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Advances in Mathematics 224 (2010) 827–862

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Jeu de taquin and a monodromy problem for Wronskians of polynomials

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Received 31 March 2009; accepted 16 December 2009

Available online 22 January 2010

Communicated by Ravi Vakil

Abstract

The Wronskian associates to d linearly independent polynomials of degree at most n, a non-zero polynomial of degree at most $d(n-d)$. This can be viewed as giving a flat, finite morphism from the Grassmannian $\mathrm{Gr}(d,n)$ to projective space of the same dimension. In this paper, we study the monodromy groupoid of this map. When the roots of the Wronskian are real, we show that the monodromy is combinatorially encoded by Schützenberger's jeu de taquin; hence we obtain new geometric interpretations and proofs of a number of results from jeu de taquin theory, including the Littlewood–Richardson rule.

Keywords: Schubert calculus; Grassmannian; Wronskian; Jeu de taquin; Geometric Littlewood–Richardson rule

1. Introduction

1.1. The Wronski map

For any non-negative integer m, let $\mathbb{F}_m[z]$ denote the $(m+1)$-dimensional vector space of polynomials of degree at most m over a field $\mathbb{F}$:

$$\mathbb{F}_m[z] := \left\{ f(z) \in \mathbb{F}[z] \;\middle|\; \deg f(z) \leqslant m \right\}.$$

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0001-8708/$ – see front matter
doi:10.1016/j.aim.2009.12.013

Throughout, we fix integers $0 < d < n$. Let $X := \mathrm{Gr}_d(\mathbb{C}_{n-1}[z])$ be the Grassmannian whose points represent d-dimensional linear subspaces of $\mathbb{C}_{n-1}[z]$. Let $N := d(n-d) = \dim X$ be its dimension.

Given polynomials $f_1(z), \dots, f_d(z) \in \mathbb{C}_{n-1}[z]$, the Wronskian

$$\mathrm{Wr}_{f_1,\dots,f_d}(z) := \begin{vmatrix} f_1(z) & \cdots & f_d(z) \\ f_1'(z) & \cdots & f_d'(z) \\ \vdots & \vdots & \vdots \\ f_1^{(d-1)}(z) & \cdots & f_d^{(d-1)}(z) \end{vmatrix}$$

is a polynomial of degree at most N. If $f_1, \dots, f_d$ are linearly dependent, the Wronskian is zero; otherwise up to a constant multiple, $\mathrm{Wr}_{f_1,\dots,f_d}(z)$ depends only on the linear span $\langle f_1(z), \dots, f_d(z)\rangle \subset \mathbb{C}_{n-1}[z]$. Thus the Wronskian gives a well defined morphism of schemes $\mathrm{Wr}: X \to \mathbb{P}(\mathbb{C}_N[z])$, called the ***Wronski map***. For $x \in X$ we write $\mathrm{Wr}(x; z)$ for any representative of $\mathrm{Wr}(x)$ in $\mathbb{C}_N[z]$.

This morphism turns out to be extremely well behaved. It appears in algebraic geometry in a number of different guises. In the context of enumerating rational curves with prescribed ramifications, Eisenbud and Harris proved the following theorem [5]:

Theorem 1.1. $\mathrm{Wr}: X \to \mathbb{P}(\mathbb{C}_N[z])$ *is a flat, finite morphism of schemes.*

A point $x \in X$ is ***real*** if the subspace of $\mathbb{C}_{n-1}[z]$ represented by x has a basis $f_1(z), \dots, f_d(z) \in \mathbb{R}_{n-1}[z]$. In 1995, B. Shapiro and M. Shapiro made a remarkable conjecture concerning the reality of the fibres of $\mathrm{Wr}(x; z)$, which has been a source of inspiration for much of the work relating to the Wronski map. The conjecture (as refined by Sottile [18]) has two parts, the first of which is given below and was proved in two papers by Mukhin, Tarasov and Varchenko [13,14] (see also [8]).

Theorem 1.2. *Let* $g(z) \in \mathbb{R}_N[z]$ *be a polynomial with* N *distinct real roots. Then the fibre* $\mathrm{Wr}^{-1}(g(z))$ *is reduced and every point in the fibre is real.*

Although the reality of the fibres is prominent in their proof, the more pertinent fact for us is that these fibres are reduced; the reality statement is a relatively simple consequence of this [17]. The second part of the Shapiro–Shapiro conjecture concerns the multiplicities of the fibre when the roots of $g(z)$ are real but not distinct (see Remark 2.7).

In this paper, we study the monodromy groupoid of the Wronski map over the base of points where the fibre is reduced. Specifically we will be looking at a subgroupoid, which describes the lifting of certain interesting paths and loops. Our main goal is to show that these liftings are fundamentally related to Schützenberger's jeu de taquin [16]. Through this relationship, we will see that much of the combinatorial structure in jeu de taquin theory can be attributed to the geometric structure of the Wronski map.

1.2. Outline of paper

It is a classical result, originating with work of Castelnuovo [2], that the fibres of the Wronski map can be interpreted as intersections of Schubert varieties. We review this and other relevant background material in Section 2. From this interpretation, one can see that the degree of the map Wr is given by counting standard Young tableaux whose shape is a $d \times (n-d)$ rectangle, a calculation which dates back to Schubert [15]. We denote the set of all such tableaux by $\mathsf{SYT}(\Box)$.

Eremenko and Gabrielov [6] showed that for suitable base points in $\mathbb{P}(\mathbb{C}_N[z])$, there is in fact a natural way to index the points in the fibre of Wr by $\mathsf{SYT}(\Box)$. Using Theorem 1.2, the notion of a suitable base point can be extended to any polynomial with N distinct real roots. We will give a generalised and more explicit reformulation of this correspondence, which will allow us to describe the monodromy for certain loops and paths in $\mathbb{P}(\mathbb{C}_N[z])$ in terms of tableaux. To facilitate such a description, it will be helpful to modify our notion of standard Young tableau slightly, to allow entries in a field $\mathbb{F}$ with a norm. As explained in Section 3, these enhancements allow us to speak of paths of tableaux, which, when $\mathbb{F} = \mathbb{R}$, can be viewed as a mild extension of jeu de taquin.

In Section 4, we state and establish our formulation of the correspondence. Briefly, this works as follows: the Plücker coordinates of a point $x \in X$ are described in terms a tableau whose entries are the roots of $\mathrm{Wr}(x; -z)$. If we work over the field of Puiseux series $\mathbb{C}\{\{u\}\}$, the tableau tells us the leading terms of the Plücker coordinates; over the complex numbers, this becomes an approximation. Our approach is related to the types of arguments found in [6,17], in that it can be interpreted as an asymptotic analysis over the real or complex numbers.

Using this correspondence, we can identify certain paths of tableaux with paths in X. The most important example of this directly relates the monodromy problem to jeu de taquin theory. We will show that for paths in $\mathbb{P}(\mathbb{C}_N[z])$ of polynomials whose roots are all real, the monodromy of Wr is described (in the sense outlined in Section 3) by a sequence of Schützenberger slides. This result is formulated in Section 3.2, and proved in Section 5.

A secondary example, also discussed in Section 5, is the following. For any positive integers k, L such that $1 \leqslant k < N$, and $L \geqslant 2$, we can define a permutation $s_{k,L} : \mathsf{SYT}(\Box) \to \mathsf{SYT}(\Box)$, as follows. For $\mathsf{T} \in \mathsf{SYT}(\Box)$, $s_{k,L}(\mathsf{T})$ is the tableau obtained by swapping entries k and $k+1$ in T, if the total of the horizontal and vertical distance between k and $k+1$ equals L; otherwise $s_{k,L}(\mathsf{T}) = \mathsf{T}$. We will show that there exist loops in $\mathbb{P}(\mathbb{C}_N[z])$, such that the monodromy of Wr is given by $s_{k,L}$.

These two results allow us, in Section 6, to give geometric interpretations and proofs of a number of combinatorial theorems involving jeu de taquin. Among these is the Littlewood–Richardson rule. Our geometric interpretation of the Littlewood–Richardson rule is notably different from those of Vakil [20] and Coskun [3]: whereas their approaches involve degenerations of an intersection of two Schubert varieties, we begin by considering a general fibre of the Wronski map, which can be regarded as an intersection of N Schubert varieties, and degenerating to a special fibre, supported on a union of intersections of Schubert varieties (cf. (2.4)). We deduce the Littlewood–Richardson rule by showing that the combinatorics keeps track of multiplicities in each individual intersection of Schubert varieties comprising this union.

2. Background on the Wronski map

2.1. *Roots of the Wronskian and* $\mathrm{SL}_2(\mathbb{C})$*-action*

If $\mathbf{a}$ is a multiset and S is a set, we say $\mathbf{a}$ is a ***multisubset*** of S and write $\mathbf{a} \Subset S$ if every element of $\mathbf{a}$ is an element of S. We write $\mathbf{a} \subset S$ if every element of $\mathbf{a}$ has multiplicity 1, i.e. $\mathbf{a}$ is a set.

As is suggested by Theorem 1.2, it will be convenient to regard $\mathrm{Wr}(x; z)$ in terms of the multiset of its roots. If the degree of $\mathrm{Wr}(x; z)$ is strictly less than N, we will think of $\mathrm{Wr}(x; z)$ as having $N - \deg \mathrm{Wr}(x; z)$ roots at infinity. If $\mathrm{Wr}(x; z) = \prod_{i=1}^{k}(z+a_i)$, let $\pi(x) := \{a_1, \dots, a_N\} \Subset \mathbb{CP}^1$, viewed as a multiset, where $a_{k+1} = \cdots = a_N = \infty$ if $k < N$. Thus $\pi(x)$ is the multiset of roots of $\mathrm{Wr}(x; -z)$.

The group $\mathrm{SL}_2(\mathbb{C})$ acts on everything. If $\phi = \left(\begin{smallmatrix}\phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22}\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$, we have the usual action on $\mathbb{CP}^1$,

$$\phi(w) := \frac{\phi_{11}w + \phi_{12}}{\phi_{21}w + \phi_{22}}$$

for $w \in \mathbb{CP}^1$, and hence an action on multisubsets of $\mathbb{CP}^1$. On $\mathbb{C}_m[z]$, we define the action as follows:

$$\phi f(z) := (\phi_{21}z + \phi_{11})^m f\left(\frac{\phi_{22}z + \phi_{12}}{\phi_{21}z + \phi_{11}}\right)$$

for $f(z) \in \mathbb{C}_m[z]$. The action on $\mathbb{C}_{n-1}[z]$ induces an action on X. With these definitions, the following proposition is straightforward to check.

Proposition 2.1. *For $\phi \in \mathrm{SL}_2(\mathbb{C})$ and $x \in X$ we have, $\phi(\pi(x)) = \pi(\phi(x))$.*

We will use the following notation to describe the fibres of the Wronski map. For a multiset $\mathbf{a} = \{a_1, \dots, a_N\} \Subset \mathbb{CP}^1$, let $X(\mathbf{a}) := \pi^{-1}(\mathbf{a}) = \{x \in X \mid \pi(x) = \mathbf{a}\}$. Thus $X(\mathbf{a})$ is the fibre of the map Wr at the point $\prod_{a_i \neq \infty}(z + a_i)$. If $\mathbf{a}_t$, $t \in [0, 1]$, is a path in the space of N-element multisubsets of $\mathbb{CP}^1$ such that the fibre $X(\mathbf{a}_t)$ is reduced for all $t \in [0, 1]$, we write $x_t \in X(\mathbf{a}_t)$ to describe a lifting of this path to X. If x_0 is specified, this lifting is unique. In particular, we associate to each $x_0 \in X(\mathbf{a}_0)$ a point $x_1 \in X(\mathbf{a}_1)$. The ***monodromy*** of the path $\mathbf{a}_t$ is the bijection $X(\mathbf{a}_0) \to X(\mathbf{a}_1)$ defined by this process.

When the roots of the Wronskian are real, we will generally restrict the action of $\mathrm{SL}_2(\mathbb{C})$ to the subgroup $\mathrm{SL}_2(\mathbb{R})$, as exemplified in the following important corollary of Theorem 1.2.

Corollary 2.2. *Let $\mathbf{a}_t$, $t \in [0, 1]$ be a loop in the space of N-element subsets of $\mathbb{RP}^1$. Suppose there exists some $w \in \mathbb{RP}^1$ such that $w \notin \mathbf{a}_t$ for all $t \in [0, 1]$. Then the monodromy of $\mathbf{a}_t$ is trivial, i.e. the identity map.*

Proof. First suppose $w = \infty$. Let $Z \subset \mathbb{P}(\mathbb{C}_N[z])$ be the topological subspace of polynomials with exactly N distinct real roots. Then $\mathbf{a}_t$ encodes a path in Z, which is a simply connected space. Since the fibres of the map $\mathrm{Wr}: \mathrm{Wr}^{-1}(Z) \to Z$ are reduced by Theorem 1.2, the monodromy is necessarily trivial.

For other w, there exists $\phi \in \mathrm{SL}_2(\mathbb{R})$ such that $\phi(w) = \infty$. From the first case, we know that the monodromy of the loop $\phi(\mathbf{a}_t)$ is trivial, and the result follows from Proposition 2.1. □

2.2. *Partitions and Plücker coordinates on X*

Let Λ denote the set of all partitions whose diagrams fit inside a $d \times (n-d)$ rectangle. Formally, these are decreasing sequences of integers $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_d)$, where $n - d \geqslant \lambda_1$ and $\lambda_d \geqslant 0$. We will draw the diagram of $\lambda \in \Lambda$ in the English convention, with λ_1 boxes left justified in the top row of a $d \times (n-d)$ rectangle, λ_2 in the next row, etc. For $\lambda \in \Lambda$, the number of boxes in the diagram of λ is denoted $|\lambda| := \lambda_1 + \cdots + \lambda_d$. If $|\lambda| = k$, we say λ is a partition of k, and write $\lambda \vdash k$. The set Λ is partially ordered by inclusion of diagrams: we write $\lambda \geqslant \mu$ iff $\lambda_i \geqslant \mu_i$ for all i, and $\lambda \succ \mu$ iff $\lambda > \mu$ and $|\lambda| = |\mu| + 1$.

The empty partition $0 \geqslant \cdots \geqslant 0$ is denoted $\emptyset$. We denote the unique partition of 1 by $\square$, since its diagram consists of a single box. The largest partition in Λ, $n-d \geqslant \cdots \geqslant n-d$, is denoted $\sqsubset\!\sqsupset$.

Partitions whose diagrams fit inside $\sqsubset\!\sqsupset$ are in bijection with d-element subsets of $\{1, \ldots, n\}$: for $\lambda \in \Lambda$, set

$$J(\lambda) := \{ j + \lambda_{d+1-j} \mid 1 \leqslant j \leqslant d \}.$$

The ***Plücker coordinates*** of a point $x \in X$ are the homogeneous coordinates $[p_\lambda(x)]_{\lambda \in \Lambda}$, defined as follows. Suppose the subspace of $\mathbb{C}_{n-1}[z]$ represented by x is the linear span of polynomials $f_1(z), \ldots, f_d(z)$. Consider the $d \times n$ matrix $A_{ij} := [z^{j-1}] f_i(z)$, whose entries are the coefficients of the polynomials $f_i(z)$. Then $p_\lambda(x) := A_{J(\lambda)}$ is the maximal minor of A with column set $J(\lambda)$.

For all $\lambda \in \Lambda$, define q_λ to be the Vandermonde determinant

$$q_\lambda := \begin{vmatrix} 1 & \cdots & 1 \\ k_1 & \cdots & k_d \\ \vdots & \vdots & \vdots \\ k_1^{d-1} & \cdots & k_d^{d-1} \end{vmatrix} = \prod_{1 \leqslant i < j \leqslant d} (k_j - k_i), \tag{2.1}$$

where $k_j = j + \lambda_{d+1-j}$. In particular, note that $q_\lambda > 0$.

Proposition 2.3. *The Wronskian* $\mathrm{Wr}(x; z)$ *is (up to a scalar multiple) given explicitly in terms of the Plücker coordinates of* x *by*

$$\mathrm{Wr}(x; z) = \sum_{\lambda \in \Lambda} q_\lambda p_\lambda(x) z^{|\lambda|}. \tag{2.2}$$

Proof. Consider the $d \times n$ matrix $B_{ij} = (\frac{d}{dz})^{i-1} z^{j-1}$. We have $(BA^t)_{ij} = f_j^{(i-1)}(z)$. Moreover, it is not hard to calculate that the maximal minor of B with column set $J(\lambda)$ is $q_\lambda z^{|\lambda|}$. Thus, using the Cauchy–Binet determinant formula,

$$\mathrm{Wr}(x; z) = \det\bigl(BA^t\bigr) = \sum_{\lambda \in \Lambda} A_{J(\lambda)} B_{J(\lambda)} = \sum_{\lambda \in \Lambda} p_\lambda(x) q_\lambda z^{|\lambda|}. \qquad \square$$

2.3. Schubert varieties

For $a \in \mathbb{CP}^1$, we define full flags

$$F_\bullet(a) = \{0\} \subset F_1(a) \subset \cdots \subset F_{n-1}(a) \subset \mathbb{C}_{n-1}[z]$$

in $\mathbb{C}_{n-1}[z]$. If $a \in \mathbb{C}$,

$$F_i(a) := (z+a)^{n-i} \mathbb{C}[z] \cap \mathbb{C}_{n-1}[z]$$

is the set of polynomials in $\mathbb{C}_{n-1}[z]$ divisible by $(z+a)^{n-i}$. For $a=\infty$, we set

$$F_i(\infty) := \mathbb{C}_{i-1}[z].$$

It is straightforward to verify that $F_\bullet(\infty) = \lim_{a\to\infty} F_\bullet(a)$.

For every $\lambda \in \Lambda$, we have a ***Schubert cell*** relative to the flag $F_\bullet(a)$:

$$X^\circ_\lambda(a) := \{x \in X \mid \dim x \cap F_i(a) = |J(\lambda) \cap \{n-i+1, \ldots, n\}|\}.$$

Its closure, $X_\lambda(a) := \overline{X^\circ_\lambda(a)}$ is the ***Schubert variety***. The codimension of $X_\lambda(a)$ in X is $|\lambda|$. When the codimension is 1, i.e. $\lambda = \square$, we call $X_\square(a)$ a ***Schubert divisor***.

The Schubert varieties $X_\lambda(0)$ and Schubert cells $X^\circ_\lambda(0)$ can be characterised in terms of the Plücker coordinates on X.

Lemma 2.4. *Let $x \in X$ be a closed point. Then*

(i) $x \in X_\lambda(0)$ *if and only if* $p_\mu(x) = 0$ *for all* $\mu \not\geqslant \lambda$;
(ii) $x \in X^\circ_\lambda(0)$ *if and only if* $p_\lambda(x) \neq 0$, *and* $p_\mu(x) = 0$ *for all* $\mu \not\geqslant \lambda$.

The proof is straightforward, using the fact that $x \in X^\circ_\lambda(0)$ iff the pivots of the matrix A are in columns $J(\lambda)$. In fact it is true that the conditions of Lemma 2.4(i) define $X_\lambda(0)$ scheme-theoretically (see [11]), but we will not need this.

