Stieltjes transform of some measure is important and difficult.

**Corollary 3.7.** Let \( G \) be a locally compact abelian group with connected dual group, and let \( \varphi \) be a continuous function on \( \hat{G} \) such that \( \|\varphi\|_\infty \leq 1 \). If \( \varphi = \hat{\mu} \) for some power bounded \( \mu \in M(G) \), then \(|\varphi(x)| < 1 \) almost everywhere on \( \hat{G} \).

4. The main structure theorem

In this section we establish the structure theorem, alluded to in the introduction (Theorem 4.5). Together with Theorem 5.3, Theorem 4.5 forms the main body of the paper. These results lead to a precise description of the power bounded elements \( u \) of \( B(G) \) in terms of sets in \( \mathcal{R}_c(G) \), affine maps and the range of \( u \).
Lemma 4.1. Let \( u \in B(G) \) be power bounded and let \( \theta = w^* - \lim_{n \to \infty} (\frac{1+u}{2})^n \). Then we have

(i) \( \langle \theta, f \rangle = \int_{F_u} f(x) \, dx \) for all \( f \in L^1(G) \);
(ii) \( \theta = 0 \) if and only if \( F_u \) has measure zero.

Proof. (i) Let \( v = \frac{1+u}{2} \). It is clear that \( F_v = E_v = F_u \). For any \( f \in L^1(G) \), we therefore have

\[
\langle v^n, f \rangle = \int_{G \setminus F_u} v(x)^n f(x) \, dx + \int_{F_u} f(x) \, dx.
\]

Since \( |v(x)^n f(x)| \leq |f(x)| \) for all \( x \in G \) and \( v(x)^n f(x) \to 0 \) pointwise on \( G \setminus F_u \), the Lebesgue dominated convergence theorem implies that

\[
\langle \theta, f \rangle = \lim_{n \to \infty} \langle v^n, f \rangle = \int_{F_u} f(x) \, dx + \lim_{n \to \infty} \int_{G \setminus F_u} v(x)^n f(x) \, dx = \int_{F_u} f(x) \, dx.
\]

This proves (i), and (ii) is an immediate consequence of (i) since \( u \) is power bounded and \( L^1(G) \) is dense in \( C^*(G) \).

Lemma 4.2. Let \( u \in B(G) \) be power bounded. Then

(i) \( w^* - \lim_{n \to \infty} |u|^n = 0 \) if and only if \( E_u^o = \emptyset \).
(ii) If \( w^* - \lim_{n \to \infty} |u|^n = 0 \), then \( w^* - \lim_{n \to \infty} u^n = 0 \).

Proof. (i) Notice first that \( F_{|u|} = E_u \) and assume that \( E_u^o \neq \emptyset \). Then, for any \( g \in C^+_c(G) \) with \( \text{supp} \, g \subseteq E_u^o \),

\[
\int_G g(x) \, dx = \int_G g(x) |u(x)|^n \, dx = \langle |u|^n, g \rangle.
\]

Thus \( w^* - \lim_{n \to \infty} |u|^n = 0 \) forces \( E_u^o = \emptyset \). Conversely, suppose that \( E_u^o = \emptyset \). Then \( |E_u| = 0 \) by Proposition 3.5, and hence for every \( f \in L^1(G) \),

\[
\langle |u|^n, f \rangle = \int_{G \setminus E_u} |u(x)|^n f(x) \, dx.
\]

As \( |u(x)| < 1 \) for all \( x \in G \setminus E_u \), this integral converges to 0 as \( n \to \infty \). Since \( L^1(G) \) is dense in \( C^*(G) \) and \( |u| \) is power bounded, it follows that \( \langle |u|^n, f \rangle \to 0 \) for all \( f \in C^*(G) \).

Finally, (ii) follows immediately from the power boundedness of \( u \), the denseness of \( L^1(G) \) in \( C^*(G) \) and the fact that

\[
\int_{G} |u(x)|^n f(x) \, dx = \langle |u|^n, |f| \rangle \to 0
\]

for each \( f \in L^1(G) \).

In Lemma 4.2(ii), the reverse implication does not hold. Indeed, compare Section 5 or simply take for \( G \) the circle group \( \mathbb{T} \) and for \( u \) the character \( \gamma(z) = z, z \in \mathbb{T} \).
In general, the multiplication in $B(G)$ is not jointly $w^*$-continuous. It is not even true that $w^*\lim_{n \to \infty} (u^n v^n) = 0$ whenever $w^*\lim_{n \to \infty} u^n = 0$ and $w^*\lim_{n \to \infty} v^n = 0$. In fact, if $G$ is connected and $u$ is a nontrivial character of $G$, then $w^*\lim_{n \to \infty} u^n = w^*\lim_{n \to \infty} r^n = 0$ (compare Section 5), but $u \alpha = 1_G$. Concerning $w^*$-convergence to 0 of the sequence $(u^n v^n)_{n \in \mathbb{N}}$, the following result seems to be the best possible.

**Corollary 4.3.** Let $u$ and $v$ be elements of $B(G)$ such that $w^*\lim_{n \to \infty} |u|^n = 0$ and $v$ is power bounded. Then $w^*\lim_{n \to \infty} (u^n v^n) = 0$.

