The Boolean Pivot Operation, M-matrices, and Reducible Matrices

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ABSTRACT

Let $A$ be a Boolean $n \times n$ matrix, and let $G = (N, U)$ be the corresponding digraph, where $N := \{1, \ldots, n\}$ is the set of vertices and $U \subset N \times N$ is the set of arcs of $G$. For any $R \subset N$, combining the steps $r \in R$ of Warshall's algorithm for determining the reachability matrix of $A$ yields the Boolean pivot operation $\mathcal{B}_R$. The matrix $\vec{A} := \mathcal{B}_R A$ is the so-called $R$-reachability matrix of $A$: $\vec{a}_{ij} = 1$ if and only if $a_{ij} = 1$ or there is a connection between $i$ and $j$ via vertices belonging to $R$. We also have $\mathcal{B}_R \mathcal{B}_S = \mathcal{B}_{R \cup S}$ for any $R, S \subset N$.

The Boolean pivot operation is closely related to the principal pivot operation for real matrices. So we obtain the Boolean analogues to the formula for inverting a real block matrix and for the Sherman-Morrison-Woodbury formulae for updating the inverse of a real matrix. Using the close connection between nonsingular $M$-matrices and the corresponding Boolean matrices, we obtain a flexible algorithm for excluding and including vertices one at a time in $G$ while retaining the original connections between the vertices of the current digraph. We derive some criteria for irreducibility of Boolean block matrices and of partitioned $Z$-matrices and give a condensed form of Warshall's algorithm for testing whether a Boolean matrix is irreducible. Moreover, we correct two recent results by R. L. Smith on irreducibility of real block matrices.

1. INTRODUCTION

If $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ ($A$ is a real $m \times n$ matrix), we denote its transpose by $A^T$ and define $|A| = [|a_{ij}|]$. The inequalities $A > 0$, $A \geq 0$, etc. are elementwise. If $R \subset \{1, \ldots, m\}$ and $S \subset \{1, \ldots, n\}$, we let $A_{RS}$ stand for the submatrix of $A$ induced by rows with indices in $R$ and columns with indices...
in $S$. If $A$ is square, we write $\det A$ for its determinant. By a principal permutation of a square matrix we mean simultaneous permutation of the rows and the columns. The identity matrix of order $n$ is denoted by $I_n$ (or sometimes just $I$). Any vector $x \in \mathbb{R}^n$ is interpreted as an $n \times 1$ matrix and denoted by $x = (x_1, \ldots, x_n)^T$. We let $x_R$ stand for the subvector of $x$ consisting of the components with indices from $R$. We denote the empty set by $\emptyset$ and define $N = \{1, \ldots, n\}$. The cardinality of a finite set $R$ is denoted by $|R|$. We abbreviate $R \setminus \{r\}$ by $R - r$, and $R \cup \{s\}$ by $R + s$. We use the symbol $\equiv$ (or $=\!:\!$) for definition.

If $A \in \mathbb{R}^{n \times n}$, $\emptyset \neq R \subset N$, and $A_{RR}$ is nonsingular, then the principal pivotal operation $\mathcal{P}_R$ with the pivot $A_{RR}$ exchanges the variables $y_R$ and $x_R$ in the equation $y = Ax$; see e.g. Keller (1973). Denote the matrix of the resulting equation by $\tilde{A}$. The operation under which the matrix $A$ is transformed into $\tilde{A}$ is also denoted by $\mathcal{P}_R$, and we write $\tilde{A} = \mathcal{P}_R A$. The matrix $\tilde{A}$ is obtained as follows. Denoting $E = N \setminus R$ (which may be empty), there is a principal permutation $\mathcal{C}$ such that

$$\mathcal{C} A = \begin{bmatrix} A_{RR} & A_{R \bar{R}} \\ A_{\bar{R} R} & A_{\bar{R} \bar{R}} \end{bmatrix} =: M.$$

Then $\tilde{A} = \mathcal{C}^{-1} M$, where

$$\tilde{M} = \begin{bmatrix} A_{RR}^{-1} & -A_{RR}^{-1} A_{R \bar{R}} \\ A_{\bar{R} R} A_{RR}^{-1} & A_{\bar{R} R} - A_{R \bar{R}} A_{RR}^{-1} A_{R \bar{R}} \end{bmatrix}. \tag{1.1}$$

The operation $\mathcal{P}_R$ followed by deleting rows and columns with indices in $R$ is denoted by $\mathcal{P}_R^k$ (principal pivotal condensation). If $R = \emptyset$, then $\mathcal{P}_R$ is defined as the identity operation. If $A = [A_{ij}]$ is a block matrix and $R$ is the set of row (and column) indices of its principal submatrix $A_{rr}$, then $\mathcal{P}(r)$ will stand for $\mathcal{P}_R$ and we shall speak of a principal block pivotal operation with pivot $A_{rr}$. The Schur complement of $A_{RR}$ in $A$ equals $\mathcal{P}_R^* A$ and is denoted by $[A/A_{RR}]$. If $R = \{r\}$, then the principal pivotal operation is called single; it will be denoted by $\mathcal{P}_r$, and $a_{rr}$ is its pivot. We have $\tilde{A} = \mathcal{P}_r A$, where

$$\tilde{a}_{rr} = 1/a_{rr},$$
$$\tilde{a}_{ir} = a_{ir}/a_{rr}, \quad i \neq r,$$
$$\tilde{a}_{ej} = -a_{ej}/a_{rr}, \quad j \neq r,$$
$$\tilde{a}_{ij} = a_{ij} - a_{ir} a_{ej}/a_{rr}, \quad i, j \neq r. \tag{1.2}$$
We list some basic properties of the principal pivotal operation.

**Theorem 1.1.** Let $A \in \mathbb{R}^{n \times n}$, $R, S \subseteq N$, and $H = (R \cup S) \setminus (R \cap S)$. Then

(i) $P_R^2 A = A$, provided that $A_{RR}$ is nonsingular;

(ii) $P_R^{} P_S^{} A = P_S^{} P_R^{} A = P_H^{} A$, provided that all the operations are defined;

(iii) if $B := L^? A$ is defined, then $\det A_{HH} = \det A_{RR} \det B_{SS}$;

(iv) if $R \cap S = \emptyset$, then $P_R^{} P_S^{} A = P_S^{} P_R^{} A = P_{R \cup S}^{} A$, provided that all the operations are defined;

(v) if $A$ is nonsingular, then $P_N^{} A = A^{-1}$.