Theorem 2.5. *Let $x \in X$ be a closed point, $a \in \mathbb{CP}^1$, and $k \geqslant 0$ an integer. Then $a \in \pi(x)$ with multiplicity at least k if and only if $x \in X_\lambda(a)$ for some $\lambda \vdash k$.*

Proof. By the $\mathrm{SL}_2(\mathbb{C})$-equivariance of the Wronski map (Proposition 2.1), it is enough to prove this for $a=0$. If $x \in X_\lambda(0)$, then by Lemma 2.4(i), all Plücker coordinates $p_\lambda(x)$ for $|\lambda| < k$ are zero, and hence by (2.2), z^k divides $\mathrm{Wr}(x; z)$.

To prove the converse, we proceed by induction. The result is trivially true for $k=0$; assume $0 \in \pi(x)$ has multiplicity $k > 0$, and the result is true for $k-1$. Then $x \in X_\lambda(0)$ for some $\lambda \vdash k-1$; hence by Lemma 2.4(i), $p_\mu(x) = 0$ for all $\mu \not\geqslant \lambda$, in particular for all $|\mu| \leqslant k-1$ apart from $\lambda = \mu$. But then by (2.2),

$$\mathrm{Wr}(x; z) = q_{\Box} p_{\Box}(x) z^N + \cdots + q_\lambda p_\lambda(x) z^{k-1}.$$

Since $\mathrm{Wr}(x; z)$ is divisible by z^k, we see that $p_\lambda(x) = 0$. Thus by Lemma 2.4(ii), $x \notin X^\circ_\lambda(0)$. Hence $x \in X_\lambda(0) \setminus X^\circ_\lambda(0)$, i.e. $x \in X_{\lambda'}$ for some $\lambda' > \lambda$. $\square$

In particular if $\mathbf{a} = \{a_1, \ldots, a_N\}$ has N distinct elements, then

$$X(\mathbf{a}) = \bigcap_{i=1}^{N} X_\square(a_i).$$

By Theorem 1.1, this intersection is proper, and hence the number of intersection points counted with multiplicities is given by the Schubert intersection number

$$\int_X [X_\square]^N,$$

where $[X_\square] \in H^2(X)$ denotes the cohomology class of a Schubert divisor $X_\square(a)$ (which is independent of $a \in \mathbb{CP}^1$). It is a basic result in Schubert calculus that this intersection number is the number of standard Young tableaux of shape $\boxed{}$ (see e.g. [7]).

More generally if $\mathbf{a}$ is a multiset, then set-theoretically we have

$$X(\mathbf{a}) = \bigcap_{a \in \mathbf{a}} \bigcup_{\lambda \vdash m(a)} X_\lambda(a), \tag{2.3}$$

where $m(a)$ denotes the multiplicity of $a \in \mathbf{a}$. However, by considering the total multiplicity of both sides, it is easy to see that in general this is not true scheme-theoretically. For example, if $\mathbf{a} = \{a, a, \ldots, a\}$, then the right hand side consists of the single reduced point $X_{\boxed{}}(a)$, whereas on the left hand side this point has multiplicity $\deg \mathrm{Wr} = |\mathsf{SYT}(\boxed{})|$. In fact we can say more about the multiplicities in general. Scheme theoretically, $X(\mathbf{a})$ is defined by

$$(z+a)^{m(a)} \,\big|\, \mathrm{Wr}(x; z), \tag{2.4}$$

for $a \in \mathbf{a}$, which is a system of linear equations in the Plücker variables. In general, each equation (2.4) defines a non-reduced scheme supported on a union of Schubert varieties.

Corollary 2.6. *Let $a \in \mathbb{CP}^1$, and let k be a positive integer. Consider the subscheme $X(a^{(k)})$ of X defined by the equations asserting that $(z+a)^k$ divides $\mathrm{Wr}(x; z)$. Then the cycle defined by $X(a^{(k)})$ is*

$$\sum_{\lambda \vdash k} \big|\mathsf{SYT}(\lambda)\big| \cdot X_\lambda(a),$$

where $\mathsf{SYT}(\lambda)$ is the number of standard Young tableaux of shape λ.

Proof. The cycles $[X_\lambda]$ form a basis for the Chow group of X. By Theorem 2.5, $X(a^{(k)})$ has support $\bigcup_{\lambda \vdash k} X_\lambda(a)$. Thus it is enough to show that the multiplicity of $[X_\lambda(a)]$ in $[X(a^{(k)})]$ is $|\mathsf{SYT}(\lambda)|$. Since Wr is flat, $X(a^{(k)})$ is rationally equivalent to $\bigcap_{i=1}^k X_\square(a_i)$ for any distinct $\{a_1, \ldots, a_k\} \subset \mathbb{CP}^1$. Thus $[X(a^{(k)})] = [X_\square]^k = \sum_{\lambda \vdash k} |\mathsf{SYT}(\lambda)| \cdot [X_\lambda(a)]$, as required. $\square$

Remark 2.7. Mukhin, Tarasov and Varchenko have recently shown [14] that the intersection on the right hand side of (2.3) is always reduced if the elements of $\mathbf{a}$ are real. It follows from Corollary 2.6 that for $\mathbf{a} \Subset \mathbb{RP}^1$, the multiplicity of a point $x \in X(\mathbf{a})$ is exactly $\prod_{a \in \mathbf{a}} |\mathsf{SYT}(\lambda(x, a))|$, where $\lambda(x, a) \vdash m(\mathbf{a})$ denotes the partition for which $x \in X_{\lambda(x,a)}(a)$. This reducedness theorem is the second part of the Shapiro–Shapiro conjecture; however, we will not need it in this paper.

We conclude this expository section with a quick proof of Theorem 1.1, using Theorem 2.5.

Proof of Theorem 1.1. Every positive dimensional subvariety Y of X satisfies $[Y] \cdot [X_\square] \neq 0$ in $H^*(X)$, since $[Y]$ is a positive linear combination of Schubert classes. Thus if $\dim Y > 0$, $Y \cap X_\square(a) \neq \emptyset$ for all $a \in \mathbb{CP}^1$.

Consider a fibre $X(\mathbf{a})$. If $a_0 \in \mathbb{C} \setminus \mathbf{a}$, then $z + a_0$ does not divide $\mathrm{Wr}(x; z)$ for all $x \in X(\mathbf{a})$. By Theorem 2.5, this means $X(\mathbf{a}) \cap X_\square(a_0) = \emptyset$. Thus $X(\mathbf{a})$ is 0-dimensional. Since Wr is projective, this implies that it is a finite morphism. Flatness now follows from the fact that Wr is a finite, projective morphism of non-singular varieties [10, Chapter III, Exercise 9.3(a)]. □

3. Jeu de taquin theory revisited

3.1. Standard Young tableaux with values in $\mathbb{F}$

A ***skew partition diagram*** λ/μ is a difference of partition diagrams λ and μ, where $\lambda \geqslant \mu$. Let λ/μ be a skew partition diagram which fits inside a $d \times (n-d)$ rectangle, i.e. for which $\lambda, \mu \in \Lambda$. We write μ^c for the skew partition $\square/\mu$, and $\mu^\vee := (n-d-\mu_d \geqslant \cdots \geqslant n-d-\mu_1)$ for the partition diagram obtained by rotating μ^c by 180°. As with partitions, $|\lambda/\mu| := |\lambda| - |\mu|$ is the number of boxes in λ/μ.

By an ordinary ***standard Young tableau*** of shape λ/μ, we will mean the usual notion: a filling of the boxes of λ/μ with entries $1, \ldots, |\lambda/\mu|$, each used once, where the entries increase along rows and down columns. The set of all such tableaux is denoted $\mathsf{SYT}(\lambda/\mu)$. We assume some basic familiarity with the combinatorics of tableaux, and refer the reader to [7].

For our purposes, it will be convenient to have a slightly enhanced notion of a standard Young tableau on λ/μ. Let $\mathbb{F}$ be a field, with a norm $\|\cdot\| : \mathbb{F} \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$ that is multiplicative and satisfies the triangle inequality. We extend $\|\cdot\|$ to $\mathbb{FP}^1$ by setting $\|\infty\| = +\infty$. Let $\mathbf{a} = \{a_1, \ldots, a_{|\lambda/\mu|}\} \subset \mathbb{FP}^1$ be a subset of cardinality $|\lambda/\mu|$. We think of $\mathbf{a}$ as a multiset, whose elements happen to be distinct. We impose the following restrictions, which will appear throughout this section and Section 4:

(I) For all pairs of elements $\{a_i, a_j\}$ with $i \neq j$, we have $\|a_i\| \neq \|a_j\|$.
(II) If $\mu \neq \emptyset$, then $0 \notin \mathbf{a}$.
(III) If $\lambda \neq \square$, then $\infty \notin \mathbf{a}$.

For many of our purposes λ/μ will be the entire rectangle $\square$, in which case restrictions (II) and (III) are irrelevant.

Definition 3.1. A ***standard Young tableau*** with ***values*** in $\mathbf{a}$ and ***shape*** λ/μ is a filling of the boxes of λ/μ with the elements of $\mathbf{a}$, where each element is used once and the norm of the entries is increasing along rows and down columns. The set of all standard Young tableaux with values in $\mathbf{a}$ and shape λ/μ is denoted $\mathrm{SYT}(\lambda/\mu; \mathbf{a})$.

Let $T \in \mathrm{SYT}(\lambda/\mu; \mathbf{a})$. By replacing the smallest entry (in norm) of T by 1, the second smallest by 2, and so forth, we obtain an ordinary standard Young tableau. We denote this tableau by $\mathrm{ord}(T) \in \mathsf{SYT}(\lambda/\mu)$.

3.2. Sliding

We now introduce an operation on our enhanced standard Young tableaux, called *sliding*. To define sliding, we must assume $\mathbb{F} = \mathbb{R}$, with norm $\|\cdot\| = |\cdot|$.

Let $T_0 \in \mathrm{SYT}(\lambda/\mu; \mathbf{a}_0)$. We can imagine $\mathbf{a}_0$ varying continuously along a path $\mathbf{a}_t$, $t \in [0,1]$ in the space of $|\lambda/\mu|$-element multisubsets of $\mathbb{RP}^1$. If we insist that $\mathbf{a}_t$ satisfy restrictions (I)–(III) above for all t, then $\mathbf{a}_t$ is in fact always a set, and there is a canonical way to define a tableau $T_t \in \mathrm{SYT}(\lambda/\mu; \mathbf{a}_t)$ over the point $\mathbf{a}_t$, namely so that the entries of the family T_t vary continuously, or equivalently so that $\mathrm{ord}(T_t)$ is independent of t.

We wish to extend this definition of T_t for paths $\mathbf{a}_t$ that include multisets and violations of restriction (I). (It is tempting to relax restrictions (II) and (III) also; unfortunately, this does not lead to well-behaved combinatorial structures.) The tableau T_t will not be defined at these points of violation, but it will be defined at all other points.

First suppose $\mathbf{a}_t$, $t \in [0,1]$ is a *generic* smooth path in the space of $|\lambda/\mu|$-element multisubsets of $\mathbb{RP}^1$. A generic path may be assumed to have the following form. For every $t \in [0,1]$, $\mathbf{a}_t$ is a set, and at finitely many points $t_1, \dots, t_l \in (0,1)$ there will be a violation of restriction (I) of the mildest possible sort: namely, $\mathbf{a}_{t_i} = \{a_1, \dots, a_{|\lambda/\mu|}\}$ with $a_1 = -a_2 \notin \{0, \infty\}$, and restriction (I) holds for all other pairs of elements $\{a_i, a_j\} \neq \{a_1, a_2\}$. Other sorts of violations of restriction (I), such as multisets, do not arise generically, as they can be avoided by perturbing the path (see Example 3.4).

In this case we define T_t for t near t_i as follows. If a_1 and a_2 are not in the same row or column define T_t by changing the entries continuously. If a_1 and a_2 are in the same row or column define T_t so that $\mathrm{ord}(T_t)$ is independent of t in a neighbourhood of t_i. (Note that in the former case, $\mathrm{ord}(T_t)$ will normally change at $t = t_i$; in the latter case, the entries of T_t will normally be discontinuous at $t = t_i$.) Another way to think of this is that a_1 and a_2 swap places if and only if they are forced to swap in order to maintain row and column strictness in the tableau.

Definition 3.2. Let $\mathbf{a}, \mathbf{a}' \subset \mathbb{RP}^1$ be $|\lambda/\mu|$-element subsets satisfying restrictions (I)–(III) above, which can be joined by a path satisfying restrictions (II) and (III). Define $\mathrm{slide}_{\mathbf{a}'} : \mathrm{SYT}(\lambda/\mu; \mathbf{a}) \to \mathrm{SYT}(\lambda/\mu; \mathbf{a}')$ as follows. If $T_0 \in \mathrm{SYT}(\lambda/\mu; \mathbf{a})$, $\mathrm{slide}_{\mathbf{a}'}(T_0)$ is the tableau T_1 obtained by following T_0 over any generic smooth path $\mathbf{a}_t$ interpolating $\mathbf{a}_0 = \mathbf{a}$ and $\mathbf{a}_1 = \mathbf{a}'$.

Theorem 3.3. *The tableau* $\mathrm{slide}_{\mathbf{a}'}(T)$ *depends only on the homotopy class of the path* $\mathbf{a}_t$ *in Definition* 3.2.

Proof. This will be an immediate consequence of Theorem 3.5 (below) and Corollary 2.2. □

For the main applications we consider in this paper, there will be additional constraints on our paths, which ensure that the homotopy class of $\mathbf{a}_t$ in Definition 3.2 is unique. For this reason, we have chosen to suppress the dependence on this homotopy class from our notation. In general, changing the homotopy class of $\mathbf{a}_t$ does have a non-trivial effect (see Remark 3.7).

In light of Theorem 3.3, the path $\mathbf{a}_t$ does not need to be generic in order to define T_t. We simply put $T_t := \mathrm{slide}_{\mathbf{a}_t}(T_0)$.

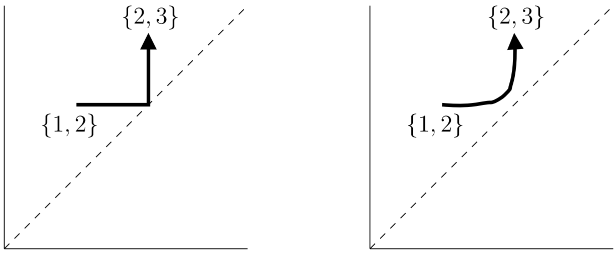


Fig. 1. The path $\mathbf{a}_t$ in Example 3.4 (left), and a slight perturbation (right).

Example 3.4. Let $\mathbf{a}_t = \{1 + 2t, 2\}$ for $t \in [0, 1]$, and let

$$T_0 = \begin{array}{|c|c|} \hline & 1 \\ \hline 2 & \\ \hline \end{array}\ .$$

The path $\mathbf{a}_t$ is not generic, since $\mathbf{a}_t$ is a multiset when $t = \frac{1}{2}$; however by perturbing the path slightly to avoid this behaviour (see Fig. 1), we see that

$$T_t = \mathrm{slide}_{\mathbf{a}_t}(T_0) = \begin{cases} \begin{array}{|c|c|} \hline & 1{+}2t \\ \hline 2 & \\ \hline \end{array} & \text{if } 0 \leqslant t < \frac{1}{2}\,, \\[2ex] \begin{array}{|c|c|} \hline & 2 \\ \hline 1{+}2t & \\ \hline \end{array} & \text{if } \frac{1}{2} < t \leqslant 1\,. \end{cases}$$

Note that $\mathrm{ord}(T_t)$ is independent of t; this will always be the case when the entries all have the same sign. For an illustration of the case with mixed signs, see Example 3.6.

We can now state one of our main theorems, which relates the operation of sliding to the Wronski map.

Theorem 3.5. *For* $\mathbf{a} \subset \mathbb{R}$ *satisfying restriction* (I)*, there is a correspondence* $x \leftrightarrow T_x$ *between points* $x \in X(\mathbf{a})$ *and tableaux* $T_x \in \mathrm{SYT}(\lambda/\mu; \mathbf{a})$*. Under this correspondence, if* $\mathbf{a}_t \Subset \mathbb{RP}^1$*,* $t \in [0, 1]$ *is a generic real path satisfying restrictions* (II) *and* (III)*, and* $x_t \in X(\mathbf{a}_t)$ *is any lifting of* $\mathbf{a}_t$ *to* X*, then* $T_{x_1} = \mathrm{slide}_{\mathbf{a}_1}(T_{x_0})$.

A precise statement of the correspondence is given in Section 4, and the proof is given in Section 5.

3.3. Subtableaux and jeu de taquin

We now explain the connection between the sliding operation of Definition 3.2 and the usual notion of a slide in jeu de taquin theory.

Let λ/μ be a skew partition, and let $\mathbf{a} = \{a_1, \ldots, a_{|\lambda/\mu|}\} \subset \mathbb{RP}^1$, with $|a_1| < \cdots < |a_{|\lambda/\mu|}|$. Let $\mathbf{b} \subset \mathbf{a}$. If $T \in \mathrm{SYT}(\lambda/\mu; \mathbf{a})$, we denote the set of boxes of T whose entries are in $\mathbf{b}$ by $T|_{\mathbf{b}}$. If $\mathbf{b} = \{a_i, a_{i+1}, \ldots, a_j\}$ for some $i < j$, then $T|_{\mathbf{b}}$ is a standard Young tableau with values in $\mathbf{b}$ of some shape λ'/μ'. In this case we say $T|_{\mathbf{b}}$ is a ***subtableau*** of T, and we also denote this subtableau by $T|_{\lambda'/\mu'}$.

Let $\mathbf{b} = \{a_1, \ldots, a_j\}$, and $\mathbf{c} = \{a_{j+1}, \ldots, a_{|\lambda/\mu|}\}$. Suppose that all elements of $\mathbf{b}$ are positive and that all elements in $\mathbf{c}$ are negative. Let $a'_1, \ldots, a'_j$ be positive real numbers such that $a'_1 > \cdots > a'_j > -a_{|\lambda/\mu|}$, and set $\mathbf{b}' = \{a'_1, \ldots, a'_j\}$ and $\mathbf{a}' = \mathbf{b}' \cup \mathbf{c}$. Note that the elements of $\mathbf{b}$ are smaller in absolute value than the elements of $\mathbf{c}$, which are in turn smaller than those of $\mathbf{b}'$.

In a mild abuse of notation, define

$$\mathrm{slide}_{T|_{\mathbf{b}}}(T|_{\mathbf{c}}) := \mathrm{slide}_{\mathbf{a}'}(T)|_{\mathbf{c}}.$$

By switching signs everywhere, we can also perform this construction if the elements of $\mathbf{b}$ are negative and the elements of $\mathbf{c}$ are positive. Similarly, we define $\mathrm{slide}_{T|_{\mathbf{c}}}(T|_{\mathbf{b}})$ by reversing the roles of $\mathbf{b}$ and $\mathbf{c}$ (and reversing the inequalities) in this construction.

