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Linearization of Hamiltonian Systems, Jacobi Varieties and Representation Theory*

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1. INTRODUCTION

The purpose of this paper is twofold: first we show that all the systems discussed in Adler and van Moerbeke [2] (paper I) in connection with Kac–Moody Lie algebras can be linearized according to a general scheme common to all of them reminiscent of Mumford and van Moerbeke's treatment of the Toda lattice [16]. The methods reflect the decomposition of the Lie algebra as explained in I and therefore are divided into two different sections dealing on the one hand with Toda-type flows and on the other hand with flows of spinning top type (Sections 2 and 3).

The second part of the paper adresses the following problem: each Lie algebra representation leads to a different curve and therefore a different Jacobi variety; therefore one might expect the linearization to depend on the representation; i.e. the same flow would lead to essentially different solution by quadratures, depending on the representation. We show this is not the case, because the Jacobi varieties corresponding to higher-order representations all contain one or several copies of the Jacobi variety going with the fundamental representation. To show this, use is made of the theory of correspondences: a correspondence is established between the fundamental curve and the curves corresponding to higher-dimensional representations; the latter curves are Galois extensions of the fundamental curve. Such a correspondence induces a homomorphism between the associated Jacobi varieties, which is shown to be different from zero and injective; the first statement follows from an inequality of Castelnuovo and the second from the irreducibility of the Jacobians for Toda-like

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curves. In an appendix we sketch the main concepts and theorems of the theory of correspondences in a style reminiscent of the classic Italian geometers.

2. Linearization of the Toda-Type Flows

In I, Sect. 4, we discussed special orbits of the classical Lie groups of minimal dimension. For $sl(n)$, these orbits gave rise to the classical Toda systems. However, for the other groups, discussed in I, Section 4, they give rise to generalizations of this system; their linearization will be dealt with in this section. We first state the main theorems which emphasize the analogies and the differences between these cases. It is interesting to point out that the distinction between the Lie algebras reflect themselves at the level of the curves and/or the way the isospectral set is embedded in the Jacobi variety.

We first sketch the proof of the theorem for $sl(n)$ (cf. Moerbeke [15]), then we linearize the problem in detail for $so(n, n+1)$ and sketch the proofs for the other groups; the case G_2 is covered by $so(3, 4)$ and some additional structure. As compared to $sl(n)$ the methods involve particular attention to the special symmetries and degeneracies inherent in the description of the Lie algebras; they reflect themselves in Prym subvarieties on which the flows evolve.

Consider a matrix A_h in the orbit corresponding to any of the Toda matrices (I, Sect. 4) associated to the classical groups and G_2, specifically (4.24), (4.25), (4.26), (4.27), (4.28). Remember that to each matrix A_h is associated the curve X given by

$$Q(z, h) \equiv \det(A_h - zI) = 0.$$

For every group above, define the isospectral set $\mathscr{A}(X)$, defined by a given curve X, and the functions z and h on it: $\mathscr{A}(X) = \{A_h \mid \det(A_h - zI) = Q(z, h)\}$ modulo a discrete group action. This action takes on the following form for

(1)[1] $sl(2n)$: conjugation by diagonal matrices with entries ± 1.

(2) $so(n, n+1)$:
- (i) the conjugation by diagonal matrices of the form $\operatorname{diag}(a, 1, b)$ where $a = \operatorname{diag}(\pm 1, \pm 1, \ldots, \pm 1)$ of size n and b is the same diagonal matrix read right to left
- (ii) interchanging a_1 and a_{n+1} and also $-b_1$ and b_1 simultaneously.

(3) $so(n, n)$:
- (i) the conjugation by diagonal matrices of the form $\operatorname{diag}(a, b)$ where $a = \operatorname{diag}(\pm 1, \ldots, \pm 1)$ of size n and b is the same diagonal matrix read right to left.
- (ii) interchanging a_1 and $-a_{n+1}$ and also $-b_1$ and b_1 simultaneously.
- (iii) interchanging a_n and a_{n-1} and also $-b_n$ and b_n.

[1] For analogy we have picked the even-dimensional case which is not different from the odd case.

(4) $sp(n)$: conjugation by the same matrices as in (3i)

(5) G_2 conjugation by the same matrices as in ($3i$) with $n = 3$ and a as in the above, but in addition restricted to be of the form $(d_1, d_2, d_1 d_2)$.

THEOREM 1. (a) *Any matrix of the form A_h with all $a_k \neq 0$ leads to a hyperelliptic curve X (with hyperelliptic involution τ) of genus $g = 2n - 1$ having two distinguished non-constant meromorphic functions h and z and distinguished points $P, Q = P^\tau, R, S = R^\tau$ such that $(z) = -P - Q + R + S$.*

Except for $sl(n)$, it has the extra-involution σ defined by

$$\sigma\colon (z, h) \to (-z, h).$$

The set of matrices $\mathscr{A}(X)$ maps one-to-one onto a set (containing a Zariski open set) in the variety Y defined below; let J be the natural embedding of Y in Jac(X).

Lie Algebra	$sl(2n)$	$so(n, n+1)$
Genus g	$2n - 1$	$2n - 1$
(h)	$-(g+1)P + (g+1)Q$	$-(g+1)P + (g+1)Q$
Y	$S^g(X)$	$\left\{\begin{matrix} \mathscr{D} \in S^{g+1}(X) \\ \mathscr{D} = \mathscr{D}^{\tau\sigma} \end{matrix}\right\}$
J linear[3] *variety in* Jac(X) *defined by*	Jac (X)	$\omega_{2k}(\mathscr{D}) + \omega_{2k}(\mathscr{D}^\sigma) \equiv 0$, $1 \leqslant k \leqslant n-1, 0(\mathscr{D}) = g + 1$
dim $\mathscr{A}(X) =$ dim $Y =$ dim J	g	$(g+1)/2$

$so(n, n)$	$sp(n)$	G_2
$2n - 1$	$2n - 1$	5
$-(g-1)P + (g-1)Q - 2R + 2S$	$-(g+1)P + (g+1)Q$	$-6P + 6Q$
$\left\{\begin{matrix} \mathscr{D} \in S^{g+1}(X) \\ \mathscr{D} \equiv \mathscr{D}^{\tau\sigma} \end{matrix}\right\}^2$	$\left\{\begin{matrix} \mathscr{D} \in S^{g}(X) \\ \mathscr{D} \equiv \mathscr{D}^{\tau\sigma} \end{matrix}\right\}$	*hypersurface of the* $so(3,4)$ *case*
$\omega_{2k}(\mathscr{D}) + \omega_{2k}(\mathscr{D}^\sigma) \equiv 0$, $1 \leqslant k \leqslant n-1, 0(\mathscr{D}) = g+1$	$\omega_{2k}(\mathscr{D}) + \omega_{2k}(\mathscr{D}^\sigma) \equiv 0$, $1 \leqslant k \leqslant n-1, 0(\mathscr{D}) = g$	*Abelian variety of* $so(3, 4)$ *case*
$(g+1)/2$	$(g+1)/2$	2

(b) *Every linear flow on Y*

$$\int_{\mathscr{D}(0)}^{\mathscr{D}(t)} \omega_k = a_k t, \qquad a_k = 0 \text{ for } k \text{ even (except for } sl(2n)),$$

[2] $\equiv$ refers to equivalence in terms of Abelian sums.

[3] $\omega_j = z^{j-1}y^{-1}dz$, y hyperelliptic irrationality. For a divisor $\mathscr{D} = \sum_1^l \mu_i$, $\omega_j(\mathscr{D}) = \sum_{i=1}^l \int^{\mu_i} \omega_j = \int^{\mathscr{D}} \omega_j$ for some appropriate choice of origin.

is associated with a polynomial $T(z)$ of degree $\leqslant g$, which except for the case of $sl(n)$ is odd; $T(z)$ is chosen such that

$$a_k = \mathrm{Res}_P(\omega_k T(z)).$$

This flow translates into a system of differential equations given by

$$\dot{A} = [A, P_K(T(A))], \qquad P_K(T(A)) = T(A)^+$$

(*strictly "upper triangular" part of $T(A)$ in the sense of Kac–Moody* (I, *Sect.* 3)).

Remark 1. The algebras differ from $sl(2n)$ by the presence of the involution σ. The algebra $so(n, n)$ differs from the other algebras from the point of view of the function h, which imposes different conditions on the curve. The algebras $sl(2n)$, $so(n, n+1)$, and $sp(n)$ all give rise to the same curve via the relation between h and z. In addition, the cases $so(n, n+1)$ and $sp(n)$ give rise to the Prym variety σ on $\mathrm{Jac}(X)$. However, at the level of Y (divisors) things are different, as is seen from the table.

Remark 2. Except for $sl(2n)$ and $sp(n)$, the various cases present degeneracies and seemingly non-uniqueness of the map $\mathscr{A}(x) \to Y$ due to $O(\mathscr{D}) = g + 1 > g$. However, the symmetries inherent in the matrices eliminate the non-uniqueness. The more complicated "boundary conditions" suggest the use of a Wronskian-type function.

Sketch of the proof for $sl(N)$. The curve X of genus g, defined by $Q(z, h) \equiv \det(A_h - zI) = 0$ has the above properties, where h and z are regarded as meromorphic functions on X. The elements of the eigenvector $f = (f_0, f_1, \ldots, f_{N-1})^{\ddagger}$ defined by $A_h f = zf$ and normalized at $f_0 = 1$ are considered to be meromorphic functions on the curve; they are defined by[4]

$$f_k = \frac{\Delta_{N,k}}{\Delta_{NN}} h,$$

where Δ_{ij} stands for the minor of $A_h - zI$ corresponding to the i, jth entry; one further defines $f_{k+jN} = h^j f_k$, $j \in \mathbb{Z}$; one verifies that

$$(f_k) \geqslant -\mathscr{D} - kP + kQ \tag{1}$$

with equality at P and Q and $\mathscr{D}$ minimal; moreover $\mathscr{D}$ has order g and dim $\mathscr{L}(\mathscr{D} + kP - (k+1)Q) = 0$, $k \in \mathbb{Z}$. Given a divisor of order g satisfying the latter property, one constructs uniquely a set of functions f_k satisfying (1) and these define the matrix A_h above uniquely via the relation

$$zf = A_h f.$$

[4] $\Delta_{N0} = \Delta_{NN} h^{-1}$.

Consider the flow

$$\dot{A}' = [A, P_K(T(A))] = [A, B], \tag{2}$$

where $T(z)$ is a polynomial of degree $\leqslant g$. As an ordinary differential equation for the entries a_k and b_k, this Lipchitzian flow exists for $0 \leqslant t < \epsilon$. For ϵ small enough, the set of divisors $\{\mathscr{D}(t) \mid 0 \leqslant t < \epsilon\}$ projects into a compact set in $\mathbb{C}^g$, on which $|h|$ is bounded by M. Let $U(t)$ be the finite matrix, analytic in h and t, defined by[5]

$$\dot{U}(t) = UB, \qquad U(0) = I, \qquad |h| \leqslant M, 0 \leqslant t \leqslant \epsilon' < \epsilon.$$

Hence

$$\begin{aligned} U(t) &= I + tB(0) + O(t^2), \qquad |h| \leqslant M, \\ &= I + t\hat{B}, \qquad \hat{B} = B(0) + O(t), \end{aligned}$$

where, by considerations of I, Section 4, the infinite-dimensional representation of $\hat{B}$ is a strictly upper triangular matrix; hence $\hat{B}$ is analytic in h and t, never containing h^{-1}. Since $U(t)$ makes the eigenvectors move, we have[5] that $(I + t\hat{B})$ $f(0)$ and $f(t)$ are proportional vectors; so let

$$(I + t\hat{B})f(0) = (1 + tg)f(t), \tag{3}$$

where, by equating the 0th entry of both vectors, we compute that

$$\begin{aligned} 1 + tg &= 1 + t(\hat{B}f(0))_0 \\ &= 1 + t(Bf(0))_0 + O(t^2); \\ &= 1 + t\sum_{\iota\geqslant 1} B_{0\iota}(0)f_\iota(0) + t\sum_{\iota\geqslant 1} O(t)f_\iota(0) \\ &= 1 + tg(0) + \sum_{\iota\geqslant 1} O(t^2)f_\iota(0) \end{aligned} \tag{4}$$

with $O(t^2)$ analytic in h. Since $\mathscr{D}$ is the minimal set of poles on $X_0 = X\backslash P \cup Q$ of the left-hand vector in (3) and since the new divisor $\mathscr{D}(t)$ of $f(t)$ (after flowing by (2)) is generally distinct from $\mathscr{D}$, the divisor of poles of $1 + tg$ must be $\mathscr{D}$, at least within X_0. Since the minimal divisor of poles of $f(t)$ is $\mathscr{D}(t)$ and since the left generally does not have poles at $\mathscr{D}(t)$, we conclude that

$$(1 + tg)_0 = \mathscr{D}(t) - \mathscr{D}$$

and from the last equality in (4),

$$(1 + tg(0))_0 = \mathscr{D}'(t) - \mathscr{D}$$

[5] $f(0) = f|_{t=0}$, $B(0) = B|_{t=0}$.

for some divisor $\mathscr{D}'(t)$ near $\mathscr{D}$. By manipulating power series and using the fact that the above functions differ by $O(t^2)$, it is easy to show that pointwise

$$\mathscr{D}(t) - \mathscr{D}'(t) = O(t^2). \tag{5}$$

Since $g(0) = \sum_{i\geqslant 1} B_{0i}(0) f_i(0)$, which is a finite sum extending up to K, for instance,

$$\left(\frac{1}{t} + g(0)\right) = -KP - \mathscr{D} + \mathscr{D}'(t) + \sum_i^K P_i(t),$$

where the points $P_1(t)$ are near P for t small. Using Abel's theorem, for every holomorphic differential

$$\int_{\mathscr{D}}^{\mathscr{D}'(t)} \omega = -\sum_i^K \int_P^{P_i(t)} \omega,$$

which by a lemma in Mumford and Moerbeke [16, Sect. 4.2]

$$\begin{aligned} &= -t \operatorname{Res}_P(\omega g(0)) + O(t^2) \\ &= -t \operatorname{Res}_P(\omega T) + O(t^2); \end{aligned}$$

the latter equality holds, because $(P_K(T(A))f)_0 = \sum_{i>0} B_{0i}(0) f_i$ contains the polar part (at P) of $(T(A)f)_0 \equiv T(z)$. Therefore,

$$\frac{d}{dt} \int_{\mathscr{D}}^{\mathscr{D}(t)} \omega \,\bigg|_{t=0} = -\operatorname{Res}_P(\omega T).$$

Proof of Theorem 1 *for so*$(n, n+1)$. Let σ, applied to matrices, be the operation of taking the transpose about the minor diagonal. X shall be the curve defined by

$$Q(z, h) \equiv \det(A_h - zI) = 0.$$

Then

$$Q(-z, h) = \det(A_h + zI) = \det(A_h + zI)^\sigma = (-1)^{2n+1}\det(A_h - zI) = -Q(z, h),$$

i.e., $Q(z, h)$ is odd in z and therefore $Q(z, h)$ has a z-factor. Simple inspection of the matrix shows that

$$Q(z, h) = (-1)^{n+1} z\, [2A(h + h^{-1}) + (-1)^n q(z^2)], \tag{6}$$

where g is a monic polynomial of degree n and

$$A = a_1 \prod_2^n a_i^2 a_{n+1}\,.$$

This hyperelliptic curve X of genus $g = 2n - 1$ (odd) clearly has the two involutions

$$\tau: (z, h) \to (z, h^{-1})(\text{hyperelliptic}) \qquad \text{and} \qquad \sigma: (z, h) \to (-z, h).$$

Moreover, by (6)

$$(h) = -2nP + 2nQ \qquad \text{and} \qquad (z) = -P - Q + \text{two zeros}.$$

The eigenvector $f = (f_{-n}, f_{-n+1}, \ldots, f_{-1}, f_0, f_1, \ldots, f_{n-1}, f_n)^{\neq}$ of A_h with $f_0 = 1$ can be extended as usual to an infinite vector using the relation

$$h^j f_k = f_{k+j(2n+1)}, \qquad j \in \mathbb{Z}.$$

The meromorphic functions

$$f_k = \frac{\Delta_{n+1,n+1+k}}{\Delta_{n+1,n+1}} = \frac{\Delta_{n+1+k,n+1+k}}{\Delta_{n+1+k,n+1}} \tag{7}$$

have the properties

$$(f_k) \geqslant -\mathscr{D} - kP + kQ, \qquad -n \leqslant k \leqslant n, \quad \text{with equality at } P \text{ and } Q, \tag{8}$$

where $\mathscr{D}$, a positive divisor of order $g + 1 = 2n$, satisfies $\mathscr{D}^\tau = \mathscr{D}^\sigma$ and is minimal with regard to (8). The proof breaks up into several lemmas.

LEMMA 1. *The order of the pole of f_k at P equals the order of zero at Q.*

First we observe that[6]

$$(\Delta_{jj})_\infty = -2nP - 2nQ. \tag{9}$$

This is so, because Δ_{jj} is a monic polynomial of degree $2n$ in z, except for $j = 1$ or $2n + 1$ where it contains an extra term in $(h + h^{-1})$ with coefficient A, instead of $2A$. Second, let

$$(\Delta_{n+1,n+1+k})_\infty = -\alpha P - \beta Q, \qquad \alpha, \beta \in \mathbb{Z},$$

Then on the one hand,[7] since $(\Delta_{ij})^\tau = \Delta_{ji}$ and $P^\tau = Q$,

$$(\Delta_{n+k+1,n+1})_\infty = -\beta P - \alpha Q, \tag{10}$$

[6] $(\ )_\infty = (\ )_{P \cup Q}$.
[7] $\Delta(z, h)^\tau = \Delta(z, h^{-1})$.

and on the other hand, because of (7), (9) and (10)

$$\begin{aligned}(\Delta_{n+k+1,n+1})_\infty &= (\Delta_{n+1+k,n+1+k})_\infty + (\Delta_{n+1,n+1})_\infty - (\Delta_{n+1,n+k+1})_\infty \\ &= -(4n-\alpha)P - (4n-\beta)Q.\end{aligned} \tag{11}$$

Comparing (10) and (11), $\alpha + \beta = 4n$ holds and therefore, since

$$\begin{aligned}(f_k)_\infty &= (\Delta_{n+1,n+1+k})_\infty - (\Delta_{n+1,n+1})_\infty \\ &= -(\alpha - 2n)P + (-\beta + 2n)Q,\end{aligned}$$

the order of the pole of f_k at P equals the order of zero at Q.

