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Tensor Products of Multilinear Operators

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1. INTRODUCTION

The analysis of nonlinear operators frequently involves their expansion in a series of multilinear operators. In this context, tensor products of multilinear operators as defined here appear naturally. This tensor product construction is nontrivial; tensor products of bounded operators may or may not be bounded. J. Aguirre proved that the tensor product of the Calderon Commutator with itself is bounded. His proof used the duality of H^1 and BMO on the bi-disc. A direct proof of the same fact is given here using techniques developed by R. R. Coifman and Y. Meyer in their work on the Cauchy Integral Operator. Finally, a counterexample shows that tensor products of bounded operators are not always bounded. This counterexample is an adaptation of an example due to Y. Meyer.

2. TENSOR PRODUCTS

Given two functions f and g their tensor product is the function of two variables $f \otimes g$ defined below

$$f \otimes g(x, y) = f(x) \cdot g(y).$$

Suppose $A_1(f, g)$ and $A_2(f, g)$ are bilinear operators for f in a Banach space B_1 and g in a Banach space B_2. Let f_1 and f_2 be in B_1 and let g_1 and g_2 be in B_2, and define the tensor product

$$A_1 \otimes A_2(f_1 \otimes f_2, g_1 \otimes g_2) = A_1(f_1, g_1) \otimes A_2(f_2, g_2).$$

This defines the tensor product on the space of finite linear combinations of tensor products, since we can extend it linearly. The question is whether $A_1 \otimes A_2$ can be extended boundedly to a more useful or natural space. The same question can be asked if the operators are multilinear. As an example we look at the tensor product of the Calderon Commutator with itself.

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The Calderon Commutator is defined as follows:

$$C(a, f) = \text{p.v.} \int \frac{A(x) - A(y)}{(x-y)^2} f(y)\, dy,$$

where A is a primitive of a, i.e., $A'(x) = a(x)$. This is a bilinear operator for a in L^∞ and f in L^2, and the following holds

$$\|C(a, f)\|_2 \leqslant C\, \|a\|_\infty\, \|f\|_2.$$

Furthermore, if $a(x, y) = a_1(x)\, a_2(y)$ and $f(x, y) = f_1(x)\, f_2(y)$ we obtain

$$\begin{aligned} C \otimes C(a, f) &= C(a_1, f_1) \otimes C(a_2, f_2) \\ &= \text{p.v.} \iint \frac{A(x_1, x_2) - A(x_1, y_2) - A(y_1, x_2) + A(y_1, y_2)}{(x_1 - y_1)^2 (x_2 - y_2)^2} \\ &\quad \times f(y_1, y_2)\, dy_1\, dy_2, \end{aligned}$$

where $A = A_1 \otimes A_2$ and A_i is a primitive of a_i. From now on we denote the expression in the numerator $\Delta_2(A)$, the second difference of A. It is also clear that we can use the expression involving the principal value integral to define $C \otimes C$ not just for tensor products but for any a in $L^\infty(R^2)$ and any f in $L^2(R^2)$. But is this operator bounded when defined on those natural spaces? In this case the answer is yes, and this is the result of Aguirre mentioned previously.

We conclude this section with the following elementary result:

LEMMA. *The symbol of the tensor product of two multilinear operators commuting with simultaneous translations of their arguments is equal to the tensor product of the symbols of the two operators.*

Proof.

$$\begin{aligned} &A \otimes B(a_1 \otimes b_1, ..., a_n \otimes b_n) = A(a_1, ..., a_n) \otimes B(b_1, ..., b_n) \\ &= \left(\int e^{ix_1(u_1 + \cdots + u_n)} \sigma_1(u_1, ..., u_n)\, \hat{a}_1(u_1) \cdots \hat{a}_n(u_n)\, du_1 \cdots du_n \right) \\ &\quad \times \left(\int e^{ix_2(v_1 + \cdots + v_n)} \sigma_2(v_1, ..., v_n)\, \hat{b}_1(v_1) \cdots \hat{b}_n(v_n)\, dv_1 \cdots dv_n \right) \\ &= \int e^{ix \cdot (w_1 + \cdots + w_n)} \sigma_1 \otimes \sigma_2(w_1, ..., w_n)\, \widehat{a_1 \otimes b_1}(w_1) \cdots \widehat{a_n \otimes b_n}(w_n)\, dw_1 \cdots dw_n, \end{aligned}$$

where $w_i = (u_i, v_i)$, and $dw_i = du_i\, dv_i$. The proof easily extends to multiple tensor products of operators.