**Proof.** (iii) is due to A. W. Tucker [see Parsons (1970, Theorem 1)], and (v) is obvious. The rest follows from Parsons (1966, Lemma 1.1); cf. Väliaho (1985, Theorem 2.1). \[\square\]

A matrix $A \in \mathbb{R}^{n \times n}$ is reducible if there is a principal permutation $\mathcal{P}$ such that

$$\mathcal{P} A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where the blocks $A_{11}$ and $A_{22}$ are nonvacuous and square. If $A$ is not reducible, it is irreducible. Note that all $1 \times 1$ matrices are irreducible by definition. In any case there is a principal permutation $\mathcal{P}$ such that

$$\mathcal{P} A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ 0 & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{ss} \end{bmatrix},$$

where the blocks $A_{ii}$ are nonvacuous and irreducible. The matrix (1.3) is called a Frobenius normal form of $A$, and the blocks $A_{ii}$ are termed the constituents of $A$. To get the constituents unique we assume in the sequel that they are principal submatrices of $A$. The matrix $A$ is irreducible if and only if $s = 1$ in (1.3).

**Remark 1.1.** To derive a Frobenius normal form of $A \in \mathbb{R}^{n \times n}$ it suffices to determine the constituents of $A$. Indeed, in $A$ there must be a
constituent $A_{R,R}$ such that $A$ has only zero elements in columns with indices in $R$ outside of rows with indices in $R$. Define $A_{11} = A_{n,n}$. Then delete rows and columns with indices in $R$ from $A$, and continue recursively to obtain $A_{22}, \ldots, A_{ss}$.

We say that $A, B \in \mathbb{R}^{n \times n}$ have the same (off-diagonal) zero pattern if for all $i, j \in \mathbb{N}$ ($i \neq j$) we have $a_{ij} = 0 \iff b_{ij} = 0$. If $A$ and $B$ have the same off-diagonal zero pattern, they have Frobenius normal forms of the same structure, being simultaneously reducible or irreducible. So we may consider, instead of $A \in \mathbb{R}^{n \times n}$, the corresponding Boolean matrix, obtained from $A$ by replacing the nonzero elements with one; see e.g. Kim (1982). The addition and the multiplication of the elements 0, 1 in the Boolean algebra are defined as follows:

$$0 + 0 = 0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0 = 0, \quad 1 + 0 = 0 + 1 = 1 + 1 = 1 \cdot 1 = 1.$$ 

The set of all $m \times n$ Boolean matrices will be denoted by $\mathbb{B}^{m \times n}$. The $m \times n$ Boolean zero matrix and the $n \times n$ Boolean identity matrix are defined in an obvious way. The Boolean universal matrix $J \in \mathbb{B}^{n \times n}$ is defined as the matrix all of whose elements are one. Moreover, we define the universal vector $e = (1, \ldots, 1)^T \in \mathbb{B}^{1 \times n}$. Letting $A, B \in \mathbb{B}^{n \times n}$, by $A \leq B$ we mean that if $a_{ij} = 1$ then $b_{ij} = 1$ for every $i$ and $j$. We say that $A \in \mathbb{B}^{n \times n}$ has positive diagonal if $a_{ii} = 1$ for all $i$.

To any Boolean matrix $A \in \mathbb{B}^{n \times n}$ there corresponds a directed graph, or digraph, $G = G(A) = (N, U)$, where $U = \{(i,j) \mid i, j \in \mathbb{N}, a_{ij} \neq 0\}$. Here $N$ is the set of vertices and $U$ the set of arcs of the digraph. (In fact, there is a one-to-one correspondence between Boolean matrices and digraphs; the Boolean matrix corresponding to a digraph is called the adjacency matrix of the digraph.) The vertices are identified by their numbers. An arc of the form $(i, i)$ is a loop. A sequence $(i_0, \ldots, i_k)$ such that $(i_{j-1}, i_j) \in U, j = 1, \ldots, k$, is a walk of length $k$ from $i_0$ to $i_k$. If $i_0, \ldots, i_k$ are pairwise distinct, then the walk is a path. The walk $(i_0, \ldots, i_{k-1}, i_0)$ is a cycle if $i_0, \ldots, i_{k-1}$ are pairwise distinct. A loop is considered as a cycle of length 1. Vertex $j$ is reachable from vertex $i$ (to be indicated by $i \rightarrow j$) if either (a) $i = j$ and there is a cycle from $i$ to $i$, or (b) $i \neq j$ and there is a path from $i$ to $j$. If $R \subset N$, then $j$ is $R$-reachable from $i$ ($i \rightarrow j$ via $R$) if there is a (possibly empty) set $\{i_1, \ldots, i_k\} \subset R$ such that either (a) $i = j$ and $(i, i_1, \ldots, i_k, i)$ is a cycle from $i$ to $i$ or (b) $i \neq j$ and $(i, i_1, \ldots, i_k, j)$ is a path from $i$ to $j$. If $(i, j) \in U$, we say that $j$ is directly reachable from $i$ ($i \rightarrow j$ directly).

The reachability matrix of $A \in \mathbb{B}^{n \times n}$ is the matrix $T = T(A) \in \mathbb{B}^{n \times n}$.
such that \( t_{ij} = 1 \) if and only if \( i \to j \) in \( G(A) \). The \textit{k-reachability matrix} of \( A \) is the matrix \( T_k(A) \in \mathbb{B}^{n \times n} \) whose \((i, j)\)th element is one if and only if there is a path or a cycle of length \( \leq k \) from \( i \) to \( j \). It can easily be seen that

\[
T_k(A) = A + A^2 + \cdots + A^k,
\]

\[
T(A) = A + A^2 + \cdots + A^n,
\]

and it is known that

\[
T_k(I + A) = I + T_k(A) = (I + A)^k,
\]

\[
T(I + A) = I + T(A) = (I + A)^{n-1}.
\]

So, if \( A \) has positive diagonal, then

\[
T_k(A) = A^k, \quad T(A) = A^{n-1}.
\]

If \( R \subseteq N \), the \textit{R-reachability matrix} \( T_R(A) \in \mathbb{B}^{n \times n} \) of \( A \) is the matrix whose \((i, j)\)th element is one if and only if \( i \to j \) via \( R \). Note that \( A \leq T_R(A) \leq T_k(A) \) with \( k = |R| + 1 \), and \( T_N(A) = T(A) \).

We define an equivalence relation \( \sim \) on \( N \) as follows: \( i \sim j \) if and only if either \( i \to j \) or else \( i \to j \) and \( j \to i \) in \( G \). The equivalence classes under this relation are the \textit{strong components} of \( G \). If \( G \) is composed of one strong component only, it is \textit{strongly connected}. There is a one-to-one correspondence between the constituents of \( A \) and the strong components of \( G(A) \).