LEMMA 2. *Statement* (8) *holds.*

To begin with, observe that

$$(f_k)_\infty = -(f_{-k})_\infty\,, \tag{12}$$

as follows from

$$(f_k)^{\sigma\circ\tau} = \left(\frac{\Delta_{n+1,n+k+1}}{\Delta_{n+1,n+1}}\right)^{\sigma\circ\tau} = \left(\frac{\Delta_{n+k+1,n+1}}{\Delta_{n+1,n+1}}\right)^{\sigma} = \frac{\Delta_{n+1,n+1-k}}{\Delta_{n+1,n+1}} = f_{-k}\,. \tag{13}$$

From the relation (12) and (see I (4.26))

$$z = zf_0 = a_n f_{-1} - a_n f_1$$

either f_1 or f_{-1} has a simple pole at P. First, assume it to be f_1 ; then from the relation

$$zf_1 = -a_n f_0 - b_n f_1 - a_{n-1} f_2$$

it follows that f_2 has a double pole at P; so, in general, using a similar argument,

$$(f_k)_\infty = -kP + kQ, \qquad -n+1 \leqslant k \leqslant n-1.$$

From the relation

$$zf_{n-1} = -a_2 f_{n-2} - b_2 f_{n-1} - a_1 f_n + a_{n+1} f_{n+1}$$

it follows that

$$(a_1 f_n - a_{n+1} f_{n+1})_P = -nP. \tag{14}$$

Next we show that f_n and f_{n+1} both have a pole of order n at P. Indeed, assume

$$(f_n)_P = -\beta P,$$

hence, using (12) in the second equality

$$(f_{n+1})_P = (hf_{-n})_P = -(2n-\beta)P;$$

If $\beta \neq n$, then of the two numbers β and $2n-\beta$, one would be larger than and the other smaller than n, contradicting (14). There remains the possibility that f_{-1} has a simple pole at P; then using the linear relations, as before,

$$(f_k)_\infty = kP - kQ, \qquad -n+1 \leqslant k \leqslant n-1,$$

and, according to the relation

$$zf_{-n+1} = -a_{n+1}f_{-n-1} + a_1 f_{-n} + b_2 f_{-n+1} + a_2 f_{-n+2}$$

we have that

$$(-a_{n+1}f_{-n-1} + a_1 f_{-n})_P = -nP. \tag{15}$$

Let

$$(f_{-n})_P = -\beta P;$$

then, again using (12),

$$(f_{-n-1})_P = (h^{-1}f_n)_P = (\beta + 2n)P.$$

Either $\beta \geqslant n$ or $\beta < n$; in the first case, in view of (14), $\beta + 2n = -\beta$, i.e., $\beta = -n$, which contradicts $\beta > n$. In the second case, $\beta + 2n = -n$, i.e., $\beta = -3n$. Then, the relation

$$zf_{-n} = a_{n+1}f_{-n-2} + b_1 f_{-n} + a_1 f_{-n+1}$$

and the fact that $f_{-n-2} = h^{-1}f_{n-1}$ leads to the contradiction

$$-(zn+1)P = (f_{-n-2})_P = (h^{-1}f_{n-1})_P = (2n + (n-1))P.$$

This establishes (8) at the points P and Q. Let $\mathscr{D}$ be the minimal positive divisor satisfying (8); it has finite order, because every function $f_k (k \in \mathbb{Z})$ is obtained from the finite list f_j $(-n \leqslant j \leqslant n)$ by multiplication with an appropriate power of h. Moreover, since $f_{-k} = f_k^{\tau\sigma}$, we have $\mathscr{D}^{\tau\sigma} = \mathscr{D}$; the fact that $O(\mathscr{D}) = 2n$ will be established later.

Define

$$Y_0 \equiv \{\mathscr{D} \mid \mathscr{D} \geqslant 0, \mathscr{D}^\tau = \mathscr{D}^\sigma, O(\mathscr{D}) \leqslant 2n\} \subseteq \bigcup_{i=1}^{2n} S^i(X)$$

and

$$\mathscr{A}(X) = \{A_h \mid \text{of type above; } \det(A_h - zI) = Q(z,h)\}$$

mod the discrete group action. So far we know that A_h maps into a divisor $\mathscr{D}$

where the functions are uniquely determined by the relations

$$\begin{aligned} (\hat{g}_2)_\infty &= -P + Q, & (\hat{g}_2 - z)_P &\geqslant P, & (\hat{g}_2)_0 &\geqslant -\mathscr{D} \\ (\hat{h}_2)_\infty &= Q, & (\hat{h}_2 - 1)_P &\geqslant P, & (\hat{h}_2)_0 &\geqslant -\mathscr{D} \end{aligned} \tag{28}$$

and similarly for $\hat{g}_{-1}$ and $\hat{h}_{-1}$ with P and Q interchanged. Equation (25) will now be used to determine b_1 and α. As before, we make the generic assumption that b_1 and $b_1{}^\tau$ are not in $\mathscr{D}$. Therefore from (25)

$$\begin{aligned} \alpha g_{-1}(-b_1) + \alpha^{-1} g_2(-b_1) &= 0, \\ \alpha g_{-1}(-b_1{}^\tau) + \alpha^{-1} g_2(-b_1{}^\tau) &= 0, \end{aligned} \tag{29}$$

which implies that

$$(g_{-1}g_2{}^\tau - g_{-1}^\tau g_2)(-b_1) = (g_{-1}g_2{}^\tau - g_{-1}^\tau g_2)(-b_1{}^\tau) = 0.$$

Hence,

$$W(z) \equiv (\hat{g}_{-1} + z\hat{h}_{-1})(\hat{g}_2{}^\tau + z\hat{h}_2{}^\tau) - (\hat{g}_{-1}^\tau + z\hat{h}_{-1}^\tau)(\hat{g}_2 + z\hat{h}_2)$$

vanishes at $-b_1$ and $-b_1{}^\tau$. From a similar argument where f_0 is normalized at 1, instead of f_1 (thus the roles of f_0 and f_1 and b_1 and $-b_1$ are interchanged), we have that $W(z)$ also vanishes at b_1 and $b_1{}^\tau$. Moreover from (28),

$$(W(z)) \geqslant -\mathscr{D} - \mathscr{D}^\tau + b_1 + b_1{}^\tau + b_1{}^\sigma + b_1^{\sigma\tau} - 2P - 2Q + \text{all } 4n \text{ branch points}$$

and hence equality; then we conclude $o(\mathscr{D}) = 2n$ and the zeros of $W(z)$ are precisely the zeros specified. Therefore α is determined uniquely by (29) and so is f_0, g_{-1}, g_2. Using the next relation in the set of relations $A_h f = zf$, one shows that f_2 is uniquely determined up to a constant factor by an argument similar to Lemma 4 and in the fashion for f_k $(-n + 2 \leqslant k \leqslant n + 1)$. First by the periodicity relations $h^j f_k = f_{k+2nj}$ for $-(n-2) \leqslant k \leqslant (n-1)$, are determined. Using the nth and $(n+1)$th relation in $A_h f = zf$, and by an argument precisely the same as that used to determine f_n, f_{n+1} in Lemma 4, we see that indeed f_n, f_{n+1} are uniquely determined (modulo the group action). We have thus determined the f_i uniquely up to the map $f_i \to c_i f_i$; however, the symmetry of the A_h shows that $c_i{}^2 = c_j{}^2$ for all i, j, hence the matrix A_h is indeed uniquely determined modulo the group action. The linearization argument proceeds exactly as for $sl(n)$ with the modifications made for $so(n\ n+1)$.

Proof of Theorem 1 *for* $sp(n)$. The curve X is the same as the one for $sl(2n)$ with the additional σ involution. The example is of course a subcase of $sl(2n)$;

and the fact that $\Delta_{n+1,n+1}$ is fixed under τ and σ, the zeros of $\Delta_{n+1,n+1}$ have the structure

$$(\Delta_{n+1,n+1}) = \sum_1^n \mu_i + \sum_1^n \mu_i^\tau + \sum_1^n \mu_i^\sigma + \sum_1^n \mu_i^{\sigma\tau} - 2n(P+Q).$$

We show that, possibly after relabeling,

$$\mathscr{D} = \sum_1^s \mu_i^\sigma + \sum_1^s \mu_i^\tau \qquad (s \leqslant n).$$

First we define the divisors $\sum_i^{4n} \alpha_i$, $\sum_i^{2n} \xi_i$, by

$$(\Delta_{n+1,n+2}) = \sum_1^{4n} \alpha_i - 2nP - 2nQ,$$

$$(\Delta_{n+2,n+2}) = \sum_1^{2n} \xi_i + \sum_1^{2n} \xi_i^\tau - 2nP - 2nQ.$$

The latter is permitted because of the generic assumption

(G^τ) $\Delta_{n+2,n+2}$ and $q(z^2)^2 - 4A^2$ do not vanish simultaneously. Using the fact that

$$f_1 = \frac{\Delta_{n+1,n+2}}{\Delta_{n+1,n+1}} = \frac{\Delta_{n+2,n+2}}{\Delta_{n+2,n+1}}$$

and the relation $\Delta^\tau_{n+1,n+2} = \Delta_{n+2,n+1}$, we have that

$$\begin{aligned}\sum_1^{4n} \alpha_i - \sum_1^n \mu_i - \sum_1^n \mu_i^\tau - \sum_1^n \mu_i^\sigma - \sum_1^n \mu_i^{\tau\sigma} &= \sum_1^{2n} \xi_i + \sum_1^{2n} \xi_i^\tau - \sum_1^{4n} \alpha_i^\tau \\ &= (f_1)_0 = \mathscr{D}' - \mathscr{D},\end{aligned} \tag{17}$$

where the points of the positive divisor $\mathscr{D}'$ are distinct from those of $\mathscr{D}$. Cancellation must take place on the left-hand side, because otherwise, say, μ_1 would appear among the α_i^τ's and μ_1^τ would cancel with one of the α_i's. So by possibly relabeling, let $\mu_1 = \alpha_1$; then by (G_2), $\mu_1 \notin \mathscr{D}$ and $\mu_1^{\sigma\tau} \notin \mathscr{D}^{\sigma\tau} = \mathscr{D}$. Hence $\mu_1^{\sigma\tau}$ must also cancel on the left-hand side; let $\mu_1^{\sigma\tau} = \alpha_2$. Therefore, among the points μ_1, μ_1^τ, μ_1^σ, $\mu_1^{\tau\sigma}$ only $\mu_1^\tau + \mu_1^\sigma = \alpha_1^\tau + \alpha_2^\tau$ could appear in $\mathscr{D}$. Then, after canceling this expression on the left- and right-hand sides of the first equality (17) one gets

$$\sum_3^{4n} \alpha_i - \sum_2^n \mu_i - \sum_2^n \mu_i^\tau - \sum_2^n \mu_i^\sigma - \sum_2^n \mu_i^{\tau\sigma} = \sum_1^{2n} \xi_i + \sum_1^{2n} \xi_i^\tau - \sum_3^{4n} \alpha_i^\tau;$$

then also, on the right-hand side, two of the ξ_i's, say ξ_1 and $\xi_2 \neq \xi_1^\tau$ (because of

(G_3)) must cancel two of the α_i's, say $\alpha_3{}^\tau$ and $\alpha_4{}^\tau$; so we get after cancellation of $\alpha_3 + \alpha_4 = \xi_1{}^\tau + \xi_2{}^\tau$ in the equality above

$$\sum_5^{4n} \alpha_i - \sum_2^n \mu_i - \sum_2^n \mu_i^\tau - \sum_2^n \mu_i^\sigma - \sum_2^n \mu_i^{\tau\sigma} = \sum_3^{2n} \xi_i + \sum_3^{2n} \xi_i^\tau - \sum_5^{4n} \alpha_i^\tau,$$

which puts us in the same situation as before; so, we proceed by induction until nothing is left, concluding that among the μ_i's, only $\mu_i{}^\tau + \mu_i{}^\sigma$ can appear in $\mathscr{D}$. Hence $\mathscr{D} \subset \sum_1^n \mu_i{}^\tau + \sum_1^n \mu_i{}^\sigma$; since $\mathscr{D} = \mathscr{D}^{\tau\sigma}$ and because of (G_2), $\mathscr{D} = \sum_1^s \mu_i{}^\tau + \sum_1^s \mu_i{}^\sigma$ for some $1 \leqslant s \leqslant n$; hence $O(\mathscr{D}) = 2s \leqslant 2n = g + 1$, establishing Lemma 3.

We now turn to

LEMMA 4. *The map in Lemma* 3 *is one-to-one, for a possibly smaller Zariski open set in* $\mathscr{A}(X)$; *moreover* $O(\mathscr{D}) = 2n$.

Proof. Let two matrices A_h and A_h' in the Zariski open set defined by Lemma 3 lead to the same divisor; then we show that $A_h \equiv A_h'$ and that $f_k \equiv f_k'$ modulo the group action above.

To begin with, f_1 and f_1' satisfy the properties

$$(z + a_n f_1)_\infty \geqslant P - Q \quad \text{and} \quad (z + a_n' f_1')_\infty \geqslant P - Q,$$
$$(f_1)_\infty \geqslant -P + Q \quad \text{and} \quad (f_1')_\infty \geqslant -P + Q.$$

Hence, $a_n f_1 - a_n' f_1' \in \mathscr{L}(\mathscr{D} - P - Q)$. Moreover by the Riemann–Roch theorem

$$\dim \mathscr{L}(\mathscr{D} - P - Q) = \dim \Omega(-\mathscr{D} + P + Q) + 2s - (2n - 1) - 1 \quad (18)$$

and

$$-\dim \Omega(-\mathscr{D} + P + Q) = 2(n - s), \quad (19)$$

because $\omega \in \Omega(-\mathscr{D} + P + Q)$ has the form

$$\omega = R(z)\, y^{-1}\, dz, \qquad y^2 = (q(z^2))^2 - 4A^2$$

with $R(z)$ a polynomial of degree at most $g = 2n - 1$ vanishing at the $2s$ distinct points in $\mathscr{D}$, distinct from a branch point. Therefore combining (18) and (19), $\mathscr{L}(\mathscr{D} - P - Q) = \{0\}$ and $f_1' = c_1 f_1$ with $c_1 = a_n/a_n'$.

Since then

$$a_n f_{-1} = z f_0 + a_n f_1 = z f_0 + a_n' f_1' = a_n' f_{-1}',$$

also f_{-1} and f_{-1}' are proportional. Next we show $f_2' = c_2 f_2$; indeed, since

$$(z f_1 + a_n f_0 + b_n f_1) \geqslant -2P + 2Q - \mathscr{D}$$

and

$$(zf_1' + a_n' f_0 + b_n' f_1') = (zf_1 + c_1^{-1} a_n' f_0 + b_n' f_1) \geqslant -2P + 2Q - \mathscr{D},$$

we conclude $a_n = c_1^{-1} a_n' = a_n'^2 a_n^{-1}$ and $b_n = b_n'$, i.e., $a_n = c_1^{-1} a_n'$ and $c_1 = \pm 1$; also

$$-a_{n-1} f_2 = zf_1 + a_n f_0 + b_n f_1 = zf_1 + c_1 a_n' f_0 + b_n f_1 = -c_1 a_{n-1}' f_2'.$$

Proceeding in this fashion, we show that $f_k' = c_k f_k$ for $-n+1 \leqslant k \leqslant n-1$ and $a_{n+1-k} = c_{k-1} c_k a_{n+1-k}'$, $c_k = \pm 1$, for $1 \leqslant k \leqslant n-1$; moreover $c_0 = 1$ and $c_{-k} = c_k$. Finally, $f_{\pm n}$ and $f_{\pm n}'$ satisfy the first (multiplied by h) and last recursion relations:

$$(z - b_1) f_{n+1} = a_{n+1} f_{n-1} + a_1 f_{n+2}, \qquad (z - b_1') f_{n+1}' = a_{n+1}' f_{n-1}' + a_1' f_{n+2}', \tag{20}$$
$$(z + b_1) f_n = -a_1 f_{n-1} - a_{n+1} f_{n+2}, \qquad (z + b_1') f_n' = -a_1' f_{n-1}' - a_{n+1}' f_{n+2}',$$

where $f_{n-1}' = c_{n-1} f_{n-1}$ and $f_{n+2} = c_{n-1} f_{n+2}$. We show that the respective coefficients in the above relations and so the $f_{\pm n}$'s are equal modulo the group action. Remember that by (G_1), $\mathscr{D}$ does not contain b_1, b_1^τ, b_1^σ, $b_1^{\tau\sigma}$. Therefore each of the expressions on the right-hand side vanishes and breaks up as follows:

$$a_{n+1} f_{n-1}(b_1) + a_1 f_{n+2}(b_1) = 0, \qquad -a_1 f_{n-1}(b_1^\sigma) - a_{n+1} f_{n+2}(b_1^\sigma) = 0. \tag{21}$$
$$a_{n+1} f_{n-1}(b_1^\tau) + a_1 f_{n+2}(b_1^\tau) = 0, \qquad -a_1 f_{n-1}(b_1^{\sigma\tau}) - a_{n+1} f_{n+2}(b_1^{\sigma\tau}) = 0.$$

and similarly for the primed expression. Since a_1 and a_{n+1} do not vanish, the determinant

$$W(z) \equiv f_{n-1}(z) f_{n-2}(z^\tau) - f_{n-1}(z^\tau) f_{n+2}(z)$$

vanishes at b_1, b_1^τ, b_1^σ and $b_1^{\sigma\tau}$, which are all distinct by

(G_4) $b_1 \neq 0$ and $q(b_1^2)^2 - 4A^2 \neq 0$.

Moreover

$$(W) \geqslant -\mathscr{D} - \mathscr{D}^\tau + b_1 + b_1^\tau + b_1^\sigma + b_1^{\sigma\tau} - 2P - 2Q + \text{all } 4n \text{ branch points}$$
$$\text{with equality at } P \text{ and } Q, \tag{22}$$

because

$$(f_{n-1} f^{\tau}_{n+2})_P = -(n-1)P + (n+1)P = 2P$$

and

$$(f^{\tau}_{n-1} f_{n+2})_P = (n-1)P - (n+1)P = -2P.$$

Therefore

$$\# \text{ of poles of } W \leqslant 4n + 4$$

and

$$\# \text{ of zeros of } W \geqslant 4n + 4;$$

hence equality. As a by-product, equality holds in (22) and $o(\mathscr{D}) = 2n$; therefore

$$\mathscr{D} = \sum_1^n \mu_i^{\sigma} + \sum_1^n \mu_i^{\tau}.$$

For the primed equations (20), we have the same $W(z)$ up to a constant and hence the same zeros; in particular

$$b_1 + b_1^{\tau} + b_1^{\sigma} + b_1^{\sigma\tau} = b_1' + b_1'^{\tau} + b_1'^{\sigma} + b_1'^{\sigma\tau}.$$

Going back to numerical values, we get that $b_1 = \pm b_1'$. First assume that $b_1 = b_1'$. Then Eqs. (21) for the primed and nonprimed quantities yield

$$\frac{a_{n+1}}{a_1} = -\frac{f_{n+2}(b_1)}{f_{n-1}(b_1)} = -\frac{f'_{n+2}(b_1)}{f'_{n-1}(b_1)} = \frac{a'_{n+1}}{a'_1}.$$

Let $c = a_1'/a_1 = a'_{n+1}/a_{n+1}$. Then

$$
\begin{aligned}
(z - b_1) f'_{n+1} &= (z - b_1') f'_{n+1} \qquad \text{because } b_1 = b_1' \\
&= a'_{n+1} f'_{n-1} + a_1' f'_{n+2} \qquad \text{by (21) for the primed relation} \\
&= c_{n-1} c[a_{n+1} f_{n-1}(z) + a_1 f_{n+2}(z)] \\
&\qquad\qquad \text{using } f'_{n-1} = c_{n-1} f_{n-1} \text{ and } f'_{n+2} = c_{n-1} f_{n+2} \\
&= c_{n-1} c(z - b_1) f_{n+1}(z) \qquad \text{by (20) for the unprimed relation}
\end{aligned}
$$

and

$$
\begin{aligned}
(z + b_1) f'_n &= (z + b_1') f'_n \\
&= -a_1' f'_{n-1} - a'_{n+1} f'_{n+2} \\
&= c_{n-1} c(-a_1 f_{n-1} - a_{n+1} f_{n+2}) \\
&= c_{n-1} c(z + b_1) f_n.
\end{aligned}
$$

Therefore f_n' and $f_{n+1}' = hf_n'$ are proportional to f_n and $f_{n+1} = hf$, respectively, with the same proportionality constant. Since

$$a_1'a_{n+1}' = \frac{A'}{\prod_2^n a_i'^2} = \frac{A}{\prod_2^n a_i^2} = a_1 a_{n+1}\,, \text{ as } A' = A$$

we have that $c^2 = 1$, so that $a_1' = \pm a_1$, $a_{n+1}' = \pm a_{n+1}$, and $f_n' = \pm c_{n-1} f_n$ simultaneously. Second, we assume $b_1 = -b_1'$; as a result of (21) the same holds, except that in the latter relations f_n and f_{n+1} and also a_1 and $-a_{n+1}$ become interchanged. This finishes the proof of Lemma 4.