3. Tensor Products of the Higher Commutators

These are the commutators that appear in the analysis of the Cauchy Integral Operator and they will be defined below. Their tensor products are important because they can be viewed as Gateaux differentials at 0 of a generalized Cauchy Integral Operator. In this section we prove a general result yielding a certain integral representation of these operators based on a similar one in one variable. The nth commutator is defined as follows:

$$c_n(a_1,..., a_n, f) = \text{p.v.} \int \frac{\prod(A_i(x) - A_i(y))}{(x-y)^{n+1}} f(y)\, dy,$$

where the A_i are primitives of the a_i. We take the tensor product of c_n with itself and we obtain the following formula:

$$\begin{aligned} C_n &= c_n \otimes c_n(a_1,..., a_n, f) \\ &= \text{p.v.} \int \frac{\prod \Delta_2 A_i(x_1, x_2, y_1, y_2)}{(x_1 - y_1)^{n+1}(x_2 - y_2)^{n+1}} f(y_1, y_2)\, dy_1\, dy_2, \end{aligned}$$

where $D_1 D_2 A_i(x, y) = a_i(x, y)$ and

$$\Delta_2 A_i(x_1, x_2, y_1, y_2) = A_i(x_1, x_2) - A_i(x_1, y_2) - A_i(y_1, x_2) + A_i(y_1, y_2).$$

The following formula is known for the restriction of c_n on the diagonal (see [3]):

$$c_n(a,..., a, f) = \text{p.v.} \int (1 + itD)^{-1}(a(1 + itD)^{-1})^n f\, dt/t.$$

Here a also stands for the operation of multiplication by a. We let k_n denote the multilinear version of the above integral expression.

$$k_n(a_1,..., a_n, f) = \text{p.v.} \int Sa_1 \cdots Sa_n Sf\, dt/t,$$

where $S = (1 + itD)^{-1}$. Then we form the tensor product $K_n = k_n \otimes k_n$. Let $S_i = (1 + it_i D_i)^{-1}$ for $i = 1, 2$. We have

$$\begin{aligned} &K_n(a_1 \otimes b_1,..., a_n \otimes b_n, f_1 \otimes f_2) \\ &\quad = \left(\text{p.v.} \int S_1 a_1 \cdots a_n S_1 f_1\, dt_1/t_1\right)\left(\text{p.v.} \int S_2 b_1 \cdots b_n S_2 f_2\, dt_2/t_2\right) \\ &\quad = \text{p.v.} \iint S_1 S_2 a_1 \otimes b_1 \cdots a_n \otimes b_n S_1 S_2 f_1 \otimes f_2\, dt_1\, dt_2/t_1 t_2. \end{aligned}$$

It is natural to investigate the relationship between K_n and C_n. Note that $C_n(w_1,\dots, w_n, f)$ is symmetric in the arguments $w_1,\dots, w_n$, whereas $K_n(w_1,\dots, w_n, f)$ is not. The following holds:

PROPOSITION. *$C_n = K_n^\#$ where*

$$K_n^\#(w_1,\dots, w_n, f) = \frac{1}{n!} \sum_{p \text{ in } S_n} K_n(w_{p(1)},\dots, w_{p(n)}, f)$$

is the symmetrization of K_n, and S_n is the symmetric group on n elements.

Proof. $K_n^\#$ and C_n are equal when restricted to the diagonal $w_1 = \cdots = w_n$. By the polarization identity two symmetric linear operators that agree on the diagonal must agree everywhere.

COROLLARY. *Formally the following identity holds:*

$$\text{p.v.} \iint \frac{f(y_1, y_2)}{(x_1-y_1)(x_2-y_2)-(A(x_1, x_2)-A(x_1, y_2)-A(y_1, x_2)+A(y_1, y_2))}\, dy_1\, dy_2$$
$$= \sum_{k=0}^{\infty} \text{p.v.} \iint S_1 S_2 (aS_1 S_2)^k f\, dt_1\, dt_2/t_1 t_2,$$

where $D_1 D_2 A = a$, and the S_i are the same as before.

The identity of this corollary is analogous to the one used to prove the boundedness of the Cauchy Integral Operator. The 0th term is the Hilbert Transform in the second variable followed by the Hilbert Transform in the first variable and is bounded. The second term is the tensor product of the Calderon commutator with itself and is also bounded.

4. THE TENSOR PRODUCT OF THE CALDERON COMMUTATOR

Let $p(x) = \frac{1}{2} \exp(-|x|)$ and $q(x) = (i/2) \operatorname{sgn}(x) \exp(-|x|)$. Then

$$\hat{p}(u) = \frac{1}{1+u^2} \qquad \text{and} \qquad \hat{q}(u) = \frac{u}{1+u^2}.$$

Let $p_t(x) = p(x/t)/t$ and $q_t(x) = q(x/t)/t$. Define

$$P_t(f) = f * p_t = \int f(x-y)\, p_t(y)\, dy \qquad \text{and} \qquad Q_t(f) = f * q_t.$$

For $f=f(x_1, x_2)$ a function of two variables we define

$$P_1(f)=p_{t_1} * f_{x_2}(x_1)=\int f(y_1, x_2)\, p((x_1-y_1)/t_1)\, dy_1/t_1,$$

$$P_2(f)=p_{t_2} * f_{x_1}(x_2),$$

$$Q_1(f)=q_{t_1} * f_{x_2}(x_1), \qquad Q_2(f)=q_{t_2} * f_{x_1}(x_2).$$

For $A_t=Q_t(I-2P_t)$ the following identities hold (see [3]):

$$(*) \quad t\frac{\partial}{\partial t} A_t = -Q_t + 8Q_t^3,$$

$$(**) \quad t\frac{\partial}{\partial t} P_t = -2Q_t^2.$$

Given a function $b(x)$ in L^∞ we let b stand for the operator of multiplication by $b(x)$.

$$(***) \quad QbP = PP(b)\,Q + Q(b)\,P - QQ(b)\,Q.$$

To prove these identities one simply has to check that the symbols satisfy certain algebraic identities. We will soon see that the significance of (***) is that it allows us to shift Q which carries the cancellation properties from the left of a multiplication operator to its right.

