# Erdős-Ko-Rado theorems for simplicial complexes 

Russ Woodroofe<br>Department of Mathematics, Washington University in St. Louis, St. Louis, MO 63130, United States

## A R T I C L E I N F O

## Article history:

Received 1 April 2010
Available online 19 January 2011

## Keywords:

Erdős-Ko-Rado
Algebraic shifting
Cohen-Macaulay
Shellable
Depth
Independence complex


#### Abstract

A recent framework for generalizing the Erdős-Ko-Rado theorem, due to Holroyd, Spencer, and Talbot, defines the Erdős-Ko-Rado property for a graph in terms of the graph's independent sets. Since the family of all independent sets of a graph forms a simplicial complex, it is natural to further generalize the Erdős-Ko-Rado property to an arbitrary simplicial complex. An advantage of working in simplicial complexes is the availability of algebraic shifting, a powerful shifting (compression) technique, which we use to verify a conjecture of Holroyd and Talbot in the case of sequentially Cohen-Macaulay near-cones.


© 2010 Elsevier Inc. All rights reserved.

## 1. Introduction

A family $\mathcal{A}$ of sets is intersecting if every pair of sets in $\mathcal{A}$ has non-empty intersection, and is an $r$-family if every set in $\mathcal{A}$ has cardinality $r$. A well-known theorem of Erdős, Ko, and Rado bounds the cardinality of an intersecting $r$-family:

Theorem 1.1. (See Erdős, Ko and Rado [11].) Let $r \leqslant \frac{n}{2}$ and $\mathcal{A}$ be an intersecting $r$-family of subsets of $[n]$. Then $|\mathcal{A}| \leqslant\binom{ n-1}{r-1}$.

Given a simplicial complex $\Delta$ (defined in Section 2) and a face $\sigma$ of $\Delta$, we define the link of $\sigma$ in $\Delta$ to be

$$
\operatorname{link}_{\Delta} \sigma=\{\tau: \tau \cup \sigma \text { is a face of } \Delta, \tau \cap \sigma=\emptyset\}
$$

An $r$-face of $\Delta$ is a face of cardinality $r$. We further let $f_{r}(\Delta)$ be defined as the number of $r$-faces in $\Delta$, and the tuple $\left(f_{0}(\Delta), f_{1}(\Delta), \ldots, f_{d+1}(\Delta)\right)$ (where $d$ is the dimension of $\Delta$ ) is called the $f$ vector of $\Delta$.

[^0]Note 1.2. We follow Swartz [28] in our definition of $r$-face and $f_{r}$. Other sources define an $r$-face to be a face with dimension $r$ (rather than cardinality $r$ ) which shifts the indices of the $f$-vector by 1 .

We restate Theorem 1.1 using this language:
Theorem 1.3. Let $r \leqslant \frac{n}{2}$ and $\mathcal{A}$ be an intersecting $r$-family of faces of the simplex with $n$ vertices. Then $|\mathcal{A}| \leqslant$ $f_{r-1}\left(\operatorname{link}_{\Delta} v_{1}\right)$.

Let $G$ be a graph with edge set $E(G)$ and vertex set $V(G)$. The independence complex of $G$, denoted $I(G)$, is the simplicial complex consisting of all independent sets of $G$. For a vertex $v \in V(G)$, the closed neighborhood $N[v]$ consists of $v$ and all its neighbors. We notice that link ${ }_{I(G)} v=$ $I(G \backslash N[v])$.

Following Holroyd, Spencer, and Talbot [17, Section 1], we define:
Definition 1.4. A simplicial complex $\Delta$ is $r$-EKR if every intersecting $r$-family $\mathcal{A}$ of faces of $\Delta$ satisfies $|\mathcal{A}| \leqslant \max _{v \in V(\Delta)} f_{r-1}\left(\operatorname{link}_{\Delta} v\right)$. Equivalently, $\Delta$ is $r$-EKR if the set of all $r$-faces containing some $v$ has maximal cardinality among all intersecting families of $r$-faces.

Holroyd and Talbot [18] further investigated the problem of when graphs are $r$-EKR, and made the following conjecture:

Conjecture 1.5. (See Holroyd and Talbot [18, Conjecture 7].) If $G$ is a graph where the minimal facet cardinality of $I(G)$ is $k$, then $I(G)$ is $r-E K R$ for $r \leqslant \frac{k}{2}$.

The natural extension was made by Borg [5]:
Conjecture 1.6. (See Borg [5, Conjecture 1.7].) If $\Delta$ is a simplicial complex having minimal facet cardinality $k$, then $\Delta$ is $r-E K R$ for $r \leqslant \frac{k}{2}$.

Remark 1.7. A different version of an EKR property for simplicial complexes was studied by Chvátal [8], who conjectured that if $\mathcal{A}$ is an intersecting family of faces (of possibly differing dimensions) then

$$
|\mathcal{A}| \leqslant \max _{v \in V(\Delta)}\left(\sum_{r} f_{r}\left(\operatorname{link}_{\Delta} v\right)\right)
$$

i.e., that the set of all faces containing some $v$ has maximal cardinality among all intersecting families of faces. We notice that Conjecture 1.6 is an analogue of Chvátal's Conjecture for uniform intersecting families.

We refer the reader to [6] for additional background on the $r$-EKR property in graphs, and to [5] for further relationships with more general intersection problems.