Define

$$Y = \{\mathscr{D} \mid \mathscr{D} \geqslant 0, \mathscr{D}^\tau = \mathscr{D}^\sigma, o(\mathscr{D}) = 2n\} \subset Y_0\,.$$

LEMMA 5. *For generic curves X, a Zariski open set in $\mathscr{A}(X)$ maps one-to-one onto a Zariski open subset of Y.*

Proof. Indeed

$$\begin{aligned}\dim \mathscr{A}(X) &= \#\text{ parameters in } A_h - \#\text{ relations defining a given curve } X\\ &\geqslant (2n+1) - (n+1) = n\end{aligned}$$

and (see Lemma 3, for the definition of E).

$$\dim E \leqslant n;$$

since the map from $\mathscr{A}(X)$ to E is one-to-one analytic, it follows that

$$\dim \mathscr{A}(X) = \dim E = n.$$

Since E is a Zariski open subset in Y, the result follows.

We now indicate why (G_2) is a generic statement; indeed from the form of the matrix A_h, letting $a_k \to 0$ $(1 \leqslant k \leqslant n+1)$, we see that $\Delta_{n+1,n+1} \simeq \prod_1^n (z^2 - b_i^2)$ and hence for $b_i \neq 0$, the zeros of $\Delta_{n+1,n+1}$ are all distinct on X, modulo the hyperelliptic involution τ. Another limiting argument will show that a branch point of X does not need to be a zero of $\Delta_{n+1,n+1}$. The other generic statements follow in a similar fashion.

To conclude the proof of statement (a) of Theorem 2, we observe that by the continuous variation of the roots of a polynomial and its coefficients we may extend the map above from the Zariski open set in $\mathscr{A}(X)$ to the whole of $\mathscr{A}(X)$.

As to the linearization in (b), the proof previously given (Theorem 1) can be taken word by word; we only observe that the Lax equation preserves the form of A, since the polynomials $T(z)$ are taken to be of odd degree. So, any Lax flow of the type described in Theorem 1 translates into a flow

$$\sum_1^{2n} \int_{\nu_i(0)}^{\nu_i(t)} \omega_k = t\, \mathrm{Res}_P(\omega_k T(z)) \qquad \text{(which vanishes for } k \text{ even)},$$

which automatically preserves Y by (a). To see that J is the embedding of Y in Jac(X), observe that any divisor $\mathscr{D} = \mathscr{E}^\sigma + \mathscr{E}^\tau$ in E trivially satisfies

$$\mathscr{D} + \mathscr{D}^\sigma \equiv 0 \text{ in } \operatorname{Jac}(X)$$

with a τ-invariant base point. By closing up Y, this shows that Y embeds in J. In fact $\dim Y = \dim J$, because $\dim Y = n$; indeed the condition $\mathscr{D} + \mathscr{D}^\sigma \equiv 0$ in Jac(X) is trivially satisfied for $\omega_k = z^{k-1}y^{-1}\,dz$ for k odd; so, also

$$Y = \left\{\mathscr{D} \mid \mathscr{D} \geqslant 0, 0(\mathscr{D}) = 2n, \int^{\mathscr{D}} \omega_k + \int^{\mathscr{D}^\sigma} \omega_k \equiv 0, k \text{ even}\right\}$$

and $\dim Y = g - (n-1) = n$. This concludes the proof of Theorem 2.

Remark. From the arguments above we cannot exclude the fact that $\mathscr{A}(X)$ is a multiple covering of J.

Sketch of proof of Theorem 1 ($so(n, n)$). The hyperelliptic curve X is defined by (see I (4.27))

$$Q(z, h) \equiv \det(A_h - zI) = 4Az^2(h + h^{-1}) + (-1)^n g(z^2) = 0,$$

where $A = a_1 a_{n+1} a_{n-1} a_n \prod_2^{n-2} a_k^2$ and q is a monic polynomial of degree n. The curve X has the same involutions τ and σ as before and again $g = 2n - 1$; the functions h and z clearly have the divisors given in the statement of the theorem. The eigenvector $f = (f_1, \dots, f_{2n})^t$ of A_h with $f_1 = 1$ can be extended by the formula

$$h^j f_k = f_{k+2nj}, \qquad j \in \mathbb{Z},$$

to an infinite vector. The meromorphic functions

$$f_k = \frac{\Delta_{1,k}}{\Delta_{1,1}} = \frac{\Delta_{k,k}}{\Delta_{k,1}}$$

have the properties

$$\begin{aligned}(f_k) \geqslant{} & -\mathscr{D} - (k-1)P + (k-1)Q, && 1 \leqslant k \leqslant n \\ & -\mathscr{D} - (k-2)P + (k-2)Q, && n+1 \leqslant k \leqslant 2n \\ & && \text{with equality at } P \text{ and } Q, \qquad (23)\end{aligned}$$

where $\mathscr{D}$ is a positive divisor in Y and is minimal with regard to (23). The proof of this statement follows, roughly speaking, the same lines as those for Theorem 2; we shall point out the relevant difference: the structure of the divisor $\mathscr{D}$ is generically[8] given by the divisor of f_{2n} on X_0:

[8] All generic statements can be taken care of as in Theorem 2.

$$
\begin{aligned}
(f_{2n})_0 = \left(\frac{\Delta_{1,2n}}{\Delta_{2n,2n}}\right)_0 &= (\mathscr{E}_1 + \mathscr{E}_2 + \mathscr{E}_1{}^\sigma + \mathscr{E}_2{}^\sigma) - (\mathscr{D}_1 + \mathscr{D}_2 + \mathscr{D}_1{}^\tau + \mathscr{D}_2{}^\tau) \\
= \left(\frac{\Delta_{1,1}}{\Delta_{2n,1}}\right)_0 &= (\mathscr{D}_1{}^\sigma + \mathscr{D}_2{}^\sigma + \mathscr{D}_1^{\tau\sigma} + \mathscr{D}_2^{\tau\sigma}) - (\mathscr{E}_1{}^\tau + \mathscr{E}_2{}^\tau + \mathscr{E}_1^{\sigma\tau} + \mathscr{E}_2^{\sigma\tau}),
\end{aligned}
\tag{24}
$$

where $\mathscr{E}_i$ and $\mathscr{D}_i$ are positive divisors of order n; this is a consequence of the facts: $\Delta_{11}^\tau = \Delta_{11} = \Delta_{2n,2n}^\sigma$, $\Delta_{1,2n}^\tau = \Delta_{2n,1}$. Only the formal aspect of the argument will be given; as in the previous case, an induction argument is necessary. If no cancellation occurs in (24), we may assume without loss of generality that $\mathscr{D}_1 = \mathscr{E}_1{}^\tau$, which implies cancellation does occur and so we shall in fact assume $\mathscr{D}_1 = \mathscr{E}_1{}^\tau$, which leaves us with the relation above without the index 1. So we start again, assuming $\mathscr{D}_2 = \mathscr{E}_2{}^\tau$; from the above, we conclude $\mathscr{D} = \mathscr{D}_1 + \mathscr{D}_2 = \mathscr{E}_1{}^\tau + \mathscr{E}_2{}^\tau$ and so we have

$$(\Delta_{2n,2n}) = \mathscr{D} + \mathscr{D}^\tau - (2n-1)P - (2n-1)Q - R - S$$

and

$$(\Delta_{1,2n}) = \mathscr{D}^\tau + \mathscr{D}^{\tau\sigma} - (4n-3)P - Q - 3R + S;$$

therefore taking the difference of their corresponding Abelian sums $\mathscr{D} - \mathscr{D}^{\tau\sigma} \equiv (h) \equiv 0$, which shows that $\mathscr{A}(X)$ maps into Y.

We now sketch the uniqueness argument; we show that two matrices A_h leading to the same divisor $\mathscr{D}$ are equal modulo the group action, i.e., that the a_k's and b_k's are uniquely defined by $A_h f = zf$. In particular, because of I (4.27),

$$
\begin{aligned}
-a_1 f_{-1} - a_{n+1} f_2 &= (z + b_1) f_0 , \\
a_{n+1} f_{-1} + a_1 f_2 &= (z - b_1) f_1 \qquad (f_1 = 1);
\end{aligned}
$$

we first determine b_1 and the ratio $\alpha \equiv a_1/a_{n+1}$. Letting $g_{-1} \equiv a_{n+1} f_{-1}$ and $g_2 = a_1 f_2$, we rewrite the linear equations above as

$$\alpha g_{-1} + \alpha^{-1} g_2 = -(z + b_1) f_0 , \tag{25}$$

$$g_{-1} + g_2 = (z - b_1); \tag{26}$$

from (26) and the generic statements[9] dim $\mathscr{L}(\mathscr{D} - P - Q) = 0$, dim $\mathscr{L}(\mathscr{D} - Q) = 1$, dim $\mathscr{L}(\mathscr{D} - P) = 1$, dim $\mathscr{L}(\mathscr{D} - 2Q) = 0$, dim $\mathscr{L}(\mathscr{D} - 2P) = 0$, it follows that g_2 and g_{-1} can be uniquely written as an identity in b_1:

$$
\begin{aligned}
g_2(z) &= \hat{g}_2(z) - b_1 \hat{h}_2(z), \\
g_{-1}(z) &= \hat{g}_{-1}(z) - b_1 \hat{h}_{-1}(z),
\end{aligned}
\tag{27}
$$

[9] Their genericity is proved in a fashion similar to that in Theorem 2.

where the functions are uniquely determined by the relations

$$\begin{aligned} (\hat{g}_2)_\infty &= -P + Q, & (\hat{g}_2 - z)_P &\geqslant P, & (\hat{g}_2)_0 &\geqslant -\mathscr{D} \\ (\hat{h}_2)_\infty &= Q, & (\hat{h}_2 - 1)_P &\geqslant P, & (\hat{h}_2)_0 &\geqslant -\mathscr{D} \end{aligned} \tag{28}$$

and similarly for $\hat{g}_{-1}$ and $\hat{h}_{-1}$ with P and Q interchanged. Equation (25) will now be used to determine b_1 and α. As before, we make the generic assumption that b_1 and $b_1{}^\tau$ are not in $\mathscr{D}$. Therefore from (25)

$$\begin{aligned} \alpha g_{-1}(-b_1) + \alpha^{-1} g_2(-b_1) &= 0, \\ \alpha g_{-1}(-b_1{}^\tau) + \alpha^{-1} g_2(-b_1{}^\tau) &= 0, \end{aligned} \tag{29}$$

which implies that

$$(g_{-1} g_2{}^\tau - g^\tau_{-1} g_2)(-b_1) = (g_{-1} g_2{}^\tau - g^\tau_{-1} g_2)(-b_1{}^\tau) = 0.$$

Hence,

$$W(z) \equiv (\hat{g}_{-1} + z\hat{h}_{-1})(\hat{g}_2{}^\tau + z\hat{h}_2{}^\tau) - (\hat{g}^\tau_{-1} + z\hat{h}^\tau_{-1})(\hat{g}_2 + z\hat{h}_2)$$

vanishes at $-b_1$ and $-b_1{}^\tau$. From a similar argument where f_0 is normalized at 1, instead of f_1 (thus the roles of f_0 and f_1 and b_1 and $-b_1$ are interchanged), we have that $W(z)$ also vanishes at b_1 and $b_1{}^\tau$. Moreover from (28),

$$(W(z)) \geqslant -\mathscr{D} - \mathscr{D}^\tau + b_1 + b_1{}^\tau + b_1{}^\sigma + b_1^{\sigma\tau} - 2P - 2Q + \text{all } 4n \text{ branch points}$$

and hence equality; then we conclude $o(\mathscr{D}) = 2n$ and the zeros of $W(z)$ are precisely the zeros specified. Therefore α is determined uniquely by (29) and so is f_0, g_{-1}, g_2. Using the next relation in the set of relations $A_h f = zf$, one shows that f_2 is uniquely determined up to a constant factor by an argument similar to Lemma 4 and in the fashion for f_k $(-n + 2 \leqslant k \leqslant n + 1)$. First by the periodicity relations $h^j f_k = f_{k+2nj}$ for $-(n-2) \leqslant k \leqslant (n-1)$, are determined. Using the nth and $(n+1)$th relation in $A_h f = zf$, and by an argument precisely the same as that used to determine f_n, f_{n+1} in Lemma 4, we see that indeed f_n, f_{n+1} are uniquely determined (modulo the group action). We have thus determined the f_i uniquely up to the map $f_i \to c_i f_i$; however, the symmetry of the A_h shows that $c_i{}^2 = c_j{}^2$ for all i, j, hence the matrix A_h is indeed uniquely determined modulo the group action. The linearization argument proceeds exactly as for $sl(n)$ with the modifications made for $so(n\ n+1)$.

Proof of Theorem 1 *for* $sp(n)$. The curve X is the same as the one for $sl(2n)$ with the additional σ involution. The example is of course a subcase of $sl(2n)$;

so we need only prove the statement concerning Y and J. The argument is done as follows: let $N = 2n$; in view of

$$f_1 = \frac{\Delta_{N1}}{\Delta_{NN}} = \frac{\Delta_{11}}{\Delta_{1N}} \qquad (f_N = 1)$$

and $\Delta_{N1}^{\sigma} = \Delta_{1N}$, $\Delta_{ii}^{\tau} = \Delta_{ii}$, $\Delta_{NN}^{\sigma} = \Delta_{11}$ we have that generically for some divisors $\mathscr{D}$ and $\mathscr{E}$ of order g

$$(\Delta_{IN})_0 = \mathscr{D} + \mathscr{D}^{\sigma}, \qquad (\Delta_{N1})_0 = \mathscr{D}^{\tau} + \mathscr{D}^{\tau\sigma},$$
$$(\Delta_{NN})_0 = \mathscr{E} + \mathscr{E}^{\tau}, \qquad (\Delta_{11})_0 = \mathscr{E}^{\sigma} + \mathscr{E}^{\tau\sigma},$$

from which it follows from an argument previously given that, possibly after relabeling, one has $\mathscr{D} = \mathscr{E}$ and $\mathscr{D}$ is minimal. Therefore by Abel's theorem applied to

$$(\Delta_{IN}) = \mathscr{D} + \mathscr{D}^{-} - 2gP$$

we have for $\mathscr{D} = \sum_1^g \mu_\iota$

$$\sum \int_P^{\mu_i} \omega + \sum \int_P^{\mu_i^{\sigma}} \omega \equiv 0,$$

which proves the assertion concerning Y. Nothing is special about the linearization.

Proof of Theorem 1 *for* G_2. From the considerations in I, Section 4, the matrix A_h is treated with the $so(3, 4)$ case; however one checks that there are three curve invariants, hence $\dim \mathscr{A}(X) = 2$, while $\dim Y$ for $SO(3, 4)$ equals 3, and so $\mathscr{A}(X)$ maps onto a two-dimensional subvariety of Y, which moreover is linear by the fact that two independent linear flows on J preserve it. We also note that by the considerations in $so(3, 4)$, any linear flow is given by a specific odd polynomial of degree at most five. Since from the Lie algebra considerations, z leads to a linear flow on the subvariety J corresponding to the G_2 case, another independent flow must be generated by a specific linear combination of z^3 and z^5.

3. Linearization of Flows of Spinning Top Type

In I, Section 4, various problems were shown to be related to isospectral deformations of polynomials with matrix coefficients[10]

$$A = \sum_{s=0}^{\nu} A_s h^s.$$

For such matrix polynomials we have the following theorem.

[10] $e^k = (\delta_{kj})_{1 \leqslant j \leqslant n}$.

THEOREM 1. *There is a one-to-one correspondence between*

(i) *a polynomial* $A = \sum_{s=0}^{\nu} A_s h^s$ *with matrix coefficients (modulo conjugation by complex diagonal matrices), having the propertie,* $A_\nu = \operatorname{diag}(a_1, \ldots, a_n)$, $a_i \in \mathbb{C}^*$, $\prod_{i<j}(a_i - a_j) \neq 0$, *and* $(A_{\nu-1})_{1,k} \neq 0$ $(k \neq 1)$; *moreover* A *has in the limit* $h \to 0$ *distinct eigenvectors all not perpendicular*[11] *to* e^k *for some* k.

(ii) *a curve* X *of genus* $g = (n(n-1)/2)\nu - n + 1$ *with* $2n$ *distinct points* $P_1, \ldots, P_n$, $Q_1, \ldots, Q_n$ *and a general positive divisor* $\mathscr{D}$ *on* X *of degree* g *not containing any of the points* P_i *or* Q_i; *the points above have the following properties: for some meromorphic functions* h *and* z *on* X

$$(h) = -\sum_1^n P_i + \sum_1^n Q_i$$

and

$$(z) = -\nu \sum_1^n P_i + n\nu \text{ zeros, distinct from the } P_i .$$

Moreover any polynomial function $u = P(z, h, h^{-1})$ *on* X *leads to an isospectral deformation of* A

$$\dot{A} = [A, P(A, h, h^{-1})_+],$$

where $P(A, h, h^{-1})_+$ *denotes the polynomial part (in* h*) of* $P(A, h, h^{-1})$. *The flow above is a linear flow on* $\operatorname{Jac}(X)$ *defined by*

$$\sum_{i=1}^{g} \int_{\nu_i}^{\nu_i(t)} \omega = \sum_{l=1}^{n} \operatorname{Res}_{P_l}(\omega u)t.$$

In particular the flows (cf. I (4.43))

$$\dot{A} = [A, (f'(Ah^{-j})h^{k-j})_+]$$

are linear flows on $\operatorname{Jac}(X)$; *they are equivalent to one of the polynomial flows above.*

Proof. As a first step, the curve X, defined by the algebraic equation

$$Q(z, h) \equiv \det(A - zI),$$

is shown to have the properties stated in (ii). For z, h large,

$$Q(z, h) = \prod_1^n (a_i h^\nu - z) + C_1 h^{n\nu-1} + C_2 z^{n-1} + \text{lower-order terms},$$

[11] Dubrovin *et al.* [6] have considered similar matrix polynomials.

whence, using the local parameter $t = h^{-1}$ at ∞,

$$z = a_i t^{-\nu} + \text{lower-order terms}, \qquad 1 \leqslant i \leqslant n.$$

Therefore, covering ∞, there are n distinct branch points $P_1, \ldots, P_n$ of order $\nu - 1$. The spectrum of A, in the limit $h \to 0$, defines n distinct points $Q_1, \ldots, Q_n$. So,

$$(h) = -\sum_1^n P_i + \sum_1^n Q_i$$

and

$$(z) = -\nu \left(\sum_1^n P_i \right) + n\nu \text{ zeros, distinct from the points } P_i .$$

So, the ramification index V_∞ at ∞ over z equals

$$V_\infty = n(\nu - 1),$$

while the ramification index V_0 for the affine part equals the total order of pole of the expression

$$\begin{aligned} \left.\frac{\partial Q}{\partial h}\right|_{P_i} &= \left.\frac{d}{dh} \prod (z - a_j h^\nu)\right|_{P_i} + \text{lower-order terms} \\ &= -a_i \nu h^{\nu-1} \prod_{j \neq i} (z - a_j h^\nu) + \text{lower-order terms} \end{aligned}$$

at the points P_i $(1 \leqslant i \leqslant n)$, i.e.,

$$V_0 = n((\nu - 1) + (n - 1)\nu).$$

Hence

$$V = V_0 + V_\infty = n(\nu - 1) + n((\nu - 1) + (n - 1)\nu) = 2n\nu + 2g - 2.$$

and

$$g = \frac{n(n-1)}{2} \nu - n + 1. \tag{1}$$

Let ϕ^s be the eigenvector of A near the points P_s, normalized at $(\phi^s)_s = 1$. Since, at P_s, the spectral problem reads

$$(Ah^{-\nu}) e^s = a_s e^s, \qquad h = \infty,$$

ϕ^s can be expanded around e^s and $zh^{-\nu}$ around a_s

$$\phi^s(t) = e^s + vt + O(t^2) \qquad \text{with} \quad v \in C^n,\ v_s = 0, \text{ and } O(t^2)_s = 0,$$

and

$$zh^{-\nu} = a_s + b_s t + O(t^2).$$

To find v and b_s, set up the spectral problem

$$\begin{aligned}(A_\nu + A_{\nu-1}t + O(t^2))(e^s + vt + O(t^2))\\ = (a_s + b_s t + O(t^2))(e^s + vt + O(t^2)),\end{aligned}$$

which to the first order yields the linear equation

$$A_{\nu-1}e^s + A_\nu v = a_s v + b_s e^s$$

or, componentwise,

$$(A_{\nu-1})_{is} + a_i v_i = a_s v_i\,, \qquad i \neq s.$$

Therefore, solving for v_i,

$$\begin{aligned}(\phi^s)_i &= \frac{(A_{\nu-1})_{is}}{a_s - a_i}\,t + O(t^2), \qquad & i \neq s,\\ &= 1, & i = s.\end{aligned}$$

Consider now the eigenvector $f = (f_1, ..., f_n)^t$ of A, normalized at $f_1 = 1$, at each of the points P_s; then

$$\begin{aligned}k \neq 1 \qquad f_k = \frac{(\phi^s)_k}{(\phi^s)_1} &= \frac{(A_{\nu-1})_{k1}}{a_1 - a_k}\,t + O(t^2) & \text{at} \quad P_1\\ &= \frac{(A_{\nu-1})_{ks}}{(A_{\nu-1})_{1s}} \cdot \frac{a_s - a_1}{a_s - a_k} + O(t) & \text{at} \quad P_s\,, s \neq k,\\ &= \frac{a_s - a_1}{(A_{\nu-1})_{1s}}\,t^{-1} + O(1) & \text{at} \quad P_k\,,\end{aligned}$$

which makes sense, since by assumption $(A_{\nu-1})_{1s} \neq 0$ for $s \neq 1$ and $\prod_{i<j}(a_i - a_j) \neq 0$. Therefore[12]

$$(f_k)_\infty \geqslant P_1 - P_k \qquad \text{with equality at } P_k\,.$$

Let $\mathscr{D}$ be the minimal positive divisor such that

$$(f_k) \geqslant -\mathscr{D} + P_1 - P_k\,, \qquad 1 \leqslant k \leqslant n.$$

[12] $(\ )_\infty \equiv (\ )_{\Sigma P_i}$.