THEOREM. *If $D_1 D_2 A = a$ and a is in $L^\infty(R^2)$ then the operator*

$$B(a, f) = \text{p.v.} \iint S_1 S_2 a S_1 S_2 f \, dt_1\, dt_2/t_1 t_2$$

is bounded for f in $L^2(R^2)$ and the following estimate holds:

$$\|B(a, f)\|_2 \leqslant C\, \|a\|_\infty\, \|f\|_2,$$

where C is a constant, and $S_i = (1 + it_i D_i)^{-1}$ for $i = 1, 2$.

Proof. We will express the S_i in terms of P and Q and we will exploit the special properties of these operators.

$$\frac{I}{I+itD} = \frac{I}{I+t^2D^2} - \frac{itD}{I+t^2D^2} = P_t - iQ_t.$$

By using the above identity we write $B(a, f)$ as follows:

$$B(a, f) = \text{p.v.} \iint (P_1 - iQ_1)(P_2 - iQ_2)\, a (P_1 - iQ_1)(P_2 - iQ_2)\, f\, dt_1\, dt_2/t_1 t_2.$$

By carrying out the multiplications we see that the integrand involves the following terms:

(1) $P_1P_2aP_1P_2$ (2) $P_1P_2aQ_1Q_2$ (3) $P_1P_2aQ_1P_2$ (4) $P_1P_2aQ_2P_1$
(5) $Q_1P_2aP_1P_2$ (6) $Q_1P_2aQ_1Q_2$ (7) $Q_1P_2aQ_1P_2$ (8) $Q_1P_2aP_1Q_2$
(9) $Q_1Q_2aP_1P_2$ (10) $Q_1Q_1aQ_1Q_2$ (11) $Q_1Q_2aQ_1P_2$ (12) $Q_1Q_2aP_1Q_2$
(13) $P_1Q_2aP_1P_2$ (14) $P_1Q_2aQ_1Q_2$ (15) $P_1Q_2aQ_1P_2$ (16) $P_1Q_2aP_1Q_2$.

Since all integrals here are principal value integrals, any term above which is even in t_1 or t_2 (or both) corresponds to an integral which is equal to 0. This applies to the following terms: 1, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 16. For the remaining terms the principal value is no longer needed, so we can take the range of integration from 0 to ∞. The boundedness of the remaining terms follows from the following two propositions:

PROPOSITION 1. *$B_1(a, f) = \int_0^\infty \int_0^\infty Q_1Q_2aP_1P_2f\, dt_1\, dt_2/t_1t_2$ is bounded on $L^2(R^2)$ when a is in $L^\infty(R^2)$.*

PROPOSITION 2. *$B_2(a, f) = \int_0^\infty \int_0^\infty P_1Q_2aQ_1P_2f\, dt_1\, dt_2/t_1t_2$ is bounded on $L^2(R^2)$ when a is in $L^\infty(R^2)$.*

Moreover we have the estimates

$$\|B_i(a, f)\|_2 \leqslant C\,\|a\|_\infty\,\|f\|_2, \qquad i = 1, 2.$$

Proof of Proposition 1. We use the identities (*) and (**) to obtain

$$\begin{aligned} Q_1Q_2aP_1P_2 &= 64Q_1^3Q_2^3aP_1P_2 - 8t_1\left(\frac{\partial}{\partial t_1}A_1\right)Q_2^3aP_1P_2 \\ &\quad - 8t_2\left(\frac{\partial}{\partial t_2}A_2\right)Q_1^3aP_1P_2 + t_1t_2\left(\frac{\partial}{\partial t_1}A_1\right)\left(\frac{\partial}{\partial t_2}A_2\right)aP_1P_2 \\ &= 64O_1 - 8O_2 - 8O_3 + O_4. \end{aligned}$$

By symmetry we need only prove the boundedness of O_1, O_2, and O_4. First, we prove that O_1 is bounded. We estimate the norm of O_1 by duality.

$$\left(\int gQ_1^3Q_2^3aP_1P_2f(dt_1\, dt_2/t_1t_2)\, dx_1\, dx_2\right)^2$$
$$\leqslant \|Q_1^2Q_2^2g\|^2_{L^2(R^4_+)}\,\|Q_1Q_2aP_1P_2f\|^2_{L^2(R^4_+)}.$$

Here we have applied Schwarz's inequality on the space

$$L^2(R^4_+, (dt_1\, dt_2/t_1t_2)\, dx_1\, dx_2) \qquad \text{with} \quad R^4_+ = R^2_+ \times R^2_+.$$

Plancherel's theorem easily gives the following estimate:

$$\|Q_1^2 Q_2^2 g\|^2 = C\ \|g\|^2_{L^2(R^2)}.$$

Therefore we need to estimate $\|Q_1 Q_2 a P_1 P_2 f\|^2_{L^2(R^4_+)}$.