\textbf{Remark 1.2.} The constituents of \( A \in \mathbb{B}^{n \times n} \) can be determined as follows. Let \( T' = I + T(A) \). Then the constituent of \( A \) containing \( a_{ii} \) is induced by the rows and columns \( j \in \{ j \in N \mid t'_{ij} = t'_{ji} = 1 \} \); see Kim (1982, Proposition 5.1.2). After finding a constituent \( A_{R,R} \) of \( A \), delete rows and columns with indices in \( R \) from \( T' \) and continue recursively to obtain all the constituents.

The organization of the paper is as follows. In Section 2 we introduce the Boolean pivot operation. It arises from Warshall's (1962) algorithm for determining the reachability matrix of \( A \in \mathbb{B}^{n \times n} \). This algorithm consists of \( n \) similar steps. We call step \( r \) of this algorithm (where vertex \( r \) is processed) a \textit{single Boolean pivot operation} \( \mathscr{B}_r \). Combining steps \( \mathscr{B}_r, r \in R \), yields the
general Boolean pivot operation \( B_R \). We investigate the properties of this operation. It turns out that, for example, \( B_R A = T_R(A) \) and \( B_R S = B_S B_R = B_{R \cup S} \) for any \( R, S \subseteq N \). We derive a formula for the Boolean block pivot operation. There is a close relationship between the Boolean pivot operation \( B_R \) and the principal pivotal operation \( P_R \) for real matrices. So we obtain the Boolean analogues to the formula for inverting a real block matrix and to the Sherman-Morrison-Woodbury formulae for updating the inverse of a real matrix.

In Section 3 we derive, with the aid of Boolean pivot operations, some criteria for irreducibility of Boolean block matrices and a necessary condition for reducibility of a Boolean (or real) matrix. We also give a condensed form of Warshall's algorithm for testing whether \( A \in \mathbb{B}^{n \times n} \) is irreducible.

Next, in Section 4, we establish a close connection between Boolean matrices and M-matrices, showing that a nonsingular M-matrix and a Boolean matrix with the same zero pattern will have the same zero pattern also after pivoting. We use this fact to obtain, as corollaries to the criteria for irreducibility of Boolean block matrices derived in Section 3, the corresponding criteria for irreducibility of partitioned Z-matrices. We also develop a flexible algorithm for excluding and including vertices in a digraph while retaining the original connections between the vertices of the current digraph. More precisely, let \( G = (N, U) \) be a digraph, let \( B \) be its adjacency matrix, and let \( R \subseteq N, \overline{R} = N \setminus R \). Then \( B_R B \), after deleting rows and columns with indices in \( R \), is the adjacency matrix of a digraph \( G' = (\overline{R}, V) \) where \((i,j) \in V \) if and only if \( j \) is \( R \)-reachable from \( i \) in \( G \). So, by applying \( B_R \) to \( B \), one can exclude the vertices \( i \in R \) from \( G \) while retaining the original connections between the vertices \( j \in \overline{R} \). Using Boolean pivots it is not possible to include in a simple way vertices which have been excluded. This can, however, be done if we construct a nonsingular M-matrix \( A \) with the same zero pattern as \( B \) and apply principal pivotal operations to \( A \). Our algorithm works for rather sparse matrices \( B \) only.

In the concluding Section 5 we correct two results by R. L. Smith (1988) on irreducibility of real block matrices.

2. THE BOOLEAN PIVOT OPERATION

Let \( A \in \mathbb{B}^{n \times n} \) and \( r \in N \). Then the single Boolean pivot operation \( B_r \), with the pivot \( a_r \), is defined by \( A = B_r A \), where

\[
\tilde{a}_{ij} = \begin{cases} 
  a_{ij} + a_{ir}a_{rj}, & i, j \neq r, \\
  a_{ij}, & \text{otherwise}.
\end{cases}
\]  

(2.1)
Equation (2.1) is analogous to (1.2). It is a step of Warshall's (1962) algorithm for determining the transitive closure of $G(A)$; see also Swamy and Thulasiraman (1981, pp. 429–432). It is easy to see that $B_r A \geq A$, $B_r^2 A = B_r A$, and $B_r B_s A = B_r B_s A$ for any $r, s \in N$. The general Boolean pivot operation $B_R$, $\emptyset \neq R \subset N$, with the pivot $A_{RR}$ is defined as follows:

$$B_R A = \left( \prod_{r \in R} B_r \right) A.$$ 

Here the pivot operations $B_r$ can be performed in any order. The operation $B_R$ followed by deleting rows and columns with indices in $R$ is denoted by $B^* _R$. If $R = \emptyset$, then $B_r$ is defined as the identity operation. If $A = [A_{ij}] \in B^{n \times n}$ is a block matrix and $R$ is the set of row (and column) indices of its principal submatrix $A_{rr}$, then $B_{(r)}$ will stand for $B_R$, and we shall speak of a Boolean block pivot operation with pivot $A_{rr}$. Basic properties of the Boolean pivot operation are as follows (cf. Theorem 1.1).

**Theorem 2.1.** If $A, B \in B^{n \times n}$ and $R, S \subset N$, then

(i) $B_R A \geq A$;
(ii) $B_R A = T_R(A)$, in particular, $B_N A = T(A)$;
(iii) if $A$ has positive diagonal, then $B_N A = A^{n-1}$;
(iv) $B_R A \leq (I + A)^{|R|+1}$;
(v) if $A$ is symmetric, then so is $B_R A$;
(vi) $B_R (I + A) = I + B_R A$;
(vii) $B_R B_S A = B_S B_R A = B_{R \cup S} A$; in particular, $B_r^2 A = B_r A$;
(viii) if $R \subset S$, then $B_R B_S A = B_S B_R A = B_S A$;
(ix) if $R \subset S$, then $A \leq B_R A \leq B_S A$;
(x) $B_R (A + B) \geq B_R A + B_R B$;
(xi) if $A \geq B$, then $B_R A \geq B_R B$.

**Proof.** We establish (ii) below. The rest is easy.

(ii): By induction on $|R|$. The result holds obviously for $R = \emptyset$. We assume it to hold for $|R| < k$ and show that it holds for $|R| = k$ where $k \geq 1$. Denoting $\bar{A} = B_R A$ we have to verify that

$$i \rightarrow j \text{ via } R \iff \bar{a}_{ij} = 1. \quad (2.2)$$

There are three cases.
(a) \(i \in R\). Denote \(B = \mathcal{S}_{R-i}A\). Then \(\bar{A} = \mathcal{S}_iB\), whence \(\bar{a}_{ij} = b_{ij}\). Using the induction hypothesis, we have that

\[i \rightarrow j \text{ via } R \iff i \rightarrow j \text{ via } R - i \iff b_{ij} = 1 \iff \bar{a}_{ij} = 1.\]

(b) \(j \in R\). Analogous to (a).