Suppose $T \in \mathrm{SYT}(\lambda/\mu; +) := \bigcup_{\mathbf{a} \subset \mathbb{R}_+} \mathrm{SYT}(\lambda/\mu; \mathbf{a})$ is a standard Young tableau with all positive real entries (or $T \in \mathrm{SYT}(\lambda/\mu; -) := \bigcup_{\mathbf{a} \subset \mathbb{R}_-} \mathrm{SYT}(\lambda/\mu; \mathbf{a})$). We can think of slide_T as an operation on tableaux which takes as input any skew tableau U with all negative (resp. positive) entries that can be placed adjacent to T to form a larger tableau, and returns a tableau with the same entries but different shape.

If the shape of T consists of a single box, it is not hard to see that $\mathrm{slide}_T(U)$ performs a Schützenberger slide or a reverse slide through U using the box of T (see Example 3.6). More generally, slide_T is the operation of performing a sequence of slides in the order dictated by the entries of T. If $T' = \mathrm{slide}_U(T)$, and $U' = \mathrm{slide}_T(U)$ then the pair $(\mathrm{ord}(T'), \mathrm{ord}(U'))$ is the result of applying *tableau switching* to the pair $(\mathrm{ord}(T), \mathrm{ord}(U))$, (see [1] and the references therein). Arguments that show that tableau switching is well defined and independent of a number of choices can be used to give a combinatorial proof of Theorem 3.3.

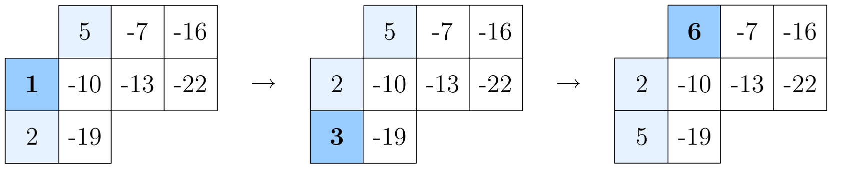
Example 3.6. Let $\mathbf{b} = \{1, 2, 5\}$, $\mathbf{c} = \{-7, -10, -13, -16, -19, -22\}$. Let T be the standard Young tableau with values in $\mathbf{a} = \mathbf{b} \cup \mathbf{c}$ shown below.

$T =$

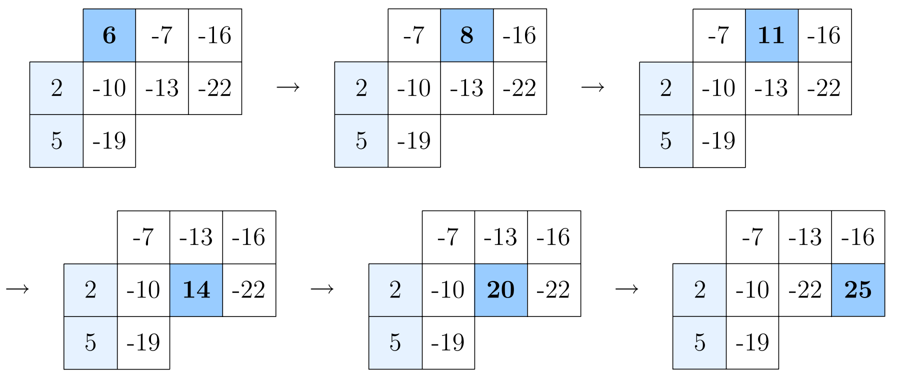
	5	-7	-16
1	-10	-13	-22
2	-19		

.

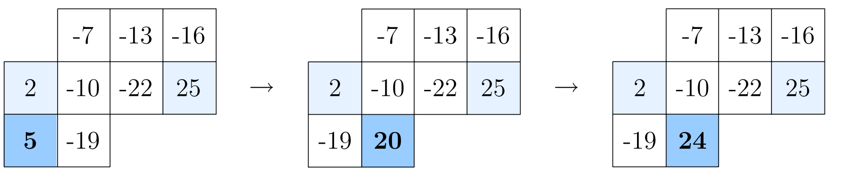
We compute $\mathrm{slide}_{T|_{\mathbf{b}}}(T|_{\mathbf{c}})$, by increasing the entries of $\mathbf{b}$ one at a time until we reach $\mathbf{b}' = \{23, 24, 25\}$. The order in which we do this does not affect the answer. We choose to begin by increasing the entry 1, shown highlighted below. As its value climbs past the other positive entries in the tableau it swaps places with them, hence $\mathrm{ord}(T)$ does not change (see Example 3.4).



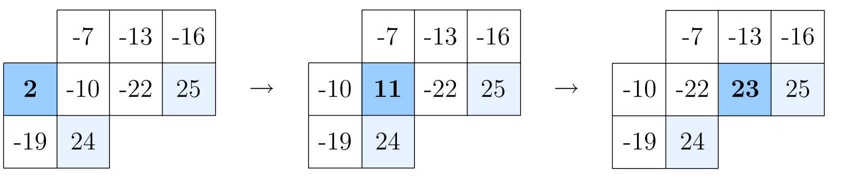
As the highlighted entry continues to increase, it switches places with the next smallest negative entry if only if the two entries are adjacent, thereby performing a Schützenberger slide through $T|_{\mathbf{c}}$.



Next we increase the entry 5 until it is larger than 22.



Finally we increase the entry 2.



Thus we find,

$$\mathrm{slide}_{T|_{\mathbf{b}}}(T|_{\mathbf{c}}) = \begin{array}{|c|c|c|c|} \cline{2-4} \multicolumn{1}{c|}{} & -7 & -13 & -16 \\ \hline -10 & -22 & \multicolumn{2}{c}{} \\ \cline{1-2} -19 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} .$$

Moreover, the relative order of the positive entries in this final tableau tells us,

$$\mathrm{slide}_{T|_{\mathbf{c}}}(T|_{\mathbf{b}}) = \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 1 & 5 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} .$$

Remark 3.7. A special case of sliding is when $\mathbf{a}_t = \{(a_1)_t, \ldots, (a_N)_t\}$ is a loop that cyclically rotates the elements of $\mathbf{a}_0$. Suppose each $(a_i)_t$ is a cyclically decreasing path in $\mathbb{RP}^1$, and

$$0 < (a_1)_0 = (a_2)_1 < (a_2)_0 = (a_3)_1 < \cdots < (a_N)_0 = (a_1)_1.$$

Let $T \in \mathrm{SYT}(\square; \mathbf{a}_0)$. By sliding T using the path $\mathbf{a}_t$, we perform one step of Schützenberger's evacuation on T: the smallest entry performs a slide through the tableau, becoming the largest entry. This procedure defines a $\mathbb{Z}$-action on standard Young tableaux. Since every loop is homotopic to some power of this basic loop, the evacuation action completely describes the monodromy of the sliding operation on real valued standard Young tableaux of shape $\square$. By Theorem 3.5, this is also the monodromy of the Wronski map for real polynomials with N or $N-1$ distinct real roots.

3.4. Equivalence relations on tableaux

We will need to adopt some additional notions from ordinary jeu de taquin theory.

Definition 3.8. If $T \in \mathrm{SYT}(\lambda/\mu; \pm)$, then the ***rectification*** of T is defined to be $\mathrm{rect}(T) := \mathrm{slide}_U(T)$, where $U \in \mathrm{SYT}(\mu; \mp)$ can be placed adjacent to T to form a larger standard Young tableau. The ***rectification shape*** of T is the shape of $\mathrm{rect}(T)$.

Definition 3.9. If $T \in \mathrm{SYT}(\lambda/\mu; \pm)$ and $T' \in \mathrm{SYT}(\lambda'/\mu'; \pm)$, we say that T and T' are ***equivalent***, and write $T \sim T'$, if $\mathrm{rect}(T) = \mathrm{rect}(T')$.

Definition 3.10. We say that $T, T' \in \mathrm{SYT}(\lambda/\mu; \pm)$ are ***dual equivalent***, and write $T \sim^* T'$, if slide_T and $\mathrm{slide}_{T'}$ are identical as operations on tableaux.

If we replace T, T' by $\mathrm{ord}(T), \mathrm{ord}(T')$, these definitions become the usual notions of rectification, equivalence [16], and dual equivalence [9] on standard Young tableaux. A classical theorem of Schützenberger states that $\mathrm{rect}(T)$ does not depend on the choice of the tableau $U \in \mathrm{SYT}(\mu; \mp)$ [16].

It is not hard to see that if either $T \sim T'$ or $T \sim^* T'$, then T and T' have the same rectification shape. Thus it makes sense to speak of the rectification shape of an equivalence class or a dual equivalence class. The interaction between the equivalence and dual equivalence relations is governed by the following fact: there is a unique tableau in the intersection of any equivalence class of tableaux with a dual equivalence class of the same rectification shape.

The Littlewood–Richardson rule can be formulated in a variety of different ways. For us, the formulation below in terms of dual equivalence classes is the most convenient.

Theorem 3.11 *(Littlewood–Richardson rule). The Littlewood–Richardson number*

$$c^{\lambda}_{\mu\nu} := \int_X [X_{\lambda^\vee}][X_\mu][X_\nu]$$

is the number of dual equivalence classes in $\mathsf{SYT}(\lambda/\mu)$ *with rectification shape* ν.

Alternatively, $c^{\lambda}_{\mu\nu}$ is the number of tableaux in $\mathsf{SYT}(\lambda/\mu)$ in any single equivalence class with rectification shape ν. That this statement and Theorem 3.11 are interchangeable follows from the relationship between equivalence and dual equivalence classes of tableaux.

In Section 6, we will see that rectification shape, equivalence, dual equivalence and many combinatorial facts pertaining to them have natural interpretations in terms of the Wronski map. Based on these, in Section 6.3 we give a new proof of the Littlewood–Richardson rule.

4. Labelling points in a Grassmannian by tableaux

4.1. Fibres of the Wronski map over a non-archimedian field

Let $\mathcal{K} := \mathbb{C}\{\{u\}\} = \bigcup_{n \geqslant 1} \mathbb{C}((u^{\frac{1}{n}}))$ be the field of Puiseux series over $\mathbb{C}$. In this section, we formulate a correspondence between tableaux and points in the fibre of the Wronski map, working over $\mathcal{K}$. In Section 4.5, we will show how this can be used to obtain a correspondence over $\mathbb{C}$ when the roots of the Wronskian are real.

Let $\mathcal{X} := \mathrm{Gr}(d, \mathcal{K}_{n-1}[z])$, be the Grassmannian defined over $\mathcal{K}$. As over $\mathbb{C}$, we denote the Wronski map by $\mathrm{Wr} : \mathcal{X} \to \mathbb{P}(\mathcal{K}_N[z])$, and its fibre at $\prod_{a_i \neq \infty}(z + a_i)$ by $\mathcal{X}(\mathbf{a})$, where $\mathbf{a} = \{a_1, \ldots, a_N\}$. The Schubert varieties $\mathcal{X}_\lambda(a)$, for $a \in \mathbb{P}^1(\mathcal{K})$, are also defined analogously.

If $g(u) = c_\ell u^\ell + \sum_{r>\ell} c_r u^r \in \mathcal{K}^\times$, the ***valuation*** of $g(u)$ is defined to be

$$\mathrm{val}\big(g(u)\big) := \ell.$$

The ***leading term*** $\mathrm{LT}(g(u))$ and ***leading coefficient*** $\mathrm{LC}(g(u))$ are

$$\mathrm{LT}\big(g(u)\big) := c_\ell u^\ell, \qquad \mathrm{LC}\big(g(u)\big) := \big[u^\ell\big]g(u) = c_\ell.$$

Additionally, we set $\mathrm{val}(0) := +\infty$, $\mathrm{val}(\infty) := -\infty$ and $\mathrm{LT}(0) := 0$. Let $\mathcal{K}_+ = \{g(u) \in \mathcal{K} \mid \mathrm{val}(g(u)) \geqslant 0\}$.

For any $0 < \varepsilon < 1$, we can define a norm on $\mathcal{K}$, by

$$\big\|g(u)\big\| := \varepsilon^{\mathrm{val}(\tau)}.$$

It therefore makes sense to consider standard Young tableaux with values in $\mathbf{a} \subset \mathbb{P}^1(\mathcal{K})$. Clearly this notion does not depend on the choice of ε. Note that in such a tableau, the valuation of the entries *decreases* along rows and down columns.

Since our analysis will need to deal with cases where $\mathbf{a}$ is a multiset, we introduce some mild generalisations of standard Young tableaux, called *weakly increasing* and *diagonally increasing* tableaux. Let λ/μ be a skew partition fitting inside $\Box$, and let $\mathbf{a} = \{a_1, \dots, a_{|\lambda/\mu|}\} \Subset \mathbb{P}^1(\mathcal{K})$ be a $|\lambda/\mu|$-element multisubset satisfying restrictions (II) and (III), but not necessarily (I).

Definition 4.1. A ***weakly increasing tableau*** with shape λ/μ and values in $\mathbf{a}$ is a filling of the boxes of λ/μ with the elements of $\mathbf{a}$ (each used as many times as its multiplicity) such that entries weakly increase in norm along rows and down columns. A weakly increasing tableau is ***diagonally increasing*** if the entries are also *strictly* increasing in norm diagonally right and downward. The set of all diagonally increasing tableaux with shape λ/μ and values in $\mathbf{a}$ is denoted $\mathrm{DIT}(\lambda/\mu; \mathbf{a})$. (Note that both definitions coincide with Definition 3.1, if (I) holds.)

Before we can formulate the correspondence between points in $\mathcal{X}$ and tableaux (Theorem 4.2), we must introduce some notation.

For $T \in \mathrm{DIT}(\lambda/\mu; \mathbf{a})$, write $\mathrm{val}(T) := \mathrm{val}(a_1) + \cdots + \mathrm{val}(a_{|\lambda/\mu|})$ for the sum of the valuation of the entries. In degenerate cases where T has an empty shape, $\mathrm{val}(T) := 0$. The reader will note that this definition is problematic if 0 and ∞ are both entries of T. As we explain in Section 4.2, the trouble this causes is always resolvable by an appropriate renormalisation. For now, we will state our results under the assumption that $\infty \notin \mathbf{a}$.

Put

$$\mathbf{a}^+ := \mathbf{a} \cup \{\underbrace{0, \dots, 0}_{|\mu|}, \overbrace{\infty, \dots, \infty}^{N-|\lambda|}\},$$

so that $|\mathbf{a}^+| = N$. The reader should imagine that the extra zeros and infinities are there to fill the boxes of μ and λ^c inside $\Box$, which do not already have entries from T (see Theorem 4.5). Let

$$E_i(\mathbf{a}) := \sum_{k_1 < \cdots < k_{|\lambda/\mu|-i}} a_{k_1} \cdots a_{k_{|\lambda/\mu|-i}}$$

be the $(|\lambda/\mu| - i)$th elementary symmetric function, and put

$$e_i(\mathbf{a}) := \left[u^{\ell_i}\right] E_i(\mathbf{a}),$$

where $\ell_i = \min \mathrm{val}(a_{k_1} \cdots a_{k_{|\lambda/\mu|-i}})$ is the minimum of the valuations of the terms in the sum. Thus $e_i(\mathbf{a})$ equals either the leading coefficient of $E_i(\mathbf{a})$ or 0.

For $0 \leqslant i \leqslant |\lambda/\mu|$, define sets of partitions

$$M_i(T) := \left\{ \nu \in \Lambda \;\middle|\; \begin{array}{l} \mu \leqslant \nu \leqslant \lambda,\ \nu \vdash |\mu| + i, \text{ and} \\ \mathrm{val}(T|_{\lambda/\nu}) \leqslant \mathrm{val}(T|_{\lambda/\nu'}) \text{ for all } \nu' \vdash |\mu| + i \end{array} \right\}.$$

Let $\omega_1, \dots, \omega_{|\lambda/\mu|}$ be complex variables. Fill a skew diagram of shape λ/μ with entries $\omega_1, \dots, \omega_{|\lambda/\mu|}$, in such a way that the position of ω_i matches the position of a_i. Let Ω_ν denote the product of all the variables ω_i which are outside of ν in this filling, if $\mu \leqslant \nu \leqslant \lambda$. Put $\Omega_\nu := 0$ for all other ν.

Finally, recall the definition of q_ν from (2.1).

Theorem 4.2. *Let $T \in \mathrm{DIT}(\lambda/\mu; \mathbf{a})$. Assume that $\omega_1, \ldots, \omega_{|\lambda/\mu|}$ are such that the Jacobian condition below holds:*

$$\det J \neq 0, \quad \textit{where } J_{ij} = \frac{\partial}{\partial \omega_j} \sum_{\nu \in M_{i-1}(T)} q_\nu \Omega_\nu, \; i, j = 1, \ldots, |\lambda/\mu|. \tag{4.1}$$

There is a point $x \in \mathcal{X}(\mathbf{a}^+)$ with Plücker coordinates $[p_\nu(x)]_{\nu \in \Lambda}$ satisfying

$$\mathrm{LT}\big(p_\nu(x)\big) = \Omega_\nu u^{\mathrm{val}(T|_{\lambda/\nu})} \quad \textit{for all } \nu \in \Lambda, \tag{4.2}$$

if and only if $\omega_1, \ldots, \omega_{|\lambda/\mu|}$ satisfy

$$\sum_{\nu \in M_i(T)} q_\nu \Omega_\nu = q_\lambda e_i(\mathbf{a}) \quad \textit{for } 0 \leqslant i < |\lambda/\mu|. \tag{4.3}$$

In other words, to find points in $\mathcal{X}(\mathbf{a}^+)$ corresponding to $T \in \mathrm{SYT}(\lambda/\mu; \mathbf{a})$, we solve the system of Eqs. (4.3) for $\omega_1, \ldots, \omega_{|\lambda/\mu|}$, and check that the solution satisfies (4.1). The following example illustrates the details of this process.

Example 4.3. With $n = 4$, $d = 2$, $\lambda = (2 \geqslant 2)$, $\mu = (1 \geqslant 0)$, and $\mathbf{a} = \{4u + 2u^2, 1, 1\}$, let $T \in \mathrm{DIT}(\lambda/\mu; \mathbf{a})$ be the tableau

$$T = \begin{array}{|c|c|} \hline & 4u{+}2u^2 \\ \hline 1 & 1 \\ \hline \end{array} \,.$$

We will apply Theorem 4.2 to the tableau T.