**Proof.** For any $x \in G$ we have $|u(x)v(x)| = 1$ if and only if $|u(x)| = 1$ and $|v(x)| = 1$. Hence $E_{uv} = E_u \cap E_v$. Since $w^*\lim_{n \to \infty} |u|^n = 0$, the set $E_u$ has empty interior by Lemma 4.2(i). Thus $E_{uv}^o = \emptyset$ and since $uv$ is power bounded, using Lemma 4.2 again, it follows that $w^*\lim_{n \to \infty} (u^n v^n) = 0$. □

If $u \in B(G)$ is power bounded, then the spectral radius $r(u)$ of $u$ is $\leq 1$. The converse fails to hold whenever $G$ is nondiscrete. This can be seen as follows and was also shown in [11] by a somewhat different method. Let $G$ be a nondiscrete locally compact group. Then $G$ contains compact $G_\delta$-sets $K$ with empty interior but positive Haar measure. Employing regularity of $A(G)$ and the fact that $K$ is a $G_\delta$-set, it is easy to find $u \in A(G)$ such that $0 \leq u \leq 1$ and $F_u = K$. Thus $r(u) \leq 1$, but $u$ cannot be power bounded. Indeed, since by hypothesis $|K| > 0$ and $K = \partial F_u$, by Corollary 3.4 and Lemma 3.1 $u$ cannot be power bounded. Thus we have

**Corollary 4.4.** Let $K$ be a compact $G_\delta$-subset of $G$ of positive measure and with empty interior. Then there exists $u \in A(G)$ with $r(u) \leq 1$ and $F_u = K$, but $u$ is not power bounded. In particular, for every nondiscrete locally compact group $G$, there exists $u \in A(G)$ such that $r(u) \leq 1$, but $u$ is not power bounded.

We are now ready for the structure theorem mentioned at the outset of this section.

**Theorem 4.5.** Let $G$ be an arbitrary locally compact group and $u \in B(G)$. Then $u$ is power bounded if and only if $u$ decomposes as $u = u_1 + u_2$, where $u_1, u_2 \in B(G)$ have the following properties:

1. $u_1 u_2 = 0$, $E_{u_1} = \partial E_u$ and $E_{u_2} = E_u^o$.
2. $w^*\lim_{n \to \infty} u_1^n = 0$.
3. There exist pairwise disjoint open sets $F_1, \ldots, F_m$ in $\mathcal{R}(G)$ with $E_u^o = \bigcup_{k=1}^m F_k$, open subgroups $H_k$ of $G$ and $a_k \in G$ such that $F_k \subseteq a_k H_k$, and characters $\gamma_k$ of $H_k$ and $\lambda_k \in \mathbb{T}$, $k = 1, \ldots, m$, such that
   \[ u_2 = \sum_{k=1}^m \lambda_k 1_{F_k} L_{a_k} \gamma_k. \]

In particular, if $u \in B(G)$ is such that $|u(x)| = 1$ for all $x \in G$, then $u$ is power bounded precisely when $u = u_2$ and $\bigcup_{k=1}^m F_k = G$.

**Proof.** If $u = u_1 + u_2$, where $u_1$ and $u_2$ satisfy (1), (2) and (3), then $u$ is power bounded. Indeed, for all $n \in \mathbb{N}$,
$u^n = u_1^n + u_2^n = u_1^n + \sum_{k=1}^m \lambda_k^n 1_{F_k} (L_{a_k} \gamma_k)^n$

and hence $\|u^n\| \leq \|u_1^n\| + m$. Since $w^*\text{-}\lim_{n \to \infty} u_1^n = 0$, $u_1$ is power bounded and hence so is $u$.

Conversely, suppose that $u$ is power bounded. Then, as observed prior to Proposition 3.5, $|u|$ is power bounded and $\theta = w^*\text{-}\lim_{n \to \infty} (\frac{1+|u|}{2})^n$ exists. By Corollary 3.4, $\theta$ is the characteristic function of the set $F^0_u = E^0_u$. Let $u_1 = (1 - \theta)u$ and $u_2 = \theta u$. Then $u_1$ and $u_2$ are power bounded, $u = u_1 + u_2$ and $u_1 u_2 = 0$. Moreover, for $x \in G$,

$$|u_1(x)| = 1 \iff (1 - \theta(x)) \cdot |u(x)| = 1 \iff \theta(x) = 0 \text{ and } |u(x)| = 1,$$

whence $E_{u_1} = (G \setminus E^0_u) \cap E_u = \partial E_u$. In particular $E^0_{u_1} = \emptyset$ and therefore $w^*\text{-}\lim_{n \to \infty} u_1^n = 0$ by Lemma 4.2, (i) and (ii). For $u_2$, we have

$$E_{u_2} = \{ x \in G : \theta(x) |u(x)| = 1 \} = E^0_u.$$ 

Now Theorem 3.4 of [19] implies that there exist $F_k, H_k, a_k, \gamma_k$ and $\lambda_k$, $k = 1, \ldots, m$, as in (3) above such that $u_2(x) = \lambda_k \gamma_k (a_k^{-1}x)$ for all $x \in F_k$. Thus, since $u_2 = 0$ outside of $E^0_u$,

$$u_2 = \sum_{k=1}^m \lambda_k 1_{F_k} (L_{a_k} \gamma_k).$$

This shows that $u_1$ and $u_2$ satisfy (1), (2) and (3). \qed

**Corollary 4.6.** Let $G$ be a connected locally compact group and let $u \in B(G)$. Then $u$ is power bounded if and only if $w^*\text{-}\lim_{n \to \infty} u^n = 0$ or $u = a \gamma$ for some $a \in \mathbb{T}$ and some character $\gamma$ of $G$.

**Proof.** If $u$ is power bounded then, since $G$ is connected, either $E^0_u = G$ or $E^0_u = \emptyset$. In the first case, $u = u_2$ and, by Theorem 4.5(3), there exist $a \in \mathbb{T}$ and a character $\gamma$ of $G$ such that $u(x) = a \gamma(x)$ for all $x \in G$. If $E^0_u = \emptyset$, then $u = u_1$ and $w^*\text{-}\lim_{n \to \infty} u^n = 0$ by Theorem 4.5. The converse is trivial. \qed

It is worth pointing out that the two conditions in Corollary 4.6 don’t exclude each other. For instance any nontrivial character $\gamma$ of the circle group $\mathbb{T}$ satisfies $w^*\text{-}\lim_{n \to \infty} \gamma^n = 0$ (compare Theorem 5.3 below). Theorem 4.5 fairly quickly leads to characterizations of real valued and of positive power bounded functions in $B(G)$.