Neither of the points P_j or Q_j appears in $\mathscr{D}$. If $\mathscr{D}$ contained P_j then

$$(f_k) \geqslant -(\mathscr{D} - P_j) + P_1 - P_k$$

because f_k never has a pole at P_j ($j \neq k$) and has a pole of order 1 at P_k. Then $\mathscr{D}$ would not be minimal. If $\mathscr{D}$ contained Q_j, then at least one of the f_k's would have to have a pole at Q_j. This is impossible, because the eigenvector of A, when $h \to 0$, has a nonzero first component by assumption. In general, for any $k + \alpha n \in \mathbb{Z}$

$$f_{k+\alpha n} \equiv h^\alpha f_k$$

and

$$(f_{k+\alpha n}) \geqslant -\mathscr{D} + P_1 - P_k + \alpha \sum_1^n (Q_i - P_i).$$

The proof that $\mathscr{D}$ is general or, equivalently, dim $\mathscr{L}(\mathscr{D}) = 1$ follows the lines of Moerbeke and Mumford [16]; consider the spaces

$$\begin{aligned} R \equiv \frac{\mathbb{C}[h, h^{-1}, z]}{Q(h, z)} \subset \mathscr{L}_1 &\equiv \{\text{linear span of } f_k\text{'s}\} \\ \subset \mathscr{L} &= \{f \text{ meromorphic with } (f) + \mathscr{D} \geqslant \text{any} \\ &\qquad \text{linear combination of } P_k \text{ and } Q_k \text{ with } k \in \mathbb{Z}\}. \end{aligned}$$

Since $\mathscr{L}_1$ is an R-module (indeed zf_k and $hf_k \in \mathscr{L}_1$), $\mathscr{L}_1 = \mathscr{L}$ must hold, because otherwise all functions in $\mathscr{L}$ would vanish at some point $p \in X$, which $1 \in \mathscr{L}_1$ clearly violates. Hence any function of $f \in \mathscr{L}(\mathscr{D})$ is a linear combination of f_k's; no f_k with $k \geqslant 2$ can occur, because any such linear combination would always have at least a pole at the points P_k; therefore

$$f = b_1 f_1 + h^{-1}\left(\sum_{i=1}^n f_i b_i^1\right) + \cdots + h^{-\alpha}\left(\sum_{i=1}^n f_i b_i^\alpha\right), \qquad \alpha \geqslant 1,$$

and

$$fh^\alpha(Q_j) = b_1 f_1 h^\alpha(Q_j) + h^{\alpha-1}\left(\sum_{i=1}^n b_i^1 f_i\right)(Q_j) + \cdots + \sum_{i=1}^n b_i^\alpha f_i(Q_j), \quad 1 \leqslant j \leqslant n,$$

i.e.

$$\sum_{i=1}^n b_i^\alpha f_i(Q_j) = 0, \qquad 1 \leqslant j \leqslant n.$$

Now, it is assumed that the eigenvectors of A, in the limit $h \to 0$, are distinct. Hence the n eigenvectors $(f_i(Q_j))_{1 \leqslant i \leqslant n}$ are independent and $\det(f_i(Q_j))_{1 \leqslant i,j \leqslant n} \neq 0$, implying $b_i^\alpha = 0$, $1 \leqslant i \leqslant n$. In the same way $b_i^j = 0$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant \alpha$, proving that $f = b_1 f_1 = b_1$. Therefore $\mathscr{L}(\mathscr{D}) = \{\text{constants}\}$ and dim $\mathscr{L}(\mathscr{D}) = 1$.

To show that $o(\mathscr{D}) = g$, we show that $\dim \mathscr{L}(\mathscr{E}) = ni + 1$ for $\mathscr{E} = \mathscr{D} + i\sum_1^n P_k$ with i large enough. Clearly $\dim \mathscr{L}(\mathscr{E}) \geqslant ni + 1$, since every function f_k $(1 \leqslant k \leqslant ni + 1)$ belongs to that space. But every function $f \in \mathscr{L}(\mathscr{E})$ is a linear combination of f_k's $(1 \leqslant k \leqslant ni + 1)$, because at each P_n it has a pole of order at most i. Therefore, since $\dim \Omega(-\mathscr{E}) = 0$, by Riemann–Roch

$$ni + o(\mathscr{D}) - g + 1 = \dim \mathscr{L}(\mathscr{E}) = ni + 1$$

and

$$o(\mathscr{D}) = g.$$

Conversely, let $\mathscr{D}$ be a general divisor not containing P_i's or Q_i's on the curve X as described in (ii). Then

$$1 \leqslant \dim \mathscr{L}(\mathscr{D} - P_1 + P_k) \leqslant \dim \mathscr{L}(\mathscr{D} - P_1) + 1 = 1;$$

the first inequality is a consequence of Riemann–Roch, the second a consequence of the fact that adding a pole increases the dimension by at most one, and the third (equality) a consequence of the fact that $\dim \mathscr{L}(\mathscr{D}) = 1$. Then the unique function $f_k \in \mathscr{L}(\mathscr{D} - P_1 + P_k)$ must have a pole of exact order one at P_k, because otherwise $f_k \neq$ constant would belong to $\mathscr{L}(\mathscr{D})$, contradicting $\dim \mathscr{L}(\mathscr{D}) = 1$. Then

$$(zf_k) \geqslant -\mathscr{D} - (\nu - 1)P_1 - \nu \sum_{\substack{i \neq 1 \\ i \neq k}} P_i - (\nu + 1)P_k$$

with equality at P_k. Then, for some $a_k \in \mathbb{C}^*$

$$(zf_k - a_k h^\nu f_k) \geqslant -\mathscr{D} - (\nu - 1)P_1 - \nu \sum_{i=2}^{n} P_i .$$

From the expression in brackets, subtract some multiple of $h^{\nu-1}f_j$ $(j \neq 1)$ if it has a pole of exact order ν at some P_j $(j \neq 1)$. Proceeding in this way, zf_k is seen to be expressible as a linear combination

$$zf_k = a_k h^\nu f_k + \sum_{s=0}^{\nu-1} h^s \sum_{j=1}^{n} (A_s)_{kj} f_j$$

defining a matrix

$$A = A_\nu h^\nu + \sum_{s=0}^{\nu-1} A_s h^s, \qquad A_\nu = \operatorname{diag}(a_1, \ldots, a_n), \qquad \prod_1^n a_k \neq 0.$$

Also, $\prod_{i<j} (a_i - a_j) \neq 0$ must hold, because otherwise the n points $P_1, \ldots, P_n$ would not be distinct. Moreover $(A_{\nu-1})_{1,k} \neq 0$ $(k \neq 1)$, because otherwise f_j $(2 \leqslant j \leqslant n)$ would not have a pole of order higher than 1 at P_j. Finally A_0 must

have in the limit distinct eigenvalues, for otherwise the points Q_i would not be distinct. The divisor $\mathscr{D}$ not containing Q_i's forces the A_0 eigenvectors to all have non-zero components.

Note that since the functions f_k can be multiplied with a scalar, the matrix A is determined modulo conjugation by diagonal matrices.

As in Moerbeke and Mumford [16], the isospectral flows derive from polynomials u in z, h and h^{-1}, which admit the splitting

$$u = uf_1 = \sum_{i=-N'}^{0} c_i f_i + \sum_{i=1}^{N} c_i f_i = g_- + g_+ .$$

Abel's theorem applied to the functions $1 + tg_+$ and $1 - tg_-$ for small t leads to

$$\sum_1^g \int_{\nu_i}^{\nu_i(t)} \omega = -\sum_{i=1}^{n} \int_{P_i}^{P_i(t)} \omega = t \sum \operatorname{Res}_{P_i} \omega u + O(t^2)$$

and

$$\sum_1^g \int_{\nu_i}^{\nu_i'(t)} \omega = -\sum_{i=1}^{n} \int_{Q_i}^{Q_i(t)} \omega = -t \sum \operatorname{Res}_{Q_i} \omega u + O(t^2),$$

where $\mathscr{D}(t) = \sum_1^g \nu_i(t)$ in the divisor of zeros of $1 + tg_+$ and $\mathscr{D}'(t) = \sum \nu_i'(t)$ of $1 - tg_-$. For the function $f_k(t)$ associated with the divisor $\mathscr{D}(t)$,

$$((1 + tg_+) f_k(t)) \geqslant -\mathscr{D} - \sum P_i + \sum Q_i ,$$

where the sum in $\sum P_i$ extends to those P_i as if $(1 + tg_+)$ were replaced by u. Let $f = (f_1 ,..., f_n)^{\ddagger}$; then

$$(1 + tg_+) f_k(t) = \sum_{s=0}^{\mu} (B_s f)_k h^s$$

and

$$(1 - tg_-) f_k(t) = \sum_{s=-\mu'}^{-1} (B_s f)_k h^s$$

for some matrices B_s. Comparing the two formulas and using the fact that $g_+ + g_- = u$, we have that

$$\left(\sum_{s=0}^{\mu} h^s B_s \right)_{ki} = tC[u]_{ki} = t(P(A, h, h^{-1})_+)_{ki} .$$

Moreover, for any basis $(\omega_k)_{1 \leqslant k \leqslant g}$ of holomorphic differentials, we have that

$$\left\{ \left(\sum_i \operatorname{Res}_{P_i}(u\omega_k) \right)_{1 \leqslant k \leqslant g} \,\middle|\, u \in \mathbb{C}[z, h, h^{-1}] \right\} = \mathbb{C}^g$$

because the polar part of u at P_i can be prescribed by allowing arbitrarily large poles at the points Q_i. This implies that every linear flow on Jac(X) can be obtained by the right choice of $u \in C[z, h, h^{-1}]$. Remark that, after possible cancellation, u must contain h^{-1} in order for the flow to be nontrivial.[13]

From the group theoretical argument (I), the dimension of the orbit corresponding to A equals $n(n-1)\nu$, implying the existence of $n(n-1)\nu/2$ independent Hamiltonian flows. The set of coefficients of the polynomial $Q(z, h)$ provides a basis for these Hamiltonians:

(i) $g = [n(n-1)/2]\nu - (n-1)$ among these coefficients lead to independent flows on the orbit modulo conjugation by diagonal matrices, according to the argument above.

(ii) The flows on A generated by conjugation by diagonal matrices can be spanned by the flows

$$\dot{A} = [A, \operatorname{diag}(a_1^k, \ldots, a_n^k)]; \qquad k = 1, \ldots, n-1.$$

They are independent since

$$\det \begin{pmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-1} \\ \vdots & & & \vdots \\ 1 & a_n & & a_n^{n-1} \end{pmatrix} = \prod_{i<j} (a_i - a_j) \neq 0.$$

These flows derive from the Hamiltonians (see I (3.6)) $\langle (Ah^{-\nu})^k, h^\nu \rangle_1$ $k = 2, \ldots, n$, which are among the coefficients of $Q(z, h)$.

This finishes the proof of Theorem 1.

COROLLARY. *Any smooth function of the Hamiltonians, appearing as coefficients of $Q(z, h)$, leads to a linear flow on* Jac(X).

Proof. Let $\lambda_1, \ldots, \lambda_N$ be the set of coefficients of the polynomials $Q(z, h)$; then for any function $H = H(\lambda_1, \ldots, \lambda_N)$, the vector field, defined by

$$X_H(g) = \{g, H\}$$

can be written

$$X_H(g) = \sum_1^N \{g, \lambda_i\} \frac{\partial H}{\partial \lambda_i}(g) = \sum_1^N \frac{\partial H}{\partial \lambda_i} X_{\lambda_i}(g).$$

Since the λ_i's are constants of the motion, $\partial H/\partial \lambda_i$ are also constants of the motion; therefore X_H is a constant linear combination of the vector fields defined by $\lambda_1, \ldots, \lambda_N$.

[13] The linearization argument of Theorem 1 could have also been used, but the one given here proves more as it does not assume the Lax equation but in fact deduces it.

EXAMPLE. In view of later applications, it is more practical to choose for Hamiltonians nonpolynomial functions of A. For instance the Hamiltonian $\langle f(Ah^{-\nu}), h^{\nu+1}\rangle_1$ with

$f(x) = [2/(\nu+2)]x^{(\nu+2)/2}$ leads to the linear flow generated by

$$(A^{1/\nu})^{+} = A_{\nu}^{1/\nu}h + \Gamma \equiv Bh + \Gamma,$$

where (see I (4.44))[14]

$$\Gamma = \mathrm{ad}_B\, \mathrm{ad}_{A_\nu}^{-1}((I - P_D)\, A_{\nu-1}) + (\nu/2)(A_\nu)^{\nu/2-1}\, P_D(A_{\nu-1})$$

We now discuss several special cases of physical or geometrical interest:

1. *$A = A_1h + A_0$ with $A_0 + A_0^{\ddagger} = 0$ and the Properties of Theorem* 1

The set of such isospectral matrices maps one-to-one onto the general points $\mathscr{D}$ of the $[(n-1)^2/4]$-dimensional variety in Jac(X) defined by the relations

$$\int^{\mathscr{D}} \omega = 0 \qquad \text{for} \quad \omega^{\tau} = \omega,$$

where τ is the involution

$$(z, h) \to (-z, -h).$$

The only flows preserving the antisymmetry of A_0 are those for which $u^{\tau} = -u$, i.e., linear combinations of $z^i h^{-k}$ such that $i + k$ is odd; they linearize on Prym(X). In particular, the flow generated by the Hamiltonian $H = \frac{2}{3}\langle (Ah^{-1})^{3/2}, h^2\rangle_1$, namely,

$$\dot{A} = [A, ((Ah^{-1})^{1/2}h)_{+}] = [A_0 + A_1h, E + \beta h]$$

(Euler's rigid body motion), where $\beta = A_1^{1/2} = \mathrm{diag}(a_1^{1/2},\ldots, a_n^{1/2})$ and $E_{ij} = (\beta_i + \beta_j)^{-1}(A_0)_{ij}$, linearizes on Prym($X$).

Proof. The map τ is an involution on the curve X, because whenever $\det(A_1h + A_0 - zI) = 0$, we have that

$$\begin{aligned}\det(-A_1h + A_0 + zI) &= \det(-A_1h + A_0 + zI)^{\ddagger} = \det(-A_1h - A_0 + zI)\\ &= (-1)^n \det(A_1h + A_0 - zI) = 0.\end{aligned}$$

According to Theorem 1, the matrix A maps into a general divisor $\mathscr{D}$ on X of degree g, which because of the property $A_0 + A_0^{\ddagger} = 0$ belongs to the Prym variety for the involution τ. More precisely, since $\Delta_{NN}^{\tau} = \Delta_{NN}$ (mod ± 1), an argument

[14] $P_D(B) \equiv$ diagonal part of B.

similar to the one in Moerbeke and Mumford [16, Sect. 3, Theorem 2] leads to the fact that

$$(\Delta_{NN}) = \mathscr{D} + \mathscr{D}^\tau + \text{poles at } P_i \text{ and } Q_i .$$

Therefore, from Abel's theorem, the following relation holds for some choice of origin and for every holomorphic differential,

$$\int^{\mathscr{D}} \omega + \int^{\mathscr{D}^\tau} \omega = 0. \tag{2}$$

It is trivially satisfied for the holomorphic differentials ω such that $\omega^\tau = -\omega$ and leads to genuine relations for the ones such that $\omega^\tau = \omega$. Now, the dimension of their space (i.e., with $\omega^\tau = \omega$) is g_0-dimensional, where g_0 is the genus of the quotient curve $X_0 = X/\tau$. Therefore the variety of divisors $\mathscr{D}$ in Jac(X) satisfying (2) is $(g - g_0)$-dimensional; let it be Prym(X). Conversely, every general divisor in Prym(X) leads to a matrix $A = A_1 h + A_0$ with $A_0 + A_0{}^\ddagger = 0$, modulo conjugation by diagonal matrices with entries ± 1.