We apply identity (***) first on $Q_2 a P_2$ and then on $Q_1(P_2 a) P_1$ and on $Q_1(Q_2 a) P_1$ that appear after the first application of (***):

$$\begin{aligned}\|Q_1 Q_2 a P_1 P_2 f\|_2 \leqslant\ & \|P_1 P_2(P_1 P_2 a)\, Q_1 Q_2 f\|_2 + \|P_2(Q_1 P_2 a)\, P_1 Q_2 f\|_2 \\ & + \|P_1(P_1 Q_2 a)\, Q_1 P_2 f\|_2 + \|Q_1 P_2(Q_1 P_2 a)\, Q_1 Q_2 f\|_2 \\ & + \|(Q_1 Q_2 a)\, P_1 P_2 f\|_2 + \|Q_1(Q_1 Q_2 a)\, Q_1 P_2 f\|_2 \\ & + \|P_1 Q_2(P_1 Q_2 a)\, Q_1 Q_2 f\|_2 + \|Q_2(Q_1 Q_2 a)\, P_1 Q_2 f\|_2 \\ & + \|Q_1 Q_2(Q_1 Q_2 a)\, Q_1 Q_2 f\|_2 = \sum_1^9 T_i.\end{aligned}$$

The estimate for T_1 follows easily from Plancherel's theorem:

$$\|P_1 P_2(P_1 P_2 a)\, Q_1 Q_2 f\|_2 \leqslant C\ \|a\|_\infty\ \|Q_1 Q_2 f\|_2 = C\ \|a\|_\infty\ \|f\|_2.$$

We estimate T_2 as follows:

$$T_2^2 \leqslant \|(Q_1 P_2 a)\, P_1 Q_2 f\|_2^2 = \int |Q_1 P_2 a|^2\ |P_1 Q_2 f|^2 (dx_1\, dt_1/t_1)(dx_2\, dt_2/t_2).$$

But, $|Q_1 P_2 a|^2 (dx_1\, dt_1/t_1)$ is a Carleson measure, therefore

$$T_2^2 \leqslant C \int \|P_2 a\|_\infty^2\ |(Q_2 f)^{*,1}|^2\, dx_1 (dx_2\, dt_2/t_2),$$

where $(\cdot)^{*,1}$ denotes the Hardy–Littlewood Maximal Function in the first variable. By the Maximal theorem and Plancherel's theorem

$$T_2^2 \leqslant C\ \|a\|_\infty^2 \int |Q_2 f|^2\, dx_1\, dx_2\, dt_2/t_2 \leqslant C\ \|a\|_\infty^2\ \|f\|_2^2.$$

T_4, T_7, and T_9 are estimated in the same way as T_1. T_3 is estimated the same way as T_2. The remaining terms involve multiplication by $Q_1 Q_2 a$, and the estimate follows by observing that

$$|Q_1 Q_2 a|^2 \frac{dx\, dx_2\, dt_1\, dt_2}{t_1 t_2}$$

is a Generalized Carleson measure on $R^2_+ \times R^2_+$.

We now move on to O_2. An integration by parts yields

$$\int_0^\infty \int_0^\infty t_1 \frac{\partial}{\partial t_1} A_1 Q_2^3 a P_1 P_2 f \, dt_1 \, dt_2/t_1 t_2 = -\int_0^\infty \int_0^\infty A_1 Q_2^3 a \frac{\partial}{\partial t_1} P_1 P_2 \, dt_1 \, dt_2/t_2.$$

By (**) the right-hand side becomes

$$\int_0^\infty \int_0^\infty Q_1(1-2P_1)\, Q_2^3 a Q_1^2 P_2 f \, dt_1 \, dt_2/t_1 t_2.$$

Since we have $Q_1 Q_2$ on the left of the integrand we can apply the duality argument as before to reduce this to an estimate for the following:

$$\|Q_2 a Q_1^2 P_2 f\|_{L^2(R_+^4)}.$$

By applying (***) we obtain terms that can be treated in exactly the same way as the T_i. We omit the details of the estimate for O_4 since it is similar to that of O_2.