This paper is organized as follows. In Section 2 we review the necessary background on shifted complexes, algebraic shifting, the Cohen-Macaulay property, and near-cones. We also characterize the graphs $G$ such that $\operatorname{Shift} I(G)$ is the independence complex of some other graph, and the graphs such that $I(G)$ is a near-cone. In Section 3 we present and prove our main theorem, Theorem 3.3. In Section 4 we give applications of Theorem 3.3 to Conjecture 1.5. In particular, we recover the main result of [19], and many of the results of [17]. We close in Section 5 with further questions regarding the strict $r$-EKR property.

## 2. Shifting

We will need some basic simplicial complex language: An (abstract) simplicial complex $\Delta$ is a system of sets (called faces) on base set $V(\Delta)$ (called vertices) such that if $\sigma$ is a face then every subset
of $\sigma$ is also a face. We assume that every vertex is contained in some face. A facet of $\Delta$ is a face that is maximal under inclusion. It is well known that any abstract simplicial complex has a geometric realization, a geometric simplicial complex with the same face incidences; so we can use terms from geometry such as dimension to describe a simplicial complex.

If $\mathcal{F}$ is some family of sets, then the simplicial complex $\Delta(\mathcal{F})$ generated by $\mathcal{F}$ has faces consisting of all subsets of all sets in $\mathcal{F}$. For a simplicial complex $\Delta$, the $r$-skeleton $\Delta^{(r)}$ consists of all faces of $\Delta$ having dimension at most $r$, while the pure $r$-skeleton is the subcomplex generated by all faces of $\Delta$ having dimension exactly $r$. The join of disjoint simplicial complexes $\Delta$ and $\Sigma$ is the simplicial complex $\Delta * \Sigma$ with faces $\tau \cup \sigma$, where $\tau$ is a face of $\Delta$ and $\sigma$ a face of $\Sigma$.

A simplicial complex $\Delta$ with ordered vertex set $\left\{v_{1}, \ldots, v_{n}\right\}$ is shifted if whenever $\sigma$ is a face of $\Delta$ containing vertex $v_{i}$, then $\left(\sigma \backslash\left\{v_{i}\right\}\right) \cup\left\{v_{j}\right\}$ is a face of $\Delta$ for every $j<i$. An $r$-family $\mathcal{F}$ of subsets of $\left\{v_{1}, \ldots, v_{n}\right\}$ is shifted if it generates a shifted complex.

A general approach to proving theorems similar to Theorem 1.3 is to define a shifting operation or compression operation which replaces a non-shifted set system with a shifted system obeying some of the same combinatorial properties. Erdős, Ko, and Rado pioneered this technique in [11], and their operation is now called combinatorial shifting. Combinatorial shifting is discussed in the survey article [13], particularly as applied to intersection theorems.

### 2.1. Algebraic shifting

The specific shifting operation we will use is called exterior algebraic shifting (with respect to a field $F$ ), and we denote the (exterior) algebraic shift of $\Delta$ by Shift $\Delta$, or $\operatorname{Shift}_{F} \Delta$ if we want to emphasize the field. Algebraic shifting was first studied by Kalai, and unless otherwise stated the facts we present here were first proved by him.

The precise definition of Shift $\Delta$ will not be important for us, but can be found in Kalai's survey article [22]. Rather than working with the definition, we examine Shift $\Delta$ using a series of lemmas collected in [22]. The following basic properties we will use without further mention:

Lemma 2.1. (See [22].) Let $\Delta$ be a simplicial complex with $n$ vertices. Then:
(1) Shift $\Delta$ is a shifted simplicial complex on an ordered vertex set $\left\{v_{1}, \ldots, v_{n}\right\}$.
(2) If $\Delta$ is shifted, then Shift $\Delta \cong \Delta$.
(3) $f_{i}($ Shift $\Delta)=f_{i}(\Delta)$ for all $i$.

Shifting respects subcomplexes, at least in a weak sense:
Lemma 2.2. (See [22, Theorem 2.2].) If $\Sigma \subset \Delta$ are simplicial complexes, then Shift $\Sigma \subset \operatorname{Shift} \Delta$.
Example 2.3. Let $\Delta$ be the simplicial complex with facets $\{1,2\}$ and $\{3,4\}$. Then it is easy to see that the unique shifted complex with the same $f$-vector has facets $\{1,2\},\{2,3\}$, and $\{4\}$, hence that this complex is Shift $\Delta$. Then $\operatorname{Shift}\{1,2\}=\operatorname{Shift}\{3,4\}=\{1,2\} \subset \operatorname{Shift} \Delta$.

Corollary 2.4. If $\Delta$ is a simplicial complex, then $\operatorname{Shift}\left(\Delta^{(r)}\right)=(\operatorname{Shift} \Delta)^{(r)}$.
Proof. Lemma 2.2 gives that $\operatorname{Shift}\left(\Delta^{(r)}\right) \subseteq(\operatorname{Shift} \Delta)^{(r)}$, and by Lemma 2.1 part (3) the $f$-vectors are equal.

Notice that if $\Delta$ is a shifted complex, then it is immediate that $\Delta^{(r)}$ is shifted for every $r$.
If $\mathcal{A}$ is some $r$-family of sets, then $\operatorname{Shift} \mathcal{A}$ is the pure $r$-skeleton of Shift $\Delta(\mathcal{A})$. (Kalai's equivalent definition actually defines Shift $\Delta$ as a union of the shift of its $r$-faces [22, Section 2.1].) Kalai proves:

Lemma 2.5. (See [22, Corollary 6.3].) If $\mathcal{A}$ is an intersecting $r$-family, then $\operatorname{Shift} \mathcal{A}$ is an intersecting $r$-family.