(c) \(i, j \not\in R\). It suffices to consider the case \(a_{ij} = 0\). Without loss of generality we may assume that \(R = \{1, \ldots, k\}\).

\[\Rightarrow:\] If \(i \neq j (i = j)\), there is a path (a cycle) \((i, p, \ldots, q, j)\) from \(i\) to \(j\) where \(S = \{p, \ldots, q\} \subset R\). Here \(S \neq \emptyset\) because \(a_{ij} = 0\). Denote \(B = \mathcal{S}_{R-p}A\); then \(\bar{A} = \mathcal{S}_pB\). Because \(p \rightarrow j\) via \(R - p\), we have \(b_{pj} = 1\). Moreover, \(a_{ip} = 1\), implying \(b_{ip} = 1\). So \(\bar{a}_{ij} = b_{ij} + b_{ip}b_{pj} = 1\).

\[\Leftarrow:\] Let \(B_0 = A\) and \(B_r = \mathcal{S}_rB_{r-1}, r = 1, \ldots, k\), and let \(B_h = [b_{pq}^h]\) be the first of these matrices having the \((i, j)\)th element one. Then \(b_{ij}^{h-1} = 0\) and \(b_{ij}^h = 1\), whence, by (2.1), \(b_{ij}^{h-1} = b_{ij}^h = 1\). So \(i \rightarrow h\) and \(h \rightarrow j\), both via \(\{1, \ldots, h - 1\}\) (which is empty if \(h = 1\)). It follows that \(i \rightarrow j\) via \(\{1, \ldots, h\} \subset R\).

Theorem 2.1(ii) can also be proved by slightly modifying the proof of Warshall's (1962) theorem. Swamy and Thulasiraman (1981, Theorem 14.1), have proved the implication from left to right in (2.2).

We next give a formula for the Boolean block pivot operation.

**Theorem 2.2.** Let \(A = [A_{ij}] \in \mathbb{B}^{n \times n}, i, j \in \{1, 2\}, \) where \(A_{11}\) is square, and let \(T = T(A_{11}), T' = I + T\). Then

\[
\mathcal{S}_1 \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} T & T'A_{12} \\ A_{21}T' & A_{22} + A_{21}T'A_{12} \end{bmatrix}. \tag{2.3}
\]

Note that (2.3) is analogous to (1.1).

**Proof.** Let \(A_{11} \in \mathbb{B}^{r \times r}\) and \(R = \{1, \ldots, r\}\), and denote the right-hand side of (2.3) by \(\bar{A} = [\bar{a}_{ij}], i, j \in \{1, 2\}\). We have to show that \(\bar{A} = T_R(A)\), i.e.,

\[i \rightarrow j \text{ via } R \iff \bar{a}_{ij} = 1.\]

Clearly it suffices to consider the case \(a_{ij} = 0\). There are four subcases.
(a) $i, j \in R$. Obvious.

(b) $i \in R$, $j \notin R$.
\[
\Rightarrow: \text{There is a path } (i, p, \ldots, q, j) \text{ from } i \text{ to } j \text{ where } S := \{p, \ldots, q\} \subset R - i. \text{ Here } S \neq \emptyset \text{ because } a_{ij} = 0. \text{ So } i \rightarrow q \text{ via } R, \text{ and } q \rightarrow j \text{ directly, whence } t'_{iq} = t_{iq} = 1 \text{ and } a_{qj} = 1, \text{ implying } \bar{a}_{ij} = 1.
\]
\[
\Leftarrow: \text{There is a } p \in R \text{ such that } t'_{ip} = a_{pj} = 1. \text{ Here } p \neq i \text{ because } a_{ij} = 0. \text{ So } i \rightarrow p \text{ via } R, \text{ and } p \rightarrow j \text{ directly, whence } a_{ij} = 0. \text{ So } t \rightarrow p \text{ via } R, \text{ and } p \rightarrow j \text{ directly, whence } a_{ij} = 0.
\]

(c) $i \notin R$, $j \in R$. Analogous to (ii).

(d) $i, j \notin R$.
\[
\Rightarrow: \text{If } i \neq j (i = j), \text{ there is a path (a cycle) } (i, p, \ldots, q, j) \text{ from } i \text{ to } j \text{ where } S := \{p, \ldots, q\} \subset R. \text{ Here } S \neq \emptyset \text{ because } a_{ij} = 0. \text{ If } p = q \text{ (or, equivalently, } |S| = 1), \text{ then } i \rightarrow p \text{ directly and } p \rightarrow j \text{ directly, whence } a_{ip} = a_{pj} = 1, \text{ implying } \bar{a}_{ij} = 1 \text{ (because } t'_{pp} = 1). \text{ If } p \neq q \text{ (or, equivalently, } |S| \geq 2), \text{ then } i \rightarrow p \text{ directly, } p \rightarrow q \text{ via } R, \text{ and } q \rightarrow j \text{ directly, whence } a_{ip} = t'_{pq} = t_{pq} = a_{qj} = 1, \text{ implying } \bar{a}_{ij} = 1.
\]
\[
\Leftarrow: \text{There are } p, q \in R \text{ such that } a_{ip}t'_{pq}a_{qj} = 1. \text{ If } p = q, \text{ then } i \rightarrow p \text{ directly and } p \rightarrow j \text{ directly, whence } i \rightarrow j \text{ via } R. \text{ If } p \neq q, \text{ then } i \rightarrow p \text{ directly, } p \rightarrow q \text{ via } R, \text{ and } q \rightarrow j \text{ directly, whence } i \rightarrow j \text{ via } R.
\]

COROLLARY 2.1. Let $A$ be as in Theorem 2.2, and let $A_{11}$ be of order $r$ and have positive diagonal. Then
\[
\mathcal{B}^{(1)} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11}^{-1} & A_{11}^{-1}A_{12} \\ A_{21}A_{11}^{-1} & A_{22} + A_{21}A_{11}^{-1}A_{12} \end{bmatrix}.
\]

REMARK 2.1. Above we have noted that the principal pivotal operation for real matrices and the Boolean pivot operation are analogous concepts. The main similarity between them is that for disjoint sets $R, S \subset N$ Theorem 1.1(iv) and Theorem 2.1(vii) are closely related. The main difference between these two kinds of pivotal operations is that $\mathcal{P}_R$ is involutory while $\mathcal{B}_R$ is not. So, if $R$ is a proper subset of $S$, then for $A \in \mathbb{R}^{n \times n}$,
\[
\mathcal{P}_S A = \mathcal{P}_S \setminus \mathcal{P}_R A \quad \text{and} \quad \mathcal{P}_R A = \mathcal{P}_S \setminus \mathcal{P}_S A
\]