We now compute g_0. To begin with,

$$\begin{aligned} Q(h, z) = \prod_1^n (a_i h - z) + Q_{n-2}(a_1 h - z,..., a_n h - z) \\ + Q_{n-4}(a_1 h - z,..., a_n h - z) + \cdots = 0, \end{aligned}$$

where $Q_k(t_1 ,..., t_n)$ are homogeneous polynomials in t_i $(1 \leqslant i \leqslant n)$ of degree k. The substitution $t = h/z$ turns the equation above into

$$\begin{aligned} z^n \prod (a_i t - 1) + z^{n-2} Q_{n-2}(a_1 t - 1,..., a_n t - 1) + \cdots + Q_0 = 0 \quad &\text{for } n \text{ even} \\ + \cdots + z Q_1 = 0 \quad &\text{for } n \text{ odd}. \end{aligned}$$

In the *even* (resp. *odd*) case, the curve X_0 is obtained by putting $u = z^2$ in the latter equation

$$u^{n/2} \prod_1^n (a_i t - 1) + u^{(n-2)/2} Q_{n-2}(a_1 t - 1,..., a_n t - 1) + \cdots + Q_0 = 0 \tag{3}$$

(resp. $u^{(n-1)/2} \prod (a_i t - 1) + u^{(n-3)/2} Q_{n-2}(a_1 t - 1,..., a_n t - 1) + \cdots + Q_1 = 0$); it is a $[n/2]$-sheeted covering of the t-plane with branch points at the zeros of the discriminant

$$\Delta = [n/2] u^{[(n-2)/2]} \prod_1^n (a_i t - 1) + \left[\frac{n-2}{2}\right] u^{[(n-4)/2]} Q_{n-2}(a_1 t - 1,..., a_n t - 1) + \cdots$$

and possibly also at the poles of Δ. Dividing (3) by $\prod_1^n (a_i t - 1)$, we find that, near $t = \infty$, it reads

$$u^{n/2}+\left(\frac{A_1}{t^2}+\cdots\right)u^{n/2-1}+\left(\frac{B_1}{t^4}+\cdots\right)u^{n/2-2}+\cdots+\frac{L_1}{t^n}=0$$

$$\left(\text{resp. } u^{(n-1)/2}+\left(\frac{A_1}{t^2}+\cdots\right)u^{n/2-1}+\left(\frac{B_1}{t^4}+\cdots\right)u^{n/2-2}+\cdots+\frac{L_1}{t^{n-1}}+\cdots=0\right).$$

Therefore u has the following expansion with generically distinct constants C_i

$$u=\frac{C_i}{t^2}+O\left(\frac{1}{t}\right),\qquad 1\leqslant i\leqslant\left[\frac{n}{2}\right],\ t \text{ near } \infty;$$

So, u has a double zero at $[n/2]$ points, labeled R_i. Moreover, the function u has a simple pole at $t=a_i^{-1}$ $(1\leqslant i\leqslant n)$. So Δ has poles at

$$2\sum_1^{n/2}R_i+\frac{n-4}{2}\sum_1^n P_i$$

$$\left(\text{resp. } 3\sum_1^{n-1/2}R_i+\frac{n-5}{2}\sum_1^n P_i\right)$$

and therefore vanishes at

$$\frac{n(n-4)}{2}+n=\frac{n(n-2)}{2}\qquad\left(\text{resp. }\frac{n(n-5)}{2}+\frac{3(n-1)}{2}=\frac{n(n-2)-3}{2}\right)$$

points on X_0. Let V_0 be this number; then from the Riemann–Hurwicz formula,

$$g_0=\frac{V_0}{2}-\frac{n}{2}+1=\frac{(n-2)^2}{4}$$

$$\left(\text{resp. } g_0=\frac{V_0}{2}-\frac{n-1}{2}+1=\frac{(n-2)^2-1}{4}\right).$$

Observe that formula (1) for the genus g of the curve X can be recovered; indeed X is a double covering of the curve X_0 branched at the points where u has an odd pole or zero; this only occurs at the n points P_i and one extra point where n is odd. So

$$g=2g_0+\left[\frac{n-1}{2}\right]=\frac{n(n-1)}{2}\nu-(n-1)\qquad\text{for}\quad \nu=1.$$

We conclude that

$$\begin{aligned}\dim\operatorname{Prym}=g-g_0&=\frac{n(n-2)}{4},\qquad n\text{ even},\\&=\frac{(n-1)^2}{4},\qquad n\text{ odd}.\end{aligned}$$

The statement about linearization follows at once from Theorem 1.

2. *The Special Orbits* (a), (b), (c) of I, *Theorem* 4.4 (*geodesic Flow on Ellipsoids, Neumann Problem, etc* ... .)

The curves defined by $\det(A - zI) = 0$, where[15]

(a) $A = \alpha h + \Lambda_{xy}$, $\alpha = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$, α_i all distinct,

(b) $A = \alpha h^2 + h\Lambda_{xy} - \Gamma_{xx}$,

(c) $A = \alpha h^2 + h\Lambda_{xy} + (\Delta_{xy} - \alpha)$

are hyperelliptic of the form

(a) $h^2 = -(|x|^2 |y|^2 - (x, y)^2) \prod_{i=1}^{n-2} (t - \sigma_i)\, a(t)^{-1}$ with $t = zh^{-1}$,

(b) $h^2 = -|x|^2 \prod_{i=1}^{n-1} (t - \mu_i)\, a(t)^{-1}$,

(c) $h^2 - 1 = 2(x, y) \prod_{i=1}^{n-1} (t - \nu_i)\, a(t)^{-1}$

of genera $g = n - 2$, $n - 1$ and $n - 1$, respectively. The function h has the property

$$(h) = \sum_1^n Q_i - \sum_1^n P_i ,$$

where the distinct points P_i correspond to $h = \infty$, $t = \alpha_i$. One assumes that for some $1 \leqslant i \leqslant n$, $x_i y_j - x_j y_i \neq 0$ for all $j \neq i$. The eigenvectors of Ah^{-2} in the limit $h \to 0$ are all distinct, because they are the eigenvectors of the matrix L considered in I, (4.54). For instance in the geodesic flow case, the eigenvectors are the components of the Chasles frame. Then from Theorem 1, the matrices of the form above parametrize a part or all of the Jacobi variety. We now show they parametrize a Zariski open set of the Jacobi variety using a dimension count. Consider

(a) $\mathscr{S} = \{A \mid \alpha \text{ fixed}, \ |x| = 1, \ (x, y) = 0, \ \det(A - zI) = Q(z, h)\}$ modulo the rotation

$$x_i \to x_i \cos\theta - \frac{y_i}{|y|} \sin\theta,$$

$$y_i \to x_i \sin\theta + \frac{y_i}{|y|} \cos\theta.$$

The variety has dimension $(2n - 3) - (n - 1) = n - 2$, the same as $\operatorname{Jac}(X)$.

(b) $\mathscr{S} = \{A \mid \alpha \text{ fixed}, (x, y) = 0, \det(A - zI) = Q(z, h)\}$[16] has dimension $(2n - 1) - n = n - 1 = \dim \operatorname{Jac}(X)$.

(c) $\mathscr{S} = \{A \mid \alpha \text{ fixed}, |x| = 1, \det(A - zI) = Q(z, h)\}$[16] has dimension $2n - 1 - n = n - 1 = \dim \operatorname{Jac}(X)$.

[15] $\Lambda_{xy} = x \otimes y - y \otimes x$, $\Gamma_{xx} = x \otimes x$, $\Delta_{xy} = x \otimes y + y \otimes x$.

[16] The isospectral set is obtained by dividing out by the discrete action $(x_i, y_i) \to (-x_i, -y_i)$.

Finally the linearization of the flows described in I, Section 4, is straightforward by Theorem 1.

3. *The Lagrange Top*[17]

It describes the motion of a rigid symmetric body ($I_1 = I_2$) under gravity, with center of gravity belonging to the principal axis. As was explained in I, Sections 2 and 4, the equations of motion take the form

$$\frac{d}{dt}(\gamma + hM + h^2 I_1 l) = [\gamma + hM + h^2 I_1 l, \Omega + lh], \tag{4}$$

where

$$\gamma = \begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & -I_3\Omega_3 & I_2\Omega_2 \\ I_3\Omega_3 & 0 & -I_1\Omega_1 \\ -I_2\Omega_2 & I_1\Omega_1 & 0 \end{pmatrix},$$

$$l = \begin{pmatrix} 0 & -z_0 & 0 \\ z_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}.$$

We perform a unitary operation on $A' \equiv \gamma + hM + R^2 I_1 l$ in order to make the leading coefficient $I_1 l$ diagonal. Let

$$U = \begin{pmatrix} 0 & i/2^{1/2} & 1/2^{1/2} \\ 0 & 1/2^{1/2} & i/2^{1/2} \\ 1 & 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} A &\equiv U^{-1}A'U \\ &= \begin{pmatrix} 0 & -\frac{i}{2^{1/2}}(A'_{13} - iA'_{23}) & -\frac{1}{2^{1/2}}(A'_{13} + iA'_{23}) \\ -\frac{i}{2^{1/2}}(A'_{13} + iA'_{23}) & -iA'_{12} & 0 \\ \frac{1}{2^{1/2}}(A'_{13} - iA'_{23}) & 0 & iA'_{12} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \beta & i\beta^* \\ -\beta^* & -\omega & 0 \\ i\beta & 0 & \omega \end{pmatrix} \end{aligned}$$

[17] See Ratiu and van Moerbeke [20] and especially Ratiu [19].

with

$$\beta = -\frac{i}{2^{1/2}}(A'_{13} - iA'_{23}) = y + hx, \qquad \beta^* = \bar{y} + h\bar{x},$$

$$y = \frac{1}{2^{1/2}}(\gamma_1 - i\gamma_2), \qquad x = \frac{I_1}{2^{1/2}}(\Omega_1 - i\Omega_2)$$

and

$$\omega = iA'_{12} = i(-z_0 I_1 h^2 - I_3 \Omega_3 h - \gamma_3). \tag{5}$$

Then the curve X is defined by

$$Q(z, h) = \det(A - zI) = -z[z^2 - \omega^2 + 2\beta\beta^*] = 0,$$

which amounts to the elliptic curve

$$z^2 = P_4(h) \equiv \omega^2 - 2\beta\beta^*$$

with two points P and Q covering $h = \infty$ and R_1 and R_2 covering $h = 0$; so

$$\begin{aligned}(h) &= -P - Q + R_1 + R_2\,,\\ (z) &= -2P - 2Q + 4 \text{ other zeros.}\end{aligned}$$

The eigenvectors $f = (f_1, f_2, f_3)$ normalized at $f_1 = 1$ equal

$$\begin{aligned} f_2 &= \frac{\Delta_{12}}{\Delta_{11}} = \frac{\beta^*(\omega - z)}{\omega^2 - z^2} = \frac{\beta^*}{\omega + z} = \frac{\omega - z}{2\beta}, \\ f_3 &= \frac{\Delta_{13}}{\Delta_{11}} = \frac{i\beta(\omega + z)}{-(\omega^2 - z^2)} = -\frac{i\beta}{\omega - z} = \frac{\omega + z}{2i\beta^*}; \end{aligned} \tag{6}$$

observe that $f_2 \cdot f_3 = -i/2$. In view of (5) and the expansion

$$z = \pm i z_0 I_1 h^2 \left(1 + \frac{I_3\Omega_3}{z_0 I_1}\frac{1}{h} + O\left(\frac{1}{h^2}\right)\right) \text{ about } P \text{ and } Q,$$

we have that

$$\begin{aligned} \omega + z &= O(1) & &\text{at } P \\ &= -2iz_0 I_1 h^2 + O(h) & &\text{at } Q \end{aligned}$$

so that

$$(f_2)_\infty = -P + Q, \qquad (f_3)_\infty = P - Q.$$

According to (6), the function f_2 has a pole at the point ν defined by $\omega + z = 0$ and $\beta = 0$, i.e., $h = -y/x$ and $z = \omega\,|_{h=-y/x}$ and has a zero at the point $\bar{\nu}$

defined by $\omega - z = 0$ and $\beta^* = 0$, i.e., $h = -\bar{y}/x$ and $z = \omega\,|_{h=-\bar{y}/x}$. Therefore,

$$(f_2) = -P + Q - \nu + \bar{\nu} \quad \text{and} \quad (f_3) = P - Q + \nu - \bar{\nu}.$$

This defines a map $A_h \to \nu$ from

$$\begin{aligned}\mathscr{A}(X) = \{A_h \mid\ & A_h \text{ of the form above, } z_0 \neq 0 \text{ and} \\ & \det(A_h - zI) = Q(z, h)\} \text{ mod the rotation} \\ & x \to xe^{i\theta} \text{ and } y \to ye^{i\theta}\end{aligned}$$

to the curve X. To show that the map is onto, we show that it is one-to-one. From the curve one determines uniquely the coefficients of the polynomial $\omega^2 - 2\beta\beta^*$ in h, in particular $z_0 I_1$ and $I_3\Omega_3$. The divisor ν defines $-y/x$ and ω evaluated at $h = -y/x$; therefore, from (5),

$$\gamma_3 = i\omega - z_0 I_1 h^2 - I_3\Omega_3 h\,|_{h=-y/x}$$

and also ω, as a polynomial in h, are known. Hence the polynomial in h

$$z^2 - \omega^2 = -2\beta\beta^* = -2(|\,y\,|^2 + |\,x\,|^2 h^2 + (y\bar{x} + \bar{y}x)h)$$

is known, i.e., $|\,y\,|^2$, $|\,x\,|^2$, $(y\bar{x} + \bar{y}x)$ and $\bar{x}y = (y/x)\,|\,x\,|^2$. This implies that x and y are known, up to a common rotation $x \to xe^{i\theta}$, $y \to ye^{i\theta}$.

The linearization statement can be established by direct computation or by the general method, somewhat adapted. The direct computation cannot be generalized to higher dimensions, while the general method could easily be extended to higher dimensions. Any linear flow of the point

$$\begin{aligned}\nu &= (h, z) \\ &= \left(-\frac{y}{x}, z\,\Big|_{h=-y/x}\right) \\ &= \left(-\frac{y}{x}, \pm\omega\,\Big|_{h=-y/x}\right)\end{aligned}$$

on the elliptic curve $z^2 = \omega^2 - 2\beta\beta^*$, reads

$$\begin{aligned}\dot{h} &= \pm c\omega\,|_{h=-y/x} = \pm cix^{-2}(-z_0 I_1 y^2 + I_3\Omega_3 yx - \gamma_3 x^2) \\ &= x^{-2}(\dot{x}y - \dot{y}x).\end{aligned}$$

Since, from the differential equations (4)

$$\dot{x} = -i\left(\frac{I_3 - I_1}{I_1}\right)\Omega_3 x + iz_0 y, \qquad \dot{y} = i\Omega_3 y - \frac{i\gamma_3}{I_1}x,$$

and

$$\dot{x}y - \dot{y}x = \frac{i}{I_1}(z_0 I_1 y^2 - I_3 \Omega_3 yx + \gamma_3 x^2)$$

we have that $c = \pm I_1^{-1}$; hence, the Lagrange top motion is a linear flow on $\mathrm{Jac}(X) = X$.

We now apply the general method to this case. Consider the non-singular curve X_ϵ of genus $= 4$ defined by

$$\begin{aligned} Q_\epsilon(z, h) = \det(A^{(\epsilon)} - zI) &= \det \begin{pmatrix} \epsilon h^2_{-z} & \beta & i\beta^* \\ -\beta^* & -\omega - z & 0 \\ i\beta & 0 & \omega - z \end{pmatrix} \\ &= (\epsilon h^2 - z)(z^2 - \omega^2) - 2\beta\beta^* z \\ &= -z^3 + \epsilon h^2 z^2 + (-2\beta\beta^* + \omega^2)z - \epsilon h^2\omega^2 = 0, \end{aligned}$$

with three points P_1, P_2, P_3 defined by $z = \infty$, $h = \infty$ such that

$$\begin{aligned} \frac{z}{h^2} &\simeq a_1 = \epsilon && \text{at } P_1 \\ &\simeq a_2 = -iz_0 I_1 && \text{at } P_2 \\ &\simeq a_3 = iz_0 I_1 && \text{at } P_3 \end{aligned}$$

where $\epsilon \to 0$, X_ϵ tends to a reducible curve X_0 containing X; the points P_2 and P_3 go over into P and Q. The nonzero flow generated by the meromorphic function $I_1^{-1}zh^{-1}$ linearizes, as a consequence of Theorem 1, on the four-dimensional variety $\mathrm{Jac}(X_\epsilon)$. The Lax point corresponding to this flow reads

$$\dot{A}^{(\epsilon)} = [A^{(\epsilon)}, I_1^{-1}(A_1^{(\epsilon)} + A_2^{(\epsilon)}h)].$$

When $\epsilon \searrow 0$, it converges to $\dot{A} = [A, \tilde{B}]$, where

$$\tilde{B} = \begin{pmatrix} 0 & I_1^{-1}x & iI_1^{-1}\bar{x} \\ -I_1^{-1}\bar{x} & iI_3 I_1^{-1}\Omega_3 & 0 \\ iI_1^{-1}x & 0 & -iI_3 I_1^{-1}\Omega_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & iz_0 & 0 \\ 0 & 0 & -iz_0 \end{pmatrix} h$$

and the linear flow on $\mathrm{Jac}(X_\epsilon)$ goes over into a linear flow on $\mathrm{Jac}(X_0)$ and, more specifically, on its compact piece $\mathrm{Jac}(X)$. Since the matrix A can always be conjugated by a diagonal matrix depending on t, without modifying the flow on $\mathrm{Jac}(X_0)$, we may modify the diagonal entries of B so as to get

$$\begin{aligned} B &= \begin{pmatrix} 0 & I_1^{-1}x & iI_1^{-1}\bar{x} \\ -I_1^{-1}\bar{x} & i\Omega_3 & 0 \\ iI_1^{-1}x & 0 & -i\Omega_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & iz_0 & 0 \\ 0 & 0 & -iz_0 \end{pmatrix} h \\ &= U^{-1}(\Omega + lh)U, \end{aligned}$$

which establishes the linearity of the original flow (4).

Remark. It is interesting to further explore some phenomena of Example 2. In the ellipsoidal problem (see, for instance, Arnold [24]) or the Neumann problem (see, for instance, Devaney [25]), one knows that one has hyperbolic phenomena. Associated to geodesic motion on the ellipsoid

$$\frac{x_1^2}{a} + \frac{x_2^2}{b} + \frac{x_3^2}{c} = 1, \qquad a < b < c,$$

we have a curve of the form X: $h^2 = z(z - \mu_1)/(z - a)(z - b)(z - c)$, but the hyperbolic periodic orbit U about $x_2 = 0$, corresponds to the case $\mu_1 = b$, and it linearizes on the real part of the Jacobian of the curve $h^2 = z/(z - a)(z - c)$. Orbits entering the stable manifold of U wind into U as time $t \to \infty$, while these orbits actually hook up to the unstable orbit as $t \to -\infty$, and hence wind backwards into U. These orbits in fact are characterized as those passing through the umbilics of the ellipsoid (see Arnold [24]. Such asymptotically behaved orbits could not possibly be linearized on a torus and so must correspond to the case where the curve X is singular i.e. (see I, Section 2, for definition of the F_i's)

$$X : h^2 = \frac{z^2}{(z - a)(z - b)(z - c)} = \frac{F_1(x, y)}{z - a} + \frac{F_2(x, y)}{z - b} + \frac{F_3(x, y)}{z - c} = \Phi_z$$

From I, Section 2, and the above it follows that the stable–unstable manifold is the two-dimensional real variety V in R^6 given by

$$V = \left\{(x, y) \;\middle|\; \frac{F_1}{a} + \frac{F_2}{b} + \frac{F_3}{c} = 0, (b + c)F_1 + (a + c)F_2 + (a + b) F_3 = 0, \right.$$
$$\left. | x | = 1 \quad \langle x, y\rangle = 0\right\}.$$

Thus our isospectral set maps into the generalized Jacobian of X, which is $\mathbb{C} \times$ torus, whose real part is a cylinder. The geodesic flow corresponding to the stable–unstable variety V is linearized on a cylinder, with exponentials playing a role, as in fact is intuitively suggested by the asymptotic orbits "screwing" forward and backwards in time into the hyperbolic orbit U. It would be interesting to explicitly show from our machinery that the coefficients of the exponentials are the Floquet multipliers, as they must be. A similar discussion applies to the Neumann problem, which may have equilibrium points[18] (as none of the μ_i's need be zero) and thus as Devaney [25] discussed, the Neumann flow is far from ergodic near hyperbolic equilibria points. In general the linearizations will take place on Real Jacobian(X) $= R^k \times T^j$ for appropriate j, k, again with exponentials playing a role.