### 2.2. Near-cones

A simplicial complex $\Delta$ is a near-cone with respect to an apex vertex $v$ if for every face $\sigma$, the set $(\sigma \backslash\{w\}) \cup\{v\}$ is also a face for each vertex $w \in \sigma$. Equivalently, the boundary of every facet of $\Delta$ is contained in $v * \operatorname{link}_{\Delta} v$; another equivalent condition is that $\Delta$ consists of $v * \operatorname{link}_{\Delta} v$ union some set of facets not containing $v$ (but whose boundary is contained in $\operatorname{link}_{\Delta} v$ ). If $\Delta$ is a cone with apex vertex $v$, then obviously $\Delta=v * \operatorname{link}_{\Delta} v$, thus every cone is a near-cone.

Because the apex vertex is always the vertex with the largest link, near-cones are relatively easy to work with in the context of intersection theorems, as has been previously noticed in e.g. [27,4]. In particular:

Lemma 2.6. If $\Delta$ is a near-cone with apex vertex $v$, then $f_{r}\left(\operatorname{link}_{\Delta} w\right) \leqslant f_{r}\left(\operatorname{link}_{\Delta} v\right)$ for any vertex $w$ and all $r$.
Proof. For every ( $r+1$ )-face $\sigma$ containing $w$, either $v \in \sigma$ or else $(\sigma \backslash w) \cup v$ is an $r$-face containing $v$ but not $w$.

Notice that $\Delta$ being a near-cone with apex vertex $v$ essentially says that $\Delta$ is "shifted with respect to $v$ ". In particular, any shifted complex is a near-cone with apex vertex $v_{1}$. Nevo examined the algebraic shift of a near-cone, showing:

Lemma 2.7. (See Nevo [25, Theorems 5.2 and 5.3].) If $\Delta$ is a near-cone with apex $v$, let us consider Shift $\left(\operatorname{link}_{\Delta} v\right.$ ) as having ordered vertex set $\left\{v_{2}, \ldots, v_{n}\right\}$. Then:
(1) $\operatorname{link}_{\text {shift }} \Delta v_{1}=\operatorname{Shift}\left(\operatorname{link}_{\Delta} v\right.$ ).
(2) Shift $\Delta=\left(v_{1} * \operatorname{Shift}\left(\operatorname{link}_{\Delta} v\right)\right) \cup \mathcal{B}$, where $\mathcal{B}$ is a set of facets not containing $v_{1}$.

Corollary 2.8. If $\Delta$ is a near-cone with apex $v$, then $f_{r}\left(\operatorname{link}_{\Delta} v\right)=f_{r}\left(\operatorname{link}_{\text {Shift }} \Delta v_{1}\right)$ for all $r$.

### 2.3. Pure complexes, Cohen-Macaulay complexes, and depth

A simplicial complex $\Delta$ is pure if all facets of $\Delta$ have the same dimension. Graphs with a pure independence complex are sometimes called well-covered.

Let $F$ be either any field, or the ring of integers. A simplicial complex $\Delta$ is Cohen-Macaulay over $F$ if its homology satisfies $\tilde{H}_{i}\left(\operatorname{link}_{\Delta} \sigma ; F\right)=0$ for all $i<\operatorname{dim}\left(\operatorname{link}_{\Delta} \sigma\right)$ and all faces $\sigma$ of $\Delta$ (including $\sigma=\emptyset$ ). It is well known that every Cohen-Macaulay complex is pure, and that every skeleton of a Cohen-Macaulay complex is Cohen-Macaulay.

A simplicial complex is sequentially Cohen-Macaulay over $F$ if the pure $r$-skeleton of $\Delta$ is Cohen-Macaulay (over F) for all $r$. Thus, a pure sequentially Cohen-Macaulay complex is CohenMacaulay.

When we simply say that a simplicial complex $\Delta$ is (sequentially) Cohen-Macaulay, with no mention of $F$, then we mean that $\Delta$ is (sequentially) Cohen-Macaulay over all $F$. For example, every "shellable" or "vertex decomposable" complex is sequentially Cohen-Macaulay over any $F[2,3]$.

The main relationships between the Cohen-Macaulay property and shifting are the following:
Lemma 2.9. (See [3, Theorem 11.3].) If $\Delta$ is shifted, then $\Delta$ is "vertex decomposable", hence sequentially Cohen-Macaulay (over any F).

Lemma 2.10. (See [22, Theorem 4.1], see also [1, Proposition 8.4].) $\operatorname{Shift}_{F} \Delta$ is pure if and only if $\Delta$ is CohenMacaulay (over F).

Duval [9] also examined the algebraic shift of a sequentially Cohen-Macaulay complex, and more generally of Cohen-Macaulay skeletons. A result of his that will be of particular interest to us is:

Corollary 2.11. (See Duval [9, Corollary 4.5].) The minimum facet dimension of $\operatorname{Shift}_{F} \Delta$ is $\geqslant d$ if and only if $\Delta^{(d)}$ is Cohen-Macaulay (over F).

Corollary 2.11 suggests the definition of the depth of $\Delta$ over $F$ as

$$
\operatorname{depth}_{F} \Delta=\max \left\{d: \Delta^{(d)} \text { is Cohen-Macaulay over } F\right\} .
$$

Thus, $\operatorname{depth}_{F} \Delta$ is the minimum facet dimension of $\operatorname{Shift}_{F} \Delta$. We note that depth ${ }_{F} \Delta$ is one less than the ring-theoretic depth of the "Stanley-Reisner ring" $F[\Delta]$ [26, Theorem 3.7]. Thus, just as in the ring-theoretic situation, $\Delta$ is Cohen-Macaulay over $F$ if and only if $\operatorname{depth}_{F} \Delta=\operatorname{dim} \Delta$. If $\Delta$ is sequentially Cohen-Macaulay over $F$ then $\operatorname{depth}_{F} \Delta$ is the minimum facet dimension of $\Delta$.