(2.5)
provided that all the operations are defined, while for $B \in \mathbb{B}^{n \times n}$,

$$\mathcal{B}_S B = \mathcal{B}_{S \setminus R} \mathcal{B}_R B \quad \text{but} \quad \mathcal{B}_R B \neq \mathcal{B}_{S \setminus R} \mathcal{B}_S B = \mathcal{B}_S B. \quad (2.6)$$

If $B$ has positive diagonal, then $A^{-1}$ and $B^{n-1}$ are analogous concepts; cf. Theorem 1.1(v) and Theorem 2.1(iii). Moreover, letting again $A \in \mathbb{B}^{n \times n}$, the block $\bar{A}_{22} := \mathcal{B}_{(1)}^n A = A_{22} + A_{21} T' A_{12}$ in (2.3) is analogous to the Schur complement of a leading principal submatrix in a real matrix. If the $A_{11}$ in (2.3) is of order $r$ and $R = \{1, \ldots, r\}$, $\bar{R} = N \setminus R$, then $\bar{A}_{22}$ is the adjacency matrix of the digraph $G_1 - (\bar{R}, V)$, where $(i, j) \in V$ if and only if $i \to j$ via $R$ in $G = G(A)$. So by applying $\mathcal{B}_R^w$ to $A$ we may remove vertices $i \in R$ from $G$ while retaining the original connections between the vertices $j \in \bar{R}$ in $G_1$. This result generalizes immediately to any set $R \subset N$.

As an application of Theorem 2.2 we obtain an analogue to the formula for inverting a real block matrix.

**Theorem 2.3.** Let $A$ be as in Theorem 2.2 and let

$$T_1 = T(A_{11}), \quad T'_1 = I + T_1,$$
$$T_2 = T(A_{22} + A_{21} T_1' A_{12}), \quad T'_2 = I + T_2.$$  

Then

$$T(A) = \begin{bmatrix} T_1 + T'_1 A_{12} T'_2 A_{21} T_1' & T'_1 A_{12} T'_2 \\ T_2 A_{21} T_1' & T_2 \end{bmatrix}. \quad (2.7)$$

**Proof.** Note that $\mathcal{B}_N A = \mathcal{B}_{(2)} \mathcal{B}_{(1)} A$, and apply Theorem 2.2. \hfill \blacksquare

**Corollary 2.2.** Let $A$ be as in Theorem 2.2, let $A_{11}$ be of order $r$ and have positive diagonal, and let $B := A_{22} + A_{21} A_{12}^{-1} A_{11}$ have positive diagonal. Then

$$A^{n-1} = \begin{bmatrix} A_{11}^{r-1} + A_{11}^{-1} A_{12} B^{n-r-1} A_{21} A_{11}^{r-1} & A_{11}^{-1} A_{12} B^{n-r-1} \\ B^{n-r-1} A_{21} A_{11}^{r-1} & B^{n-r-1} \end{bmatrix}. \quad \text{(2.8)}$$

The following two theorems are analogues to the Sherman-Morrison-Woodbury formulae for updating the inverse of a matrix; see Hager (1989).
THEOREM 2.4. Let $A \in \mathbb{B}^{n \times n}$, $B \in \mathbb{B}^{n \times r}$, $C \in \mathbb{B}^{r \times n}$, $T_1 = T(A)$, $T'_1 = I + T_1$, and $T_2 = T(I + CT'_1 B)$. Then

$$T(A + BC) = T_1 + T'_1 BT_2 CT'_1.$$  

In particular, if $A$ has positive diagonal, then

$$(A + BC)^{-1} = A^{-1} + A^{-1} B (I + CA^{-1} B)^{-1} CA^{-1}.$$  

Proof. Define

$$D = \begin{bmatrix} A & B \\ C & I \end{bmatrix},$$

and note that the blocks $(1, 1)$ of the matrices $D_2 D_1 D$ and $D_1 D_2 D$ are identical.

Taking $r = 1$ in Theorem 2.4 yields the following result.

THEOREM 2.5. Let $A \in \mathbb{B}^{n \times n}$, $T = T(A)$, $T'_1 = I + T$, and let $B$ arise by principal permutation of the matrix

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{B}^{n \times n}.$$  

Then

$$T(A + B) = T + T'_1 BT'_1.$$  

In particular, if $A$ has positive diagonal, then

$$(A + B)^{-1} = A^{-1} (I + BA^{-1}).$$  

The computational complexity of algorithms pertaining to Boolean matrices will be indicated by means of the maximal number of comparisons of elements required (the additional computational effort is roughly proportional to this number). It can be shown that computing $r_r A$ with $A \in \mathbb{B}^{n \times n}$ requires at most $n(n - 1)$ comparisons.
3. REDUCIBLE MATRICES

The basic result on reducibility of Boolean matrices is as follows.

**Theorem 3.1.** If $A \in \mathbb{B}^{n \times n}$, then the following properties are equivalent:

(i) $A$ is irreducible,
(ii) $I + T(A) = J$,
(iii) $(I + A)^{n-1} = J$,
(iv) $\mathcal{B}_n(I + A) = I + \mathcal{B}_n \cdots \mathcal{B}_1 A = J$,
(v) $G(A)$ is strongly connected.

**Proof.** Refer to (1.4), Theorem 2.1, Kim (1982, Proposition 5.1.1 and Remark 5.1.2), and Fiedler (1986, Theorem 3.6).

Then we prove some results concerning reducibility of Boolean block matrices.

**Theorem 3.2.** Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{B}^{n \times n}, \quad (3.1)$$

where $A_{11}$ is square, $A_{12}$ has no zero rows, and $A_{21}$ has no zero columns. Then $A$ is irreducible if and only if $\mathcal{B}_{(1)} A$ is irreducible.

**Proof.** Let $T_1'$ and $T_2'$ be as in Theorem 2.3. Using Theorems 2.2–2.3 and 3.1, we have that

$$A \text{ irreducible} \iff I + T(A) = J \iff T_2' = J \iff \mathcal{B}_{(1)} A \text{ irreducible}.$$ 

Finally, $T_2' = J \Rightarrow I + T(A) = J$ because $T_1' A_{12}$ has no zero rows and $A_{21} T_1'$ has no zero columns; see (2.7).

Along the same lines one can verify the following.