There is one technical hitch to this argument, namely the assumption that if the curve is nonsingular the flow is actually linearized on the torus, with no

[18] This must be related to the n-soliton solution of the KdV equation using the well known relation between the KdV equation and the Neumann problem.

exceptional behavior. To see that, one must check that for this case, the map $\mathscr{A}(X) \mapsto \mathrm{Jac}(X)$ is defined for all $A \in \mathscr{A}(X)$. For the case $n = 3$, geodesic flow, it is easy to see that the criteria of Theorem 1 are satisfied, for if there is one zero element in every row, then we might as well assume the first row and column of $A_1 = A_{\nu-1} = A_{xy}$ to be zero. In that case, it is easy to see, since $|x| \neq 0$, that either $x_i = y_i = 0$ for some i, in which case we are in the $n = 2$ (periodic orbit) case, or else $A_{xy} = 0$. In that case by the formula X: $h^2 = z(z - \mu_1)/(z - a)(z - b)(z - c)$, at most one of the x_i may equal zero, in which case we are again in the $n = 2$ case. If none of the x_i equal zero, by the equations of motion (I, Theorem 4.4) for geodesic flow, we see that A_{xy} instantly has all non-zero entries, and so we can use the flow to complete the map $\mathscr{A}(X) \to \mathrm{Jac}(X)$, for these special cases. Thus the hypotheses of Theorem 1 are verified and so asymptotic geodesic flow is linearized on a cylinder. This argument fails for $n > 3$, but nonetheless singular curves and hence asymptotic behavior associated with the joining of the stable and unstable manifold abound. We just cannot rigorously prove there is a one-to-one relation, although it is undoubtedly true.

4. Independence of the Linearization on the Representation

We have seen in Sections 2 and 3 how to linearize the various differential equations of this paper. We noted that the linearization takes place on the Jacobian of the curve X defined by the Kac–Moody recipe. Moreover it was noted in I, Section 3, that the curve X is representation dependent. The recipe for linearization, although given for the classical representation (minimal dimensional), extends to the higher-dimensional representations. Since the ordinary differential equations and their isospectral sets are representation independent, one would expect that the Jacobians of the curves associated with the different representations have a common abelian subvariety on which the linearization takes place. This is in fact the case; the purpose of this section is to prove this fact.

Consider the hyperelliptic curve

$$F(z, t) \equiv A(t + t^{-1}) + P(z) = \prod_1^N (z - \lambda_i(t)) = 0, \tag{1}$$

where A is a non-zero number and $P(z)$ an arbitrary monic polynomial of degree N. Fix $1 \leqslant \tau \leqslant N$; let α_i $(1 \leqslant i \leqslant \tau)$ be integers and let $l_i(j_1, ..., j_n) \equiv j_i$. Suppose

$$\alpha_1 = \cdots = \alpha_{a_1}, \alpha_{a_1+1} = \cdots = \alpha_{a_1+a_2}, ..., \alpha_{a_1+\cdots+a_{l-1}+1} = \cdots = \alpha_{a_1+\cdots+a_l}$$

with $\sum_1^l a_i = \tau$. Let π denote a permutation of the set $(1, ..., N)$.

Consider now the curve X', defined by

$$G(z, t) \equiv \prod \left(z - \sum_{i=1}^{\tau} \alpha_i \lambda_{l_i(\pi)}(t) \right) = 0, \tag{2}$$

where the product extends over the orbit of $\sum_{i=1}^{\tau} \alpha_i \lambda_i(t)$ under the action of the permutation group $\{\pi\}$. X' will be irreducible with X, by I, Lemma 3.1; the latter is generically irreducible. Observe that the curves associated with any higher-order representation involve curves of the type X' in the case of the Toda systems. This section is concerned with proving the following theorem.

THEOREM 1. *When X is non-singular, the following decomposition holds:*

$$\mathrm{Jac}(X') = \mathrm{Jac}(X) \oplus B,$$

where B is an abelian variety and where $\oplus$ is understood to be a direct sum modulo a finite group of translations (isogenies) and therefore, whatever be the representation the linearization can always take place on the $\mathrm{Jac}(X)$ *piece of the* $\mathrm{Jac}(X')$, *and hence the linearization is indeed representation independent.*

Remark. The above theorem deals with the Toda system of $sl(n)$, but can be extended to the other classical groups; this point will be discussed at the end.

Proof of the theorem. X is an N-sheeted covering of the t-plane, while X' is an M'-sheeted covering with

$$M' = \frac{N!}{a_1! \cdots ! \, a_l! \, (N - \tau)!} \, .$$

For notation, definitions and theorems about correspondences, the reader is referred to the Appendix. Consider now the correspondence $C[n, n']$ between X and X' defined by

$$(t, \lambda_m(t)) \to \left\{ \left(t, \sum_{1}^{\tau} \alpha_i \lambda_{l_i(\pi)}(t) \right) \;\middle|\; \text{over all permutations } \pi \text{ such that } l_i(\pi) = m \text{ for some } i \right\}$$

and the inverse correspondence between X' and X defined by

$$\left(t, \sum_{1}^{\tau} \alpha_i \lambda_{l_i(\pi)}(t) \right) \to \{ (t, \lambda_{l_i(\pi)}(t)) \mid 1 \leqslant i \leqslant \tau \}.$$

This is indeed a correspondence; the two equations can be exhibited.[19]

[19] They take a particularly simple form when $G(z, t) = \prod_{i<j} (z - (\lambda_i + \lambda_j))$; to wit, $F(z, t) = 0$, $t = t'$, $F(z' - z, t) = 0$, $G(z', t') = 0$. In the general case, the expressions are more complicated.

The indices of the correspondence are $n = \tau$ and

$$n' = \sum_{m=1}^{l} \frac{(N-1)!}{a_1! \cdots ! (a_m - 1)!\, a_{m+1}! \cdots ! a_l!\, (N-\tau)!} = \frac{M'}{N} \sum_1^l a_m = M' \frac{\tau}{N} \quad (3)$$

because the number of elements in the orbit above containing a fixed element λ_i can be obtained by adding $\alpha_{a_1+\cdots+a_m}\lambda_i$ to each of the

$$\binom{N-1}{a_1, \dots, a_m - 1, \dots, a_l}$$

permutations of $\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_N$, where m ranges between 1 and l. We now state two lemmas, whose proofs will be delayed.

LEMMA 1. *The correspondence* $C[n, n']$ *between* X *and* X' *is singular.*

LEMMA 2. *Generically* $\mathrm{Jac}(X)$ *is irreducible, i.e.,* $\mathrm{Jac}(X)$ *has no nontrivial subabelian varieties.*

Let R be the homomorphism between $\mathrm{Jac}(X)$ and $\mathrm{Jac}(X')$ induced by the correspondence $C[n, n']$ between X and X'. Fix a generic curve X, for which X is nonsingular and irreducible. By Lemma 1, this homomorphism is nontrivial. Then, according to Weil [22, p. 22, Theorem 112], there is an abelian subvariety

$$A \subsetneqq \mathrm{Jac}(X)$$

and a finite number of points $a_i \in \mathrm{Jac}(X)$, forming a group such that

$$\ker h = \bigcup_i (A + a_i).$$

But since $\mathrm{Jac}(X)$ is irreducible, $\ker h$ is a finite group; so $h(\mathrm{Jac}(X))$ can be regarded as a subabelian variety A' of $\mathrm{Jac}(X')$, modulo the finite group accounting for isogenies. By the "Poincaré reducibility theorem" (Weil [22, p. 176, Theorem 26]), there is another abelian variety $B' \subset \mathrm{Jac}(X')$ such that

$$A' \cap B' = \text{finite subgroup of } \mathrm{Jac}(X'),$$
$$\forall x \in \mathrm{Jac}(X'),\ x = y + z \qquad \text{with} \quad y \in A',\ z \in B'.$$

Observe that the decomposition above is not unique; if $x = y + z = y' + z'$, then $y - y' = z' - z \in A' \cap B'$, which is a finite set. Therefore the decomposition can be chosen in a consistent fashion so that

$$\mathrm{Jac}(X') = A' \oplus B',$$
$$A' \text{ is isogenic to } \mathrm{Jac}(X).$$

For the nonsingular curves X for which $\mathrm{Jac}(X)$ is reducible, one uses a different argument, invoking the principle of specialization, to be explained below. Consider the coefficient of $P(z)$ in the expression $F(z, t)$ to be the indeterminates $u_1, \ldots, u_N$, which take on values in $U = \mathbb{C}^N \backslash \{$points where X becomes singular$\}$. If $u = (u_1, \ldots, u_N)$ is specialized to $u^0 = (u_1^0, \ldots, u_N^0) \in U$, the curve X_u specializes to X_{u_0}. With Chow [4] and Matsusaka [12] construct the Jacobi variety $\mathrm{Jac}(X_u)$ (thought of as defined over the indeterminates $u_1, \ldots, u_N$); it has the property that

$$\mathrm{Jac}(X_u)|_{u=u_0} = \mathrm{Jac}(X_{u_0}).$$

Similarly, the equation $G(z, t) = 0$ is a polynomial in the indeterminates $u_1, \ldots, u_N$ and the equation defining the correspondence between X_u and X_u' as well. Let Γ_u be the graph of the associated homomorphism h_u between $\mathrm{Jac}(X_u)$ and $\mathrm{Jac}(X_u')$:

$$\Gamma_u \subset \mathrm{Jac}(X_u) \times \mathrm{Jac}(X_u');$$

Γ_u is an algebraic subvariety; since h_u is a homomorphism, Γ_u has the group structure induced by $\mathrm{Jac}(X_u) \times \mathrm{Jac}(X_u')$. Therefore, according to [4] and [12] Γ_u is a subabelian variety over the field generated by $u_1, \ldots, u_N$. By putting $u = u_0$, Γ_{u_0} is a closed algebraic subset of $\mathrm{Jac}(X_{u_0}) \times \mathrm{Jac}(X_{u_0}')$, which is also a group. Therefore by

$$\Gamma_u |_{u=u_0} = \Gamma_{u_0}.$$

Also

$$\mathrm{proj}_{\mathrm{Jac}(X_u')} \Gamma_u = A_u$$

has the same dimension as Γ_u. Under the theorem of specialization (Matsusaka [13] or Shimura [21])

$$A_u |_{u=u_0} = A_{u_0}$$

and in the specialization the dimension does not drop; so therefore $\dim h_{u_0}(\mathrm{Jac}\, X_{u_0}) = \dim A_{u_0} = \dim A_u =$ the dimension of $\{$the generic A_{u_1} for $u_1 \in U\} = g$ and the result that $\mathrm{Jac}(X)$ is a subabelian variety of $\mathrm{Jac}(X')$ holds even for a nonsingular X, for which $\mathrm{Jac}(X)$ is reducible. This establishes Theorem 1.

We now prove Lemmas 1 and 2:

Proof of Lemma 1. Theorem 1 of the Appendix will be used to show that the correspondence between X and X' is nonsingular; it will suffice to show that

$$\eta < 2n(n' + g' - 1),$$

where $n = \tau$, $n' = M'\tau/N$ and g' is the genus of curve X'. We have mentioned that the correspondence $C[n, n']$ was given by two equations; this does not suffice to conclude nonsingularity of C, because possibly these two equations could be reduced to one in the usual or in some other coordinates. Therefore we establish the inequality above. Unfortunately η cannot be computed by inspection, because most coincidence points $p \in X$ (where $C(p)$ has at least double points) are all branch points of X with regard to the t-plane; also for these points, $C(p)$ contains branch points of X', which makes it a delicate matter. Instead of this argument, we use Zeuthen's formula (Lemma 1 of the Appendix) to compute η.

Introduce the graph curve X'' of the correspondence $C[n, n']$, whose sheet number equals $n'N$. Let g'' be its genus. There is a $C[1, n']$ correspondence between X and X''; hence, according to Lemma 2 (Zeuthen formula) of the Appendix:

$$\eta = 2(g'' - 1) - 2n'(g - 1).$$

So, it suffices to show that

$$(g'' - 1) - n'(g - 1) - \tau(g' - 1) < \tau n'. \tag{4}$$

Let V, V' and V'' denote the ramification indices of X, X' and X'' respectively. Then according to the Riemann–Hurwicz formula,

$$g'' - 1 = \frac{V''}{2} - n'N,$$

$$g - 1 = \frac{V}{2} - N,$$

$$g' - 1 = \frac{V'}{2} - M'.$$

Combining the relations (3) and (4), it suffices to show that

$$\frac{V''}{2} - n'\frac{V}{2} - \tau\frac{V'}{2} < -(N - \tau)n'. \tag{5}$$

Generically, X has $2(N - 1)$ simple branch points for $t \in \mathbb{C}^*$ and a branch point of index $N - 1$ at $t = 0$ and at $t = \infty$, so that

$$V = 4(N - 1).$$

The curve X' will have branch points for those values of $t \in \mathbb{C}^*$ which produce branch points for X, at $t = 0$ and at $t = \infty$. Whenever $\lambda_i = \lambda_j$, one has two types of branch points. First obtained by adjoining λ_i or λ_j to all the different combinations of $\tau - 1$ symbols (taken from the $N-2$ symbols $\lambda_1, \ldots, \hat{\lambda}_i, \ldots, \hat{\lambda}_j, \ldots, \lambda_N$) in the "box" corresponding to a_m over all m ($1 \leqslant m \leqslant l$)

$$\sum_{m=1}^{l} \frac{(N-2)!}{a_1! \cdots ! (a_m - 1)! \cdots ! a_l! (N - \tau - 1)!} = \frac{(N-\tau)}{(N-1)} n'.$$

Second, those obtained by adjoining λ_i *and* λ_j to all the different combinations of $\tau - 2$ symbols (taken from the $N-2$ symbols $\lambda_1, \ldots, \hat{\lambda}_i, \ldots, \hat{\lambda}_j, \ldots, \lambda_N$) in the "boxes" corresponding to a_m and $a_{m'}$ with $m \neq m'$:

$$\sum_{m \neq m'} \frac{(N-2)!}{a_1! \cdots ! (a_m - 1)! \cdots ! (a_{m'} - 1)! \cdots ! a_l! (N-\tau)!} = \frac{n'}{\tau(N-1)} \sum_{m \neq m'} a_m a_{m'}$$

$$= \frac{n'}{\tau(N-1)} \left(\frac{\tau^2}{2} - \sum_1^l \frac{a_m{}^2}{2} \right)$$

Assume that the roots $\lambda_1 \cdots \lambda_N$ of F (see (1)) are labeled such that winding once around $t = 0$ or $t = \infty$ amounts to going from sheet λ_i to sheet λ_{i+1}. This transformation $i \to i + 1$ extends to

$$\sum_{i=1}^{\tau} \alpha_i \lambda_{l_i} \to \sum_{i=1}^{\tau} \alpha_i \lambda_{l_i + 1} \qquad \text{with} \quad \lambda_{N+1} \equiv \lambda_1 .$$

Let r_k be the number of elements in each orbit under this transformation, where $1 \leqslant k \leqslant \#$ orbits. Then the ramification index at $t = 0$ and $t = \infty$ combined equals

$$2 \sum_{1 \leqslant k \leqslant \# \text{orbits}} (r_k - 1) = 2M' - 2 \# \text{ orbits}.$$

Therefore we conclude that, using (3), i.e., $n' = M'\tau/N$,

$$\begin{aligned} \frac{\tau V'}{2} &= 2(N-1) \frac{\tau}{2} \left[\frac{N-\tau}{N-1} n' + \left(\frac{\tau^2}{2} - \sum_1^l \frac{a_i{}^2}{2} \right) \frac{1}{\tau(N-1)} n' \right] \\ &\quad + \frac{1}{2} (2\tau M' - 2\tau \# \text{ orbits}) \\ &= \tau(N-\tau) n' + \left(\frac{\tau^2}{2} - \sum_1^l \frac{a_i{}^2}{2} \right) n' + n'N - \tau \# \text{ orbits}. \end{aligned}$$

The points of the graph curve are given by

$$\left\{ \left(\lambda_i(t), \sum_{k=1}^{\tau} \alpha_k \lambda_{l_k}(t) \right) \middle| \; \text{over all } i \text{ and all} \right.$$
$$\left. \text{permutations such that } \lambda_{l_k} = \lambda_i \text{ for some } k \right\}$$

over the t-plane. For those $t \in \mathbb{C}^*$ for which $\lambda_i(t) = \lambda_j(t)$, branch points occur in three different fashions.

(i) Every point of the type $(\lambda_i(t), \sum_1^\tau \alpha_n \lambda_{l_n}(t))$ equals some point of the type $(\lambda_j(t), \sum \alpha_n' \lambda_{l_n'}(t))$ and vice versa; so taking into account all the affine branch points of X this contributes

$$2n'(N-1)$$

to the ramification index.

(ii) Within the group of points $(\lambda_h(t), \sum_1^\tau \alpha_k \lambda_{l_k}(t))$, $h \neq i, j$, there are some points containing λ_i and not λ_j and some λ_j and not λ_i. Whenever $\lambda_i = \lambda_j$, some point in the first group equals a corresponding point in the second group and vice versa. Its number equals the number of different combinations $\sum_1^\tau \alpha_k \lambda_{l_k}(t)$ containing λ_h and λ_i and not λ_j for fixed h, i.e.,

$$\begin{aligned}
&\sum_{s,t} \frac{(N-3)!}{a_1! \cdots ! \, (a_s-1)! \cdots ! \, (a_t-1)! \cdots ! \, a_l! \, (N-\tau-1)!} \\
&= \frac{(N-\tau)}{(N-1)(N-2)\tau} n' \left(\sum_{s \neq t} a_s a_t + \sum_s a_s(a_s-1) \right) \\
&= \frac{N-\tau}{(N-1)(N-2)} n'(\tau^2-\tau) \\
&= \frac{(N-\tau)(\tau-1)}{(N-1)(N-2)} n'.
\end{aligned}$$

Let now $t \in \mathbb{C}^*$ run over all branch points of X and let $h \neq i, j$ run from 1 to N; then the total contribution amounts to

$$2(N-1)(N-2) \frac{(N-\tau)(\tau-1)}{(N-1)(N-2)} n' = 2(N-\tau)(\tau-1)n'.$$

(iii) Within the group of points $(\lambda_h(t) \sum \alpha_k \lambda_{l_k}(t))$, $h \neq i, j$, there are some points containing λ_i and λ_j in different "boxes." For a given such group, permuting λ_i and λ_j leads to a different combination; but, when $\lambda_i = \lambda_j$, they are the same. The number of such combinations is given by the different combinations $\sum \alpha_k \lambda_{l_k}(t)$ containing λ_h and λ_i, λ_j in different boxes. Its number equals

$$\sum_{\substack{t<u \\ \text{all } s}} \frac{(N-3)!}{a_1! \cdots ! \, (a_s-1)! \cdots ! \, (a_t-1)! \cdots ! \, (a_u-1)! \cdots ! \, a_l! \, (N-\tau)!}.$$

When h takes on all values $\neq i, j$ and $t \in \mathbb{C}^*$ runs over all the branch points of X, the total number of such points equals

$$2(N-1)(N-2)\frac{1}{(N-1)(N-2)\tau}n'\Bigg(\sum_{\substack{s\\ t<u\\ t,u\neq s}} a_s a_t a_u + \sum_{\substack{s\\ t<s}} a_t a_s(a_s-1) + \sum_{\substack{s\\ s<u}} a_s(a_s-1)a_u\Bigg)$$

$$= 2\frac{n'}{\tau}\Bigg[\sum_s a_s\Bigg(\sum_{\substack{t<u\\ t,u\neq s}} a_t a_u + \sum_{t<s} a_t a_s + \sum_{s<u} a_s a_u\Bigg) - \sum_s a_s \sum_{t<s} a_t - \sum a_s \sum_{u>s} a_u\Bigg]$$

$$= 2\frac{n'}{\tau}\Bigg[\tau\Bigg(\sum_{t<u} a_t a_u\Bigg) - \sum_s a_s\Bigg(\sum_t a_t - a_s\Bigg)\Bigg]$$

$$= 2\frac{n'}{\tau}\Bigg[\tau\Bigg(\frac{\tau^2}{2} - \sum\frac{a_s^2}{2}\Bigg) - \tau^2 + \sum a_s^2\Bigg]$$

$$= 2n'\Bigg(\frac{\tau^2}{2} - \sum\frac{a_s^2}{2}\Bigg) - 2\tau n' + 2\sum\frac{a_s^2}{\tau}n'.$$

Finally, both at $t = 0$ and $t = \infty$, there are n' branch points each of index $N - 1$, because winding around $t = 0$ or $t = \infty$ has the effect of raising each of the n' sheets associated to λ_i into each of the n' sheets associated to λ_{i+1} in X''; this contributes

$$2n'(N-1)$$

to the ramification index. Adding up these contributions,

$$\frac{V''}{2} = 2n'(N-1) + (N-\tau)(\tau-1)n' + n'\Bigg(\frac{\tau^2}{2} - \sum\frac{a_s^2}{2}\Bigg) - \frac{n'}{\tau}\Bigg(\tau^2 - \sum a_s^2\Bigg).$$

Putting these expressions for V, V' and V'' into the equality (5), we are led to the inequality

$$\#\,\text{orbits} < \frac{n'N}{\tau} + \frac{n'}{\tau^2}\Big(\tau^2 - \sum a_s^2\Big) = M' + \frac{n'}{\tau^2}\Big(\tau^2 - \sum a_s^2\Big)\,;$$

this inequality now is obvious, since the number of orbits is clearly smaller than M'. Note that whenever $\tau = a_1$ $(l = 1)$, the right-hand side equals M'. This finishes the proof of Lemma 1.