Proof of Proposition 2. We start by computing the symbol of $B_2(a, f)$.

$$B_2(a, f)(x) = \int e^{ix(y+u)} W(y, u)\, \hat{a}(y)\, \hat{f}(u)\, dy\, du,$$

where $u = (u_1, u_2)$, $y = (y_1, y_2)$, and $W(y, u) = W_1(y_1, u_1)\, W_2(y_2, u_2)$ with

$$W_1(y_1, u_1) = \int_0^\infty \frac{1}{1 + t_1^2(y_1 + u_1)^2} \frac{t_1 u_1}{1 + t_1^2 u_1^2} \frac{dt_1}{t_1},$$

$$W_2(y_2, u_2) = \int_0^\infty \frac{t_2(y_2 + u_2)}{1 + t_2^2(y_2 + u_2)^2} \frac{1}{1 + t_2^2 u_2^2} \frac{dt_2}{t_2}.$$

By making the change of variables $t_i' = t_i u_i$, $i = 1, 2$, and by setting $s_i = (y_i + u_i)/u_i$, $i = 1, 2$, we obtain

$$W_1(y_1, u_1) = W_1(s_1) = \int_0^\infty \hat{p}(t_1 s_1)\, \hat{p}(t_1)\, dt_1,$$

$$W_2(y_2, u_2) = W_2(s_2) = \int_0^\infty \hat{q}(t_2 s_2)\, \hat{p}(t_2)\, dt_2/t_2.$$

We integrate using the method of partial fractions to obtain

$$W_1(s_1) = c \frac{1}{1 + |s_1|},$$

$$W_2(s_2) = c \operatorname{sgn}(s_2) \left(1 - \frac{1}{1 + |s_2|}\right) = \operatorname{sgn}(s_2)(c - W_1(s_2)).$$

Hence $W(s_1,s_2)=W_1(s_1)W_2(s_2)=cW_1(s_1)\mathrm{sgn}(s_2)-W_1(s_1)W_1(s_2)\mathrm{sgn}(s_2)$. Since $W_1(s_1)\,W_2(s_2)$ is the symbol of $\int_0^\infty\int_0^\infty P_1P_2aQ_1Q_2f\,dt_1\,dt_2/t_1t_2$ and since $W_1(s_1)$ is the symbol of $\int_0^\infty P_1aQ_1f\,dt_1/t_1$ we have proved the following identity:

$$\int_0^\infty\int_0^\infty P_1Q_2aQ_1P_2f\frac{dt_1\,dt_2}{t_1t_2}$$
$$=cH_2\int_0^\infty P_1aQ_1(H_2f)\frac{dt_1}{t_1}-H_2\int_0^\infty\int_0^\infty P_1P_2aQ_1Q_2(H_2f)\frac{dt_1\,dt_2}{t_1t_2},$$

where H_2 is the Hilbert Transform in the second variable. Proposition 2 now follows from Proposition 1.

5. The Counterexample

In this section we prove that the tensor product of bounded operators is not necessarily bounded. We start with a technical lemma. The idea behind its proof is the following: the outcome of a convolution $a*p$ of an oscillatory function a and a smooth function p depends on the relative size of the support of p and the period of a; if the support of p is small the convolution essentially reproduces a, if the support of p is large then the outcome is close to 0.

Let $p(x)$ be in $C_0^\infty(R)$, with support in the interval $(-1,0)$, and with $\int p(t)\,dt=1$. For all positive u set $p_u(t)=p(t/u)/u$. When $u=2^{-m^2}$ write $p_u=p_m$. Let $a(x)$, in $C^\infty(R)$, be equal to 1 for x negative and equal to $\exp(i2^{x^2})$ for x bigger than 1.

Lemma 1. *$a*p_m=a(x)\,w_m(x)$ with $w_m(x)$ satisfying*

$$|w_m(x)-1|\leqslant Cm2^{-2m}\quad\textit{and}\quad|w_m'(x)|\leqslant c2^{-m}m^2\quad\textit{for}\quad x\leqslant m-1,$$
$$|w_m(x)|\leqslant C2^{m^2-x^2}/x\quad\textit{for}\quad x>m+1.$$

Proof. Assume that m is large enough so that $m<10^{-6}2^{m^2}$.

$$J_m(x)=a*p_m(x)=\int a(x-t)\,p(t/u)\,dt/u$$
$$=\int a(x-ut)\,p(t)\,dt\qquad\text{with}\quad u=2^{-m^2}.$$

Therefore, $J_m(x)=\int\exp(i2^{(x-ut)^2})\,p(t)\,dt$ and

$$2^{(x-ut)^2}=2^{x^2}2^{-2uxt+u^2t^2}=2^{x^2}(1-2uxs)\qquad\text{with}\quad s=\frac{1-2^{-2uxt+u^2t^2}}{2ux}.$$

We see that s is given in terms of t by a diffeomorphism which is uniformly bounded for $xu < 10^{-6}$. Therefore,

$$J_m(x) = \exp(i2^{x^2}) \int \exp(-2iuxs2^{x^2})\, p^{\#}(s)\, ds \qquad \text{with} \quad p^{\#}(s) = p(t(s))\, t'(s).$$