**Corollary 4.7.** Let $u \in B(G)$ be real valued. Then $u$ is power bounded if and only if $u = u_1 + \theta_1 - \theta_2$, where $w^*\text{-}\lim_{n \to \infty} u_1^n = 0$, $\theta_1$ and $\theta_2$ are idempotents in $B(G)$ and $u_1, \theta_1$ and $\theta_2$ are pairwise orthogonal.

**Proof.** Suppose that $u$ is power bounded and let $\theta = w^*\text{-}\lim_{n \to \infty} (\frac{1+|u|}{2})^n$. As $u = u_1 + (1 - \theta)u$, it is enough to show that $(1 - \theta)u = \theta_1 - \theta_2$ where $\theta_1$ and $\theta_2$ are orthogonal idempotents in $B(G)$. Now $E_{(1-\theta)u} = E^0_u$ and hence, since $u$ is real valued, $(1 - \theta)u$ can attain only the values $1$ and $-1$ on $E^0_u$. This implies that $E^0_u = F^0_u \cup F^0_{-u}$. Let $\theta_1$ and $\theta_2$ be the characteristic function of $F^0_u$ and $F^0_{-u}$, respectively. Then $\theta_1$ and $\theta_2$ are idempotents in $B(G)$ and, since $(1 - \theta)u$ is zero on $G \setminus E^0_u$, we have $(1 - \theta)u = \theta_1 - \theta_2$.

The converse is obvious. \qed
Corollary 4.8. Let $u$ be a positive function in $\mathcal{B}(G)$. Then $u$ is power bounded if and only if $u$ is of the form $u = u_1 + \vartheta$, where $\vartheta = 1_E = 0$ and $w^* \cdot \lim_{n \to \infty} u_1^n = 0$. In this case, $w^* \cdot \lim_{n \to \infty} u^n$ exists.

Proof. Suppose that $u$ is power bounded. Retaining the notation of Corollary 4.7, $\vartheta_2 = 0$ and so $u = u_1 + \vartheta_1$. The statement follows from Theorem 4.5(2). The converse is clear by the uniform boundedness principle.

Remark 4.9. Let $u \in \mathcal{B}(G)$ be power bounded and $u = u_1 + u_2$ as in Theorem 4.5. If $u$ belongs to some ideal $I$ of $\mathcal{B}(G)$, then $u_1, u_2 \in I$ too. Indeed, $u_2 = \vartheta u \in I$ and hence also $u_1 \in I$. In particular, if $u \in A(G)$ then $u_1, u_2 \in A(G)$ although $A(G)$ is not $w^*$-closed in $\mathcal{B}(G)$ and the decomposition of $u$ involves the $w^*$-limit of the sequence $((1 + |u|)/2)_n \in \mathbb{N}$. To conclude this section, we present two applications of Theorem 4.5. The first one characterizes the power bounded elements of the Fourier algebra for a special class of locally compact groups, whereas the second one describes the power bounded elements of $\mathcal{B}(G)$ for certain discrete groups $G$.

Corollary 4.10. Let $G$ be a locally compact group with noncompact connected component of the identity. Then, if $u \in A(G)$, $u$ is power bounded (if and only if $w^* \cdot \lim_{n \to \infty} u^n = 0$.

Proof. Let $u$ be power bounded and let $u = u_1 + u_2$ as in Theorem 4.5. Then $u_2 \in A(G)$ by Remark 4.9. If $u_2 \neq 0$, then in the decomposition of $u_2$, for at least one $k$, $F_k$ is a nonempty open and closed subset of $G$ and hence consists of cosets of $G_0$. But since $u_2$ is of absolute value one on $F_k$ and $u_2 \in A(G)$, $F_k$ has to be compact. This contradiction shows that $u_2 = 0$. Let $G$ be a discrete group and $u, v \in \mathcal{B}(G)$ such that $\|u\|_\infty, \|v\|_\infty < 1$. For brevity, let us say that $u$ and $v$ are equivalent ($u \sim v$) if $u$ and $v$ agree on a cofinite set, that is, $u(x) = v(x)$ for all but finitely many $x \in G$. We claim that if $u \sim v$, then $u$ is power bounded if and only if $v$ is power bounded. To see this, note first that if a function $w$ on $G$ with $\|w\|_\infty \leq 1$ is supported by a finite subset $F$ with $m$ elements, then $w$ is power bounded since

$$\|w^n\|_{\mathcal{B}(G)} \leq \sum_{x \in F} |w(x)|^n \cdot \|w_x^n\|_{\mathcal{B}(G)} \leq m$$

for all $m \in \mathbb{N}$. Now let $E$ be a cofinite subset of $G$ such that $u|_E = v|_E$. Then, for all $n \in \mathbb{N}$,

$$u^n = (u \cdot 1_E)^n + (u \cdot 1_{G \setminus E})^n.$$

It follows that $u$ power bounded if and only if $u \cdot 1_E$ is so, and similarly for $v$. This proves the claim.

Example 4.11. Let $G$ be an infinite discrete group such that every proper subgroup of $G$ is finite and $G$ coincides with its commutator subgroup. Then $u \in \mathcal{B}(G)$ is power bounded if and only if $u$ is equivalent to some $v \in \mathcal{B}(G)$ which is either constant of absolute value one or satisfies $w^* \cdot \lim_{n \to \infty} v^n = 0$. It is clear that if $u$ is equivalent to such a $v$, then $u$ is power bounded. Conversely, let $u$ be power bounded and note first that every $E \in \mathcal{R}(G)$ is either finite or cofinite. This follows easily from the fact that proper subgroups of $G$ are finite and from the following description of sets in $\mathcal{R}(G)$: $E \subseteq G$ belongs to $\mathcal{R}(G)$ if and only if $E$ is of the form
Lemma 5.2. The reader might wonder whether groups as considered in Example 4.11 at all exist. However, due to constructions performed by Olshanskii [23] (see also [22]), there exists a continuum of nonisomorphic groups satisfying the above conditions.