**Theorem 3.3.** Let $A$ be as in (3.1) where $A_{11}$ is irreducible, $A_{12} \neq 0$ and $A_{21} \neq 0$. Then $A$ is irreducible if and only if $\mathcal{B}_{(1)} A = A_{22} + A_{21} J A_{12}$ is irreducible.
COROLLARY 3.1. Let
\[ A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{B}^{n \times n}, \]
where \( A_{12} \neq 0 \) and \( A_{21} \neq 0 \). Then \( A \) is irreducible if and only if \( A_{22} + A_{21}A_{12} \) is irreducible.

THEOREM 3.4. Let \( A \) be as in (3.1), and let \( A_{11} \in \mathbb{B}^{r \times r} \) with \( r \geq 2 \) be irreducible. Then \( A \) is irreducible if and only if
\[ B := \begin{bmatrix} e^T A_{11} e & e^T A_{12} \\ A_{21} e & A_{22} \end{bmatrix} \]
is irreducible.

Proof. If \( A_{12} = 0 \) or \( A_{21} = 0 \), then \( A \) and \( B \) are reducible. If \( A_{12} \neq 0 \) and \( A_{21} \neq 0 \), then, by Theorem 3.3 and its Corollary 3.1,
\[ B \text{ irreducible } \iff \ A_{22} + A_{21} e e^T A_{12} = A_{22} + A_{21} J A_{12} \text{ irreducible} \]
\[ \iff \ A \text{ irreducible}. \]

Theorem 3.4 is applicable for example in the case that \( a_{ij} = a_{ji} = 1 \) for some distinct \( i, j \in \mathbb{N} \).

REMARK 3.1. If, in Theorem 3.4, the rows and columns of \( B \) are numbered \( 1, r + 1, \ldots, n \) and the constituents \( B_{11}, \ldots, B_{rr} \) of \( B \) are induced by rows and columns with indices in \( R_1, \ldots, R_r \), respectively, where \( 1 \in R_1 \), then the constituents of \( A \) are induced by rows and columns with indices in \( R_1 \cup \{2, \ldots, r\}, R_2, \ldots, R_r \).

For determining a Frobenius normal form of \( A \in \mathbb{B}^{n \times n} \) we may use Warshall’s (1962) algorithm.

PROCEDURE 3.1 (Determining a Frobenius normal form of \( A \in \mathbb{B}^{n \times n} \)).

S1: Set \( B = A \) and \( r = 1 \).
S2: Set \( B \leftarrow \mathcal{R}_2 B \) and \( r \leftarrow r + 1 \). Repeat until \( r = n + 1 \).
S3: Determine a Frobenius normal form of \( A \) from \( T(A) = B \) using Remarks 1.2 and 1.1.
REMARK 3.2. In step S2 of Procedure 3.1, \( n \) Boolean pivot operations on \( n \times n \) Boolean matrices are performed. In Remark 1.2, at most \( \frac{1}{2}n(n - 1) \) comparisons of elements are needed (the identity matrix being the worst case). As for Remark 1.1, determining \( A_{11} \) requires at most \( (n - 1)^2 \) comparisons, determining \( A_{22} \) at most \( (n - 2)^2 \) comparisons, etc.; together, less than \( \frac{1}{3}n^3 \) comparisons are needed. So Procedure 3.1 requires less than \( \frac{4}{3}n^3 \) comparisons of elements, being thus \( O(n^3) \). There are more efficient algorithms for determining a Frobenius normal form, for example the depth-first search algorithm of Tarjan (1972), which is \( O(n^2) \); see Ramamurthy (1986) and Swamy and Thulasiraman (1981, p. 436).

In Procedure 3.1, denote \( A_k = [a_{ij}^k], \ k = 0, \ldots, n, \) where \( A_0 = A \) and \( A_k = B_k A_{k-1}, \ k = 1, \ldots, n. \) If \( A \) is reducible, this becomes evident when for the first time we have for some \( k \in \{1, \ldots, n - 1\} \)

\[
a_{kj}^{k-1} = 0, \quad j = k + 1, \ldots, n, \quad \text{or} \quad a_{ik}^{k-1} = 0, \quad i = k + 1, \ldots, n. \tag{3.2}
\]

Note that (3.2) must occur for some \( k \in \{1, \ldots, n - 1\} \), because otherwise \( A \) is irreducible by Corollary 3.1. In view of Theorem 2.1(i), (3.2) is possible only if

\[
(a_{k,k+1}, \ldots, a_{kn}) = 0 \quad \text{or} \quad (a_{k+1,k}, \ldots, a_{nk}) = 0
\]

and if the zero vector in question is not affected by any pivot \( B_r, \ r \in \{1, \ldots, k - 1\}, \) as applied to \( A \). So we have the following.

THEOREM 3.5. A necessary condition for \( A \in \mathbb{B}^{n \times n} \) (or \( A \in \mathbb{R}^{n \times n} \)) to be reducible is that

(i) for some \( k \in \{1, \ldots, n - 1\}, a_{kj} = 0 \) for all \( j > k \) or \( a_{ik} = 0 \) for all \( i > k, \) and

(ii) for some \( h \in \{2, \ldots, n\}, a_{hj} = 0 \) for all \( j < h \) or \( a_{ih} = 0 \) for all \( i < h \).

Proof. (ii) follows as (i) on noting that \( T(A) = B_1 \cdots B_n A. \)
Remark 3.3. Theorem 3.5 can also be established directly from (1.3). For example, to prove (i) let

\[ p = \max\{i|a_{ii} \text{ belongs to the block } A_{11}\}, \]
\[ q = \max\{i|a_{ii} \text{ belongs to the block } A_{ss}\}, \]

where \( A_{11}, \ldots, A_{ss} \) are the constituents of \( A \); see (1.3) (note that we have defined the constituents of \( A \) as principal submatrices of \( A \)). If \( p < n \), then the latter equation in (i) is satisfied for \( k = p \). If again \( p = n \), then \( q < n \), and the former equation of (i) is satisfied for \( k = q \).

Remark 3.4. Theorem 3.5 can be strengthened. For example, (i) can be replaced by the following:

\[ a_{kj} = 0 \text{ for all } j > k \text{ and } a_{ki} \neq 0, i < k \implies a_{ij} = 0 \text{ for all } j > k, \]

or

\[ a_{ik} = 0 \text{ for all } i > k \text{ and } a_{jk} \neq 0, j < k \implies a_{ij} = 0 \text{ for all } i > k. \]

If the only information needed is whether \( A \) is irreducible or not, we may replace in step S2 of Procedure 3.1 the operation \( \mathcal{B}_r \) by \( \mathcal{B}^*_r \). If, in this condensed scheme, (3.2) does not occur, then \( A \) is irreducible. Otherwise the first occurrence of (3.2) indicates that \( A \) is reducible, and we may stop. In the following procedure the operations \( \mathcal{B}^*_r \) are performed in the order \( \mathcal{B}^*_n, \ldots, \mathcal{B}^*_2 \).