First, we determine $e_i(\mathbf{a})$ and $M_i(T)$ for $i = 0, 1, 2$. We have,

$$\begin{aligned} E_0(\mathbf{a}) &= 4u + 2u^2, \\ E_1(\mathbf{a}) &= \big(4u + 2u^2\big) + \big(4u + 2u^2\big) + 1, \\ E_2(\mathbf{a}) &= \big(4u + 2u^2\big) + 1 + 1, \end{aligned}$$

whence $e_0(\mathbf{a}) = 4$, $e_1(\mathbf{a}) = 1$, $e_2(\mathbf{a}) = 2$. Each $M_i(T)$ is a singleton: $M_i(T) = \{\alpha_i\}$, where

$$\alpha_0 = (1 \geqslant 0), \qquad \alpha_1 = (2 \geqslant 0), \qquad \alpha_2 = (2 \geqslant 1).$$

Next, we assign variables $\omega_1, \omega_2, \omega_3$ to the boxes of λ/μ as shown here

$$\begin{array}{|c|c|} \hline & \omega_1 \\ \hline \omega_2 & \omega_3 \\ \hline \end{array} \,,$$

and write down the conditions (4.1) and (4.3). We have

$$q_{\alpha_0}\Omega_{\alpha_0} = 2\omega_1\omega_2\omega_3, \qquad q_{\alpha_1}\Omega_{\alpha_1} = 3\omega_2\omega_3, \qquad q_{\alpha_2}\Omega_{\alpha_2} = 2\omega_3.$$

Thus the Jacobian matrix from (4.1) is

$$J = \begin{pmatrix} 2\omega_2\omega_3 & 2\omega_1\omega_3 & 2\omega_1\omega_2 \\ 0 & 3\omega_3 & 3\omega_2 \\ 0 & 0 & 2 \end{pmatrix},$$

and the system of Eqs. (4.3) is simply

$$2\omega_1\omega_2\omega_3 = 4,$$
$$3\omega_2\omega_3 = 1,$$
$$2\omega_3 = 2.$$

The solution, $\omega_1 = 6$, $\omega_2 = \frac{1}{3}$, $\omega_3 = 1$, is a point for which J is non-singular. Therefore, Theorem 4.2 asserts that there exists a point $x \in \mathcal{X}(\mathbf{a}^+)$ whose Plücker coordinates satisfy (4.2):

$$\mathrm{LT}\big(p_{0\geqslant 0}(x)\big) = 0, \qquad \mathrm{LT}\big(p_{2\geqslant 0}(x)\big) = \omega_2\omega_3 = \frac{1}{3},$$
$$\mathrm{LT}\big(p_{1\geqslant 0}(x)\big) = \omega_1\omega_2\omega_3 u = 2u, \qquad \mathrm{LT}\big(p_{2\geqslant 1}(x)\big) = \omega_3 = 1,$$
$$\mathrm{LT}\big(p_{1\geqslant 1}(x)\big) = \omega_1\omega_3 u = 6u, \qquad \mathrm{LT}\big(p_{2\geqslant 2}(x)\big) = 1. \tag{4.4}$$

A straightforward calculation shows that the two points in $\mathcal{X}(\mathbf{a}^+)$ are $\langle f_1(z), f_2(z)\rangle$ and $\langle g_1(z), g_2(z)\rangle$, where

$$f_1(z) = z^3 + z^2, \qquad g_1(z) = (1+u)^2 z^3 + \big(6u + 3u^2\big)z^2,$$
$$f_2(z) = z^2 + (1+u)^2 z + 2u + u^2, \qquad g_2(z) = z^2 + (1+u)^2 z + \frac{1}{3}(1+u)^2.$$

The reader can easily check that $x = \langle g_1(z), g_2(z)\rangle$ does indeed satisfy (4.4). It is a useful exercise to verify that the point $\langle f_1(z), f_2(z)\rangle$ corresponds to the other tableau in $\mathrm{DIT}(\lambda/\mu; \mathbf{a})$.

We will prove Theorem 4.2 in Section 4.4. The most fundamental case is when $\lambda/\mu = \square$ and restriction (I) holds. In this case we obtain a bijection $x \leftrightarrow T_x$ between $\mathcal{X}(\mathbf{a})$ and $\mathrm{SYT}(\square; \mathbf{a})$.

Corollary 4.4. *Let $\mathbf{a} = \{a_1, \ldots, a_N\} \subset \mathcal{K}$ satisfying restriction* (I)*. For every $T \in \mathrm{SYT}(\square; \mathbf{a})$, there is a unique point $x_T \in \mathcal{X}(\mathbf{a})$ whose Plücker coordinates $[p_\nu(x_T)]_{\nu\in\Lambda}$ satisfy*

$$\mathrm{val}\big(p_\nu(x_T)\big) = \mathrm{val}(T|_{\nu^c}) \quad \textit{for all } \nu \in \Lambda. \tag{4.5}$$

Moreover, for every point $x \in \mathcal{X}(\mathbf{a})$ there is a unique tableau $T_x \in \mathrm{SYT}(\square; \mathbf{a})$ such that $x = x_{T_x}$. In particular, the fibre $\mathcal{X}(\mathbf{a})$ is reduced.

Proof. Assume $\|a_1\| < \cdots < \|a_N\|$, and let $c_i := \mathrm{LC}(a_i)$ be the leading coefficient of a_i. Then $e_i(\mathbf{a}) = c_{i+1} \cdots c_N$.

Let α_i be the shape of $T|_{\{a_1,\ldots,a_i\}}$. Then α_i is the unique element in $M_i(T)$, and $\Omega_{\alpha_i} = \omega_{i+1} \cdots \omega_N$. Eqs. (4.3) are $q_{\alpha_i}\omega_{i+1} \cdots \omega_N = q_{\boxed{}} c_{i+1} \cdots c_N$, which has the unique solution

$$\omega_i = \frac{q_{\alpha_i} c_i}{q_{\alpha_{i-1}}}. \tag{4.6}$$

At this solution, the Jacobian matrix J is upper triangular, with non-zero entries on the diagonal; thus (4.1) is satisfied. Therefore, by Theorem 4.2, the solution (4.6) gives rise to a point x_T satisfying (4.5).

It is easy to see that if $T \neq T' \in \mathrm{SYT}(\boxed{}; \mathbf{a})$ then $\mathrm{val}(T|_{\nu^c}) \neq \mathrm{val}(T'|_{\nu^c})$ for some ν; thus we have found $|\mathsf{SYT}(\boxed{})|$ distinct points in the fibre, which is all of them, and the uniqueness follows. $\Box$

4.2. Tableau entries of 0 *and* ∞

If 0 is an entry of a tableau T, satisfying restriction (II), then T must have a straight shape λ, and 0 must be in the upper left corner. By deleting the 0, one obtains a skew tableau $\tilde{T}$ of shape $\lambda/\Box$. This new tableau $\tilde{T}$ produces the same Eqs. (4.2) and (4.3) to be solved in Theorem 4.2; hence T and $\tilde{T}$ are equivalent for practical purposes in that they correspond to the same point(s) in $\mathcal{X}$.

If ∞ is an entry of T, the situation is similar, however we must renormalised our equations in order to make sense of Theorem 4.2 and Corollary 4.4. For example, consider Eq. (4.5). If ∞ is an entry of T, there is a summand of $-\infty$ in each expression $\mathrm{val}(T|_{\nu^c})$, except for the degenerate case $\nu = \boxed{}$. Since the Plücker coordinates are only well defined up to a multiplicative constant, Eq. (4.5) should be regarded up to an additive constant. If we treat $-\infty$ as a formal symbol, and subtract it from the valuation of each Plücker coordinate, we arrive at the correct replacement for (4.5) when ∞ is an entry:

$$\mathrm{val}\big(p_\nu(x_T)\big) = \begin{cases} \mathrm{val}(T|_{\nu^c} \backslash \infty) & \text{if } \nu \neq \boxed{}, \\ +\infty & \text{if } \nu = \boxed{}, \end{cases}$$

where $T|_{\nu^c} \backslash \infty$ means $T|_{\nu^c}$ with the box containing ∞ deleted. Other cases where ∞ is an entry of T can be analysed similarly, and always one finds that the point(s) corresponding to T are exactly the same as the point(s) corresponding to $T \setminus \infty$.

The next theorem further illustrates why if $\lambda/\mu \neq \boxed{}$, the boxes of μ and λ^c should be thought of as containing entries of 0 and ∞ respectively, for purposes of Theorem 4.2.

Theorem 4.5. *Let $T \in \mathrm{DIT}(\boxed{}; \mathbf{a})$, where $\mathbf{a} = \{a_1, \ldots, a_N\} \Subset \mathbb{P}^1(\mathcal{K})$ and*

$$\|a_1\| \leqslant \cdots \leqslant \|a_{i-1}\| < \|a_i\| \leqslant \cdots \leqslant \|a_j\| < \|a_{j+1}\| \leqslant \cdots \leqslant \|a_N\|.$$

Let λ be the shape of $T|_{\{a_1,\ldots,a_j\}}$, and let μ be the shape of $T|_{\{a_1,\ldots,a_{i-1}\}}$. For all $t \in \mathcal{K}^\times$ with $\|t\| \leqslant 1$, define a tableau T_t of shape $\boxed{}$ obtained from T as follows: $T_t|_{\lambda/\mu} = T|_{\lambda/\mu}$ for all t; the entries $a_1, \ldots, a_{i-1} \in T$ are replaced by $ta_1, \ldots, ta_{i-1}$ in T_t; the entries $a_{j+1}, \ldots, a_N \in T$ are replaced by $t^{-1}a_{j+1}, \ldots, t^{-1}a_N$ in T_t.

Let $x_{T_t} \in \mathcal{X}$ be the point corresponding to T_t (as in Theorem 4.2*). Let $x' = \lim_{t\to 0} x_{T_t} \in \mathcal{X}$, $\mathbf{a}' = \{a_i, \ldots, a_j\}$, and $T' = T|_{\lambda/\mu}$. Then the following are true:*

(i) $x' \in \mathcal{X}_\mu(0) \cap \mathcal{X}_{\lambda^\vee}(\infty)$*;*
(ii) *x' corresponds to T';*
(iii) *if $\mathbf{a}'$ satisfies restriction* (I)*, then x' is the unique point corresponding to T'.*

Proof. After normalising the Plücker coordinates so that $\lim_{t\to 0} p_\nu(x_{T_t})$ is defined for all $\nu \in \Lambda$, we find that

$$\mathrm{LT}\big(p_\nu(x')\big) = \lim_{t\to 0} \mathrm{LT}\big(p_\nu(x_{T_t})\big) = \begin{cases} \mathrm{LT}(p_\nu(x)) & \text{if } \mu \leqslant \nu \leqslant \lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

Thus the fact that $x' \in \mathcal{X}_\mu(0)$ follows from Lemma 2.4, and the fact that $x' \in \mathcal{X}_{\lambda^\vee}(\infty)$ can be shown analogously, proving (i).

For (ii), we must consider (4.2) and (4.3) as they pertain to the pair (T, x) and to the pair (T', x'). To quell the notational conflicts that this naturally presents, we will use unprimed variable names and equation numbers ($\omega_1, \ldots, \omega_N$, (4.2), etc.), when referring to the context of (T, x), and primed variable names and equation numbers ($\omega'_i, \ldots, \omega'_j$, (4.2)$'$, etc.) in the context of (T', x').

We know that (4.2) holds for (T, x), where $\omega_1, \ldots, \omega_N$ are a solution to (4.3). Eq. (4.7) gives the lead terms of the Plücker coordinates of x': after normalising, these are

$$\mathrm{LT}\big(p_\nu(x')\big) = \begin{cases} \frac{\Omega_\nu}{\Omega_\lambda} u^{\mathrm{val}(T|_{\lambda/\nu})} & \text{if } \mu \leqslant \nu \leqslant \lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

Set $\omega'_k = \omega_k$ for $k = i, \ldots, j$, where ω'_k is the variable corresponding to the box of T' containing a_k. Then $\Omega'_\nu = \Omega_\nu / \Omega_\lambda$, and so from (4.8) we see that (4.2)$'$ holds for (T', x'). Furthermore, the equations in the system (4.3) include

$$\sum_{\nu \in M_{i+k}(T)} q_\nu \Omega_\nu = q_{\square} e_{i+k}(\mathbf{a}) \quad \text{for } 0 \leqslant k < |\lambda/\mu| \tag{4.9}$$

and

$$q_\lambda \Omega_\lambda = q_{\square} e_j(\mathbf{a}). \tag{4.10}$$

We deduce that (4.3)$'$ also holds for (T', x') by dividing (4.9) by (4.10), and noting that $e_{i+k}(\mathbf{a})/e_j(\mathbf{a}) = e_k(\mathbf{a}')$, $M_{k+i}(T) = M_k(T')$, and $\Omega_\nu/\Omega_\lambda = \Omega'_\nu$. Since (4.2)$'$ and (4.3)$'$ hold simultaneously, T' corresponds to x'.

Finally, for (iii), we argue as in the proof of Corollary 4.4. We have just shown that every tableau $T \in \mathrm{SYT}(\lambda/\mu; \mathbf{a}')$ corresponds to at least one point in the intersection $\mathcal{X}((\mathbf{a}')^+) \cap \mathcal{X}_\mu(0) \cap \mathcal{X}_{\lambda^\vee}(\infty)$. But from Schubert calculus, we know the number of distinct points in this intersection is at most $|\mathsf{SYT}(\lambda/\mu)|$, so the correspondence is bijective. $\square$

4.3. The Plücker ideal and its initial ideal

We recall some standard facts about the equations defining X and initial ideals, for which [12] may serve as a general reference.

Viewing $[p_\lambda]_{\lambda\in\Lambda}$ as the coordinates on $\mathbb{CP}^{\binom{n}{d}-1}$, the Plücker coordinates define a projective embedding of X; hence,

$$X = \operatorname{Proj}\mathbb{C}[\mathbf{p}]/I,$$

where $\mathbb{C}[\mathbf{p}] = \mathbb{C}[p_\lambda]_{\lambda\in\Lambda}$ has grading given by $\deg p_\lambda = 1$ for all $\lambda \in \Lambda$, and where I is the ***Plücker ideal***, consisting of all polynomial relations among the Plücker coordinates. To state the generators of this ideal, let

$$p_{i_1,\dots,i_d} := \begin{cases} \operatorname{sgn}(\sigma_{i_1,\dots,i_d})p_\lambda & \text{if } J(\lambda) = \{i_1,\dots,i_d\} \text{ for some } \lambda \in \Lambda,\\ 0 & \text{otherwise,}\end{cases}$$

where $\sigma_{i_1,\dots,i_d}$ denotes the permutation that puts the list $i_1,\dots,i_d$ in increasing order. The ideal I is generated by all quadratics of the form

$$\sum_{m=1}^{d+1}(-1)^m p_{i_1,\dots,i_{d-1},j_m}p_{j_1,\dots,\widehat{j_m},\dots,j_{d+1}},$$

for $i_1,\dots,i_{d-1},j_1,\dots,j_{d+1} \in \{1,\dots,n\}$.

Let $\mathbf{w} = (w_\lambda)_{\lambda\in\Lambda} \in \mathbb{Q}^\Lambda$ be a vector of rational numbers. The ***weight*** of a monomial $m(\mathbf{p}) = c\prod_{\lambda\in\Lambda} p_\lambda^{k_\lambda}$ with respect to $\mathbf{w}$ is

$$\mathrm{wt}_\mathbf{w}(m) := \sum_{\lambda\in\Lambda} w_\lambda k_\lambda.$$

Define a homomorphism $u^\mathbf{w} : \mathbb{C}[\mathbf{p}] \to \mathcal{K}[\mathbf{p}]$ by

$$u^\mathbf{w} m(\mathbf{p}) := u^{\mathrm{wt}_\mathbf{w}(m)} m(\mathbf{p})$$

for monomials and extending linearly to $\mathbb{C}[\mathbf{p}]$. If $h(\mathbf{p}) \in \mathbb{C}[\mathbf{p}]$, then the ***initial form*** of h with respect to $\mathbf{w}$, denoted $\mathrm{In}_\mathbf{w}(h)$, is the sum of all monomial terms in h for which the weight of the term is minimised. The ***initial ideal*** of the Plücker ideal I with respect to $\mathbf{w}$ is the ideal

$$\mathrm{In}_\mathbf{w}(I) := \{\mathrm{In}_\mathbf{w}(h) \mid h \in I\}.$$

In this context, the vector $\mathbf{w}$ is called a ***weight vector***.

The scheme $\operatorname{Proj}\mathbb{C}[\mathbf{p}]/\mathrm{In}_\mathbf{w}(I)$ can also be described as follows. Consider the ideal $u^\mathbf{w}I := \{u^\mathbf{w}h \mid h \in I\}$ in $\mathbb{C}[u^{\pm\frac{1}{\delta}};\mathbf{p}]$. Here the variable u has weight 0, and δ is a common denominator of the weights w_λ. Let $\tilde{X}$ be the closure of $\operatorname{Proj}\mathbb{C}[u^{\pm\frac{1}{\delta}};\mathbf{p}]/(u^\mathbf{w}I \otimes \mathbb{C}[u^{\pm\frac{1}{\delta}}])$ inside $\operatorname{Proj}\mathbb{C}[u^{\frac{1}{\delta}};\mathbf{p}] = \operatorname{Proj}\mathbb{C}[\mathbf{p}] \times \operatorname{Spec}\mathbb{C}[u^{\frac{1}{\delta}}]$. $\tilde{X}$ defines a flat family of projective varieties over $\operatorname{Spec}\mathbb{C}[u^{\frac{1}{\delta}}]$, whose fibre at $u^{\frac{1}{\delta}} = \varepsilon$ is denoted $\tilde{X}_\varepsilon$. Each of the fibres $\tilde{X}_\varepsilon$, $\varepsilon \neq 0$, is isomorphic to X; indeed the ring map $h \mapsto u^{-\mathbf{w}} h|_{u^{1/\delta}=\varepsilon}$ induces an isomorphism $\psi_\varepsilon : X \mapsto \tilde{X}_\varepsilon$. We put $\tilde{X}_\varepsilon(\mathbf{a}) := \psi_\varepsilon(X(\mathbf{a}))$. Note that $\tilde{X}_1$ is naturally identified with X. The special fibre $\tilde{X}_0$ is $\operatorname{Proj} \mathbb{C}[\mathbf{p}]/\mathrm{In}_{\mathbf{w}}(I)$.

The same construction can be performed with $\mathbb{C}[u^{\pm\frac{1}{\delta}}; \mathbf{p}]$ and $\mathbb{C}[u^{\frac{1}{\delta}}; \mathbf{p}]$ replaced by $\mathcal{K}[\mathbf{p}]$ and $\mathcal{K}_+[\mathbf{p}]$ respectively. Note that since $u^{\mathbf{w}}$ acts as an automorphism on $\mathcal{K}[\mathbf{p}]$, $\operatorname{Proj} \mathcal{K}[\mathbf{p}]/(u^{\mathbf{w}} I \otimes \mathcal{K}) \cong \operatorname{Proj} \mathcal{K}[\mathbf{p}]/(I \otimes \mathcal{K}) = \mathcal{X}$. Its closure in $\operatorname{Proj} \mathcal{K}_+[\mathbf{p}]$, denoted $\bar{\mathcal{X}}$, is a flat scheme over $\operatorname{Spec} \mathcal{K}_+$. Note that since $\mathbb{C}[u^{\pm\frac{1}{\delta}}] \hookrightarrow \mathcal{K}$, we have a morphism $\bar{\mathcal{X}} \to \tilde{X}$, which is an isomorphism on the fibres at $u = 0$. Though we have suppressed it from our notation, the schemes $\tilde{X}$ and $\bar{\mathcal{X}}$ depend on $\mathbf{w}$.

We will be primarily concerned with the case where the weight vector comes from a diagonally increasing tableau. Let $T \in \mathrm{DIT}(\square; \mathbf{a})$, where $\mathbf{a} \Subset \mathcal{K}^\times$. Then T gives rise to a weight vector $\mathbf{w}(T) = (w_\lambda(T))_{\lambda \in \Lambda}$, where $w_\lambda(T) := \mathrm{val}(T|_{\lambda^c})$, for $\lambda \in \Lambda$.