Remark. The reader will observe that in the above, there is actually another source of ramification, namely, the case, for instance, where for a fixed $t = t_0$, $\lambda_1 + 3\lambda_6 = \lambda_2 + 3\lambda_8$, $\lambda_1 \neq \lambda_2$, $\lambda_6 \neq \lambda_8$. This leads to ramification on X', X'', but not on X, and as we will show, such a situation leads to an increase $\delta V'$, $\delta V''$

in V', V'', such that $\delta V'' - \tau\delta V' \leqslant 0$, and so by (5), can be ignored in our computations. Suppose a sum consisting of b_i ($\leqslant a_i$) λ_i's from each of the "boxes" of a_i elements is equal to another such set, with entirely different λ_i's. Set $\sum b_i = \tau' \leqslant \tau$. This leads to ramification on X' by holding fixed in their "boxes," the first set of τ' λ_i's, and considering how many points can occur over t_0, where the rest of the λ_i's in the sum $\sum \alpha_i\lambda_{l_i(\pi)}(t_0)$ are drawn from the $N - 2\tau'$ λ_i's in neither of the two sets. Such points are coincident with the sum $\sum \alpha_i\lambda_{l_i(\pi)}$, where the role of the first two sets of τ' points are interchanged. This clearly leads to $\delta V' = [(N - 2\tau')/\prod_i (a_i - b_i)!\,(N - \tau + \tau')!]$. On the X'' curve, the above situation leads to ramification at the points $(\lambda_i(t_0), \sum \alpha_n\lambda_{l_n}(t_0))$, where $\lambda_i(t_0)$ is not one of the $2\tau'$ points in the two special sets of λ_i, and the $\sum \alpha_n\lambda_{l_n}(t_0)$ contains λ_i and one of the special sets, but no element in the other, and vice versa. This situation occurs precisely

$$\begin{aligned}\delta V'' &= (N - 2\tau') \times \left(\sum_i (N - 2\tau' - 1)!\Big/\prod_{j\neq i} (a_j - b_j)!\,(a_i - b_i - 1)!\,(N - \tau + \tau')!\right)\\ &= \delta V'(\tau - \tau') \qquad \text{number of times;}\end{aligned}$$

hence $\delta V'' - \tau\delta V' = -\tau'\delta V' \leqslant 0$, as promised. That the above situation does not occur twice at the same time can be assumed by taking a generic X.

Proof of Lemma 2. The proof of this lemma is reminiscent of a proof of Lefschetz [8] for general curves, which cannot be applied to this case. Consider a real Toda curve X with branch points $\lambda_1 < \lambda_2 < \cdots < \lambda_{2g+1}$ with homology cycles

$$\begin{aligned} c_j &= a_j, & 1 \leqslant j \leqslant g,\\ &= b_{j-g}, & g+1 \leqslant j \leqslant 2g,\end{aligned}$$

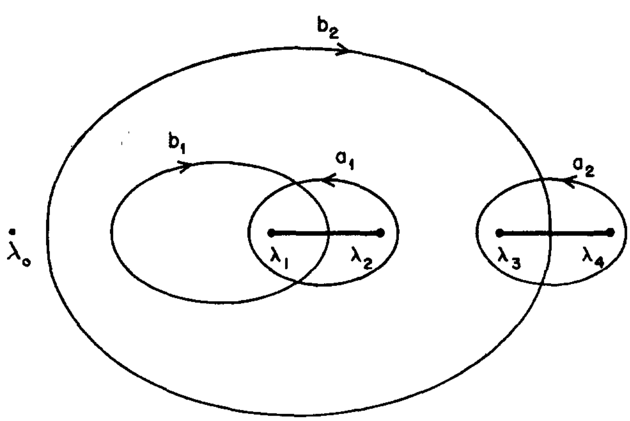

FIGURE 1

as in Fig. 1. Define $c_j(\omega) \equiv \int_{cj} \omega$. The Toda character is expressed by the fact that (see Moerbeke [14])

$$\sum_{i=1}^{g} m_i a_i(\omega_k) = N \int_p^Q \omega_k\,, \qquad 1 \leqslant k \leqslant g,\ 1 \leqslant m_1 < m_2 < \cdots < m_g < N,$$

or, equivalently,

$$\sum_{i=1}^{g} m_i \int_{\lambda_{2i-1}}^{\lambda_{2i}} \omega_k = N \int_{\lambda_{2g+1}}^{q} \omega_k\,, \qquad 1 \leqslant k \leqslant g,$$

$$q \text{ real} \leqslant \infty, \text{ different from branch points.} \tag{6}$$

As explained in McKean and Moerbeke [10], the Marcenko–Ostrovskii transformation maps the λ-plane of Fig. 1 cut along $[\lambda_1\,, \infty)$ to a slit domain with vertical spikes at the points πm_i $(1 \leqslant i \leqslant g)$. Conversely, any slit domain with vertical spikes at g positive multiples of π gets mapped into the λ-plane depicted in Fig. 1. The height of the ith spike controls the width of the corresponding band $[\lambda_{2i-1}\,, \lambda_{2i}]$ and since the height of the spikes can be chosen arbitrarily, the width of the bands can be varied independently of each other. So, in particular, any band can be squeezed to a point, while maintaining the other bands open, possibly at the expense of moving all the λ_i's. This degree of freedom in the Toda variety is the main ingredient of the proof, reminiscent of an argument by Lefschetz [8].

Next we show that the Toda character of X is maintained after squeezing one band; namely, it reduces to the Toda relation for the smooth Riemann surface X_0 of one lower genus. Indeed, let ω be an arbitrary differential vanishing at $z_j \in [\lambda_{2j-1}\,, \lambda_{2j}]$; then, when $\lambda_{2j} - \lambda_{2j-1} \to 0$,

$$\omega = \frac{P(z)(z-z_j)}{(R(z))^{1/2}}\, dz \to \omega^0 = \frac{P(z)\, dz}{(R_0(z))^{1/2}}, \qquad \text{degree } P(z) \leqslant g-1,$$

where $y^2 = R(z) \equiv (z-\lambda_0) \cdots (z-\lambda_{2g+1})$ and

$$y_0{}^2 = R_0(z) \equiv (z-\lambda_0{}^0) \cdots \widehat{(z-\lambda_{2j-1}^0)}\widehat{(z-\lambda_{2j}^0)} \cdots (z-\lambda_{2g+1}^0).$$

So, ω tends to a generic holomorphic differential ω^0 on the curve X_0 defined by $y_0{}^2 = R_0(z)$. Moreover

$$\int_{\lambda_{2i-1}}^{\lambda_{2i}} \omega \to \int_{\lambda_{2i-1}^0}^{\lambda_{2i}^0} \omega^0, \qquad i \neq j, \tag{7}$$

and[20]

[20] Using the integrals

$$\int_a^b \frac{dz}{((z-a)(b-z))^{1/2}} = \pi \qquad \text{and} \qquad \int_a^b \frac{(z-c)dz}{((z-a)(b-z))^{1/2}} = \left(\frac{a+b}{2} - c\right)\pi, \quad a < c < b.$$

$$\int_{\lambda_{2j-1}}^{\lambda_{2j}} \omega = \int_{\lambda_{2j-1}}^{\lambda_{2j}} \frac{(z - z_j)}{((z-\lambda_{2j-1})(z-\lambda_{2j}))^{1/2}} \frac{P(z)\,dz}{(\prod_{i\neq 2j,2j-1}(z-\lambda_i))^{1/2}}$$
$$= C\left(\frac{\lambda_{2j-1}+\lambda_{2j}}{2} - z_j\right)\pi \to 0. \tag{8}$$

Therefore (6) reduces to

$$\sum_{\substack{i=1\\ i\neq j}}^{g} m_i \int_{\lambda_{2i-1}}^{\lambda_{2i}} \omega_k^0 = N \int_{\lambda_{2g+1}}^{a} \omega_k^0,$$

which expresses the fact that X_0 is also Toda.

For future use, we also evaluate $b_j(\omega)$ for ω vanishing or nonvanishing at z_j where $\lambda_{2j} - \lambda_{2j-1} \to 0$. When $\omega(z_j) \neq 0$, one gets after a deformation of the integration path.

$$b_j(\omega) \simeq C + C' \int_{\lambda_{2j-1}-a}^{\lambda_{2j-1}} \frac{dz}{((\lambda_{2j-1}-z)(\lambda_{2j}-z))^{1/2}}$$
$$= \ln(2((\lambda_{2j-1}-z)(\lambda_{2j}-z))^{1/2} + 2z - \lambda_{2j-1} - \lambda_{2j})\Big|_{\lambda_{2j-1}-a}^{\lambda_{2j-1}}$$
$$= \ln \frac{\lambda_{2j-1}-\lambda_{2j}}{2(a(\lambda_{2j}-\lambda_{2j-1}+a))^{1/2} + \lambda_{2j-1} - \lambda_{2j} - 2a} \tag{9}$$
$$= -O(\ln(\lambda_{2j}-\lambda_{2j-1})) \uparrow \infty \qquad \text{logarithmically,}$$

where C and C' are functions of the other branch points and $a > 0$ is bounded away from zero. Moreover when $\omega(z_j) = 0$,

$$|\, b_j(\omega)| \simeq C + C' \int_{\lambda_{2j-1}-a}^{\lambda_{2j}} \frac{(z-z_j)\,dz}{((\lambda_{2j-1}-z)(\lambda_{2j}-z))^{1/2}}$$
$$\simeq C + C'\left(\frac{\lambda_{2j}+\lambda_{2j-1}}{2} - z_j\right) O(\ln(\lambda_{2j}-\lambda_{2j-1})) < \infty. \tag{10}$$

Assume now that every Toda curve is reducible; then, the period matrix $\Omega = (c_j(\omega_i))$, $1 \leqslant i \leqslant g$, $1 \leqslant j \leqslant 2g$, splits up in two blocks (for a proper choice of cycles and basis ω_i of holomorphic differentials), consisting of Ω_1, a matrix of order $(k, 2k)$, and Ω_2, of order $(g-k, 2(g-k))$

$$\Omega = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix}.$$

Since Ω_1 and Ω_2 correspond to abelian subvarieties, we have that for some antisymmetric integer matrices J_1 and J_2 of order $2k$ and $2(g-k)$, respectively,

$$\Omega_1 J_1 \Omega_1{}^T = 0 \qquad \text{and} \qquad \Omega_2 J_2 \Omega_2{}^T = 0$$

as a consequence of the Riemann bilinear relations. Therefore both relations,

$$\Omega \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \Omega^T = 0 \qquad \text{and} \qquad \Omega \begin{pmatrix} J_1 & 0 \\ 0 & -J_2 \end{pmatrix} \Omega^T = 0$$

are valid, i.e., Ω has a complex multiplication besides the usual Riemann bilinear relation. Now since the Toda curves of fixed genus corresponding to the same relatively prime set of integers $0 < m_1 < m_2 < \cdots < m_g < N$ form an analytic variety [11], and since there are only a denumerable number of complex multiplications, they all must share the same additional complex multiplication; let

$$\Omega\, d\Omega^T = 0$$

for an integral matrix d of size $2g$. This implies the quadratic relation

$$\sum_{i,j} c_i(\omega_k)\, d_{ij} c_j(\omega_l) = 0, \qquad 1 \leqslant l, k \leqslant g.$$

Hence, for any pair of holomorphic differentials ω and ω'

$$\sum_{i,j} c_i(\omega)\, d_{ij} c_j(\omega') = 0. \tag{11}$$

This relation now holds for all Toda curves of a given genus with fixed integers. We now show by induction on the genus that the matrix d reduces to the symplectic matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

More precisely, if this property is true for genus $= g-1$, then it holds for genus g. The quadratic relation (11) can be split up in its real and imaginary parts, which because of the reality of the branch points leads to the following two relations (A) and (B):

$$\text{(A)} \quad \sum_{i=1}^{g} b_i(\omega_1) \sum_{j=1}^{g} n_{ij} b_j(\omega_2) + \sum_{i=1}^{g} a_i(\omega_1) \sum_{j=1}^{g} m_{ij} a_j(\omega_2) = 0.$$

By successive choices of ω_1 and ω_2, the integers n_{ij} and m_{ij} will be shown to vanish.

(i) Squeeze the cycle a_g to the point z_g ($\lambda_{2g-1} < z_g < \lambda_{2g}$) and let $\omega_2(z_g)$, $\omega_1(z_g) \neq 0$. Then by (9), $b_g(\omega_1)\, b_g(\omega_2)$ blows up faster than $b_j(\omega_1)\, b_g(\omega_2)$ and $b_g(\omega_1)\, b_j(\omega_2)$ ($1 < j < g-1$) while all the remaining terms in (A) remain finite by (7) and (8). This implies $n_{gg} = 0$.

(ii) Squeeze both cycles a_i ($1 \leqslant i \leqslant g$) and a_g to the points z_i ($\lambda_{2i-1} < z_i < \lambda_{2i}$) and z_g ($\lambda_{2g-1} < z_g < \lambda_{2g}$), respectively; let

$$\omega_1(z_i) \neq 0, \quad \omega_1(z_g) = 0 \qquad \text{and} \qquad \omega_2(z_i) = 0, \quad \omega_2(z_g) \neq 0.$$

Then by (7), (8), (9), and (10), only $b_i(\omega_1)$ and $b_g(\omega_2)$ blow up; so the leading term in (A) is $n_{i,g} b_i(\omega_1)\, b_g(\omega_2)$ with the result that $n_{i,g} = 0$, $1 \leqslant i \leqslant g-1$.

(iii) The roles of ω_1 and ω_2 can be reversed, implying that also $n_{g,i} = 0$, $1 \leqslant i \leqslant g$.

(iv) Consider now two arbitrary differentials ω_1 and ω_2 both vanishing at z_g ($\lambda_{2g-1} < z_g < \lambda_{2g}$) and squeeze a_g. Then, according to (7) and (8), $a_j(\omega_i)$ and $b_j(\omega_i)$ ($1 \leqslant j \leqslant g-1$, $1 \leqslant i \leqslant 2$) lead to generic abelian integrals for a Toda curve of genus $g-1$, as pointed out before, and

$$a_g(\omega_1) \to 0 \qquad \text{and} \qquad a_g(\omega_2) \to 0.$$

So, relation (A), taking into acount (i)–(iii), goes over into a similar relation for a generic Toda curve (defined by $m_1 < m_2 < \cdots < m_{g-1} < M$) of genus $g-1$. By assumption, this generic curve has no complex multiplication, so that

$$n_{ij} = m_{ij} = 0, \qquad 1 \leqslant i, j \leqslant g-1.$$

(v) All that is left in (A) is the sum

$$\sum_{j=1}^{g-1} m_{gj} a_g(\omega_1)\, a_j(\omega_2) + \sum_{j=1}^{g} m_{jg} a_j(\omega_1)\, a_g(\omega_2) = 0. \tag{12}$$

Choose ω_1 and ω_2 such that

$$\omega_1(z_1) = \cdots = \omega_1(z_{i-1}) = \omega_1(z_{i+1}) = \cdots = \omega_1(z_g) = 0,$$

hence

$$\omega_1(z_i) \neq 0 \qquad \text{and} \qquad \omega_2(z_g) \neq 0;$$

then by (7) and (8), upon squeezing all a_i ($1 \leqslant i \leqslant g$), $a_j(\omega_1) \to 0$ ($1 \leqslant j \leqslant g$, $j \neq i$), $a_i(\omega_1) \nrightarrow 0$, $a_j(\omega_2)$ finite, and $a_g(\omega_2) \nrightarrow 0$; therefore in the limit the only surviving term in (12) is $\lim m_{ig} a_i(\omega_1)\, a_g(\omega_2) = 0$ with $\lim a_i(\omega_1) a_g(\omega_2) \neq 0$; hence $m_{ig} = 0$ ($1 \leqslant i \leqslant g$) and by interchanging the roles of ω_1 and ω_2, also $m_{gi} = 0$, $1 \leqslant i \leqslant g$; hence (A) is seen to be zero.

$$\text{(B)} \ \sum_{i=1}^{g} b_i(\omega_1) \sum_{i=1}^{g} n_{ij} a_j(\omega_2) + \sum_{i=1}^{g} a_i(\omega_1) \sum_{i=1}^{g} m_{ij} b_j(\omega_2) = 0.$$

(i) Squeeze all a_i $(1 \leqslant i \leqslant g)$ and choose ω_1 and ω_2 such that

$$\omega_1(z_1) = \omega_1(z_2) = \cdots = \omega_1(z_{g-1}) = 0, \qquad \lambda_{2i-1} \leqslant z_i \leqslant \lambda_{2i},$$
$$\omega_2(z_1) = \omega_2(z_2) = \cdots = \omega_2(z_{j-1}) = \omega_2(z_{j+1}) = \cdots = \omega_2(z_g), \quad 1 \leqslant j \leqslant g-1.$$

Then, by (7), (8), (9), and (10), and the fact that $\omega_1(z_g) \neq 0$ and $\omega_2(z_j) \neq 0$,

$$| b_i(\omega_1)| < \infty, \ 1 \leqslant i \leqslant g-1, \qquad | b_g(\omega_1)| \nearrow \infty,$$
$$| b_i(\omega_2)| < \infty, \ 1 \leqslant i \leqslant g, \ i \neq j, \qquad | b_j(\omega_2)| \nearrow \infty,$$

and

$$a_i(\omega_1) \to 0, \ 1 \leqslant i \leqslant g-1, \qquad a_g(\omega_1) \not\to 0,$$
$$a_i(\omega_2) \to 0, \ 1 \leqslant i \leqslant g, \ i \neq j, \qquad a_j(\omega_2) \not\to 0.$$

Then both $a_i(\omega_1)\, b_j(\omega_2)$, $1 \leqslant i \leqslant g-1$, $i \neq j$, and $a_j(\omega_1)\, b_j(\omega_2) \to 0$ by letting $\lambda_{2i} - \lambda_{2i-1} \to 0$ independently of $1 \leqslant i \leqslant g-1$, using estimates (7), (8), (9), and (10). Consequently, the leading term of (B), namely,

$$n_{gj} b_g(\omega_1)\, a_j(\omega_2) + m_{gj} a_g(\omega_1)\, b_j(\omega_2),$$

must vanish. Since both $\lambda_{2g} - \lambda_{2g-1}$ and $\lambda_{2j} - \lambda_{2j-1}$ can be made to vanish independently, we conclude that $n_{gj} = m_{gj} = 0$ $(1 \leqslant j \leqslant g-1)$.

(ii) By interchanging the roles of ω_1 and ω_2, also $n_{jg} = m_{jg} = 0$, $1 \leqslant j \leqslant g-1$.