**Procedure 3.2** (Testing whether \( A \in \mathbb{B}^{n \times n} \) is irreducible).

S1: Set \( B = A \) and \( r = n \).

S2: If \( b_{jr} = 0 \) for all \( j < r \) or \( b_{ir} = 0 \) for all \( i < r \), stop; \( A \) is reducible. Otherwise set \( B \leftarrow \mathcal{B}^*_r B \) and \( r \leftarrow r - 1 \). Repeat until \( r = 1 \).

S3: \( A \) is irreducible.

In step S2 of Procedure 3.2, the round with an \( r \times r \) Boolean matrix requires at most \( 2(r - 1) + r(r - 1) \) comparisons of elements, \( r = n, \ldots, 2 \). So in Procedure 3.2 less than \( \frac{1}{3}n^3 + n^2 \) comparisons are needed.
4. M-MATRICES

There is a close connection between Boolean matrices and M-matrices. To pursue this connection we state first some definitions and results pertaining to M-matrices; see Berman and Plemmons (1979), Fiedler (1986), and Väliaho (1991).

**Definition 4.1.** A ∈ ℝⁿˣⁿ is a Z-matrix if all its off-diagonal elements are nonpositive. The Z-matrices form the class Z.

**Definition 4.2.** A Z-matrix with nonnegative principal minors is an M-matrix. All the M-matrices form the class K₀, and all the nonsingular M-matrices (i.e., all the Z-matrices with positive principal minors) the class K.

**Theorem 4.1.** A ∈ ℝⁿˣⁿ is a nonsingular M-matrix if and only if it is of the form $A = tI - C$, where $C ≥ 0$ and $t > ρ(C)$, the spectral radius of $C$.

**Theorem 4.2.** If $A ∈ ℝⁿˣⁿ$ is a nonsingular M-matrix, then

(i) $A^{-1} ≥ 0$ with positive diagonal;
(ii) $A^{-1}$ has the same zero pattern as $|A|^{n-1}$;
(iii) $A^{-1} > 0$ if and only if $A$ is irreducible.

**Theorem 4.3.** If $B ∈ ℝⁿˣⁿ$, then $A := (n + 1)I - B ∈ ℝⁿˣⁿ$ (where the operations are real) belongs to K and has the same off-diagonal zero pattern as $B$. If $B$ has positive diagonal, then $A$ and $B$ have the same zero pattern.

**Proof.** It suffices to prove that $A ∈ K$. Considering $B$ as a real matrix, we have $ρ(B) ≤ ∥B∥₁ < n$, where $∥·∥₁$ stands for the maximum-row-sum norm on $ℝⁿˣⁿ$. So $A ∈ K$ by Theorem 4.1.

Väliaho (1991, Theorem 4.1) has the following corollary.

**Theorem 4.4.** Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} ∈ Z,$$  (4.1)

where $A_{11} ∈ K$. Then $A ∈ K$ if and only if $[A/A_{11}] = A_{22} - A_{21}A_{11}^{-1}A_{12} ∈ K$. 


THEOREM 4.5. Let $A$ be as in (4.1) where $A_{11} \in K$. Then there is a $t$ such that the matrix obtained from $A$ by adding $tI$ to the block $A_{22}$ is a nonsingular $M$-matrix.

Proof. Note first that $[A/A_{11} - A_{22} - A_{21}A_{11}^{-1}A_{12}] \in Z$, because $A_{22} \in Z$, $A_{11}^{-1} \geq 0$, $A_{12} \leq 0$, and $A_{21} \leq 0$. Let $t$ be such that the $Z$-matrix $[A/A_{11} + tI]$ belongs to $K$ (the existence of such a $t$ follows easily from Theorem 4.1). Then the proof is completed by Theorem 4.4.

The following theorem reveals a close relationship between Boolean matrices and $M$-matrices.

THEOREM 4.6. Let $A \in \mathbb{R}^{n \times n}$ be a Z-matrix, let $R \subset N$, let $A_{RR} \in K$, and let $B \in \mathbb{B}^{n \times n}$ have the same off-diagonal zero pattern as $A$. Then $\mathcal{P}_R A$ and $\mathcal{B}_R B$ have the same off-diagonal zero pattern. If $A \in K$ and $B$ has the same zero pattern as $A$, then $\mathcal{P}_R A$ and $\mathcal{B}_R B$ have the same zero pattern.

Proof. In view of Theorems 4.5 and 2.1(vi) it suffices to verify the latter part in the case $R = \{1, \ldots, r\}$. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where $A_{11}$ and $B_{11}$ are $r \times r$. Now, by Theorem 4.2(ii), $A_{11}^{-1}$ has the same zero pattern as $|A_{11}|^{-1}$ and thus the same zero pattern as $B_{11}^{-1}$. The rest follows from (1.1) and (2.4).

Theorem 4.6 may also be proved by induction on $|R|$.

Let $G = (N, U)$ be a digraph with $(i, i) \in U$ for all $i \in N$, let $B \in \mathbb{B}^{n \times n}$ be the adjacency matrix of $G$, let $R, S \subset N$ and $\bar{R} = N \setminus R$, $\bar{S} = N \setminus S$, and let $G_1 = (\bar{R}, V_1)$ be the digraph corresponding to $\mathcal{B}^*_R B$ and $G_2 = (S, V_2)$ the digraph corresponding to $\mathcal{B}^*_S B$; cf. Remark 2.1. If $R$ is a proper subset of $S$, we have $T_S(B) = \mathcal{B}^*_S \mathcal{T}_R(B)$ but cannot derive $T_S(B)$ from $T_S(B)$; see (2.6). However, we may use the $A \in K$ defined in Theorem 4.3. Note by Theorem 4.6 that $T_R(B) = \mathcal{B}_R B$ is the Boolean matrix corresponding to $\mathcal{P}_R A$. Moreover, we have $\mathcal{P}_S A = \mathcal{P}_S \setminus R \mathcal{P}_R A$ and $\mathcal{P}_R A = \mathcal{P}_S \setminus R \mathcal{P}_S A$; see (2.5). So, using matrices $\mathcal{P}_H A$ instead of $\mathcal{B}_H B$, $H \subset N$, we obtain a method for passing from $G_1 = (\bar{R}, V_1)$ to $G_2 = (\bar{S}, V_2)$ for any $R, S \subset N$. Thus, starting from $G$, we may exclude and include vertices flexibly, one at a time, retaining in the current digraph the original connections between the vertices of the current digraph.
The above procedure for excluding and including vertices in a digraph is numerically infeasible: because of rounding errors it is often impossible to discern whether an element of $\mathcal{P}_R A$ is zero or not. We derive an improved procedure. Note first that if $A$ is an integral matrix, then the elements of $\mathcal{P}_R A$ are integral multiples of $(\det A_R)^{-1}$. This is seen from (1.1) by taking into account that the elements of $A_{R^{-1}}$ are of this kind. So we use the integral matrices $(\det A_{R^{-1}})\mathcal{P}_R A$ instead of the matrices $\mathcal{P}_R A$. A step in this modified procedure is as follows: Let $R \subset N$, $d = \det A_{RR}$, $C = d\mathcal{P}_R A$, $r \in N$; let $S = R + r$ or $R - r$ according as $r \notin R$ or $r \in R$; and let $d' = \det A_{SS}$ and $C' = d'\mathcal{P}_S A$ (initially $R = \emptyset$, $d = 1$, and $C = A$). Then, using (1.2) and Theorem 1.1(iii),