Lemma 4.6. *For any $T \in \mathrm{DIT}(\square; \mathbf{a})$, the initial ideal $\mathrm{In}_{\mathbf{w}(T)}(I) \subset \mathbb{C}[\mathbf{p}]$ is generated by quadratic binomials*

$$p_\lambda p_{\lambda'} - p_{\lambda \vee \lambda'} p_{\lambda \wedge \lambda'}, \tag{4.11}$$

for all $\lambda, \lambda' \in \Lambda$. Here, $\wedge$ and $\vee$ are the meet and join operators on Λ respectively.

Thus $\mathbf{c} = [c_\lambda]_{\lambda \in \Lambda}$ represents a point in $\tilde{X}_0$ if and only if

$$c_\lambda c_{\lambda'} = c_{\lambda \vee \lambda'} c_{\lambda \wedge \lambda'}, \tag{4.12}$$

for all $\lambda, \lambda' \in \Lambda$. In this case, $\tilde{X}_0$ is the Gel'fand–Tsetlin toric variety.

4.4. Proof of Theorem 4.2

Lemma 4.7. *For $T \in \mathrm{DIT}(\square; \mathbf{a})$, $\mathbf{a} \Subset \mathcal{K}^\times$, let $x \in \mathcal{X}$ be a point satisfying*

$$\mathrm{LT}\big(p_\nu(x)\big) = c_\nu u^{w_\nu(T)} \tag{4.13}$$

for some $[c_\nu]_{\nu \in \Lambda} \in \mathbb{C}^\Lambda$. Then $\mathbf{c} = [c_\nu]_{\nu \in \Lambda}$ satisfies the relations (4.12)*. Conversely, if $\mathbf{c}$ satisfies* (4.12)*, then there is a point $x \in \mathcal{X}$ for which* (4.13) *holds.*

Proof. It is a general fact that a zero of the initial ideal over $\mathbb{C}$ lifts in this way to a zero of the original ideal over $\mathcal{K}$. See [19, Corollary 2.2]. $\square$

Lemma 4.8. *A point $[c_\nu]_{\nu \in \Lambda}$ is a solution to* (4.12) *if and only if for some skew partition λ/μ and some $\omega_1, \ldots, \omega_{|\lambda/\mu|} \in \mathbb{C}^\times$, $c_\nu = c_\lambda \Omega_\nu$ for all $\nu \in \Lambda$.*

Proof. The "if" direction is straightforward. For the "only if" direction, we note that for any solution $[c_\nu]_{\nu \in \Lambda}$ to (4.12), if $c_\nu \neq 0$ and $c_{\nu'} \neq 0$, then $c_{\nu \wedge \nu'} \neq 0$ and $c_{\nu \vee \nu'} \neq 0$. Thus the set $\{\nu \in \Lambda \mid c_\nu \neq 0\}$ has a unique maximal partition λ and a unique minimal partition μ. With this choice of λ and μ, it is now straightforward to check that one can consistently define $\omega_i := c_\alpha / c_\beta$, where $\beta \succ \alpha$ and the unique box of β/α corresponds to ω_i. Thus we have $c_\nu = c_\lambda \Omega_\nu$ for all $\nu \in \Lambda$. $\square$

Proof of Theorem 4.2. First consider the case where $\lambda/\mu = \square$. By Proposition 2.3, a point $x \in \mathcal{X}(\mathbf{a})$ is a solution to the equations:

$$h(\mathbf{p}) = 0 \quad \text{for } h(\mathbf{p}) \in I, \tag{4.14}$$

$$\sum_{\nu \vdash k} q_\nu p_\nu = q_\square E_k(\mathbf{a}) \quad \text{for } 1 \leqslant k \leqslant N. \tag{4.15}$$

If such an x exists and satisfies (4.13), then c_ν, the leading coefficient of p_ν, is of the form Ω_ν for some $\omega_1, \dots, \omega_N$, by Lemma 4.8; thus, taking the leading term of (4.15), we find that Eqs. (4.3) hold.

Conversely, suppose that we have a solution to (4.3). Then by Lemmas 4.8 and 4.7, there is a point $x' \in \mathcal{X}$ satisfying (4.2). Thus $p_\nu = p_\nu(x')$ satisfy (4.14); however, Eqs. (4.15) are only satisfied to first order, i.e. there exists a solution to (4.14) and

$$\sum_{\nu \vdash k} q_\nu p_\nu = q_\square Y_k \quad \text{for } 1 \leqslant k \leqslant N, \tag{4.16}$$

for some $(Y_1, \dots, Y_N) \in \mathcal{U}$, where

$$\mathcal{U} = \big\{(Y_1, \dots, Y_N) \in \mathcal{K}^N \bigm| \operatorname{val}(Y_k) \geqslant \ell_k,\ \big[u^{\ell_k}\big] Y_k = e_k(\mathbf{a})\big\}.$$

Since $\Omega_\nu = u^{-\mathbf{w}(T)} p_\nu(x')|_{u=0}$, we can view $(\omega_1, \dots, \omega_N)$ as the leading coefficients of local coordinates on $\mathcal{X}$ near x'. In these coordinates, the initial form of (4.15) is just (4.3); moreover the Jacobian condition required to apply Hensel's lemma to the system of Eqs. (4.16) is exactly (4.1) (see e.g. [4, Exercise 7.25]). Since this Jacobian condition is assumed to hold, by Hensel's lemma, the points $\mathbf{p}$ satisfying (4.14) and (4.16) are implicitly a function of the $Y_1, \dots, Y_N$, in the neighbourhood $\mathcal{U}$. Since $(E_1(\mathbf{a}), \dots, E_N(\mathbf{a})) \in \mathcal{U}$, there exists a solution to (4.14) and (4.15).

In the case where $\lambda/\mu \neq \square$, we consider a tableau $\tilde{T} \in \mathrm{DIT}(\square; \tilde{\mathbf{a}})$ for which the $|\mu|$ smallest elements of $\tilde{\mathbf{a}}$ form a subtableau of shape μ, the $|\lambda^c|$ largest elements form a subtableau of shape λ^c, and the remaining elements form T. Then $\tilde{T}|_{\lambda/\mu} = T$ and so the result follows from Theorem 4.5(ii). $\square$

4.5. Fibres of the Wronski map over $\mathbb{C}$ and $\mathbb{R}$

We now describe how one can deduce results over $\mathbb{C}$ and $\mathbb{R}$ from Theorem 4.2 and Corollary 4.4, which are stated over $\mathcal{K}$. We will assume implicitly here that the solutions to (4.3) are always distinct (i.e. multiplicity-free), and moreover that (4.1) holds for each solution.

As before, let $\mathbf{a} = \{a_1, \dots, a_N\} \Subset \mathcal{K}^\times$, but now suppose that each $a_i \in \mathbb{C}[u^{\pm\frac{1}{\delta}}]$ is a *Laurent polynomial* in some rational power of u. Thus it makes sense to evaluate a_i at $u^{\frac{1}{\delta}} = \varepsilon$ for $\varepsilon \in \mathbb{C}^\times$. We denote this evaluation $a_i(\varepsilon)$, and put $\mathbf{a}(\varepsilon) := \{a_1(\varepsilon), \dots, a_{|\lambda/\mu|}(\varepsilon)\}$.

We now show that for $|\varepsilon|$ sufficiently small, we can evaluate a point $x \in \mathcal{X}(\mathbf{a})$ at $u^{\frac{1}{\delta}} = \varepsilon$ to obtain a point $x(\varepsilon) \in X(\mathbf{a}(\varepsilon))$. If $x = x_T$ for $T \in \mathrm{SYT}(\square; \mathbf{a})$, then we will declare $x(\varepsilon)$ to be the point corresponding to the tableau $T(\varepsilon)$, obtained by evaluating each entry of T at $\varepsilon \approx 0$. We can make a similar declaration if $T \in \mathrm{DIT}(\square; \mathbf{a})$, in the cases where $T(\varepsilon)$ is actually a tableau; however, this is less refined, as the correspondence over $\mathcal{K}$ may not be one-to-one.

Let $T \in \mathrm{DIT}(\square; \mathbf{a})$. By Theorem 4.2, each such solution $(\omega_1, \dots, \omega_N)$ to (4.3) produces a point $x = x_T \in \mathcal{X}(\mathbf{a})$ satisfying (4.2). Letting $p_\lambda = u^{-\mathbf{w}(T)} p_\lambda(x)$, the coordinates $[p_\lambda]$ define a point $\bar{x} \in \bar{\mathcal{X}}$ over $\operatorname{Spec} \mathcal{K}_+$.

Since the entries a_i are Laurent polynomials in $u^{\frac{1}{\delta}}$, $\bar{x}$ is defined not just over $\mathcal{K}_+$, but over a finite algebraic extension of $\mathbb{C}[u^{\frac{1}{\delta}}]$. This extension is unramified at 0, since the solutions to (4.3) are distinct, and therefore $\bar{x}$ is an analytic function of $u^{\frac{1}{\delta}}$ in some neighbourhood of 0. We define $\bar{x}(\varepsilon)$ to be the evaluation at this function at $u^{\frac{1}{\delta}} = \varepsilon$, and thereby obtain our point $x(\varepsilon) := \psi_\varepsilon^{-1}(\bar{x}(\varepsilon)) \in X(\mathbf{a}(\varepsilon))$, where ψ_ε is the isomorphism $X \to \tilde{X}_\varepsilon$.

Note that $\bar{x}(0)$ has coordinates $[\Omega_\lambda]_{\lambda \in \Lambda}$, which is just the solution to (4.3) that we started with. For $\varepsilon \approx 0$, $\bar{x}(\varepsilon) \approx \bar{x}(0)$. Thus, any time we have a correspondence between points in $X(\mathbf{a})$ and $\mathrm{DIT}(\square; \mathbf{a})$ over $\mathcal{K}$, we obtain a similar correspondence over $\mathbb{C}$, wherein points in the fibre $X(\mathbf{a}(\varepsilon))$ are approximately described by solutions to (4.3), taken over all tableaux $T \in \mathrm{DIT}(\square; \mathbf{a})$. Put another way, (4.2) describes the asymptotic behaviour of $x(\varepsilon)$ as $\varepsilon \to 0$. Specifically,

$$p_\nu\big(x(\varepsilon)\big) \approx \Omega_\nu \varepsilon^{\delta w_\nu(T)},$$

for $\varepsilon \approx 0$.

If $|\varepsilon|$ is sufficiently small, $\|a_i\| < \|a_j\|$ implies that $a_i(\varepsilon)$ is of a smaller order of magnitude than $a_j(\varepsilon)$, i.e. $\log|a_i(\varepsilon)| \ll \log|a_j(\varepsilon)|$. From the proof of Corollary 4.4, we deduce the following:

Corollary 4.9. *Let* $\mathbf{a} = \{a_1, \dots, a_N\} \subset \mathbb{C}$, *with*

$$\log|a_1| \ll \cdots \ll \log|a_N|.$$

Then every tableau $T \in \mathrm{SYT}(\square; \mathbf{a})$ *corresponds to a point* x_T *satisfying*

$$p_\nu(x_T) \approx \Omega_\nu,$$

where $\omega_i = \frac{q_{\alpha_{i-1}} a_i}{q_{\alpha_i}}$, *and* α_i *is the shape of* $T|_{\{a_1,\dots,a_i\}}$.

Corollary 4.10. *Let* $\mathbf{a} = \{a_1, \dots, a_N\} \subset \mathbb{R}$, *with*

$$|a_1| < \cdots < |a_N|.$$

There is a canonical bijective correspondence between tableaux $T \in \mathrm{SYT}(\square; \mathbf{a})$ *and points* $x_T \in X(\mathbf{a})$, *which extends the correspondence of Corollary* 4.9.

Proof. By Corollary 2.2, there is no ambiguity in extending the correspondence, if the roots of the Wronskian are real. □

5. Monodromy and sliding

5.1. When two roots have the same norm

We now consider the case of Theorem 4.2 where $\mathbf{a} = \{a_1, \ldots, a_N\} \Subset \mathcal{K}^\times$, with

$$\|a_1\| < \cdots < \|a_k\| = \|a_{k+1}\| < \cdots < \|a_N\|.$$

From the discussion in Section 4.5, this analysis will describe for us what happens to a fibre $X(\mathbf{a}(\varepsilon))$, when two of the roots have the same order of magnitude, while the others have different orders of magnitude.

Let $c_i := \mathrm{LT}(a_i)u^{-\mathrm{val}(a_i)}$ be the leading coefficient of a_i. We have

$$e_i(\mathbf{a}) = \begin{cases} c_{i+1} \cdots c_N & \text{for } i \neq k, \\ (c_k + c_{k+1})c_{k+2} \cdots c_N & \text{for } i = k. \end{cases}$$

Let $T \in \mathrm{DIT}(\mathbf{a})$. We apply Theorem 4.2 to find points in the fibre $\mathcal{X}(\mathbf{a})$ corresponding to T. There are two cases: either a_k and a_{k+1} are in the same row or column of T, or they are in different rows and columns.

If a_k and a_{k+1} are in the same row or column of T, then $|M_i(T)| = 1$ for all i. Thus, as in the proof of Corollary 4.4, we have $\Omega_{\alpha_i} = \omega_{i+1} \cdots \omega_N$, where α_i is the shape of $T|_{\{a_1,\ldots,a_i\}}$ and unique element in $M_i(T)$. Thus Eqs. (4.3) become

$$q_{\alpha_i}\omega_{i+1} \cdots \omega_N = \begin{cases} q_{\square} c_{i+1} \cdots c_N & \text{if } i \neq k, \\ q_{\square}(c_k + c_{k+1})c_{k+2} \cdots c_N & \text{if } i = k. \end{cases}$$

If we assume that $c_k + c_{k+1} \neq 0$, there is a unique solution for $\omega_1, \ldots, \omega_N$, and the Jacobian condition (4.1) holds at this solution. Hence we deduce that there is a unique point x in the fibre $\mathcal{X}(\mathbf{a})$ corresponding to T, provided $c_k + c_{k+1} \neq 0$.

Unlike in Corollary 4.4, the correspondence is two-to-one. The tableau T' obtained by swapping the positions of a_k and a_{k+1} in T gives rise to the same system of equations, and hence also corresponds to x. Thus we have a choice when identifying x with a tableau T_x. However, sometimes there is a reason to prefer one choice over the other. In keeping with the idea that the entries of a tableau should be (weakly) increasing, if $\log|c_k| \ll \log|c_{k+1}|$, we will put $T_x = T$ if a_k is above or left of a_{k+1}, and $T_x = T'$ otherwise. Similarly if $\log|c_{k+1}| \ll \log|c_k|$, $T_x = T$ if a_{k+1} is above or left of a_k, and $T_x = T'$ otherwise.

If a_k and a_{k+1} are in different rows and columns, there are generally two points in the fibre corresponding to T, and for a certain locus of points of a_k, a_{k+1}, there will be a double point corresponding to T. We begin our analysis by finding this critical locus.

In this case, $|M_i(T)| = 1$ for $i \neq k$, and $M_k(T) = 2$. Let $\alpha_i \in M_i(T)$ be the unique element for $i \neq k$, and $M_k(T) = \{\alpha_k, \alpha'_k\}$. We distinguish the two elements of $M_k(T)$ by asserting that $a_k \in T|_{\alpha_k}$ and $a_{k+1} \in T|_{\alpha'_k}$. We have $\Omega_{\alpha_i} = \omega_{i+1} \cdots \omega_N$, $\Omega_{\alpha_k} = \omega_{k+1}\omega_{k+2} \cdots \omega_N$, $\Omega_{\alpha'_k} = \omega_k\omega_{k+2} \cdots \omega_N$. Thus, the system of Eqs. (4.3) is

$$q_{\alpha_i}\omega_{i+1} \cdots \omega_N = q_{\square} c_{i+1} \cdots c_N \quad \text{for } i \neq k$$

$$q_{\alpha_k}\omega_{k+1}\omega_{k+2} \cdots \omega_N + q_{\alpha'_k}\omega_k\omega_{k+2} \cdots \omega_N = q_{\square}(c_k + c_{k+1})c_{k+2} \cdots c_N,$$

which in turn simplifies to

$$\omega_i = \frac{q_{\alpha_i}}{q_{\alpha_{i-1}}} c_i \quad \text{for } i \neq k, k+1, \tag{5.1}$$

$$\frac{q_{\alpha'_k}\omega_k + q_{\alpha_k}\omega_{k+1}}{q_{\alpha_{k+1}}} = c_k + c_{k+1}, \tag{5.2}$$

$$\frac{q_{\alpha_{k-1}}}{q_{\alpha_{k+1}}}\omega_k\omega_{k+1} = c_k c_{k+1}. \tag{5.3}$$

Eqs. (5.1) give us ω_i for all $i \neq k, k+1$. Solving (5.2) and (5.3) for ω_k, we find that

$$\frac{q_{\alpha'_k}}{q_{\alpha_{k+1}}}\omega_k^2 - (c_k + c_{k+1})\omega_k + \frac{q_{\alpha_k}}{q_{\alpha_{k-1}}} c_k c_{k+1} = 0. \tag{5.4}$$

This equation has a double root when the discriminant is zero:

$$(c_k + c_{k+1})^2 - 4\frac{q_{\alpha_k} q_{\alpha'_k}}{q_{\alpha_{k-1}} q_{\alpha_{k+1}}} c_k c_{k+1} = 0. \tag{5.5}$$

Lemma 5.1. *Let L be the total horizontal and vertical distances between the two boxes in the diagram $\alpha_{k+1}/\alpha_{k-1}$. Then*

$$\frac{q_{\alpha_k} q_{\alpha'_k}}{q_{\alpha_{k-1}} q_{\alpha_{k+1}}} = 1 - L^{-2}.$$

Proof. Suppose the unique box of $\alpha_{k+1}/\alpha'_k = \alpha_k/\alpha_{k-1}$ is in row i_1, and the box of $\alpha_{k+1}/\alpha_k = \alpha'_k/\alpha_{k-1}$ is in row i_2. Then

$$L = \left|(\alpha_{k+1})_{i_1} - (\alpha_{k+1})_{i_2} + i_2 - i_1\right|.$$

We have

$$(\alpha_{k+1})_j = \left(\alpha'_k\right)_j = (\alpha_k)_j = (\alpha_{k-1})_j \quad \text{for } j \neq i_1, i_2,$$

$$(\alpha_{k+1})_{i_1} = 1 + \left(\alpha'_k\right)_{i_1} = (\alpha_k)_{i_1} = 1 + (\alpha_{k-1})_{i_1},$$

$$(\alpha_{k+1})_{i_2} = \left(\alpha'_k\right)_{i_2} = 1 + (\alpha_k)_{i_2} = 1 + (\alpha_{k-1})_{i_2}.$$