5. Characterizing $w^*$-convergence of the sequence $(u^n)_{n \in \mathbb{N}}$

As noted earlier, if $u \in B(G)$ is such that $w^*-\lim_{n \to \infty} u^n$ exists, then $u$ is power bounded by the uniform boundedness principle. Moreover, in the canonical decomposition (Theorem 4.5) of a power bounded element $u$ of $B(G)$ the summand $u_1$ satisfies $w^*-\lim_{n \to \infty} u^n_1 = 0$. It is therefore of fundamental interest to express the condition that $w^*-\lim_{n \to \infty} u^n$ exists in terms of the range of $u$. The proof of the corresponding result (Theorem 5.3 below) turns out to be fairly involved. We start with two preliminary lemmas.

Lemma 5.1. Let $H$ be an open subgroup of $G$ and let $u \in B(G)$ be power bounded. Then $w^*-\lim_{n \to \infty} u^n = 0$ if and only if $w^*-\lim_{n \to \infty} (L_x u|_H)_n = 0$ for all $x \in G$.

Proof. Since $u$ is power bounded and $C_c(G)$ is dense in $C^*(G)$, $w^*-\lim_{n \to \infty} u^n = 0$ if and only if $(u^n, f) \to 0$ for every $f \in C_c(G)$. The analogous equivalence holds for $L_x u|_H$ since $L_x u|_H$ is also power bounded. If $u^n \to 0$ in the $w^*$-topology, then taking only functions $f \in C_c(H) \subseteq C_c(G)$ in the above, it follows that $(L_x u|_H)_n \to 0$ in the $w^*$-topology of $B(H)$ for every $x \in G$.

For the converse, fix $f \in C_c(G)$ and choose $x_1, \ldots, x_m \in G$ such that $\text{supp} f \subseteq \bigcup_{j=1}^m x_j^{-1} H$ and the cosets $x_j^{-1} H$ are all distinct. Then

$$\langle u^n, f \rangle = \sum_{j=1}^m \langle (u|_{x_j^{-1} H})^n, f|_{x_j^{-1} H} \rangle = \sum_{j=1}^m \langle (L_{x_j} u|_H)^n, x_j^{-1} H \rangle,$$

which tends to 0 as $n \to \infty$. ☐

Lemma 5.2. Let $u$ be a power bounded element of $B(G)$ such that $E_u$ is open in $G$. Then $w^*-\lim_{n \to \infty} u^n = 0$ if and only if $w^*-\lim_{n \to \infty} (1_{E_u} u)^n = 0$.

Proof. As in the proof of Lemma 5.1, because both $u$ and $1_{E_u} u$ are power bounded, the necessity of the condition is obvious, and for the converse we only need to verify that $(u^n, f) \to 0$ for each $f \in C_c(G)$. Since $E_u$ is open and closed, $C_c(G) = C_c(E_u) + C_c(G \setminus E_u)$. By hypothesis, we have $(u^n, g) \to 0$ for every $g \in C_c(E_u)$. Now let $g \in C_c(G \setminus E_u)$ and put $K = \text{supp} \ g$. Then, since $|u(x)| < 1$ for all $x \in G \setminus E_u$ and $K$ is compact, $\|u|_K\|_\infty < 1$. It follows that
\[ \|u^n, g\| \leq \int_K |u(x)|^n |g(x)| \, dx \leq \|u\|_K \|g\|_1, \]

which tends to 0 as \( n \to \infty \). \( \square \)

Let \( u \) be a function on \( G \). In the proof of the next theorem we shall frequently use the following phrase. We say that \( u \) is nowhere locally constant on a subset \( M \) of \( G \) if there is no open subset \( V \) of \( G \) such that \( V \cap M \neq \emptyset \) and \( u \) is constant on \( V \cap M \).

**Theorem 5.3.** Let \( G \) be an arbitrary locally compact group and let \( u \in B(G) \). Then the following conditions are equivalent:

(i) \( w^*\)-\( \lim_{n \to \infty} u^n = 0 \).

(ii) \( u \) is power bounded and the set \( F_{\lambda u} = \{ x \in G : u(x) = \lambda \} \) has empty interior for each \( \lambda \in \mathbb{T} \).

**Proof.** (i) \( \Rightarrow \) (ii) Towards a contradiction, assume that \( u = \lambda \) on some nonempty open subset \( V \) of \( E_u \). Then \( V \subseteq F_{\lambda u} \), and \( w^*\)-\( \lim_{n \to \infty} u^n = 0 \) implies \( w^*\)-\( \lim_{n \to \infty} (\lambda u)^n = 0 \). Consequently, since \( u \) is power bounded,

\[ w^*\lim_{n \to \infty} \left( \frac{1 + \lambda u}{2} \right)^n = 0 \]

(see [19, Proposition 4.1(ii)] or the proof of Theorem 5.7 below). Thus \( F^{\circ}_{\lambda u} = \emptyset \) by Lemma 3.1(i), and this contradiction shows that (ii) holds.

To prove the implication (ii) \( \Rightarrow \) (i), we first show that \( E_u \) may be assumed to be open in \( G \). To that end, recall that the function \( |u| \) is also power bounded and that the limit \( \theta = w^*\lim_{n \to \infty} \left( \frac{1 + |u|}{2} \right)^n \) exists and that \( \theta \) is the characteristic function of the set \( F^{\circ}_{|u|} = E^{\circ}_u \).