\[
\begin{align*}
\tilde{c}_{rr} &= d, \\
\tilde{c}_{ir} &= c_{ir}, & i \neq r, \\
\tilde{c}_{rj} &= -c_{rj}, & j \neq r, \\
\tilde{c}_{ij} &= d^{-1}(c_{rr}c_{ij} - c_{ir}c_{rj}), & i, j \neq r, \\
\tilde{d} &= c_{rr}.
\end{align*}
\]  

(4.2)

We may take $A = -B$ initially and later adjust the diagonal elements so that $A_{RR}$ always belongs to $K$. More precisely, any time before performing (4.2) with $r \notin R$, we replace $c_{rr}$ by $c_{rr} + kd \in (0, d\mathbb{I})$, where $k$ is an appropriately chosen integer. This means adding $k$ to $a_{rr}$. Assume that we have a matrix (det $A_{RR}$)$\mathcal{P}_R A$, with the modified $A$, at hand in this procedure. Then $A_{RR} \in K$; see Valiaho (1991, Procedure 4.1).

If $B$ is very dense, the absolute values of the elements of (det $A_{RR}$)$\mathcal{P}_R A$ produced by the modified procedure may grow faster than $2^{2(n-1)}$, where $r = |R|$. Therefore this procedure works for rather sparse matrices $B$ only.

The step (4.2) requires at most $3(n - 1)^2$ multiplications and divisions in integers.

The following two theorems are corollaries to Theorems 3.2-3.3 (via Theorem 4.6).

**Theorem 4.7.** Let

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{Z},
\]  

(4.3)
where \( A_{11} \in K, A_{12} \) has no zero rows, and \( A_{21} \) has no zero columns. Then \( A \) is irreducible if and only if \([A/A_{11}]\) is irreducible.

**Theorem 4.8.** Let \( A \) be as in (4.3), where \( A_{11} \in K \) is irreducible, \( A_{12} \neq 0 \), and \( A_{21} \neq 0 \). Then \( A \) is irreducible if and only if \([A/A_{11}]\) is irreducible.

Theorems 4.7–4.8 can also be proved analogously to Theorems 3.2–3.3, using Theorem 4.5 and the formula for inverting a block matrix.

**Corollary 4.1.** Let

\[
A = \begin{bmatrix}
a_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} \in \mathbb{Z},
\]

where \( a_{11} > 0, A_{12} \neq 0, \) and \( A_{21} \neq 0 \). Then \( A \) is irreducible if and only if \([A/a_{11}] = A_{22} - a_{11}^{-1}A_{21}A_{12}\) is irreducible.

This corollary follows also from Johnson (1982, Theorem 1.9).

It is easy to combine Väliaho (1991, Procedure 4.1) and Corollary 4.1 to obtain a procedure for testing whether a \( Z \)-matrix is an \( M \)-matrix, and if so, whether it is irreducible.

5. **Corrections to a Paper by R. L. Smith**

In this concluding section we show that two results by R. L. Smith (1988), concerned with irreducibility of block matrices, are false, and we give corrected versions of them.

5.1

Lemma 2.1(i) of Smith states the following: Let

\[
M = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} \in \mathbb{R}^{n \times n} ,
\]  \hspace{2cm} (5.1)

where \( A_{11} \) is nonsingular, \( A_{12} \neq 0, \) and \( A_{21} \neq 0 \). Then \( M \) is irreducible if \([M/A_{11}]\) is irreducible.
This result does not hold, as is seen from the following counterexample:

\[
M = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 1
\end{bmatrix},
\]

\[
\begin{bmatrix}
M/A_{11}
\end{bmatrix} = \begin{bmatrix}
2 & -1 \\
-1 & 1
\end{bmatrix}.
\] (5.2)

Here \( A_{11} \) is nonsingular, \( A_{12} \neq 0 \), \( A_{21} \neq 0 \), and \( [M/A_{11}] \) is irreducible, but \( M \) is clearly reducible. In his proof Smith has not noted that his \( M_{11} (M_{44}) \) is vacuous if \( A_{11} \) contains his \( A \) (\( D \)) and a part of his \( D \) (\( A \)). In fact, Smith’s result holds only in the case \( A_{11} \in \mathbb{R}^{1 \times 1} \), which is in Johnson [1982, Theorem 1.9(i)]. One way to guarantee that Smith’s \( M_{11} \) and \( M_{44} \) are nonvacuous is to require that \( A_{12} \) have no zero rows and \( A_{21} \) have no zero columns. So a corrected version of Smith [1988, Lemma 2.1(i)] is as follows.

**Theorem 5.1.** Let \( M \) be as in (5.1), where \( A_{11} \) is nonsingular, \( A_{12} \) has no zero rows, and \( A_{21} \) has no zero columns. Then \( M \) is irreducible if \( [M/A_{11}] \) is irreducible.

We give also another possible version.

**Theorem 5.2.** Let \( M \) be as in (5.1), where \( A_{11} \) is nonsingular and irreducible, \( A_{12} \neq 0 \), and \( A_{21} \neq 0 \). Then \( M \) is irreducible if \( [M/A_{11}] \) is irreducible.

**Proof.** Note that now, in Smith’s proof, \( A_{11} \) must be a proper principal submatrix of his \( A \) or \( D \).

5.2

The sufficiency part of Smith (1988, Corollary 2.2) is a special case of his Lemma 2.1(i) with \( M \in K_0 \) and \( A_{11} \in K \). This result also can be refuted by means of the counterexample (5.2), where \( M \in K \) and \( A_{11} \in K \). Theorems 4.7 and 4.8 are corrected (and slightly generalized) versions of Smith’s Corollary 2.2.

**References**


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