Note that $\int p^{\#}(s)\, ds = \int p(t)\, dt = 1$ and the above formula for J_m means

$$J_m(x) = \exp(i2^{x^2})(p^{\#})^{\wedge}(22^{x^2}ux),$$

$$|J_m(x) - \exp(i2^{x^2})| = \left| \int (\exp(-2iuxs2^{x^2}) - 1)\, p^{\#}(s)\, ds \right|$$

$$\leqslant \int |2iuxs2^{x^2}|\, p^{\#}(s)\, ds \leqslant C22^{x^2}ux \leqslant Cx2^{x^2-m^2}.$$

Since $x \leqslant m-1$, $x^2 - m^2 \leqslant (m-1)^2 - m^2 = -(2m-1)$. Therefore,

$$|a * p_m(x) - a(x)| = |J_m(x) - \exp(i2^{x^2})| \leqslant Cm2^{-2m} \qquad \text{for} \quad 1 \leqslant x \leqslant m-1.$$

Setting $w_m = \int \exp(-2iuxs2^{x^2})\, p^{\#}(s)\, ds$ we obtain by a similar argument that $|w'_m(x)| \leqslant C2^{-m}m^2$. We now turn our attention to the case $x > m+1$.

We distinguish two cases according the whether $x \leqslant 10^{-6}2^{m^2}$ or $x > 10^{-6}2^{m^2}$.

Case I. $m+1 \leqslant x \leqslant 10^{-6}2^{m^2}$. We have already seen that $|J_m(x)| = |(p^{\#})^{\wedge}(2ux2^{x^2})| \leqslant C/2ux2^{x^2}$ since $(p^{\#})^{\wedge}$ is in the Schwarz Class.

Case II. $x > 10^{-6}2^{m^2}$. Let $g(x) = 2^x$ until the end of this proof.

$$\operatorname{Re}(J_m(x)) = \int \cos(g(y^2))\, p_m(x-y)\, dy$$

$$= \int \cos(g(y^2))\, g(m^2)\, p(g(m^2)(x-y))\, dy$$

$$= \int_x^{x+g(-m^2)} \cos(g(y^2))\, g(m^2)\, p(g(m^2)(x-y))\, dy.$$

Set $q(x, y) = \int_x^y \cos(g(t^2))\, dt$. A simple integration by parts yields

$$\operatorname{Re}(J_m(x)) = g(m^2) \int_x^{x+g(-m^2)} q(x, y)\, p'(g(m^2)(x-y))\, dy.$$

We now need to estimate $q(x, y)$. Let $z = g(t^2)$, then $dt = C\, dz/z(\log_2 z)^{1/2}$

and $q(x, y) = C\int_{g(x^2)}^{g(y^2)} \cos z(dz/z(\log_2 z)^{1/2})$. After another integration by parts we obtain

$$q(x, y) = \sin z \frac{1}{z(\log_2 z)^{1/2}}\Bigg|_{g(x^2)}^{g(y^2)} - \int_{g(x^2)}^{g(y^2)} \sin z(z^{-1}(\log_2 z)^{-1/2})'\, dz.$$

From the estimate $|(z^{-1}(\log_2 z)^{-1/2})'| \leqslant Cz^{-2}(\log_2 z)^{-1/2}$ and recalling that $x \leqslant t \leqslant y \leqslant x + g(-m^2)$ we obtain

$$|q(x, y)| \leqslant 2x^{-1}g(-x^2) + \frac{C}{x}\int_{g(x^2)}^{g(y^2)} z^{-2}\, dz \leqslant Cx^{-1}g(-x^2).$$

Consequently,

$$|\mathrm{Re}(J_m(x))| \leqslant Cx^{-1}g(m^2 - x^2)\int_{-1}^{0} |p'(y)|\, dy.$$

The same estimate can be obtained for $\mathrm{Im}(J_m(x))$. This completes the proof of Lemma 1.

We let $f_0(x)$ be a function in $C^\infty(R)$ which is equal to 0 for $x \leqslant 0$ and which is equal to $x^{-1/2}(\log x)^{-1}$ for $x \geqslant 5$. We let p_0 be a function in the Schwarz Class such that $\mathrm{supp}(\hat{p}_0) \subset \{|u| \leqslant 10^{-4}\}$. p and a remain as in Lemma 1. Choose k_m so that

$$2^{k_m} \leqslant m^2 e^{m^2} \leqslant 2 2^{k_m} \qquad \text{and set} \quad z_m = 2^{k_m}.$$

Define $f(x) = \bar{a}(x) f_0(x)$, where $\bar{a}$ is the complex conjugate of a. Finally, let $J(x)$ denote the characteristic function of the interval $[0, \frac{1}{2}]$ and define

$$F(x_1, x_2) = J(x_1) f(x_1 + x_2),$$
$$A(x_1, x_2) = a(x_1 + x_2),\ B(x_1, x_2) = b(x_1 + x_2)$$

with

$$b(x) = \sum_{m=1}^{\infty} \exp(iz_m x)\, p_0(x - m)$$

(note that b is in L^∞).