By the definition of simplicial homology we have $\tilde{H}_{d}\left(\Delta^{(d+1)} ; F\right)=\tilde{H}_{d}(\Delta ; F)$, hence the easy equivalent characterization:

$$
\operatorname{depth}_{F} \Delta=\max \left\{d: \tilde{H}_{i}\left(\operatorname{link}_{\Delta} \sigma ; F\right)=0 \text { for all } \sigma \in \Delta \text { and } i<d-|\sigma|\right\} .
$$

In particular, we notice that $\operatorname{depth}_{F} \Delta$ is at most the minimal facet dimension, since if $\sigma$ is a facet then $\tilde{H}_{-1}\left(\operatorname{link}_{\Delta} \sigma ; F\right)=\tilde{H}_{-1}(\varnothing ; F)=F$.

The following result about the depth of the join of complexes will be especially useful in Section 4:
Lemma 2.12. Let $F$ be a field, and $\Delta_{1}$ and $\Delta_{2}$ be simplicial complexes. Then $\operatorname{depth}_{F}\left(\Delta_{1} * \Delta_{2}\right)=\operatorname{depth}_{F} \Delta_{1}+$ $\operatorname{depth}_{F} \Delta_{2}+1$.

Proof. Faces of $\Delta_{1} * \Delta_{2}$ have the form $\sigma=\sigma_{1} \dot{U} \sigma_{2}$ where $\sigma_{i}$ is a face of $\Delta_{i}$, hence $\operatorname{link}_{\Delta_{1} * \Delta_{2}} \sigma=$ $\operatorname{link}_{\Delta_{1}} \sigma_{1} * \operatorname{link}_{\Delta_{2}} \sigma_{2}$. The result then follows from the standard algebraic topology fact [20, Corollary 4.23] that $\tilde{H}_{n+1}\left(\Delta_{1} * \Delta_{2} ; F\right)=\bigoplus_{i=-1}^{n+1} \tilde{H}_{i}\left(\Delta_{1} ; F\right) \otimes \tilde{H}_{n-i}\left(\Delta_{2} ; F\right)$.

The reader is referred to [20] for additional background on $\operatorname{depth}_{F} \Delta$ and the (sequentially) CohenMacaulay property. We will henceforth take the field $F$ to be understood, and drop it from our notation.

### 2.4. Shifting independence complexes

A graph $G$ is chordal if every induced subgraph of $G$ which is a cyclic graph has length 3 . A graph is co-chordal if its complement graph is chordal.

Since the original question of Holroyd, Spencer and Talbot was restricted to the independence complexes of graphs, one might ask when $\operatorname{Shift} I(G)$ is isomorphic to $I\left(G^{\prime}\right)$ for some graph $G^{\prime}$. The answer is easy, given the necessary machinery. The Alexander dual of $\Delta$, denoted $\Delta^{\vee}$, is the complex with facets $\{\sigma: V \backslash \sigma$ is a minimal non-face of $\Delta\}$. See e.g. [21, Section 6] for more information and background on Alexander duality.

Theorem 2.13. Let $G$ be a graph. Then Shift $I(G)$ is the independence complex $I\left(G^{\prime}\right)$ of some other graph $G^{\prime}$ if and only if $G$ is co-chordal.

Proof. We need the following three facts about Alexander duality: (1) It is clear from the definition that $\Delta$ is the independence complex of a graph if and only if $\Delta^{\vee}$ is pure ( $n-2$ )-dimensional, where $n$ is the number of vertices of $\Delta$. (2) Alexander duality and shifting commute, i.e. Shift $\Delta=\left(\operatorname{Shift}\left(\Delta^{\vee}\right)\right)^{\vee}$ [22, Section 3.5.6]. (3) If $G$ is a graph, then $I(G)^{\vee}$ is Cohen-Macaulay if and only if $G$ is co-chordal [10, Proposition 8]. The result is then immediate from Lemma 2.10.

Remark 2.14. The family of independence complexes of graphs has been extensively studied in the literature under the name of flag complexes.

The shifted flag complexes were classified by Klivans, as follows: Given a graph $G$, let $D(G)$ be $G \dot{\cup}\{v\}$ for a new vertex $v$ ( $D$ for "disjoint union"). Let $S(G)$ be the graph on vertex set $V(G) \cup\{v\}$ for a new vertex $v$ and with edge set $E(G) \cup\{w v: w \in V(G)\}$ ( $S$ for "star"). In the independence complex, we have $I(D(G))$ as the cone over $I(G)$, and $I(S(G))$ as $I(G) \dot{\cup}\{v\}$, thus if $G$ is sequentially Cohen-Macaulay then both $D(G)$ and $S(G)$ are.

A graph is threshold [24] if it is obtained from a single vertex by some sequence of $D$ and $S$ operations. Every threshold graph is both chordal and co-chordal. Since a $D$ operation adds a cone vertex, which can be taken as the initial vertex in a shifted complex; and an $S$ operation adds a disjoint vertex, which can be taken as the final vertex in a shifted complex, we have proved inductively that the independence complex of any threshold graph is shifted. Klivans [23] showed the converse result that all graphs with shifted independence complex are threshold.

We prove the following generalization of [23, Theorem 1]:
Proposition 2.15. If $G$ is a graph such that $I(G)$ is a near-cone, then $G$ is obtained from some graph $G_{0}$ by a sequence of $D$ and $S$ operations, including at least one $D$ operation.