Thus by (2.1), we have

$$\frac{q_{\alpha'_k}}{q_{\alpha_{k+1}}} = \prod_{j \neq d+1-i_1} \frac{(j + (\alpha'_k)_{d+1-j}) - (d + 1 - i_1 + (\alpha'_k)_{i_1})}{(j + (\alpha_{k+1})_{d+1-j}) - (d + 1 - i_1 + (\alpha_{k+1})_{i_1})},$$

$$\frac{q_{\alpha_{k-1}}}{q_{\alpha_k}} = \prod_{j \neq d+1-i_1} \frac{(j + (\alpha_{k-1})_{d+1-j}) - (d + 1 - i_1 - (\alpha_{k-1})_{i_1})}{(j + (\alpha_k)_{d+1-j}) - (d + 1 - i_1 + (\alpha_k)_{i_1})}.$$

For $j \neq d + 1 - i_2$, the terms in these two products are equal. Thus,

$$\begin{aligned}\frac{q_{\alpha_k}q_{\alpha'_k}}{q_{\alpha_{k-1}}q_{\alpha_{k+1}}} &= \frac{(-i_2+(\alpha_k)_{i_2}+i_1-(\alpha_k)_{i_1})(-i_2+(\alpha'_k)_{i_2}+i_1-(\alpha'_k)_{i_1})}{(-i_2+(\alpha_{k-1})_{i_2}+i_1-(\alpha_{k-1})_{i_1})(-i_2+(\alpha_{k+1})_{i_2}+i_1-(\alpha_{k+1})_{i_1})}\\ &= \frac{(-i_2+(\alpha_{k+1})_{i_2}+i_1-(\alpha_{k+1})_{i_1}+1)(-i_2+(\alpha_{k+1})_{i_2}-1+i_1-(\alpha_{k+1})_{i_1})}{(-i_2+(\alpha_{k+1})_{i_2}+i_1-(\alpha_{k+1})_{i_1})(-i_2+(\alpha_{k+1})_{i_2}+i_1-(\alpha_{k+1})_{i_1})}\\ &= \frac{(L-1)(L+1)}{L^2}\\ &= 1-L^{-2}. \qquad \square\end{aligned}$$

Lemma 5.2. *The discriminant of* (5.4) *is non-zero if and only if the solutions to* (5.1)–(5.3) *are a point at which the Jacobian condition* (4.1) *holds.*

Proof. The matrix Jacobian matrix J of (4.1) is block upper triangular, with all diagonal blocks non-zero of size 1×1, except for a 2×2 block in rows $k, k+1$. Thus (4.1) holds iff the determinant of this 2×2 block

$$\begin{pmatrix} \frac{\partial}{\partial\omega_k} q_{\alpha_{k-1}}\Omega_{\alpha_{k-1}} & \frac{\partial}{\partial\omega_{k+1}} q_{\alpha_{k-1}}\Omega_{\alpha_{k-1}} \\ \frac{\partial}{\partial\omega_k}(q_{\alpha_k}\Omega_{\alpha_k}+q_{\alpha'_k}\Omega_{\alpha'_k}) & \frac{\partial}{\partial\omega_{k+1}}(q_{\alpha_k}\Omega_{\alpha_k}+q_{\alpha'_k}\Omega_{\alpha'_k}) \end{pmatrix} = \Omega_{\alpha_{k+1}} \begin{pmatrix} q_{\alpha_{k-1}}\omega_{k+1} & q_{\alpha_{k-1}}\omega_k \\ q_{\alpha'_k} & q_{\alpha_k} \end{pmatrix}$$

is non-zero, i.e. iff $q_{\alpha_k}\omega_{k+1} \neq q_{\alpha'_k}\omega_k$.

On the other hand, if (ω_k, ω_{k+1}) is one solution to (5.2) and (5.3), then the other solution is $(q_{\alpha_k}\omega_{k+1}/q_{\alpha'_k}, q_{\alpha'_k}\omega_k/q_{\alpha_k})$. The discriminant of (5.4) is non-zero iff these two solutions are distinct, i.e. iff $q_{\alpha_k}\omega_{k+1} \neq q_{\alpha'_k}\omega_k$. $\square$

Corollary 5.3. *If $c_k, c_{k+1} \in \mathbb{R}$, then the system of Eqs.* (5.2) *and* (5.3) *has two distinct real solutions, hence there are two points in $\mathcal{X}(\mathbf{a})$ corresponding to T, i.e. satisfying* (4.2).

Proof. It is enough to check that the discriminant of (5.4) is positive. Since $q_\lambda > 0$ for all $\lambda \in \Lambda$, this is certainly true if $c_k c_{k+1} < 0$. Otherwise, we have

$$\begin{aligned}(c_k+c_{k+1})^2 - 4\frac{q_{\alpha_k}q_{\alpha'_k}}{q_{\alpha_{k-1}}q_{\alpha_{k+1}}}c_k c_{k+1} &= (c_k+c_{k+1})^2 - 4\big(1-L^{-2}\big)c_k c_{k+1}\\ &> (c_k+c_{k+1})^2 - 4c_k c_{k+1}\\ &= (c_k-c_{k+1})^2\\ &\geqslant 0.\end{aligned}$$

By Lemma 5.2, we can apply Theorem 4.2 to conclude that we have two corresponding points in the fibre $\mathcal{X}(\mathbf{a})$. $\square$

The reason T is identified with two points in $\mathcal{X}(\mathbf{a})$ rather than one is that there is a tie in the order of magnitude of the roots. As in the same-row/column case, the tableau $T' \in \mathrm{DIT}(\square; \mathbf{a})$, obtained by swapping the positions of a_k and a_{k+1} in T, produces the same system of equations, and hence is also identified with these same two points. Thus, we have a two-to-two correspondence between tableaux in $\mathrm{DIT}(\square; \mathbf{a})$ and points in $\mathcal{X}(\mathbf{a})$. Note that between this two-to-two correspondence, and the two-to-one correspondence earlier, we have found all $|\mathsf{SYT}(\square)|$ points in $\mathcal{X}(\mathbf{a})$.

Now suppose that $\log|c_k| \ll \log|c_{k+1}|$. This supposition effectively breaks the tie in the order of magnitude of the roots, which gives a natural way to identify T with one of these two points in $\mathcal{X}(\mathbf{a})$, and T' with the other. To see this, we put $c_k = \bar{u}^{v_1} b_1$ and $c_{k+1} = \bar{u}^{v_2} b_2$, with $v_1 > v_2$, and solve (5.2) and (5.3) over $\mathbb{C}\{\{\bar{u}\}\}$.

Proposition 5.4. *If $c_k = \bar{u}^{v_1} b_1$ and $c_{k+1} = \bar{u}^{v_2} b_2$, and $v_1 > v_2$, then the one solution for (ω_k, ω_{k+1}) satisfies*

$$\mathrm{LT}(\omega_k) = \frac{q_{\alpha_k} c_k}{q_{\alpha_{k-1}}}, \qquad \mathrm{LT}(\omega_{k+1}) = \frac{q_{\alpha_{k+1}} c_{k+1}}{q_{\alpha_k}}, \tag{5.6}$$

and the other satisfies

$$\mathrm{LT}(\omega_{k+1}) = \frac{q_{\alpha'_k} c_k}{q_{\alpha_{k-1}}}, \qquad \mathrm{LT}(\omega_k) = \frac{q_{\alpha_{k+1}} c_{k+1}}{q_{\alpha'_k}}. \tag{5.7}$$

Proof. By Hensel's lemma, there exists a solution for $y_1, y_2 \in \mathcal{K}_+$ to the system of equations

$$\frac{\bar{u}^{v_1 - v_2} q_{\alpha'_k} y_1 + q_{\alpha_k} y_2}{q_{\alpha_{k+1}}} = \bar{u}^{v_1 - v_2} b_1 + b_2,$$

$$\frac{q_{\alpha_{k-1}}}{q_{\alpha_{k+1}}} y_1 y_2 = b_1 b_2$$

with

$$\mathrm{LT}(y_1) = \frac{q_{\alpha_k} b_1}{q_{\alpha_{k-1}}}, \qquad \mathrm{LT}(y_2) = \frac{q_{\alpha_{k+1}} b_2}{q_{\alpha_k}}.$$

Putting $\omega_k = \bar{u}^{v_1} y_1$, $\omega_{k+1} = \bar{u}^{v_2} y_2$ gives the first solution. The other solution is obtained similarly. $\square$

Replacing ω_k, ω_{k+1} by $\mathrm{LT}(\omega_k), \mathrm{LT}(\omega_{k+1})$, these are precisely the solutions to two different systems of Eqs. (4.3) that we obtain if we perturb the norm of the entries so that $\|a_k\| \neq \|a_{k+1}\|$. The first solution (5.6) is the one that is consistent with breaking the tie so that $\|a_1\| < \cdots < \|a_k\| < \|a_{k+1}\| < \cdots < \|a_N\|$. To see this, note that if $\|a_k\| < \|a_{k+1}\|$, then $M_k(T) = \{\alpha_k\}$; thus, as in the proof of Corollary 4.4, the solution to (4.3) is given by (4.6), which is consistent with (5.6). Since $\log|c_k| \ll \log|c_{k+1}|$, this is the solution we identify with T. The second solution (5.7) corresponds to $\|a_1\| < \cdots < \|a_{k-1}\| < \|a_{k+1}\| < \|a_k\| < \cdots < \|a_N\|$, since here $M_k(T) = \{\alpha_{k+1}\}$. This solution is identified with the other tableau, T'.

In summary, suppose that either $\log|c_k| \ll \log|c_{k+1}|$ or $\log|c_k| \gg \log|c_{k+1}|$. Then a solution (ω_k, ω_{k+1}) to (5.2) and (5.3) is identified with the tableau T for which $\omega_k \approx c_k$ and $\omega_{k+1} \approx c_{k+1}$ (up to a ratio of q_αs). If $(\hat{\omega}_k, \hat{\omega}_{k+1})$ denotes the other solution to (5.2) and (5.3) then $\hat{\omega}_k \approx c_{k+1}$ and $\hat{\omega}_{k+1} \approx c_k$, and this solution corresponds to T'.

5.2. Proof of Theorem 3.5

Suppose that $\mathbf{a}_t = \{(a_1)_t, \dots, (a_N)_t\}$, $t \in [0,1]$ is a path in the space of multisubsets of $\mathbb{P}^1(\mathcal{K})$, with $\mathbf{a}_0 = \mathbf{a}$ as in Section 5.1, and for all $t \in [0,1]$, $(a_i)_t = a_i$ if $i \neq k, k+1$, and $\|a_k\| = \|(a_k)_t\| = \|(a_{k+1})_t\| = \|a_{k+1}\|$. Let $(c_i)_t := \mathrm{LC}((a_i)_t)$, and suppose $(c_k)_t$ and $(c_{k+1})_t$, $t \in [0,1]$ are paths in $\mathbb{R}^\times$. Let $x_t \in \mathcal{X}(\mathbf{a}_t)$ be a path in $\mathcal{X}$. Finally, suppose $\log|(c_k)_0| \ll \log|(c_{k+1})_0|$ and $\log|(c_k)_1| \gg \log|(c_{k+1})_1|$. From the discussion in Section 5.1, these hypotheses imply that there are unique tableaux T_{x_0} and T_{x_1} corresponding to points x_0 and x_1. Since this correspondence is defined asymptotically, for other values of $t \in (0,1)$ we do not associate a unique corresponding tableau T_{x_t}.

Theorem 5.5. *With $\mathbf{a}_t$ and x_t, as above, T_{x_0} and T_{x_1} are related as follows.*

- (i) *If $(a_k)_0$ and $(a_{k+1})_0$ are in the same row or column of T_{x_0}, or if $(c_k)_0\,(c_{k+1})_0 > 0$, then T_{x_1} is obtained from T_{x_0} by replacing $(a_k)_0$ with $(a_{k+1})_1$ and $(a_{k+1})_0$ with $(a_k)_1$.*
- (ii) *If $(a_k)_0$ and $(a_{k+1})_0$ are in different rows and columns of T_{x_0} and $(c_k)_0\,(c_{k+1})_0 < 0$, then T_{x_1} is obtained from T_{x_0} by replacing $(a_k)_0$ with $(a_k)_1$ and $(a_{k+1})_0$ with $(a_{k+1})_1$.*

Proof. Let $T_0 = T_{x_0}$, and T_t be the tableau obtained from T_0 by replacing $(a_i)_0$ with $(a_i)_t$ for all i. Let T_t' be the tableau obtained by swapping the positions of $(a_k)_t$ and $(a_{k+1})_t$ in T_t. The point x_t satisfies the conditions of Theorem 4.2 for both tableaux T_t and T_t'; this is true with one exception, noted and handled below. Thus we must have either $T_{x_1} = T_1$ or $T_{x_1} = T_1'$.

If $(a_k)_0$, $(a_{k+1})_0$ are in the same row or column of T_{x_0}, then $T_{x_1} = T_1'$ simply by definition. There is one small problem, however, which is the aforementioned exception to the fact that the point x_t corresponds to both T_t and T_t': when $(c_k)_t + (c_{k+1})_t \neq 0$, we have not established this to be true; in fact it is false. To get around this, note that Theorem 1.2 guarantees that the fibre $\mathcal{X}(\mathbf{a}_t)$ is reduced even if $(c_k)_t + (c_{k+1})_t = 0$. Thus T_{x_1} is unaffected by small perturbations of the path $\mathbf{a}_t$. We can therefore perturb the path $\mathbf{a}_t$ so that $(c_k)_t$ and $(c_{k+1})_t$ become complex paths such that $(c_k)_t + (c_{k+1})_t \neq 0$ for all t, and thence see that $T_{x_1} = T_1'$. This establishes the first case of (i).

For the remaining cases, suppose that $(a_k)_0$, $(a_{k+1})_0$ are in different rows and columns of T_{x_0}. Let $((\omega_k)_t, (\omega_{k+1})_t)$ be the solution to (5.2) and (5.3) which gives rise to the point $x_t \in \mathcal{X}(\mathbf{a}_t)$ via (4.2), and let $((\hat{\omega}_k)_t, (\hat{\omega}_{k+1})_t)$ be the second solution to these equations. In each case, we will need to determine whether x_1 corresponds to T_1 or T_1'. From the discussion at the end of Section 5.1, if x_1 corresponds to T_1, then $(\omega_k)_1 \approx (c_k)_1$ and $(\omega_{k+1})_1 \approx (c_{k+1})_1$. If x_1 corresponds to T_1', then $(\omega_{k+1})_1 \approx (c_k)_1$ and $(\omega_k)_1 \approx (c_{k+1})_1$.

Suppose that $(c_k)_0 > 0$ and $(c_{k+1})_0 > 0$. Since x_0 corresponds to T_0 rather than T_0', we have $\log(\omega_k)_0 \approx \log(c_k)_0$ and $\log(\hat{\omega}_k)_0 \approx \log(c_{k+1})_0$. Since $\log(c_k)_0 \ll \log(c_{k+1})_0$, it follows that $\log(\omega_k)_0 \ll \log(\hat{\omega}_k)_0$. By Corollary 5.3, $(\omega_k)_t \neq (\hat{\omega}_k)_t$ for all $t \in [0,1]$; thus $(\omega_k)_t > (\hat{\omega}_k)_t > 0$ for all $t \in [0,1]$. Since $\log(c_{k+1})_1 \gg \log(c_k)_1$, it must be the case that $(\omega_k)_1 \approx (c_{k+1})_1$ and $(\hat{\omega}_k)_1 \approx (c_k)_1$, rather than the other way around. Thus we see that $T_{x_1} = T_1'$. Similarly, we have $T_{x_1} = T_1'$ if $(c_k)_0 < 0$ and $(c_{k+1})_0 < 0$.

Now suppose $(c_k)_0 > 0$ and $(c_{k+1})_0 < 0$. Then we must also have $(\omega_k)_0 \approx (c_k)_0 > 0$. Since $(c_k)_t\,(c_{k+1})_t \neq 0$ for all t, by (5.3), the signs of $(\omega_k)_t$ and $(c_{k+1})_t$ are independent of t. In particular, $(\omega_k)_1$ is positive, while $(c_{k+1})_1$ is negative. Since these have opposite signs, it cannot be the case that $(\omega_k)_1 \approx (c_{k+1})_1$, hence x_1 is not identified with T_1'. We must therefore have $T_{x_1} = T_1$. Similarly, $T_{x_1} = T_1$ if $(c_k)_0 < 0$ and $(c_{k+1})_0 > 0$. $\square$

Theorem 3.5 now follows.

Proof of Theorem 3.5. Let $\mathbf{a}_t = \{(a_1)_t, \dots, (a_N)_t\} \subset \mathbb{RP}^1$, $t \in [0,1]$ is a path in the space of multisubsets of $\mathbb{RP}^1$.

First, consider the case where

$$\log\big|(a_1)_t\big| \ll \cdots \ll \log\big|(a_k)_t\big|, \qquad \log\big|(a_{k+1})_t\big| \ll \cdots \ll \log\big|(a_N)_t\big|,$$

$\log|(a_k)_0| \ll \log|(a_{k+1})_0|$ and $\log|(a_{k+1})_1| \ll \log|(a_k)_1|$. Let $x_t \in X(\mathbf{a}_t)$. Then by Theorem 5.5 and the discussion in Section 4.5 we see that $T_{x_1} = \mathrm{slide}_{a_1}(T_{x_0})$.

Second, suppose that

$$\big|(a_1)_t\big| < \cdots < \big|(a_k)_t\big|, \qquad \big|(a_{k+1})_t\big| < \cdots < \big|(a_N)_t\big|,$$

$|(a_k)_0| < |(a_{k+1})_0|$ and $|(a_{k+1})_1| < |(a_k)_1|$. There is an order and sign preserving homotopy between this case and the previous. Since the correspondence of Corollary 4.10 between points in $X(\mathbf{a})$ and $\mathrm{SYT}(\square; \mathbf{a})$ is established via such homotopies, the theorem holds in this case also.

Finally, a general path $\mathbf{a}_t \Subset \mathbb{RP}^1$ can be regarded as a concatenation of paths from the second case; thus the theorem is true for any real path. □

5.3. Monodromy around special loops

Let λ/μ be a skew partition fitting inside $\square$. Throughout the rest of this section, we will assume that k, L are positive integers with $1 \leqslant k < |\lambda/\mu|$, and $L \geqslant 2$.

For any such k, L, define a permutation $s_{k,L}$ of the set $\mathsf{SYT}(\lambda/\mu)$ as follows. For $\mathsf{T} \in \mathsf{SYT}(\lambda/\mu)$, $s_{k,L}(\mathsf{T})$ is the tableau obtained by swapping entries k and $k+1$ in T, if the total of the horizontal and vertical distance between k and $k+1$ equals L; otherwise $s_{k,L}(\mathsf{T}) = \mathsf{T}$.

If $\mathbf{a} = \{a_1, \dots, a_{|\lambda/\mu|}\} \subset \mathbb{FP}^1$, then we define $s_{k,L}(T)$ for $T \in \mathrm{SYT}(\lambda/\mu; \mathbf{a})$, to satisfy $\mathrm{ord}(s_{k,L}(T)) = s_{k,L}(\mathrm{ord}(T))$. If $\mathbf{a} \subset \mathbb{R}$, we also define $s_{k,L}(x)$ for $x \in X(\mathbf{a}^+)$, by $T_{s_{k,L}(x)} = s_{k,L}(T_x)$.