Proposition. *Let q be a function in the Schwarz Class such that $\hat{q}(u) = 1$ for u in $[\frac{2}{3}, \frac{4}{3}]$ and* $\mathrm{supp}(\hat{q}) \subset [\frac{1}{2}, \frac{3}{2}]$. *Then the operator*

$$T_{A,B}(F) = \sum_{k=0}^{\infty} q_k * ((q_k * B)\, p_{0,k} * H_2((A * p_k) F))$$

is not bounded on $L^2(R^2)$. *Here,* H_2 *denotes the Hilbert Transform in the second variable, all convolutions are taken in the first variable, and* $p_k = 2^k p(2^k x)$.

Proof. Note that $q_k * B(x_1, x_2) = q_k * b(x_1 + x_2)$. We start by proving the following lemma.

LEMMA 2. $q_k * \exp(iz_m x)\, p_0(x-m)$ *is equal to* 0 *if* $k \neq k_m$, *and it is equal to* $\exp(iz_m x)\, p_0(x-m)$ *if* $k = k_m$.

Proof. A direct computation of the Fourier Transform yields

$$(\exp(iz_m x)\, p_0(x-m))^\wedge(u) = \exp(-im(u-z_m))\, \hat{p}_0(u-z_m).$$

Hence

$$\begin{aligned} q_k &* \exp(iz_m x)\, p_0(x-m) \\ &= \int \exp(ixu)\, \hat{q}(2^{-k}u) \exp(-im(u-z_m))\, \hat{p}_0(u-z_m)\, du' \\ &= \int \exp(ix(u+z_m)\, \hat{q}(2^{-k}(u+z_m))\, \hat{p}_0(u) \exp(-imu)\, du'. \end{aligned}$$

By considering the support of $\hat{p}_0$ we see that $|u| \leqslant 10^{-4}$ and since $z_{m+1}/z_m \to \infty$ with m we have:
For $k = k_m$, $\frac{2}{3} < 2^{-k_m}(u+z_m) < \frac{4}{3}$ hence $\hat{q}(2^{-k_m}(u+z_m)) = 1$. Similarly, if $k \neq k_m$ the convolution is 0. This completes the proof of Lemma 2.

Lemma 2 implies that $q_k * B = q_k * b(x_1+x_2) = \exp(iz_m(x_1+x_2))$ $p_0(x_1+x_2-m)$ if $k = k_m$, and that $q_k * B = 0$ otherwise. Therefore the operator becomes

$$T_{A,B}(F) = \sum_{m=0}^{\infty} q_m * (\exp(iz_m(x_1+x_2)\, p_0(x_1+x_2-m)\, G_m(x_1, x_2))$$

with $G_m(x_1, x_2) = p_{0,m} * H_2(A * p_m F)$, $p_m = p_{k_m}$, $q_m = q_{k_m}$,

$$\begin{aligned} H_2(A * p_m F) &= H_2((p_m * a(x_1 + y_2))\, J(x_1)\, f(x_1 + y_2)) \\ &= J(x_1)\, H((p * a) f)(x_1 + x_2). \end{aligned}$$

Throughout the rest of the argument we will hold x_2 fixed in the interval $[m-3, m-3+\frac{1}{2}]$. All Fourier Transforms will be taken with respect to the first variable. Since $\operatorname{supp}(\hat{G}_m) \subset \{|u_1| \leqslant 10^{-4} 2^{k_m}\}$ and since

$$\int \exp(-ix_1 u_1) \exp(iz_m x_1)\, p_0(x_1 + x_2 - m)\, dx_1 = e^{i(x_2-m)(u_1-z_m)} \hat{p}_0(u_1 - z_m)$$

shows that $\operatorname{supp}((\exp(iz_m(x_1+x_2)\, p_0(x_1+x_2-m))^\wedge) \subset \{|u_1 - z_m| < 10^{-4}\}$ we obtain

$$\operatorname{supp}((\exp(iz_m(x_1+x_2))\, p_0(x_1+x_2-m)\, G_m(x_1, x_2))^\wedge) \subset \{|u_1 - z_m| \leqslant 10^{-4} + 2^{k_m} 10^{-4}\}.$$

For such u_1, $\hat{q}_m(u_1) = 1$, hence the exterior convolution by q_m in the definition of T is superfluous.

$$T_{A,B}(F) = \sum_{m=0}^{\infty} \exp(iz_m(x_1+x_2))\, p_0(x_1+x_2-m)\, G_m(x_1, x_2)$$

with

$$G_m(x_1, x_2) = \int p_{0,m}(x_1 - y_1)\, J(y_1)\, h_m(y_1 + x_2)\, dy_1,$$

where

$$h_m(x) = H((a * p_m) f) = H(f_0 w_m) \text{ by Lemma 1.}$$

LEMMA 3. *$h_m(x) \geqslant Cm^{-1/2}$ for x in $[m-3, m-2]$.*

Proof. We decompose $f_0 w_m$ into three parts by the following partition of unity: $J_1(x) + J_2(x) + J_3(x) = 1$ where the J_i are as in Fig. 1. For $m-3 \leqslant x \leqslant m-2$ we have

$$H(J_1 f_0 w_m) \geqslant \int_5^{m-5} \frac{dx}{(m-x)\, x^{1/2} \log x} \geqslant \int_{m/2}^{m-5} \frac{dx}{(m-x)\, x^{1/2} \log x}$$
$$\geqslant Cm^{-1/2} (\log m)^{-1} \int_{m/2}^{m-5} \frac{dx}{m-x} \geqslant Cm^{-1/2},$$

$$H(J_2 f_0 w_m) = \text{p.v.} \int \frac{J_2 f_0 w_m(y)}{x-y}\, dy = \text{p.v.} \int \frac{J_2 f_0 w_m(y) - J_2 f_0 w_m(y_0)}{x-y}\, dy.$$