Now \( |u| = |(1 - \theta)u| + |\theta u| \) and

\[ E_{|(1-\theta)u|} = \{ x \in G : \theta(x) = 0 \text{ and } |u(x)| = 1 \} = (G \setminus E^{\circ}_u) \cap E_u = \partial E_u, \]

whence \( E^{\circ}_{|(1-\theta)u|} = \emptyset \). Applying Lemma 4.2(i) first and then Lemma 4.2(ii), it follows that \( w^*\lim_{n \to \infty} ((1 - \theta)u)^n = 0 \). As \( (1 - \theta)u \) and \( \theta u \) are orthogonal, we have \( u^n = ((1 - \theta)u)^n + (\theta u)^n \) for all \( n \in \mathbb{N} \) and therefore

\[ w^*\lim_{n \to \infty} u^n = 0 \iff w^*\lim_{n \to \infty} (\theta u)^n = 0. \]

The crucial point here is that \( E_{\theta u} \) is open since \( E_{\theta u} = E_{\theta} \cap E_u = E^{\circ}_u \). Moreover, for \( \lambda \in \mathbb{T} \), since \( \theta = 1_{E^{\circ}_u} \),

\[ F^{\circ}_{\lambda (\theta u)} = \{ x \in G : \theta(x) u(x) = \lambda \} = E^{\circ}_u \cap F^{\circ}_{\lambda u} \subseteq F^{\circ}_{\lambda u} = \emptyset. \]

Thus, considering \( \theta u \) instead of \( u \), we may (and do so) assume that \( E_u \) is open in \( G \).

Let \( G_0 \) denote the connected component of the identity of \( G \). Since \( G/G_0 \) is totally disconnected, we can choose an open subgroup \( H \) of \( G \) such that the group \( H/G_0 \) is compact. Choose \( X \subseteq G \) such that \( X^{-1} \) is a representative system for the left cosets of \( H \) in \( G \). As

\[ E_u = \bigcup_{x \in X} (E_u \cap x^{-1} H) \]
is a decomposition of $E_u$ into open subsets of $G$, $u$ is nowhere locally constant on $E_u$ if and only if $u$ is nowhere locally constant on $E_u \cap x^{-1}H$ for each $x \in X$, or equivalently, $L_xu$ is nowhere locally constant on $H \cap xE_u = EL_xu|_H$ for each $x \in G$. On the other hand, by Lemma 5.1, $w^*\lim_{n \to \infty} u^n = 0$ if and only if $w^*\lim_{n \to \infty} (L_xu|_H)^n = 0$ for all $x \in G$. Combining these two facts, we see that in order to establish the theorem, we may henceforth assume that $G$ is almost connected.

We use the representation given by Theorem 4.5. Since $E_u$ is assumed to be open, we have $u = u_2$. Hence

$$u = \sum_{j=1}^{m} \lambda_j 1_{F_j} L_{a_j} \gamma_j.$$

We claim that for each $j = 1, \ldots, m$, the character $\gamma_j$ of $H_j$ has infinite order. In fact, if $\gamma_j^k = 1_{H_j}$ for some $k \in \mathbb{N}$, then

$$u(x)^k = \lambda_j^k \gamma_j^k (a_j^{-1}x) = \lambda_j^k$$

for all $x \in F_j$. This means that $u$ attains only finitely many values on $F_j$ and hence is locally constant on the open set $F_j$, which is a contradiction.

For $j = 1, \ldots, m$, let $N_j = \{x \in H_j: \gamma_j(x) = 1\}$, the kernel of $\gamma_j$. We show next that $H_j/N_j$ is compact. Then $\gamma_j$ can be viewed as a character of the quotient group $H_j/N_j$ and as such $\gamma_j$ is faithful and $\gamma_j((H_j/N_j)_0)$ is a connected subgroup of the circle group $\mathbb{T}$. Hence $\gamma_j((H_j/N_j)_0)$ is either equal to $\mathbb{T}$ or trivial. If $\gamma_j((H_j/N_j)_0) = \{1\}$, then $G_0 \subseteq N_j$ because $\gamma_j$ is faithful and $(H_j/N_j)_0 \supseteq (G_0N_j)/N_j$. Thus $H_j/N_j$ is compact. If $\gamma_j((H_j/N_j)_0) = \mathbb{T}$, then $\gamma_j$ being an injective continuous homomorphism of the $\sigma$-compact group $(H_j/N_j)_0$ onto $\mathbb{T}$, it is a homeomorphism [14, Theorem 5.29]. Since $H_j/G_0$ is compact and $(G_0N_j)/N_j \subseteq (H_j/N_j)_0$, it follows that $H_j/N_j$ is compact in this case as well.

Now, for any function $f$ on $E_u$, let $f_j$ denote the restriction of $f$ to $F_j$, and for a function $g$ defined on $a_j^{-1}F_j \subseteq H_j$, we denote by $\tilde{g}$ the trivial extension of $g$ to $H_j$, $j = 1, \ldots, m$, that is, $\tilde{g}(x) = g(x)$ for $x \in a_j^{-1}F_j$ and $\tilde{g}(x) = 0$ for $x \in H_j \setminus a_j^{-1}F_j$. Moreover, let $\phi_j : L^1(H_j) \to L^1(H_j/N_j)$ denote the canonical homomorphism, and normalize Haar measures on $N_j$ and on $H_j/N_j$ so that Weil’s formula holds. Then, setting $b_j = a_j^{-1}$, for any $f \in C_c(E_u)$ and $n \in \mathbb{N}$, we have

$$\langle u^n, f \rangle = \sum_{j=1}^{m} \langle u^n |_{F_j}, f_j \rangle = \sum_{j=1}^{m} \int_{F_j} u^n(x) f_j(x) \, dx$$

$$= \sum_{j=1}^{m} \int_{b_jF_j} u^n(a_jx) L_{b_j} f_j(x) \, dx$$

$$= \sum_{j=1}^{m} \lambda^n_j \int_{b_jF_j} \gamma_j(x)^n L_{b_j} f_j(x) \, dx$$

$$= \sum_{j=1}^{m} \lambda^n_j \int_{H_j} \gamma_j(x)^n L_{b_j} f_j(x) \, dx$$
\[
\sum_{j=1}^{m} \lambda_j^n \int_{H_j/N_j} \gamma_j(xN_j)^n \left( \int_{\tilde{L}_j} f_j(xn) \, dn \right) d(xN_j)
\]