Proof. Let $v$ be the apex vertex of $I(G)$, and suppose that $w v \in E(G)$. Then $w x$ is also an edge for every $x \in V(G)$, since if $w x$ were a face of $I(G)$ then $w v$ would also be independent, a contradiction. We see that $G=S^{k} D(G \backslash N[v])$, where $k$ is the number of neighbors of $v$.

Remark 2.16. Since the independence complex of $S(G)$ has an isolated vertex, its minimum facet dimension is 0 . Hence Proposition 2.15 tells us that for a graph $G$ we have that $I(G)$ is a near-cone with non-trivial minimum facet dimension if and only if $G$ has an isolated vertex.

## 3. Main theorem

The following lemma follows from Borg's more general result [5, Theorem 2.7]. We use algebraic shifting to give a short new proof of the specific result.

Lemma 3.1. (See Borg [5, Theorem 2.7].) If $\Delta$ is a shifted complex having minimal facet cardinality $k$, then $\Delta$ is $r$-EKR for $r \leqslant \frac{k}{2}$.

Proof. Let $\Delta$ have ordered vertex set $\left\{v_{1}, \ldots, v_{n}\right\}$, and let $\mathcal{A}$ be an intersecting $r$-family of faces of $\Delta$. We proceed by induction: our base cases are when $\Delta$ is a simplex (Theorem 1.3), and the trivial case where $r=1$.

If $\Delta$ is not a simplex and $r>1$, then by Lemmas 2.5 and 2.2 , we have that $\operatorname{Shift} \mathcal{A}$ is a shifted intersecting $r$-family of faces of $\Delta=\operatorname{Shift} \Delta$ with $|\operatorname{Shift} \mathcal{A}|=|\mathcal{A}|$. We decompose Shift $\mathcal{A}$ into the subfamilies $\mathcal{C}$ consisting of all $\sigma \in \operatorname{Shift} \mathcal{A}$ with $v_{n} \in \sigma$, and $\mathcal{D}=(\operatorname{Shift} \mathcal{A}) \backslash \mathcal{C}$, so that $|\mathcal{A}|=|\operatorname{Shift} \mathcal{A}|=$ $|\mathcal{C}|+|\mathcal{D}|$.

We first consider $\mathcal{C}$. Let $\mathcal{C}_{0}=\left\{\sigma \backslash\left\{v_{n}\right\}: \sigma \in \mathcal{C}\right\}$, so that $|\mathcal{C}|=\left|\mathcal{C}_{0}\right|$. Suppose that $\mathcal{C}_{0}$ is not intersecting. Then there are $\sigma, \tau \in \mathcal{C}$ such that $\sigma \cap \tau=\left\{v_{n}\right\}$, and $|\sigma \cup \tau|<r+r \leqslant k<n$. It follows that there is a $v_{\ell} \notin \sigma \cup \tau$. But then $\tau^{\prime}=\left(\tau \backslash\left\{v_{n}\right\}\right) \cup\left\{v_{\ell}\right\}$ is in Shift $\mathcal{A}$ by the definition of shiftedness, and $\sigma \cap \tau^{\prime}=\emptyset$, which contradicts that $\operatorname{Shift} \mathcal{A}$ is intersecting. We conclude that $\mathcal{C}_{0}$ is an intersecting $(r-1)$-family of faces of $\operatorname{link}_{\Delta} v_{n}$. Since $\operatorname{link}_{\Delta} v_{n}$ is a shifted complex with minimum facet cardinality at least $k-1$, we get that $|\mathcal{C}|=\left|\mathcal{C}_{0}\right| \leqslant f_{r-2}\left(\operatorname{link}_{\Delta}\left\{v_{1}, v_{n}\right\}\right)$ by induction and Lemma 2.6.

We now consider $\mathcal{D}$. It is obvious that $\mathcal{D}$ is an intersecting $r$-family contained in the shifted complex $\Delta \backslash v_{n}$. Since $\Delta$ is not a simplex, it follows easily from the definition of shiftedness that the minimum facet cardinality of $\Delta \backslash v_{n}$ is at least $k$. By induction and Lemma 2.6 we have $|\mathcal{D}| \leqslant$ $f_{r-1}\left(\operatorname{link}_{\Delta \backslash v_{n}} v_{1}\right)$.

Putting our two parts together, we have

$$
|\mathcal{A}|=|\mathcal{C}|+|\mathcal{D}| \leqslant f_{r-2}\left(\operatorname{link}_{\Delta}\left\{v_{1}, v_{n}\right\}\right)+f_{r-1}\left(\operatorname{link}_{\Delta \mid v_{n}} v_{1}\right)=f_{r-1}\left(\operatorname{link}_{\Delta} v_{1}\right)
$$

Remark 3.2. Our requirement for Lemma 3.1 on the minimum facet cardinality seems much stronger than necessary. Our essential need is for a parameter $k$ which we can control in both $\Delta \backslash v_{n}$ and $\operatorname{link}_{\Delta} v_{n}$, and such that $r \leqslant \frac{k}{2}$ forces $r<\frac{n}{2}$. Use of another such parameter might give a stronger result version of Lemma 3.1. Any strengthening of Lemma 3.1 would likely also strengthen Theorem 3.3.

Holroyd and Talbot [18, Section 3] construct several examples of independence complexes that are not $r$-EKR for various $r$, which may give some intuition about what parameters are tractable. (They in particular construct an example with maximum facet cardinality $\ell$ such that $\Delta$ is not $\left\lfloor\frac{\ell}{2}\right\rfloor$ EKR.)