Theorem 5.6. *Fix k and L as above, and let $\mathbf{a} = \{a_1, \dots, a_{|\lambda/\mu|}\} \subset \mathbb{RP}^1$. There exists a loop $\mathbf{a}_t \subset \mathbb{C}$, $t \in [0,1]$, based at $\mathbf{a}$ such that the monodromy of the Wronski map around $\mathbf{a}_t$ is given by $s_{k,L}$. That is, $\mathbf{a}_0 = \mathbf{a}_1 = \mathbf{a}$, every fibre $X(\mathbf{a}_t)$ is reduced, and if $x_t \in X(\mathbf{a}_t)$ then $x_1 = s_{k,L}(x_0)$.*

Proof. By the discussion in Section 4.5, it is enough to prove the result over $\mathcal{K}$, with $\mathbf{a}$ as in Section 5.1.

Consider a loop $\mathbf{a}_t = \{(a_1)_t, \dots, (a_N)_t\}$, $t \in [0,1]$ in the space of multisubsets of $\mathbb{P}^1(\mathcal{K})$, with $\mathbf{a}_0 = \mathbf{a}_1 = \mathbf{a}$, and for all $t \in [0,1]$, $(a_i)_t = a_i$ if $i \neq k, k+1$, and $\|a_k\| = \|(a_k)_t\| = \|(a_{k+1})_t\| = \|a_{k+1}\|$. Let $(c_i)_t := \mathrm{LC}((a_i)_t)$. Suppose that $((c_k)_t, (c_{k+1})_t) \in \mathbb{C}^2$ is a small loop around the line

$$\big\{(c_k, c_{k+1}) \bigm| c_k = \big(1 + 2L^{-2} + 2L^{-1}\sqrt{1 + L^{-2}}\big)c_{k+1}\big\}. \tag{5.8}$$

We show that if $x_t \in \mathcal{X}(\mathbf{a}_t)$, then $x_1 = s_{k,L}(x_0)$.

Recall that a tableau $T_t \in \mathrm{DIT}(\square; \mathbf{a}_t)$ corresponds to one or two points. If the distance between a_k and a_{k+1} is 1 then these entries are in the same row or column, so T_{x_t} corresponds to the single point x_t; hence $x_1 = x_0 = s_{k,L}(x_0)$. Otherwise, T_{x_t} corresponds to x_t and another point. Since the equations that give the leading terms of these two points, (5.2) and (5.3), define a quadratic map

$$(\omega_k, \omega_{k+1}) \mapsto \left(\frac{q_{\alpha'_k}\omega_k + q_{\alpha_k}\omega_{k+1}}{q_{\alpha_{k+1}}}, \frac{q_{\alpha_{k-1}}}{q_{\alpha_{k+1}}}\omega_k\omega_{k+1}\right),$$

the two points will swap places under the monodromy of the loop $\mathbf{a}_t$ if and only if the leading coefficients of $\mathbf{a}_t$ wrap around the critical locus (5.5). By Lemma 5.1, the line (5.8) is contained in the critical locus iff the distance between $(a_k)_t$ and $(a_{k+1})_t$ is L. □

5.4. Monodromy and limits

Let $\mathbf{a} = \{a_1, \dots, a_N\} \subset \mathbb{RP}^1$, with $|a_1| < \cdots < |a_N|$, and let $x \in X(\mathbf{a})$.

Definition 5.7. Suppose that we have a decomposition of $\mathbb{RP}^1$ as the disjoint union of k intervals $I_1, \dots, I_k$. We then obtain a partition $(\mathbf{b}_1, \dots, \mathbf{b}_k)$ of $\mathbf{a}$, where $\mathbf{b}_i = \mathbf{a} \cap I_i$. A partition of $\mathbf{a}$ of this form is called a ***consecutive partition*** of $\mathbf{a}$. Any number $c_i \in I_i$ is called an ***internal point*** for $\mathbf{b}_i$.

The two main cases we will consider are given in the examples below.

Example 5.8. For any $\mathbf{a}$ as above, let $\mathbf{b}_0 = \{a_1, \dots, a_i\}$, $\mathbf{b}_\infty = \{a_{i+1}, \dots, a_N\}$ for some i. Then $(\mathbf{b}_0, \mathbf{b}_\infty)$ is a consecutive partition of $\mathbf{a}$. Moreover, for any tableau $T \in \mathrm{SYT}(\square; \mathbf{a})$, $T|_{\mathbf{b}_0}$ and $T|_{\mathbf{b}_\infty}$ are both subtableaux of T.

Example 5.9. Let $\mathbf{b} = \{a_i, a_{i+1}, \dots, a_j\}$, $\mathbf{b}^c = \mathbf{a} \setminus \mathbf{b}$, for some $i \leqslant j$. Suppose that all elements of $\mathbf{b}$ have the same sign. Then $(\mathbf{b}, \mathbf{b}^c)$ is a consecutive partition of $\mathbf{a}$. In this case, for any tableau $T \in \mathrm{SYT}(\square; \mathbf{a})$, $T|_{\mathbf{b}}$ is a subtableau of T.

Suppose we have a consecutive partition $(\mathbf{b}_1, \dots, \mathbf{b}_k)$ of $\mathbf{a}$, coming from intervals $I_1, \dots, I_k \subset \mathbb{RP}^1$. Let c_i be an internal point for b_i. We define points $x_{[\mathbf{b}_i \to c_i]} \in X$, as follows.

For each fixed i, we form a path $\mathbf{a}_t \Subset \mathbb{RP}^1$, $t \in [0, 1]$ satisfying the following conditions:

(i) $\mathbf{a}_0 = \mathbf{a}$;
(ii) $\mathbf{a}_t$ is a set for $t \in [0, 1)$;
(iii) $\mathbf{a}_t \cap I_j = \mathbf{b}_j$, for $j \neq i$, $t \in [0, 1]$;
(iv) $\mathbf{a}_1 = (\bigcup_{j \neq i} \mathbf{b}_j) \cup \{c_i, c_i, \dots, c_i\}$.

Let $x_0 = x$, and $x_t \in X(\mathbf{a}_t)$. Since $\mathbf{a}_t$ is a set for $x_t \in [0, 1)$ there is a unique such path for $t \in [0, 1)$. We define $x_{[\mathbf{b}_i \to c_i]}$ to be the limit point $x_1 = \lim_{t \to 1} x_t$. By Corollary 2.2, $x_{[\mathbf{b}_i \to c_i]}$ depends only on $\mathbf{b}_i$ and c_i, not on the path chosen.

Since $\mathbf{a}_1$ is a multiset, the fibre $X(\mathbf{a}_1)$ is typically non-reduced, so there may be distinct points $x, x' \in X(\mathbf{a})$ with the same limit point $x_{[\mathbf{b}_i \to c_i]} = x'_{[\mathbf{b}_i \to c_i]}$. This defines an equivalence relation on the fibre $X(\mathbf{a})$, which we will study further in Section 6.

Theorem 5.10. *Let* $\mathbf{a}$, $(\mathbf{b}, \mathbf{b}^c)$ *be as in Example* 5.9, *and let* $x, x' \in X(\mathbf{a})$. *Suppose that* $L \geqslant 2$, *and* $k \notin \{i-1, i, \dots, j\}$. *For any internal point* c_1 *for* $\mathbf{b}$, *we have* $x_{[\mathbf{b} \to c_1]} = x'_{[\mathbf{b} \to c_1]}$ *if and only if* $s_{k,L}(x)_{[\mathbf{b} \to c_1]} = s_{k,L}(x')_{[\mathbf{b} \to c_1]}$.

Proof. In order to see what happens to the point $x_t \in X(\mathbf{a}_t)$, as t approaches 1, we need to study the fibre $X(\mathbf{a}_t)$ when $|a_i|, \dots, |a_j|$ are close to each other. Working over $\mathcal{K}$, this corresponds to looking at $\mathcal{X}(\mathbf{a})$ where, $\mathbf{a} = \{a_1, \dots, a_N\} \subset \mathcal{K}^\times$ and

$$\|a_1\| \leqslant \cdots \leqslant \|a_{i-1}\| < \|a_i\| = \cdots = \|a_j\| < \|a_{j+1}\| \leqslant \cdots \leqslant \|a_N\|. \tag{5.9}$$

Let T be a weakly increasing tableau of shape $\square$ with values in $\mathbf{a}$. Let μ denote the shape of $T|_{\{a_1,\dots a_{i-1}\}}$, and let λ be the shape of $T|_{\{a_1,\dots a_j\}}$. We will assume, moreover, that $T|_\mu$ and T_{λ^c} are diagonally increasing.

We say T corresponds to a point $x \in \mathcal{X}(\mathbf{a})$ if (4.5) holds. Since T may not be a diagonally increasing tableau, Theorem 4.2 no longer provides us with the explicit system of equations needed to find the leading terms of the points in $\mathcal{X}(\mathbf{a})$ corresponding to T. However, it is still possible to write down such a system of equations by following the same lines of argument. The main difference between this case and the analysis in the proof of Theorem 4.2 is that initial ideal of the Plücker ideal $\mathrm{In}_{\mathbf{w}(T)}(I)$ is not binomial; it represents only a partial degeneration of X to the Gel'fand–Tsetlin toric variety.

We consider the initial forms of Eqs. (4.14) and (4.15) to obtain a system of equations for the leading terms of the Plücker coordinates of a point corresponding to T. This system will necessarily be solvable, because the equations can be further degenerated to the case of Theorem 4.2, which is solvable. The initial forms of (4.14) are given by $\mathrm{In}_{\mathbf{w}(T)}(I)$. We have

$$p_\nu p_{\nu'} - p_{\nu\wedge\nu'} p_{\nu\vee\nu'} \in \mathrm{In}_{\mathbf{w}(T)}(I) \quad \text{if } \nu \leqslant \mu \text{ or } \nu \geqslant \lambda\ . \tag{5.10}$$

The other quadratic relations in $\mathrm{In}_{\mathbf{w}(T)}(I)$ are more complicated; however they only involve partitions which are between μ and λ. Moreover the initial forms of (4.15) only involve partitions in this range. From this, one can see that the system of equations one obtains for $\mathrm{LT}(p_\nu)$ for $\mu \leqslant \nu \leqslant \lambda$ depends only on $\{a_i, \dots, a_j\}$ and the shapes λ and μ.

Moreover given a solution to these equations, we can solve for all remaining $\mathrm{LT}(p_\nu)$. For $\nu \leqslant \mu$, the equations determining $\mathrm{LT}(p_\nu)$, up to a constant, are the same as those given by Theorem 4.2 applied to the tableau $T|_\mu$. The constant is determined by the fact that we already have a value for $\mathrm{LT}(p_\mu)$. Similarly, for $\nu \geqslant \lambda$, the equations for $\mathrm{LT}(p_\nu)$ are given by Theorem 4.2 applied to $T|_{\lambda^c}$. All other $\mathrm{LT}(p_\nu)$ are determined by (5.10).

Now suppose that all inequalities in (5.9) are strict, except for $\|a_k\| = \|a_{k+1}\|$. Consider a loop $\mathbf{a}_t = \{(a_1)_t, \dots, (a_N)_t\}$, $t \in [0,1]$ in the space of multisubsets of $\mathbb{P}^1(\mathcal{K})$, with $\mathbf{a}_0 = \mathbf{a}_1 = \mathbf{a}$, and for all $t \in [0,1]$, $(a_l)_t = a_l$ if $i \neq k, k+1$, $\|a_k\| = \|(a_k)_t\| = \|(a_{k+1})_t\| = \|a_{k+1}\|$. Let $x_t, x'_t \in \mathcal{X}(\mathbf{a}_t)$.

Given a sufficiently small positive real number ε, suppose x_0 and x'_0 are "close together," in that they have the following properties: for all $\nu \in \Lambda$, $\mathrm{val}(p_\nu(x_0)) = \mathrm{val}(p_\nu(x'_0))$, and

$$1 - \varepsilon < \left|\frac{\mathrm{LT}(p_\nu(x_0))}{\mathrm{LT}(p_\nu(x'_0))}\right| < 1 + \varepsilon.$$

Since the valuation of Plücker coordinates of x_t, x'_t will be independent of t, the points x_t and x'_t must correspond to the same weakly increasing tableau T_t. We claim, moreover, that

$$\frac{\mathrm{LT}(p_\nu(x_0))}{\mathrm{LT}(p_\nu(x'_0))} = \frac{\mathrm{LT}(p_\nu(x_t))}{\mathrm{LT}(p_\nu(x'_t))} \quad \text{for all } \nu \in \Lambda,\ t \in [0,1]. \tag{5.11}$$

This follows from the discussion above. If $\mu \leqslant \nu \leqslant \lambda$, (5.11) is true because $\mathrm{LT}(p_\nu(x_t))$ and $\mathrm{LT}(p_\nu(x'_t))$ are both independent of t. If $\nu \leqslant \mu$ (or $\nu \geqslant \lambda$), (5.11) is true because $\mathrm{LT}(p_\nu(x_t))$ and $\mathrm{LT}(p_\nu(x'_t))$ must come from the same solution to (4.3) for the tableau $T_t|_\mu$ (resp. $T_t|_{\lambda^c}$). For all other $\nu \in \Lambda$, the claim follows from (5.10).

In particular, we see that x_1 and x'_1 are close together. Taking our loop to be the loop whose monodromy is given by $s_{k,L}$ (as defined in the proof of Theorem 5.6), the result follows. □

6. Equivalence, dual equivalence, and the Littlewood–Richardson rule

6.1. Interpretations of equivalence and dual equivalence

Throughout this section, we assume that $\mathbf{a} = \{a_1, \dots, a_N\} \subset \mathbb{RP}^1$, with $|a_1| < \cdots < |a_N|$. We now show that in the situation in Example 5.9, the equivalence relations on $X(\mathbf{a})$ defined in Section 5.4 by $x_{[\mathbf{b}_i \to c_i]} = x'_{[\mathbf{b}_i \to c_i]}$ are combinatorially described by the equivalence and dual equivalence relations on tableaux.

We will need the following lemma:

Lemma 6.1. *Let* $(\mathbf{b}_0, \mathbf{b}_\infty)$ *be as in Example* 5.8. *Let* $T \in \mathrm{SYT}(\square; \mathbf{a})$ *and let* $x_T \in X(\mathbf{a})$ *be the corresponding point. Let* μ *be the shape of* $T|_{\mathbf{b}_0}$.

(i) *The point* $(x_T)_{[\mathbf{b}_0 \to 0]}$ *is in* $X_\mu(0)$ *and corresponds to the tableau* $T|_{\mathbf{b}_\infty}$.
(ii) *The point* $(x_T)_{[\mathbf{b}_\infty \to \infty]}$ *is in* $X_{\mu^\vee}(\infty)$ *and corresponds to the tableau* $T|_{\mathbf{b}_0}$.

Proof. This follows from Theorem 4.5. □

For $\phi \in \mathrm{SL}_2(\mathbb{R})$ and $T \in \mathrm{SYT}(\square; \mathbf{a})$, define $\phi(T) := \mathrm{slide}_{\phi(\mathbf{a})}(T)$. Here sliding is defined using any path homotopic to a path of the form $\phi_t(\mathbf{a})$, $t \in [0,1]$, where $\phi_t \in \mathrm{SL}_2(\mathbb{R})$ is any path from $\phi_0 = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ to $\phi_1 = \phi$. From Theorems 3.3 and 3.5, we have that $\phi(x_T) = x_{\phi(T)}$; hence $\phi(T)$ does not depend on the choice of ϕ_t.

Theorem 6.2. *Let* $T, T' \in \mathrm{SYT}(\square; \mathbf{a})$, *and let* $(\mathbf{b}, \mathbf{b}^c)$ *be as in Example* 5.9. *Choose any internal point* c_2 *for* $\mathbf{b}^c$. *Let* $x_T, x_{T'} \in X(\mathbf{a})$ *be the points corresponding to* T *and* T' *respectively. Then* $T|_{\mathbf{b}} \sim T'|_{\mathbf{b}}$ *if and only if* $(x_T)_{[\mathbf{b}^c \to c_2]} = (x_{T'})_{[\mathbf{b}^c \to c_2]}$.

Proof. Since the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{RP}^1$ can take any three points to any other three points in the same orientation, there exists $\phi \in \mathrm{SL}_2(\mathbb{R})$ be such that $\phi(c_2) = \infty$, $|\phi(a)| < 1$ for $a \in \mathbf{b}$, and $|\phi(a)| > 1$ for $a \in \mathbf{b}^c$.

Consider $\phi(T)$ and $\phi(T')$. We can compute these, via a path $\mathbf{a}_t$ which first rectifies $T|_{\mathbf{b}}, T'|_{\mathbf{b}}$. By Theorem 3.3, it follows that $\phi(T)|_{\phi(\mathbf{b})} = \mathrm{slide}_{\phi(\mathbf{b})}(\mathrm{rect}(T|_{\mathbf{b}}))$ and $\phi(T')|_{\phi(\mathbf{b})} = \mathrm{slide}_{\phi(\mathbf{b})}(\mathrm{rect}(T'|_{\mathbf{b}}))$. Thus we have that $T|_{\mathbf{b}} \sim T'|_{\mathbf{b}}$ if and only if $\phi(T)|_{\phi(\mathbf{b})} = \phi(T')|_{\phi(\mathbf{b})}$.

By Proposition 2.1, $(x_T)_{[\mathbf{b}^c \to c_2]} = (x_{T'})_{[\mathbf{b}^c \to c_2]}$ if and only if $(x_{\phi(T)})_{[\phi(\mathbf{b}^c) \to \infty]} = (x_{\phi(T')})_{[\phi(\mathbf{b}^c) \to \infty]}$. By Lemma 6.1(ii), this holds if and only if $\phi(T)|_{\phi(\mathbf{b})} = \phi(T')|_{\phi(\mathbf{b})}$. □

Remark 6.3. Let $\mathbf{c} = \{a_1, \dots, a_{i-1}\} \subset \mathbf{a}$ be the entries of T to the left of $T|_{\mathbf{b}}$. As an addendum to the proof of Theorem 6.2, we give a quick proof of the fact that $\mathrm{rect}(T|_{\mathbf{b}}) = \mathrm{slide}_{T|_{\mathbf{c}}}(T|_{\mathbf{b}})$ does not depend on $T|_{\mathbf{c}}$.