= \sum_{j=1}^{m} \lambda_j^n \phi_j(\tilde{L}_j f_j)(\gamma_j^{-n}) - \sum_{j=1}^{m} \lambda_j^n \int_{H_j/N_j} \gamma_j(xN_j) \left( \int_{\tilde{L}_j} f_j(xn) \, dn \right) d(xN_j)

Since the function \( \phi_j(\tilde{L}_j f_j) \) is in \( L^1(H_j/N_j) \), actually in \( C_c(H_j/N_j) \), its Fourier transform vanishes at infinity on the discrete dual group \( \hat{H}_j/N_j \) of \( H_j/N_j \). Because \( \gamma_j \) has infinite order, \( \hat{\phi}_j(\tilde{L}_j f_j)(\gamma_j^{-n}) \) is arbitrarily small for \( n \) large enough. As this holds for each \( j \), it follows that \( \lim_{n \to \infty} \langle u^n, f \rangle = 0 \). Finally, since \( u \) is power bounded and \( E_u \) is open in \( G \), an application of Lemma 5.2 shows that (i) holds.

This completes the proof of the theorem. \( \Box \)

To show the implication (i) \( \Rightarrow \) (ii) of the preceding theorem we have employed [19, Proposition 4.1], the proof of which was fairly involved using the Ishikawa iteration process [16]. It is possible, though somewhat lengthier, to verify (i) \( \Rightarrow \) (ii) using arguments similar to those in the proof of (ii) \( \Rightarrow \) (i), thus avoiding the use of [19, Proposition 4.1].

**Remark 5.4.** It seems worthwhile to point out that the condition \( F_{j,u} = \emptyset \) for each \( \lambda \in \mathbb{T} \) in Theorem 5.3 is equivalent to the condition that \( u \) is nonconstant on every open coset which is contained in \( E_u \). Clearly, if \( a \in G \) and \( H \) is an open subgroup of \( G \) such that \( aH \subseteq E_u \) and \( u = \lambda \) on \( aH \), then \( aH \subseteq F_{j,u} \).

Conversely, suppose that \( F_{j,u} \neq \emptyset \) for some \( \lambda \in \mathbb{T} \) and retain the description \( u_2 = \sum_{k=1}^{m} \lambda_k \times 1_{F_k}(L_{a_k} \gamma_k) \) in Theorem 4.5. Then \( V = F_{j,u} \cap F_k \neq \emptyset \) for some \( 1 \leq k \leq m \), and \( u \) is constant on \( V \). Since \( a_k^{-1}V \subseteq H_k \), we can choose \( b_k \in H_k \) so that \( W = (a_k b_k)^{-1}V \) is a neighborhood of the identity in \( H_k \). Then \( (WG_0)/G_0 \) is a neighborhood of \( G_0 \) in \( G/G_0 \), and hence we find an open subgroup \( K \) of \( G \) contained in \( WG_0 \).

Now the function \( x \to u(x) = \lambda_k \gamma_k(a_k^{-1}x) \) is constant on \( V \) and therefore, as in the proof of Theorem 5.3, there exists an open subgroup \( L \) of \( H_k \) such that \( \gamma_k = 1 \) on \( L \). Let \( M = L \cap K \). Then \( (a_k b_k)M \subseteq (a_k b_k)WG_0 = VG_0 \subseteq F_k G_0 = F_k \), and since \( u(a_k b_k x) = \lambda_k \gamma_k(b_k x) = \lambda_k \gamma_k(b_k) \) for all \( x \in M \subseteq L \), \( u \) is constant on the coset \( a_k b_k M \) of the open subgroup \( M \) of \( G \).

We now deduce from Theorem 4.5 and Theorem 5.3 that for every power bounded element of \( B(G) \), there is a largest open and closed set \( E \) in \( \mathcal{R}(G) \) such that \( w^* - \lim_{n \to \infty} (1_E u)^n = 0 \). Equivalently, by Host’s idempotent theorem [15], there is a largest projection \( \eta \) in \( B(G) \) such that \( w^* - \lim_{n \to \infty} (\eta u)^n = 0 \).

**Proposition 5.5.** Let \( u \in B(G) \) be power bounded. Retaining the notation of Theorem 4.5, let \( E \) be the open and closed set in \( \mathcal{R}(G) \) defined by

\[ E = \bigcup \{ F_k : 1 \leq k \leq m, \ker \gamma_k \text{ nonopen in } G \} \cup (G \setminus E_u) \]
Then $w^*-\lim_{n \to \infty}(1_E u)^n = 0$, and if $F$ is any open and closed set in $\mathcal{R}(G)$ such that $w^*-\lim_{n \to \infty}(1_F u)^n = 0$, then $F \subseteq E$.