By applying algebraic shifting to an arbitrary complex, we prove:
Theorem 3.3. If $\Delta$ is a near-cone and $F$ is an arbitrary field, then $\Delta$ is $r$-EKR for $r \leqslant \frac{\operatorname{depth}_{F} \Delta+1}{2}$.
Proof. Let $\mathcal{A}$ be an intersecting $r$-family of faces of $\Delta$. By Lemma 2.6 , we need to show that $|\mathcal{A}| \leqslant$ $f_{r-1}\left(\operatorname{link}_{\Delta} v\right)$ for the apex vertex $v$.

Apply algebraic shifting. Shift ${ }_{F} \mathcal{A}$ is an intersecting $r$-family of faces of Shift $\Delta$ with $|\operatorname{Shift} \mathcal{A}|=|\mathcal{A}|$ by Lemmas 2.5 and 2.2. By Lemma 2.11 and the following discussion, the minimum facet cardinality of $\operatorname{Shift}_{F} \Delta$ is $\operatorname{depth}_{F} \Delta+1$, hence $|\mathcal{A}| \leqslant f_{r-1}\left(\operatorname{link}_{\text {Shift }} \Delta v_{1}\right)=f_{r-1}\left(\operatorname{link}_{\Delta} v\right)$ by Lemma 3.1 and Corollary 2.8 .

To the best of my knowledge, Theorem 3.3 is the first 'new' intersection theorem to be proved by algebraic shifting.

Corollary 3.4. If $\Delta$ is a sequentially Cohen-Macaulay near-cone with minimum facet cardinality $k$, then $\Delta$ is $r$-EKR for $r \leqslant \frac{k}{2}$.

Remark 3.5. Borg's aforementioned result [5, Theorem 2.7] generalizes Lemma 3.1 to include nonuniform families (i.e., sets of different sizes), and to $t$-intersecting families (i.e., to families where $|A \cap B| \geqslant t$ ). By Kalai [22, Corollary 6.3 and following], algebraic shifting preserves the $t$-intersecting property for any $r$-family, hence a reduction to [5, Theorem 2.7] similar to that in Theorem 3.3 will show that if $\Delta$ is a $t$-fold near-cone (i.e., shifted with respect to its first $t$ elements) with depth equal to its minimum facet dimension, then [5, Conjecture 2.7] holds for uniform $r$-families of faces in $\Delta$. In particular, [5, Conjecture 2.7] holds for sequentially Cohen-Macaulay $t$-fold near-cones.

## 4. Applications

It is immediate from the definitions that if $G=G_{1} \dot{\cup} G_{2}$, then $I(G)$ decomposes as the join $I\left(G_{1}\right) * I\left(G_{2}\right)$. In particular, if $G$ has an isolated vertex $v$ then $I(G)$ is a cone over $v$, as discussed in detail in Section 2.2. We will call a graph $G$ sequentially Cohen-Macaulay if its independence complex $I(G)$ is sequentially Cohen-Macaulay. An immediate consequence of Corollary 3.4 is:

Theorem 4.1. If $G$ is a sequentially Cohen-Macaulay graph with an isolated vertex, then $G$ satisfies Conjecture 1.5 .

The family of sequentially Cohen-Macaulay graphs includes:
(1) Chordal graphs [12].
(2) Graphs with no induced cycle of length other than 3 or 5 [31].
(3) Bipartite graphs containing a vertex $v$ of degree 1 such that $G \backslash N[v]$ and $G \backslash N[w]$ (where $w$ is the unique neighbor of $v$ ) recursively satisfy the same condition [30, Corollary 3.11].
(4) Incomparability graphs of shellable posets [2].
(5) The minimal set of non-faces of the isocahedron, or any other polytope where the set of minimal non-faces forms a graph [7].
(6) Disjoint unions of sequentially Cohen-Macaulay graphs, since $I\left(G_{1} \dot{\cup} G_{2}\right)=I\left(G_{1}\right) * I\left(G_{2}\right)$.

In particular, we recover the following theorem of Hurlbert and Kamat:
Corollary 4.2. (See Hurlbert and Kamat [19, Theorem 1.22].) If G is a chordal graph with an isolated vertex, then G satisfies Conjecture 1.5.

Obviously we also have that if $G$ is e.g. a threshold graph, then $G$ satisfies Conjecture 1.5. But Remark 2.16 tells us that this result is not an interesting improvement on Corollary 4.2 , since in this case the minimum facet cardinality of $I(G)$ is 1 unless $G$ has an isolated vertex.

We apply Lemma 2.12 for a result in a slightly different direction:
Proposition 4.3. If $G=G_{1} \dot{\cup} \ldots \dot{\cup} G_{n}$ is the disjoint union of $n \geqslant 2 r$ non-empty graphs, including at least one isolated vertex, then $I(G)$ is $r$-EKR.

Proof. The 0-skeleton of any non-empty complex is Cohen-Macaulay, hence depth $I\left(G_{i}\right) \geqslant 0$, and by repeated application of Lemma 2.12 we get depth $I(G)=\operatorname{depth} I\left(G_{1}\right) * \cdots * I\left(G_{n}\right) \geqslant n-1$. The result then follows by Theorem 3.3.

Proposition 4.3 significantly improves [17, Theorem 8], which proves the result in the special case where each $G_{i}$ is a complete graph, path, or cycle. By considering graphs of depth 1 , we can do slightly better.

Lemma 4.4. The independence complex of a graph $G$ has depth $\geqslant 1$ if and only if $|G|>1$ and the complement graph $\bar{G}$ is connected.

Proof. The complement graph $\bar{G}$ forms the 1 -skeleton of $I(G)$ under the hypothesis, and a 1dimensional complex is Cohen-Macaulay if and only if it is connected.