But on the interval $[m-5, m-1]$, w_m, w_m', J_2, and J_2' are bounded by fixed constants, and for f_0 and f_0' we have the estimates:

$$|f_0| \leqslant m^{-1/2} (\log m)^{-1}, \qquad |f_0'| \leqslant cm^{-1/2} (\log m)^{-1}.$$

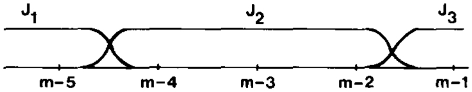

FIGURE 1

So, an application of the Mean Value Theorem yields

$$|H(J_2 f_0 w_m)| \leqslant Cm^{-1/2}(\log m)^{-1}.$$

Finally,

$$|H(J_3 f_0 w_m)| \leqslant \int_{m-2+(1/4)}^{\infty} \frac{|f_0|\,|w_m|}{y-(m-2)}\,dy \leqslant \int_{m-2+(1/4)}^{m+1} + \int_{m+1}^{\infty}.$$

The first integral is $O(m^{-1/2}(\log m)^{-1})$ and the second is even smaller because of the decay of w_m. This completes the proof of the lemma.

Remark. So far we have required p_0 to be rapidly decaying, and to have a Fourier Transform compactly supported near the origin. We now claim that we can choose p_0 so that additionally it is positive for $|x| < 4$. Let p_1 be an even function such that $\operatorname{supp}(\hat{p}_1) \subset \{|u| < \frac{1}{2}10^{-4}\}$. Define $\hat{p}_0 = \hat{p}_1 * \hat{p}_1$. Then $p_0 = p_1^2$ is nonnegative. A translation allows us to assume that $p_0(0) \neq 0$, i.e., p_0 is positive in a neighborhood of 0. Since $(p_0(\cdot/s))^\wedge(u) = s\hat{p}_0(su)$ we see that for $s > 1$ the Fourier Transform of the dilated function has smaller support. By dilating by an s large enough we obtain a p_0 that satisfies the additional requirement.

We now proceed to estimate $G_m(x_1, x_2)$ for $0 \leqslant x_1 \leqslant \frac{1}{4}$ and $m-3 \leqslant x_2 \leqslant m-3+\frac{1}{2}$. Since $0 \leqslant y_1 \leqslant \frac{1}{2}$ and $m-3 < x_2 + y_1 \leqslant m-2$,

$$G_m(x_1, x_2) \geqslant Cm^{-1/2} \int_{2^{k_m}(x_1-(1/2))}^{2^{k_m}x_1} p_0(x)\,dx \geqslant Cm^{-1/2}.$$

We observe that the terms

$$W_m(x_1, x_2) = \exp(iz_m(x_1+x_2))\,p_0(x_1+x_2-m)\,G_m(x_1, x_2)$$

are orthogonal in $L^2(R^2)$, since they have Fourier Transforms with disjoint supports. In fact we have shown that

$$\operatorname{supp}(\hat{W}_m(u_1, x_2)) \subset \{|u_1 - z_m| \leqslant 10^{-4}(1+2^{k_m})\}.$$

Consequently

$$\|T_{A,B}(F)\|_2^2 = \sum \|p_0(x_1+x_2-m)\,G_m(x_1, x_2)\|_2^2$$
$$\geqslant \sum \int_{m-3}^{m-3+(1/2)} \int_0^{1/2} |p_0(x_1+x_2-m)\,G_m(x_1, x_2)|^2\,dx_1\,dx_2.$$

Since $m-3 \leqslant x_1+x_2 \leqslant m-2$, we have $-3 \leqslant x_1+x_2-m \leqslant -2$. But we know that p_0 is larger than a positive constant on that interval. Combining this with our estimate on G_m for the rectangle specified by the limits of integration, we conclude that the infinite series diverges since it majorizes the harmonic series. This completes the proof of the proposition.

COROLLARY. *The tensor product of two bounded operators is not necessarily bounded.*

Proof. The operator $T_{A,B}(F)$ is the tensor product of the Hilbert Transform with $\sum q_m * ((q_m * b)\, p_{0,m} * (a * p_m f))$. The latter is the discrete version of the operator

$$\int_0^\infty Q_t((Q_t b)\, P_{0,t}(P_t a) f) \frac{dt}{t}$$

which is bounded.

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