**Proof.** Fix $k \in \{1, \ldots, m\}$, suppose that $\ker \gamma_k$ is not open in $G$, i.e. $F_k \subseteq E$, and let $v = 1_{F_k} u$. We claim that $w^*-\lim_{n \to \infty} v^n = 0$. By Theorem 5.3, we have to show that $F^*_\lambda v = \emptyset$ for every $\lambda \in \mathbb{T}$. Towards a contradiction, assume that $F^*_\lambda v \neq \emptyset$ for some $\lambda \in \mathbb{T}$. Now

$$F^*_\lambda v = \{ x \in F_k : \lambda u(x) = 1 \}^0 = F_k \cap F^*_\lambda u,$$

and for $x \in F_k \cap F^*_\lambda u \subseteq a_k H_k$, we have

$$\gamma_k(a_k^{-1} x) = \overline{\lambda_k} u(x) = \overline{\lambda_k} \lambda u(x) = \overline{\lambda_k} \lambda.$$

So $\gamma_k$ is constant on the nonempty open set $a_k^{-1} F^*_\lambda u$ of $H_k$, which implies that $\ker \gamma_k$ is open, contradicting the hypothesis that $F_k \subseteq E$. Hence, as claimed, $w^*-\lim_{n \to \infty}(1_{F_k} u)^n = 0$. This being true for all $k$ with $F_k \subseteq E$ and since $1_{G \setminus E_u} = u_1$, we also have $w^*-\lim_{n \to \infty}(1_{G \setminus E_u} u)^n = 0$. We conclude that $w^*-\lim_{n \to \infty}(1_E u)^n = 0$.

Now let $F \in \mathcal{R}(G)$ be any nonempty open and closed set such that $w^*-\lim_{n \to \infty}(1_E u)^n = 0$. In order to show that $F \subseteq E$, replacing $F$ by $F \cap F_k$, we may assume that $F \subseteq F_k$ for some $k$. By Theorem 5.3 again, we know that $F^*_\lambda(1_{F_k}) = \emptyset$ for every $\lambda \in \mathbb{T}$. If $\ker \gamma_k$ is open in $G$, then the set $\{ y \in H_k : \gamma_k(y) = \alpha \}$ is open in $G$ for each $\alpha \in \mathbb{T}$. Then, since $F$ nonempty and $F \subseteq F_k \subseteq a_k H_k$, the set

$$S = F \cap a_k \{ y \in H_k : \gamma_k(y) = \alpha \}$$

is nonempty for some $\alpha$. However, for $x \in S$, $u(x) = \lambda_k \gamma_k(a_k^{-1} x) = \lambda_k \alpha$, so that $F^*_\lambda \alpha(1_{F_k}) \neq \emptyset$, a contradiction. It follows that $\ker \gamma_k$ is not open and hence $F \subseteq F_k \subseteq E$ by the definition of $E$. \qed

The next proposition provides some useful information about the $w^*$-cluster points of the sequence $(u^n)_{n \in \mathbb{N}}$.

**Proposition 5.6.** Suppose that $u \in B(G)$ is power bounded. Then either $w^*-\lim_{n \to \infty} u^n = 0$ or none of the $w^*$-cluster points of $(u^n)_{n \in \mathbb{N}}$ is zero.

**Proof.** Suppose that the sequence $(u^n)_{n \in \mathbb{N}}$ does not converge to 0 in the $w^*$-topology. Then, by Theorem 5.3, $F^*_{\lambda u} \neq \emptyset$ for some $\lambda \in \mathbb{T}$. Hence, by Lemma 3.1, the idempotent

$$\theta_\lambda = w^*-\lim_{n \to \infty} \left( \frac{1 + \lambda u}{2} \right)^n = 1_{F^*_{\lambda u}}$$

is nonzero. Moreover, it satisfies $u \theta_\lambda = \overline{\lambda} \theta_\lambda$ and hence $u^n \theta_\lambda = (\overline{\lambda})^n \theta_\lambda$ for all $n \in \mathbb{N}$. Since $|\lambda| = 1$, this last equation shows that 0 cannot be a cluster point of $(u^n)_{n \in \mathbb{N}}$. \qed

We close this section by characterizing those $u \in B(G)$ for which $w^*-\lim_{n \to \infty} u^n$ exists, but in contrast to Theorem 5.3, is nonzero.

**Theorem 5.7.** Let $G$ be an arbitrary locally compact group. For $u \in B(G)$, the following are equivalent:

1. $\theta_\lambda = 0$ does not exist for any $\lambda \in \mathbb{T}$.
2. $u^n \theta_\lambda = (\overline{\lambda})^n \theta_\lambda$ for all $n \in \mathbb{N}$.
3. $\theta_\lambda$ is not zero for any $\lambda \in \mathbb{T}$.
4. $u^n \theta_\lambda = (\overline{\lambda})^n \theta_\lambda$ for all $\lambda \in \mathbb{T}$ and $n \in \mathbb{N}$.
(i) \( w^*\)-\( \lim_{n \to \infty} u^n \) exists and is different from zero.
(ii) \( u \) is power bounded and \( F^0_{\lambda u} \neq \emptyset \) if and only if \( \lambda = 1 \).

**Proof.** (i) \( \Rightarrow \) (ii) Since \( \theta = w^*\)-\( \lim_{n \to \infty} u^n \) exists, \( u \) is power bounded by the uniform boundedness principle. We claim that also \( \theta = w^*\)-\( \lim_{n \to \infty} (\frac{1+u}{2})^n \). Though the proof is analogous to the one given in Proposition 4.1(ii) of \([19]\), we include it for convenience. For \( n, k \in \mathbb{N}_0 \), let \( c_k(n) = 2^{-n(n^2)} \). Then

\[
\begin{align*}
(1) \sum_{k=0}^{\infty} c_k(n) &= \sum_{k=0}^{n} c_k(n) = 1 \text{ for all } n \in \mathbb{N}_0; \\
(2) \lim_{n \to \infty} c_k(n) &= 0 \text{ for each } k \in \mathbb{N}_0.
\end{align*}
\]