Example 4.5. Let $C_{n}$ be the cyclic graph on $n$ vertices. If $n \geqslant 5$, then $C_{n}$ satisfies the conditions of Lemma 4.4, hence depth $I\left(C_{n}\right) \geqslant 1$. But the cyclic graph $C_{4}$ on 4 vertices has disconnected complement graph, hence depth $I\left(C_{4}\right)=0$.

Proposition 4.6. Let $G=G_{1} \dot{\cup} \ldots \dot{\cup} G_{n}$ be the disjoint union of $n$ graphs, including at least one isolated vertex. Suppose that $m$ of the $G_{i}$ satisfy the conditions of Lemma 4.4. Then $I(G)$ is $r-E K R$ for $r \leqslant \frac{n+m}{2}$.

The proof is exactly as in Proposition 4.3.

## 5. Further questions

As we have discussed, for $\Delta$ to be $r$-EKR means that every maximal intersecting $r$-family of faces $\mathcal{A}$ has $|\mathcal{A}| \leqslant \max _{v \in V(\Delta)} f_{r-1}\left(\right.$ link $\left._{\Delta} v\right)$. We say that $\Delta$ is strictly $r$-EKR if every maximum cardinality intersecting $r$-family of faces $\mathcal{A}$ consists of the $r$-faces of $v *\left(\operatorname{link}_{\Delta} v\right)^{(r-1)}$ for some $v$. That is to say, every maximum intersecting $r$-family $\mathcal{A}$ satisfies $\bigcap_{A \in \mathcal{A}} A \neq \emptyset$. Hilton and Milner [15] improved Theorem 1.3 to:

Theorem 5.1. (See Hilton and Milner [15].) If $\Delta$ is the simplex with $n$ vertices, then $\Delta$ is strictly $r$-EKR for $2 \leqslant r<\frac{n}{2}$.

Holroyd and Talbot, and later Borg, actually conjectured slightly more than we stated in Conjectures 1.5 and 1.6:

Conjecture 5.2. (See Holroyd and Talbot [18, Conjecture 7], Borg [5, Conjecture 1.7].) If $\Delta$ is a simplicial complex having minimal facet cardinality $k$, then $\Delta$ is $r$-EKR for $r \leqslant \frac{k}{2}$, and strictly $r$-EKR for $r<\frac{k}{2}$.

Can an algebraic shifting (or some other) argument be adapted to prove Conjecture 5.2, optionally restricted to the case of a sequentially Cohen-Macaulay complex?

Of course, even the more restricted Conjecture 1.6 remains open for general complexes. Theorem 3.3 suggests that a counterexample to Conjecture 1.6, if one exists, should be a complex which badly fails to be Cohen-Macaulay. We discuss briefly some examples of such complexes:

Example 5.3. The facets of the boundary $\Delta_{0}$ of a simplex with $n+1$ vertices is intersecting, hence not $n$-EKR. One can increase the dimension by coning $k$ points over each facet to obtain a pure complex $\Delta$ with $(k+1) \cdot(n+1)$ points. Since $\tilde{H}_{n}(\Delta) \neq 0$, the complex is not Cohen-Macaulay, and the hope for a counterexample would be: for some $k>n$ and $n \leqslant r \leqslant\left\lfloor\frac{k+n}{2}\right\rfloor$, that the family $\mathcal{A}$ consisting of all $r$-faces that contain a facet of $\Delta_{0}$ would be larger than $f_{r-1}\left(\operatorname{link}_{\Delta} v\right)$.

But we count: if $v \in \Delta_{0}$, then $f_{r-1}\left(\right.$ link $\left._{\Delta} v\right)=n \cdot\binom{n+k}{r-1}$, while $|\mathcal{A}|=(n+1) \cdot\binom{k}{r-n}$. A straightforward computation (cancel, then match terms) yields that $f_{r-1}\left(\operatorname{link}_{\Delta} v\right) /|\mathcal{A}|>1$. Hence $|\mathcal{A}|<f_{r-1}\left(\operatorname{link}_{\Delta} v\right)$, and thus $\mathcal{A}$ and $\Delta$ are not a counterexample to Conjecture 1.6.

Example 5.4. Cyclic graphs $C_{n}$ are not sequentially Cohen-Macaulay for $n \neq 3,5$ [12, Proposition 4.1]. For example, $I\left(C_{4}\right)$ consists of two disjoint 2 -faces, while $I\left(C_{7}\right)$ is a triangulation of the Möbius strip. Nonetheless, Talbot [29] showed that the independence complex of every cyclic graph is $r$-EKR for all $r$. More recently, the independence complex of the disjoint union of two cycles [14], and of the disjoint union of an arbitrary number of cycles and a path [16] were shown to be $r$-EKR for all $r$. Conjecture 1.5 also holds [6] for the disjoint union of an isolated vertex and a somewhat wider class of non-sequentially Cohen-Macaulay graphs, including cycles and complete multipartite graphs.

## Acknowledgments

The article at Gil Kalai's blog (at http://gilkalai.wordpress.com/) on the use of algebraic shifting in intersection theorems was very helpful, especially in proving Lemma 3.1. Art Duval and Isabella Novik also answered some of my questions about algebraic shifting. Glenn Hurlbert and Vikram Kamat explained the difficulties in extending their proof of Corollary 4.2 to all vertex decomposable graphs.