Proof. Keeping the same notation, assume now that $c_2 = 0$. Consider $x_0 = (x_T)_{[\mathbf{c}\to 0]}$. By Lemma 6.1(i), x_0 corresponds to the tableau obtained by deleting the entries in $\mathbf{c}$ from T. Now, $\phi(x_0) = \phi(x_T)_{[\phi(\mathbf{c})\to\infty]}$, so by Lemma 6.1(ii), $T_{\phi(x_0)}$ is the tableau obtained by deleting the entries in $\phi(\mathbf{c})$ from $\phi(T)$. Since $\operatorname{rect}(T|_{\mathbf{b}})$ can be determined from $\phi(T)|_{\phi(\mathbf{b})}$, it can also be determined from x_0, which does not depend on $T|_{\mathbf{c}}$. $\square$

Theorem 6.4. *Let $T, T' \in \mathrm{SYT}(\square; \mathbf{a})$, and let $(\mathbf{b}, \mathbf{b}^c)$ be as in Example 5.9. Choose any internal point c_1 for $\mathbf{b}$. Let $x_T, x_{T'} \in X(\mathbf{a})$ be the points corresponding to T and T' respectively.*

(i) *If $T|_{\mathbf{b}^c} \neq T'|_{\mathbf{b}^c}$, then $(x_T)_{[\mathbf{b}\to c_1]} \neq (x_{T'})_{[\mathbf{b}\to c_1]}$.*
(ii) *If $T|_{\mathbf{b}^c} = T'|_{\mathbf{b}^c}$, then $T|_{\mathbf{b}} \sim^* T'|_{\mathbf{b}}$ if and only if $(x_T)_{[\mathbf{b}\to c_1]} = (x_{T'})_{[\mathbf{b}\to c_1]}$.*
(iii) *The point $(x_T)_{[\mathbf{b}\to c_1]}$ is in $X_\lambda(c_1)$, where λ is the rectification shape of $T|_{\mathbf{b}}$.*

(Note that $T|_{\mathbf{b}^c}$, $T'|_{\mathbf{b}^c}$ will generally not be subtableaux of T and T'.)

Proof. Let $\mathbf{a}_t = \{(a_1)_t, \ldots, (a_N)_t\} \Subset \mathbb{RP}^1$ be a path used to define $(x_T)_{[\mathbf{b}\to c_1]}$ and $(x_{T'})_{[\mathbf{b}\to c_1]}$. Let $T_t = \operatorname{slide}_{\mathbf{a}_t}(T)$, $T'_t = \operatorname{slide}_{\mathbf{a}_t}(T')$. Let $\mathbf{b}_t = \mathbf{a}_t \setminus \mathbf{b}^c$. Assume that the path of each $(a_i)_t \in \mathbf{b}_t$ is monotonic. Then for all of $t \in [0, 1)$, $T_t|_{\mathbf{b}_t} \sim^* T|_{\mathbf{b}}$, and $T'_t|_{\mathbf{b}_t} \sim^* T'|_{\mathbf{b}}$. We may therefore replace T by $T_{1-\varepsilon}$; hence we may assume that all elements of $\mathbf{b}$ are arbitrarily close to c_1.

Let $\phi \in \mathrm{SL}_2(\mathbb{R})$ be a transformation such that $\phi(c_1) = 0$, and consider $\phi(T)$ and $\phi(T')$. Since the elements of $\phi(\mathbf{b})$ are assumed to be close to zero, by Lemma 6.1(i), $\phi(T)|_{\phi(\mathbf{b}^c)} = \phi(T')|_{\phi(\mathbf{b}^c)}$ if and only if $(x_{\phi(T)})_{[\phi(\mathbf{b})\to 0]} = (x_{\phi(T')})_{[\phi(\mathbf{b})\to 0]}$. By Proposition 2.1, this holds if and only if $(x_T)_{[\mathbf{b}\to c_1]} = (x_{T'})_{[\mathbf{b}\to c_1]}$. Thus, to prove (i) and (ii), we must therefore show that $\phi(T)|_{\phi(\mathbf{b}^c)} = \phi(T')|_{\phi(\mathbf{b}^c)}$ if and only if $T|_{\mathbf{b}} \sim^* T'|_{\mathbf{b}}$ and $T|_{\mathbf{b}^c} = T'|_{\mathbf{b}^c}$.

Let $\mathbf{c} = \{a_1, \ldots, a_{i-1}\} \subset \mathbf{a}$, be the entries in the subtableau of T to the left of $T|_{\mathbf{b}}$. Let $\hat{\mathbf{c}} = \{a_{j+1}, \ldots, a_N\} \subset \mathbf{a}$, be entries in the subtableau of T to the right of $T|_{\mathbf{b}}$. Note that $\phi(T), \phi(T')$ can be computed by a path that brings the values in $\mathbf{b}$ past the values of $\mathbf{c}$ without changing their relative order. Thus by definition of dual equivalence, if $T|_{\mathbf{b}} \sim^* T'|_{\mathbf{b}}$ and $T|_{\mathbf{b}^c} = T'|_{\mathbf{b}^c}$ then $\phi(T)|_{\phi(\mathbf{b}^c)} = \phi(T')|_{\phi(\mathbf{b}^c)}$.

Conversely, suppose $\phi(T)|_{\phi(\mathbf{b}^c)} = \phi(T')|_{\phi(\mathbf{b}^c)}$. We can recover $T|_{\mathbf{b}^c}$ and $T'|_{\mathbf{b}^c}$ by sliding (the answer does not depend on $\phi(T)|_{\phi(\mathbf{b})}, \phi(T')|_{\phi(\mathbf{b})}$); hence we must have $T|_{\mathbf{b}^c} = T'|_{\mathbf{b}^c}$. Moreover, from the argument of the reverse direction, we see that $\operatorname{slide}_{T|_{\mathbf{b}}}(T|_{\mathbf{c}}) = \operatorname{slide}_{T'|_{\mathbf{b}}}(T'|_{\mathbf{c}})$. By the same reasoning with 0 replaced by ∞, we have $\operatorname{slide}_{T|_{\mathbf{b}}}(T|_{\hat{\mathbf{c}}}) = \operatorname{slide}_{T'|_{\mathbf{b}}}(T'|_{\hat{\mathbf{c}}})$.

Let λ/μ be the shape of $T|_{\mathbf{b}}$ and $T'|_{\mathbf{b}}$. To show that $T|_{\mathbf{b}} \sim^* T'|_{\mathbf{b}}$, we must show that $\operatorname{slide}_{T|_{\mathbf{b}}}(V) = \operatorname{slide}_{T'|_{\mathbf{b}}}(V)$, for any tableau V in $\mathrm{SYT}(\mu; \mathbf{c})$ or in $\mathrm{SYT}(\lambda^c; \hat{\mathbf{c}})$. Since we already know this for $V = T|_{\mathbf{c}}$ and $V = T|_{\hat{\mathbf{c}}}$, and since the operators $s_{k,L}$ act transitively on $\mathrm{SYT}(\mu; \mathbf{c})$ and on $\mathrm{SYT}(\lambda^c; \hat{\mathbf{c}})$, the result now follows from Theorem 5.10.

Finally, for (iii) we have already seen that $\phi(T)|_{\phi(\mathbf{b}^c)}$ has shape λ^c, where λ is the rectification shape of $T|_{\mathbf{b}}$. By Lemma 6.1(i), we have that $(x_{\phi(T)})_{[\phi(\mathbf{b})\to 0]} \in X_\lambda(0)$, and so by Proposition 2.1, $(x_T)_{[\mathbf{b}\to c_1]} \in X_\lambda(c_1)$. $\square$

Remark 6.5. In the proof of Theorem 6.4, we showed that if $\operatorname{slide}_{T|_{\mathbf{b}}}(V) = \operatorname{slide}_{T'|_{\mathbf{b}}}(V)$ for some $V \in \mathrm{SYT}(\mu; \mathbf{c})$, then the same is true for every $V \in \mathrm{SYT}(\mu; \mathbf{c})$. In fact our argument shows that if $\operatorname{slide}_{T|_{\mathbf{b}}}(V) = \operatorname{slide}_{T'|_{\mathbf{b}}}(V)$ for any $V \in \mathrm{SYT}(\mu; \mathbf{c})$, then $(x_T)_{[\mathbf{b}\to c_1]} = (x_{T'})_{[\mathbf{b}\to c_1]}$, whence $T|_{\mathbf{b}} \sim^* T'|_{\mathbf{b}}$. This combinatorial fact is a theorem of Haiman (see [9, Theorem 2.10]).

6.2. Combinatorial consequences

A number of other combinatorial facts about equivalence and dual equivalence can be reproved using Theorems 6.2 and 6.4.

Corollary 6.6. *The size of a dual equivalence class with rectification shape λ is $|\mathsf{SYT}(\lambda)|$.*

Proof. Let $(\mathbf{b}, \mathbf{b}^c)$, be a partition of $\mathbf{a}$, as in Example 5.9. Let c_1 be an internal point for $\mathbf{b}$, and let $\mathbf{a}_t$ be the path used to define $x_{[\mathbf{b}\to c_1]}$. By Theorem 6.4, a dual equivalence class with rectification shape λ corresponds to a point in $X(\mathbf{a}_1)$ supported on $X_\lambda(c_1)$. Since Wr is flat, the size of the dual equivalence class is the multiplicity of the point in $X(\mathbf{a}_1)$. By Corollary 2.6, the multiplicity of such a point is $|\mathsf{SYT}(\lambda)|$. □

We can also prove a fact that was used in Section 3.4 to give an alternate formulation on the Littlewood–Richardson rule.

Corollary 6.7. *There is a unique tableau in the intersection of any equivalence class of tableaux with a dual equivalence class of the same rectification shape.*

We need an additional lemma.

Lemma 6.8. *Let $(\mathbf{b}_1, \mathbf{b}_2)$ be a consecutive partition of $\mathbf{a} \subset \mathbb{RP}^1$, and let c_1, c_2 be internal points for $\mathbf{b}_1, \mathbf{b}_2$ respectively. Let $x_1 \in X(\{c_1, \dots, c_1\} \cup \mathbf{b}_2)$, and $x_2 = X(\mathbf{b}_1 \cup \{c_2, \dots, c_2\})$.*

(i) *If $x_1 \in X_\lambda(c_1)$ and $x_2 \in X_{\lambda^\vee}(c_2)$ for some $\lambda \in \Lambda$, then there exists a unique point $x \in X(\mathbf{a})$ such that $x_i = x_{[\mathbf{b}_i \to c_i]}$ for $i = 1, 2$.*
(ii) *If no such λ exists then no such point x exists.*

Proof. It suffices to prove this when $c_1 = 0$ and $c_2 = \infty$, and $|a| < |a'|$ for all $a \in \mathbf{b}_1$, $a' \in \mathbf{b}_2$. By Lemma 6.1, if $x \in X(\mathbf{a})$, then $x_{[\mathbf{b}_2\to c_2]}$ corresponds to the tableau $T_x|_{\mathbf{b}_1} = T_x|_\lambda$, and $x_{[\mathbf{b}_1\to c_1]}$ corresponds to tableau $T_x|_{\lambda^c}$. Thus $x_{[\mathbf{b}_1\to c_1]} \in X_\lambda(c_1)$ and $x_{[\mathbf{b}_2\to c_2]} \in X_{\lambda^\vee}(c_2)$, from which (ii) follows.

To prove (i), suppose that $x_1 \in X_\lambda(c_1)$ and $x_2 \in X_{\lambda^\vee}(c_2)$. Then x_1 corresponds to a tableau $T_{x_1} \in \mathrm{SYT}(\lambda^c; \mathbf{b}_2)$; x_2 corresponds to a tableau $T_{x_2} \in \mathrm{SYT}(\lambda; \mathbf{b}_1)$. There exists $T \in \mathrm{SYT}(\square; \mathbf{a})$ such that $T|_{\mathbf{b}_i} = T_{x_i}$ for $i = 1, 2$. Letting $x = x_T$, by Lemma 6.1, we have $x_i = x_{[\mathbf{b}_i\to c_i]}$ for $i = 1, 2$, as required.

To prove uniqueness, we must show that if $x, x' \in X(\mathbf{a})$ and $x_{[\mathbf{b}_i\to c_i]} = x'_{[\mathbf{b}_i\to c_i]}$ for $i = 1, 2$, then $x = x'$. By Lemma 6.1 we have $T_x|_{\mathbf{b}_i} = T_{x'}|_{\mathbf{b}_i}$ for $i = 1, 2$; hence $T_x = T_{x'}$ which implies $x = x'$. □

Proof of Corollary 6.7. We will show that if $T \in \mathsf{SYT}(\lambda/\mu; \mathbf{b})$ and $T' \in \mathsf{SYT}(\lambda'/\mu'; \mathbf{b})$ both have the same rectification shape ν, there is a unique tableau $T'' \in \mathsf{SYT}(\lambda/\mu; \mathbf{b})$ such that $T \sim^* T''$ and $T' \sim T''$.

Choose a point $x \in X$ such that $\pi(x) \subset \mathbb{RP}^1$ and $T_x|_{\mathbf{b}} = T$. Let $x_1 = x_{[\mathbf{b}\to c_1]}$, where c_1 is an internal point for $\mathbf{b}$. Choose $x' \in X$ such that $\pi(x') \subset \mathbb{RP}^1$ and $T_{x'}|_{\mathbf{b}} = T'$. Put $\mathbf{a} = \pi(x')$, $\mathbf{b}^c = \mathbf{a} \setminus \mathbf{b}$. Let $x_2 = x_{[\mathbf{b}^c\to c_2]}$, where c_2 is an internal point for $\mathbf{b}^c$.

Since T and T' have the same rectification shape ν, $x_1 \in X_\nu(c_1)$ and $x_2 \in X_{\nu^\vee}(c_2)$. Thus, by Lemma 6.8(i) there exists a unique point $x'' \in X(\mathbf{a})$ such that $x_i = x''_{[\mathbf{b}_i \to c_i]}$ for $i = 1, 2$. By Theorems 6.2 and 6.4, $T_{x''}|_{\mathbf{b}} \sim^* T$ and $T_{x''}|_{\mathbf{b}} \sim T'$. □

As a final note, recall from Remark 3.7 that evacuation defines a $\mathbb{Z}$-action on standard Young tableaux shape $\square$.

Corollary 6.9. *The evacuation action on* $\mathsf{SYT}(\square)$ *has order* N.

Proof. Let $\xi = \frac{2\pi}{N}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and note that $e^{N\xi} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Consider the loop $\mathbf{a}_t = \{(a_1)_t, \ldots, (a_N)_t\}$, where $(a_j)_t = \psi e^{(j-t)\xi}(0) \in \mathbb{RP}^1$ and $\psi \in \mathrm{SL}_2(\mathbb{R})$ is chosen so that $0 < (a_1)_0 < \cdots < (a_N)_0$. Let $\phi = \psi e^{\xi} \psi^{-1} \in \mathrm{SL}_2(\mathbb{R})$. If $T \in \mathrm{SYT}(\square; \mathbf{a}_0)$, then $\phi(T)$ is obtained by sliding T using $\mathbf{a}_t$. But since $\mathbf{a}_t$ is a loop which cyclically rotates the elements of $\mathbf{a}_0$, sliding using $\mathbf{a}_t$ gives the evacuation action on T. The result follows, since $\phi^N = \psi(e^{\xi})^N \psi^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. □

6.3. Proof of the Littlewood–Richardson rule

Fix partitions λ, μ, ν, with $|\lambda| = |\mu| + |\nu|$. Let $\mathbf{a} = \{a_1, \ldots, a_N\}$, with $0 < a_1 < \cdots < a_N$. Let $\mathbf{b}_1 = \{a_1, \ldots, a_{|\mu|}\}$, $\mathbf{b}_2 = \{a_{|\mu|+1}, \ldots, a_{|\lambda|}\}$ and $\mathbf{b}_3 = \{a_{|\lambda|+1}, \ldots, a_N\}$. Then $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ is a consecutive partition of $\mathbf{a}$. Let c_1, c_2, c_3 be internal points of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ respectively.

We wish to count the number of points (with multiplicities) in the intersection

$$Y = X_\mu(c_1) \cap X_\nu(c_2) \cap X_{\lambda^\vee}(c_3).$$

This number is the Littlewood–Richardson number $c^\lambda_{\mu\nu}$. First note that if a point $y \in Y$ has multiplicity m, then by Corollary 2.6, the point y has multiplicity $|\mathsf{SYT}(\mu)| \cdot |\mathsf{SYT}(\nu)| \cdot |\mathsf{SYT}(\lambda^\vee)| \cdot m$ in the intersection

$$\hat{Y} = X\bigl(c_1^{(|\mu|)}\bigr) \cap X\bigl(c_2^{(|\nu|)}\bigr) \cap X\bigl(c_3^{(|\lambda^\vee|)}\bigr).$$

Thus $c^\lambda_{\mu\nu} \cdot |\mathsf{SYT}(\mu)| \cdot |\mathsf{SYT}(\nu)| \cdot |\mathsf{SYT}(\lambda^\vee)|$ is the number of points in $\hat{Y}$ that are supported on Y (counted with multiplicities).

Since Wr is flat, the number of points in $\hat{Y}$ supported on Y is the number of points $x \in X(\mathbf{a})$ such that $x_{[\mathbf{b}_1 \to c_1][\mathbf{b}_2 \to c_2][\mathbf{b}_3 \to c_3]} \in Y$. For a tableau $T \in \mathrm{SYT}(\square; \mathbf{a})$, let x_T be the corresponding point in $X(\mathbf{a})$. Then by Theorem 6.4, $(x_T)_{[\mathbf{b}_1 \to c_1][\mathbf{b}_2 \to c_2][\mathbf{b}_3 \to c_3]} \in Y$ if and only if the rectification shapes of $T|_{\mathbf{b}_1}$, $T|_{\mathbf{b}_2}$, $T|_{\mathbf{b}_3}$ are μ, ν and $\lambda^\vee$ respectively. Let $S^\lambda_{\mu\nu}$ be the set of all tableaux with this property. The number of points in $\hat{Y}$ supported on Y is therefore $|S^\lambda_{\mu\nu}|$.

Note that if $T \in S^\lambda_{\mu\nu}$, then $T|_{\mathbf{b}_1}$ has shape μ, and $T|_{\mathbf{b}_3}$ has shape λ^c. Define an equivalence relation on $S^\lambda_{\mu\nu}$ by putting $T \sim_2^* T'$ if $T|_{\mathbf{b}_2} \sim^* T'|_{\mathbf{b}_2}$. But by Corollary 6.6, each equivalence class of $\sim_2^*$ has size $|[T]| = |\mathsf{SYT}(\mu)| \cdot |\mathsf{SYT}(\nu)| \cdot |\mathsf{SYT}(\lambda^\vee)|$.

Putting everything together, we have

$$\begin{aligned} c^\lambda_{\mu\nu} \cdot \bigl|\mathsf{SYT}(\mu)\bigr| \cdot \bigl|\mathsf{SYT}(\nu)\bigr| \cdot \bigl|\mathsf{SYT}\bigl(\lambda^\vee\bigr)\bigr| = \bigl|S^\lambda_{\mu\nu}\bigr| &= \sum_{[T] \in (S^\lambda_{\mu\nu}/\sim_2^*)} \bigl|[T]\bigr| \\ &= \bigl|\mathsf{SYT}(\mu)\bigr| \cdot \bigl|\mathsf{SYT}(\nu)\bigr| \cdot \bigl|\mathsf{SYT}\bigl(\lambda^\vee\bigr)\bigr| \cdot \bigl|S^\lambda_{\mu\nu}/\sim_2^*\bigr|. \end{aligned}$$

Hence $c_{\mu\nu}^{\lambda} = |S_{\mu\nu}^{\lambda}/{\sim_2^*}|$, which is precisely the statement of the Littlewood–Richardson rule, as formulated in Theorem 3.11.

Acknowledgments

I wish to thank Frank Sottile, Ravi Vakil, Alexander Varchenko, and Soroosh Yazdani for helpful discussions, and Ian Goulden for comments on the manuscript. The refereeing process improved the paper considerably, for which I am grateful. This research was partially supported by an NSERC discovery grant.

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