Since the sequence \( (u^n)_{n \in \mathbb{N}} \) is \( w^*\)-convergent and

\[
\left( \frac{1+u}{2} \right)^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} u^k = \sum_{k=0}^{\infty} c_k(n) u^k,
\]

it follows that, for all \( f \in C^*(G) \),

\[
\left( \lim_{n \to \infty} \left( \frac{1+u}{2} \right)^n \right) f = \lim_{n \to \infty} \sum_{k=0}^{\infty} c_k(n) \langle u^k, f \rangle.
\]

Now (1) and (2) show that the summation method defined by the doubly infinite matrix with entries \( c_k(n) \) is ‘regular’ in the sense of summation theory. It then follows from (3), \( w^*\)-\( \lim_{n \to \infty} u^n = \theta \) and the Toeplitz summation theorem (see \([31]\)) that

\[
\lim_{n \to \infty} \left( \left( \frac{1+u}{2} \right)^n, f \right) = \langle \theta, f \rangle
\]

for each \( f \in C^*(G) \), as was to be shown.

Thus \( \theta = 1_{F^0_u} \) by Corollary 3.4. Put \( v = (1-\theta)u \). Then, since multiplication in \( B(G) \) is separately \( w^*\)-continuous,

\[
w^*\lim_{n \to \infty} v^n = (1-\theta) \cdot w^*\lim_{n \to \infty} u^n = (1-\theta)\theta = 0.
\]

Theorem 5.3 implies that \( F^0_{\lambda v} = \emptyset \) for all \( \lambda \in \mathbb{T} \). Now, for \( \lambda \neq 1 \),

\[
F_{\lambda v} = \{ x \in G : (1-\theta(x))u(x) = \lambda \} = (G \setminus F^0_u) \cap F_{\lambda u} = F_{\lambda u}
\]

and hence \( F^0_{\lambda v} = F^0_{\lambda u} = \emptyset \) for every \( \lambda \in \mathbb{T} \), \( \lambda \neq 1 \). If also \( F^0_u = \emptyset \), then Theorem 5.3 implies that \( w^*\)-\( \lim_{n \to \infty} u^n = 0 \), a contradiction. Thus (ii) holds.

Conversely, suppose that (ii) holds. Since \( u \) is power bounded, \( \theta = w^*\)-\( \lim_{n \to \infty} (\frac{1+u}{2})^n \) exists (by Corollary 3.4) and satisfies \( \theta u = \theta \). Then \( \theta \neq 0 \) since \( \theta = 1_{F^0_u} \) and \( F^0_u \neq \emptyset \). As above, let \( v = (1-\theta)u \), so that \( F^0_{\lambda v} = F^0_{\lambda u} = \emptyset \) for \( \lambda \in \mathbb{T} \), \( \lambda \neq 1 \). Since \( F_{v} = (G \setminus F^0_u) \cap F_{u} = \emptyset \), \( F^0_{\lambda v} = F^0_{\lambda u} = \emptyset \) also. It follows from Theorem 5.3 that \( w^*\)-\( \lim_{n \to \infty} v^n = 0 \). Now, since \( \theta u = \theta \),

\[
v^n = (1-\theta)u^n = u^n - \theta u^n = u^n - \theta
\]

and therefore \( w^*\)-\( \lim_{n \to \infty} u^n = \theta + w^*\)-\( \lim_{n \to \infty} v^n = \theta \neq 0 \). \( \square \)

**6. Some problems**

We close this paper by listing a few problems.
Problem 6.1. Let $G$ be a locally compact group and $u$ a power bounded element of $B(G)$. Then $F_u$ belongs to $\mathcal{R}_c(G)$, and hence $F_u$ is a set of synthesis for $A(G)$ whenever $G$ is amenable [9, Lemma 2.2]. Is the set $F_u$ always a set of synthesis?

Problem 6.2. Let $G$ be a locally compact group and recall that, for any power bounded element $u \in B(G)$, $E_u = F_{|u|}$. Let

$$\mathcal{F}_{pb}(G) = \left\{ F \subseteq G : F = F_v \text{ for some power bounded } v \in B(G) \right\} = \left\{ E \subseteq G : E = E_u \text{ for some power bounded } u \in B(G) \right\}.$$

We know from [18, Theorem 4.1] that $\mathcal{F}_{pb}(G) \subseteq \mathcal{R}_c(G)$. The naturally arising question of which sets in $\mathcal{R}_c(G)$ are of the form $F_u$, has been addressed in Section 6 of [28] for locally compact abelian groups, but appears to be far from admitting a complete solution even in this case.

Problem 6.3. Let $H$ be a locally compact abelian group with dual group $\widehat{H}$. By the structure theory of locally compact abelian groups, $H$ is compactly generated if and only if the connected component $\widehat{H}_0$ of the identity of $\widehat{H}$ is open in $\widehat{H}$ and a Lie group [14, Theorem 9.8]. Taking $G = \widehat{H}$ (equivalently, by the Pontryagin duality theorem, $H = \widehat{G}$), Theorem 6.28 of [28] on $L^1$-algebras can then be reformulated as follows. Let $G$ be a locally compact abelian group such that $G_0$ is a Lie group and open in $G$. Then, for any closed subgroup $H$ of $G$, each power bounded function in $A(H)$ extends to a power bounded function in $A(G)$. This raises the problem of whether such an extension result is true for a wider class of locally compact groups. In this context, we remind the reader that given any closed subgroup $H$ of an arbitrary locally compact group $G$, one always has that $A(G)|_H = A(H)$ (see [21]), whereas in general $B(G)|_H$ is strictly contained in $B(H)$. In fact, as shown independently in [5] and [13], $B(G)|_H = B(H)$ if and only if $G$ has a neighborhood basis of the identity consisting of sets $V$ such that $h^{-1}Vh = V$ for all $h \in H$.

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References