## References

[1] Annetta Aramova, Jürgen Herzog, Almost regular sequences and Betti numbers, Amer. J. Math. 122 (2000) 689-719.
[2] Anders Björner, Michelle L. Wachs, Shellable nonpure complexes and posets. I, Trans. Amer. Math. Soc. 348 (1996) 12991327.
[3] Anders Björner, Michelle L. Wachs, Shellable nonpure complexes and posets. II, Trans. Amer. Math. Soc. 349 (1997) 39453975.
[4] Peter Borg, Intersecting systems of signed sets, Electron. J. Combin. 14 (2007), Research Paper $41,11 \mathrm{pp}$. (electronic).
[5] Peter Borg, Extremal $t$-intersecting sub-families of hereditary families, J. Lond. Math. Soc. (2) 79 (2009) 167-185.
[6] Peter Borg, Fred Holroyd, The Erdős-Ko-Rado properties of various graphs containing singletons, Discrete Math. 309 (2009) 2877-2885.
[7] H. Bruggesser, P. Mani, Shellable decompositions of cells and spheres, Math. Scand. 29 (1971) 197-205, (1972).
[8] Vašek Chvátal, Intersecting families of edges in hypergraphs having the hereditary property, in: Hypergraph Seminar, Proc. First, Working Sem., Ohio State Univ., Columbus, Ohio, 1972, in: Lecture Notes in Math., vol. 411, Springer, Berlin, 1974, pp. 61-66, dedicated to Arnold Ross.
[9] Art M. Duval, Algebraic shifting and sequentially Cohen-Macaulay simplicial complexes, Electron. J. Combin. 3 (1996), Research Paper 21, approx. 14 pp . (electronic).
[10] John A. Eagon, Victor Reiner, Resolutions of Stanley-Reisner rings and Alexander duality, J. Pure Appl. Algebra 130 (1998) 265-275.
[11] P. Erdős, Chao Ko, R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. (2) 12 (1961) 313320.
[12] Christopher A. Francisco, Adam Van Tuyl, Sequentially Cohen-Macaulay edge ideals, Proc. Amer. Math. Soc. 135 (2007) 2327-2337 (electronic), arXiv:math/0511022.
[13] Peter Frankl, The shifting technique in extremal set theory, in: Surveys in Combinatorics 1987, New Cross, 1987, in: London Math. Soc. Lecture Note Ser., vol. 123, Cambridge Univ. Press, Cambridge, 1987, pp. 81-110.
[14] A.J.W. Hilton, F.C. Holroyd, C.L. Spencer, King Arthur and his knights with two round tables, Q. J. Math., in press, doi:10.1093/qmath/haq005.
[15] A.J.W. Hilton, E.C. Milner, Some intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. (2) 18 (1967) 369-384.
[16] A.J.W. Hilton, C.L. Spencer, A graph-theoretical generalization of Berge's analogue of the Erdős-Ko-Rado theorem, in: Graph Theory in Paris, in: Trends Math., Birkhäuser, Basel, 2007, pp. 225-242.
[17] Fred Holroyd, Claire Spencer, John Talbot, Compression and Erdős-Ko-Rado graphs, Discrete Math. 293 (2005) 155-164.
[18] Fred Holroyd, John Talbot, Graphs with the Erdős-Ko-Rado property, Discrete Math. 293 (2005) 165-176, arXiv:math/ 0307073.
[19] Glenn Hurlbert, Vikram Kamat, Erdős-Ko-Rado theorems for chordal and bipartite graphs, J. Combin. Theory Ser. A, in press, arXiv:0903.4203.
[20] Jakob Jonsson, Simplicial Complexes of Graphs, Lecture Notes in Math., vol. 1928, Springer-Verlag, Berlin, 2008.
[21] Gil Kalai, Enumeration of $Q$-acyclic simplicial complexes, Israel J. Math. 45 (1983) 337-351.
[22] Gil Kalai, Algebraic shifting, in: Computational Commutative Algebra and Combinatorics, Osaka, 1999, in: Adv. Stud. Pure Math., vol. 33, Math. Soc. Japan, Tokyo, 2002, pp. 121-163.
[23] Caroline J. Klivans, Threshold graphs, shifted complexes, and graphical complexes, Discrete Math. 307 (2007) 2591-2597, arXiv:math/0703114.
[24] N.V.R. Mahadev, U.N. Peled, Threshold Graphs and Related Topics, Ann. Discrete Math., vol. 56, North-Holland Publishing Co., Amsterdam, 1995.
[25] Eran Nevo, Algebraic shifting and basic constructions on simplicial complexes, J. Algebraic Combin. 22 (2005) 411-433, arXiv:math/0303233.
[26] Dean E. Smith, On the Cohen-Macaulay property in commutative algebra and simplicial topology, Pacific J. Math. 141 (1990) 165-196.
[27] Hunter Snevily, A new result on Chvátal's conjecture, J. Combin. Theory Ser. A 61 (1992) 137-141.
[28] Ed Swartz, g-elements, finite buildings and higher Cohen-Macaulay connectivity, J. Combin. Theory Ser. A 113 (2006) 1305-1320, arXiv:math/0512086.
[29] John Talbot, Intersecting families of separated sets, J. London Math. Soc. (2) 68 (2003) 37-51.
[30] Adam Van Tuyl, Rafael H. Villarreal, Shellable graphs and sequentially Cohen-Macaulay bipartite graphs, J. Combin. Theory Ser. A 115 (2008) 799-814, arXiv:math/0701296.
[31] Russ Woodroofe, Vertex decomposable graphs and obstructions to shellability, Proc. Amer. Math. Soc. 137 (2009) 32353246, arXiv:0810.0311.


[^0]:    E-mail address: russw@math.wustl.edu.

    0097-3165/\$ - see front matter © 2010 Elsevier Inc. All rights reserved.
    doi:10.1016/j.jcta.2010.11.022