(iii) Squeeze a_g and put $\omega_1 = \omega_2$, with $\omega_1(z_g) \neq 0$; then using (i) and (ii), the leading term reads

$$n_{gg} b_g(\omega_1)\, a_g(\omega_1) + m_{gg} a_g(\omega_1)\, b_g(\omega_1) = (n_{gg} + m_{gg})\, a_g(\omega_1)\, b_g(\omega_1)$$

and it vanishes only if $n_{gg} = -m_{gg}$.

(iv) Consider now two arbitrary differentials ω_1 and ω_2 both vanishing at z_g. Then if a_g gets squeezed, $a_j(\omega_i)$ and $b_j(\omega_i)$ $(1 \leqslant j \leqslant g-1,\ i = 1, 2)$ lead to generic abelian integrals for a Toda curve of genus $g-1$, while $a_g(\omega_i) \to 0$ $(i = 1, 2)$ by (8) and $b_g(\omega_i) < \infty$ $(i = 1, 2)$ by (10). Therefore in that limit, relation (B) tends to a complex multiplication for a generic Toda curve of lower genus, defined by $m_1 < m_2 < \cdots < m_{g-1} < N$. Since, by assumption, such a generic Toda curve has no complex multiplication, the limiting relation must reduce to Riemann's bilinear relation. Putting the corresponding values of n_{ij} and m_{ij} in relation (B) and taking into account (i)–(iii), we find for $c = n_{gg}$,

$$\sum_{i=1}^{g-1} (b_i(\omega_1)\, a_i(\omega_2) - b_i(\omega_2)\, a_i(\omega_1))$$
$$+ c(b_g(\omega_1)\, a_g(\omega_2) - b_g(\omega_2)\, (a_g(\omega_1)) = 0.$$

The special role played by the band going with a_g could as well be played by any other band, say a_{g-1}, leading to a similar relation

$$\sum_{\substack{i=1\\ i\neq g-1}}^{g} (b_i(\omega_1)\, a_i(\omega_2) - b_i(\omega_2)\, a_i(\omega_1)) \\ + c'(b_{g-1}(\omega_1)\, a_{g-1}(\omega_2) - b_{g-1}(\omega_2)\, a_{g-1}(\omega_1)) = 0,$$

implying that $c' = c = 1$.

Remark. For the case of the Toda systems in the other classical groups and G_2, one constructs a correspondence in a similar fashion between the curves (I, 3.15, 3.16) occurring in these cases. The correspondence is given by

$$(t, \lambda_m(t)) - \left\{ t, \sum_1^{\tau} \alpha_i \lambda_{l_i(\pi)}(t) \;\middle|\; \text{over all permutations } \pi \text{ such that } \lambda_{l_i(\pi)} = \pm\lambda_m(t) \text{ for some } i \right\},$$

and one observes that the correspondence commutes with the involutions σ on both curves given by $(h, z) \to (h, -z)$. Now up to isogeny, the involutions τ on X, X' lead to the decompositions

$$\operatorname{Jac} X = x + y, \qquad \operatorname{Jac} X' = x' + y',$$

where y and y' are the Prym varieties associated with σ, etc. The correspondence C induces the complex multiplication (see the Appendix), where, for the sake of notation, we have identified Jac X and the Riemann matrix of X:

$$\operatorname{Jac} X' \cdot N = A \cdot \operatorname{Jac} X,$$

i.e.,

$$(x' \oplus y') \cdot N = A \cdot (x \oplus y). \tag{13}$$

Upon observing that, at the level of Jac X, σ induces the map $(\tilde{x}, \tilde{y}) \to (\tilde{x}, -\tilde{y})$, etc., and also that σ commutes with C, we find that acting upon (13) with σ implies[21]

$$(x' \oplus (-y')) \cdot N = A(x \oplus (-y)). \tag{14}$$

Thus upon adding and subtracting (13) and (14), we find

$$x' \cdot N = A \cdot x, \qquad y' \cdot N = A \cdot y. \tag{15}$$

One proves in the same fashion as in Lemma 1 that C is singular, and hence the complex multiplications (15) are nontrivial; thus, x', x and y', y, respectively,

[21] This involves picking a basis of holomorphic differentials of X, X', consisting of even and odd differentials with regard to σ.

have common abelian subvarieties, by Weil [22]. Thus the extra symmetry in these cases is reflected in the double factorization statement consistent with the symmetries of the Jacobians. We have that the linearizing Prym subvariety of Jac(X) is contained in all the Pryms coming from all the different representations. Thus Theorem 1 has this strengthened form for these cases. Similar remarks can be made for the other examples in I, Section 4.

Appendix

The purpose of this appendix is to provide definitions and notations to Section 4 and to provide a clear but limited exposition of the classical theory of correspondences leading up to Theorem 4.1; the theory of correspondences was developed mainly by the Italian geometers. These results can be found in Coolidge's book [5] in an obscure way or can be proved using Chern classes of line bundles and modern intersection theory. Therefore we give an exposition in the spirit of this paper.

A *correspondence* $C[n, n']$ between two curves X and X' of genera g and g', respectively, is defined by one or several algebraic relations between the coordinates of X and X' mapping a point $p \in X$ onto n' points $C(p) = q_1 + \cdots + q_{n'} \in X'$ and a point $q \in X'$ onto n points $C^{-1}(q) = p_1 + \cdots + p_n \in X$.

Remark. Let $f(x, y) = 0$ and $f'(x', y') = 0$ define the curves X and X', respectively. Thus the correspondence $C[n, n']$ will be given by

$$\phi_1(x, y, x', y') = \phi_2(x, y, x', y') = \cdots = \phi_m(x, y, x', y') = 0.$$

In fact we show that these equations $\phi_i = 0$ ($1 \leqslant i \leqslant m$) can be replaced by at most two new algebraic equations: Given a point $(x', y') \in X'$, the intersection of the curves $f = 0$, $\phi_1 = 0, \ldots$, and $\phi_m = 0$ is given by n points (x_i, y_i) ($1 \leqslant i \leqslant n$), whose abscissae can be assumed generically distinct after possibly a birational transformation of coordinates. Let $y_j(x)$ be the solutions of the polynomial $f(x, y) = 0$ in y. Form the *polynomial* in x, x', y'

$$\theta_i(x, x', y') = \prod_j \phi_i(x, y_j(x), x', y')$$

and let $\theta(x, x', y')$ be the polynomial[22] in x, x', y' which is the g.c.d. of the θ_i's. The solutions in x of this polynomial are precisely the points x_i, by the g.c.d. construction. Let $A(x)$ be the unique polynomial of degree n in x, also depending on x' and y' such that $y_i = A(x_i)$ ($1 \leqslant i \leqslant n$); then the n common solution of

$$\theta(x, x', y') = 0; \qquad y - A(x) = 0, \qquad A(x) \text{ depending on } x' \text{ and } y',$$

[22] By the Euclidean algorithm for forming the g.c.d. this still is a polynomial.

provides us precisely with the n points x_i, y_i. Moreover $A(x)$ depends *rationally* on x' and y', because $A(x)$ is a symmetric function of the x_i, which themselves are roots of the polynomial $\theta(x, x', y')$ in x, x' and y'; this uses the customary interpolation formula in the construction of $A(x)$. Possibly undoing the preparatory birational transformation we have shown that a correspondence is defined by at most two equations ϕ_i ($1 \leqslant i \leqslant 2$). A dimension count shows that if two equations are fully needed, this would be rare indeed and, in fact, implies that the two curves X and X' are intimately related.

A *singular* correspondence is one where *two equations* are needed, i.e., one equation does not suffice.

We now show, following Hurwicz, how a correspondence implies a relation between the period matrices, as explained next. Let ω_i ($1 \leqslant i \leqslant g$) and ω_i' ($1 \leqslant i \leqslant g'$) be the respective bases for holomorphic differentials on X and X'. Then

$$\sum_{j=1}^{n'} \omega_i'(q_j(p))$$

is a holomorphic differential on X, since upon integration it does not blow up and hence we have equality of the differentials

$$\sum_{j=1}^{n'} \omega_i'(q_j(p)) = \sum_{k=1}^{g} a_{ik}\omega_k(p), \qquad 1 \leqslant i \leqslant g', \tag{1}$$

for some appropriate choice of a_{ik}. Let c_i and c_i' be a basis of homology cycles on X and X', respectively. Then whenever p sweeps out a cycle c_l, then $C(p) = \sum q_j(p)$ sweeps out a cycle on X' of the form $\sum_{m=1}^{2g'} n_{ml}c_m'$ with $n_{ml} \in \mathbb{Z}$. Hence upon integrating (1), we find our statement of complex multiplication, namely,

$$\sum_{m=1}^{2g'} n_{ml}c_m'(\omega_i') = \sum_{k=1}^{g} a_{ik}c_l(\omega_k), \qquad 1 \leqslant l \leqslant 2g,\ 1 \leqslant i \leqslant g', \tag{2}$$

or, equivalently,

$$\Omega' N = A\Omega \qquad \text{with} \quad (\Omega')_{ij} = c_j'(\omega_i') \quad \text{and} \quad \Omega_{ij} = c_j(\omega_i).$$

In view of Riemann's bilinear relation $\Omega J \Omega^T = 0$, the latter implies

$$\Omega' C \Omega^T = 0, \qquad \text{where } C = NJ \text{ is an integral matrix.}$$

So a self-correspondence of X (i.e., a correspondence of X to itself) implies a quadratic integer relation between the elements of the period matrix

$$\Omega C \Omega^T = 0 \qquad \text{with } C \text{ a } 2g \times 2g \text{ integral matrix.}$$

Whenever this relation is different from Riemann's bilinear relation, it is called *complex multiplication.*

Given a correspondence C of X onto X' define the *numbers of coincidences:*[23]

$$\eta = \#\{p \in X \mid q_i = q_j \text{ for some } i \neq j\}$$

and

$$\eta' = \#\{q \in X' \mid p_i = p_j \text{ for some } i \neq j\}.$$

A self-correspondence of X is said to have *value* γ if for any two points $p, q \in X$

$$C(p) + \gamma p \equiv C(q) + \gamma q \qquad \text{in } \operatorname{Jac}(X).$$

Let ξ be the number of fixed points (i.e., such that[23] $\xi \in C(\xi)$). The Chasles–Cayley–Brill formula (Coolidge [5, p. 129, Theorem 14]) affirms that for any self-correspondence $C[n, n']$ of X,

$$\xi = n + n' + 2g\gamma.$$

Let $f_0, \ldots, f_r$ be $r + 1$ independent meromorphic functions. Then the r-dimensional family of zero divisors of the functions

$$f = \lambda_0 f_0 + \lambda_1 f_1 + \cdots + \lambda_r f_r$$

with $\lambda_k \in \mathbb{C}$ defines a linear system $g_N{}^r$, where N is the order of the minimal divisor $\mathscr{D}$ of poles of the f_i's. Then given r generic points $p_1, \ldots, p_r$ on X and since for this generic choice $\det(f_i(p_j)) \neq 0$, $1 \leqslant i, j \leqslant r$, unique constants $\lambda_1, \ldots, \lambda_r$, up to a factor, can be found such that for $1 \leqslant j \leqslant r$, and for a fired λ_0,

$$\lambda_0 f_0(p_j) + \sum_{i=1}^{r} \lambda_i f_i(p_j) = 0,$$

defining $N - r$ other zeros. In particular a linear system $g_N{}^1$ associated with $\lambda_0 f_0 + \lambda_1 f_1$ defines a self-correspondence $C[N-1, N-1]$ of X ($f(x, y) = 0$) by the $N - 1$ nontrivial solutions of

$$F_1 \equiv \frac{f_1(x, y) f_0(x, y') - f_0(x, y) f_1(x', y')}{x' - x} = 0.$$

Clearly this self-correspondence has value $\gamma = 1$ because for any choice of p and q,

$$(\lambda_0 f_0 + \lambda_1 f_1) = p + C(p) - \mathscr{D} \qquad \text{for} \quad \lambda_0/\lambda_1 = -f_1(p)/f_0(p)$$

[23] Counted with multiplicities.

and

$$(\lambda_0' f_0 + \lambda_1' f_1) = q + C(q) - \mathscr{D} \qquad \text{for} \quad \lambda_0'/\lambda_1' = -f_1(q)/f_0(q).$$

Therefore

$$\left(\frac{\lambda_0 f_0 + \lambda_1 f_1}{\lambda_0' f_0 + \lambda_1' f_1}\right) = p + C(p) - q - C(q) \equiv 0 \qquad \text{in Jac } X,$$

which implies that $\gamma = 1$. From the Chasles–Cayley–Brill formula applied to $C[N-1, N-1]$, it follows that

$$\begin{aligned} d &\equiv 2 \,\#\{\text{divisors in } g_N{}^1 \text{ with double points}\} \\ &= (N-1) + (N-1) + 2g = 2(N-1+g). \end{aligned}$$

An *algebraic system* $\gamma_N{}^1$ of index ν on a curve X is defined by a curve X' and a correspondence $C[N, \nu]$ between X and X'; so, the algebraic system $\gamma_N{}^1$ is given by the family of common zero divisors of order N of

$$\phi_1(x, y, x', y') = \phi_2(x, y, x', y') = 0 \qquad \text{in } (x, y) \in X$$

parametrized by $(x', y') \in X'$. Moreover any point $(x, y) \in X$ corresponds to the ν zero-divisor corresponding to the ν roots (x', y') of $\phi_1 = \phi_2 = 0$ on X'.

Let d have the same meaning as above but for $g_N{}^1$ replaced by $\gamma_N{}^1$; unfortunately this number cannot be computed in the above fashion; one would be tempted to replace the above formula for d by $2\nu(N-1+g)$; however, this only produces an upper bound (Lemma 4). The above recipe for providing the correspondence does not generalize to $\gamma_N{}^1$.

LEMMA 1. *A deformable algebraic system $\gamma_N{}^1$ of index 1 on X is a $g_N{}^1$.*

Proof. Consider the self-correspondence of X defined by

$$C(p) = \text{the divisor of } N \text{ points of } \gamma_N{}^1 \text{ containing } p.$$

Consider now a one-parameter (β) family of $\gamma_N{}^1$'s and the corresponding C^β; let $C^0 = C$. Then the numbers a_{ik} in (1) do not depend on the parameter in view of (2). Indeed let $\omega_1, \ldots, \omega_g$ be a normalized basis such that $c_l(\omega_k) = \delta_{lk}$ $(1 \leqslant l, k \leqslant g)$. Then from (2) with $g = g'$

$$a_{ik} = n_{ik} + \sum_{m=g+1}^{2g} n_{mk} c_m(\omega_i), \qquad 1 \leqslant i, k \leqslant g,$$

and any continuous variation of the parameter β above leaves the integers n_{ij} and the periods $c_m(\omega_i)$ fixed. Hence a_{ik} is constant with β.

Moreover for any $p' \in C^\beta(p)$, $p' \neq p$, we have that $C^\beta(p') = C^\beta(p)$; so, upon integration of (1) from p_0 to p, we have

$$\begin{aligned}\sum_{k=1}^{g} a_{ik} \int_{p_0}^{p} \omega_k &= \int_{Np_0}^{C^\beta(p)} \omega_i + d_i^\beta = \int_{Np_0}^{C^\beta(p')} \omega_i + d_i^\beta \\ &= \sum_{k=1}^{g} a_{ik} \int_{p_0}^{p'} \omega_k \qquad \text{in } \mathrm{Jac}(X),\ p' \in C^\beta(p),\end{aligned} \tag{3}$$

where d_i^β is an integration constant independent of p. Now we pick a pair of points p_1, p_2, which shall be fixed throughout the following argument. We can find a deformation C^β of $C^0 = C$ such that for $\beta = \beta_0$, $C^{\beta_0}(p_1) = C^{\beta_0}(p_2)$, and hence in (3), setting $p = p_1$, $p' = p_2$, $\beta = \beta_0$, we have

$$\begin{aligned}\int_{Np_0}^{C^{\beta_0}(p_1)} \omega_i + d_i^{\beta_0} &= \sum a_{ik} \int_{p_0}^{p_1} \omega_k \\ &= \sum a_{ik} \int_{p_0}^{p_2} \omega_k = \int_{Np_0}^{C^{\beta_0}(p_2)} \omega_i + d_i^{\beta_0} \qquad \text{in } \mathrm{Jac}(X).\end{aligned} \tag{4}$$

Since a variation in β leaves the a_{ik} unaltered, the two middle expressions are in fact also unaltered, and so the extreme expressions in (4) are also independent of β, and equal; therefore putting $\beta = 0$ we conclude

$$\int_{Np_0}^{C(p_1)} \omega_i = \int_{Np_0}^{C(p_2)} \omega_i \qquad \text{on } \mathrm{Jac}(X),$$

and therefore

$$C(p_1) \equiv C(p_2) \qquad \text{in Jac } X, \text{ for all } p_1, p_2 \in X.$$

Hence by Abel's theorem, we have a rational function on X, algebraic in λ

$$g(p, \lambda) \qquad \text{such that } (g) = \mathscr{D}_\lambda - \mathscr{D},\ \mathscr{D}_\lambda, \mathscr{D} \in \gamma_N^1,$$

λ being the parameter in γ_N^1. Since the index of γ_N^1 is one, g can be taken rational in λ, also; given $p \in X$, there is precisely one solution to $g(p, \lambda) = 0$, and so g is linear in λ, thus γ_N^1 is a g_N^1.

Corollary 1. *Given a deformable γ_N^1 of index ν, the divisors $(C^{-1} \circ C)(p)$ (i.e., sum of the groups containing p) parametrized by $p \in X$ form a $\gamma_{N\nu}^1$ of index* 1 *and hence by Lemma* 1, *a linear system $g_{N\nu}^1$.*

LEMMA 2 (Zeuthen's formula). *The following relation holds*:

$$\eta + 2n'(g - 1) = \eta' + 2n(g' - 1).$$

Proof. First, assume $n = 1$ and let X be an N-fold covering of C. Then according to the Riemann–Hurwicz formula for the ramification indices V_X and $V_{X'}$ of X and X',

$$V_X = 2(N + g - 1) \qquad \text{and} \qquad V_{X'} = 2(Nn' + g' - 1).$$

But observing that among the branch points of X', some of them come from X by the correspondence and the others come from the coincidences, we have

$$V_{X'} = n' V_X + \eta.$$

Comparing these three formulas yields

$$\eta = 2(g' - 1) - 2n'(g - 1). \tag{5}$$

Second, for arbitrary n, there is a natural correspondence $C''[1, n']$ from X to the graph curve Γ (associated with the correspondence $C[n, n']$) with genus g''; it maps $p \in X$ onto n' points $(p, q_1),..., (p, q_{n'}) \in \Gamma$. Similarly, there is a correspondence $C'''[1, n]$ from X' to Γ. Applying formula (5) to these two new correspondences and eliminating g'' from them leads to the desired result.

LEMMA 3 (Severi's formula). *For a given curve X of genus g, and $r \geqslant 1$, a linear system $g_N{}^r$ and an algebraic system $\gamma_M{}^1$ of index ν, having d coincidences*

$$\begin{aligned}&\#\{p = p_1 + \cdots + p_{r+1} \text{ such that } p + \cdots \in g_N{}^r \text{ and } p + \cdots \in \gamma_M{}^1\}\\ &\quad = 2N\binom{M-1}{r}\nu - \frac{1}{2}d\binom{M-2}{r-1}.\end{aligned}$$

Proof. Let $\phi(N, r, M)$ be this number for $r \geqslant 0$. Consider the correspondence of X onto itself mapping a point p onto a divisor $C(p)$ to be constructed as follows:

(i) To $p \in X$, associate the ν sets of points $q_2{}^j + \cdots + q_M{}^j$ $(1 \leqslant j \leqslant \nu)$ such that

$$p + q_2{}^j + \cdots + q_M{}^j = p + Q^j(p) \in \gamma_M{}^1.$$

(ii) Take any one of the $\binom{M-1}{r}$ distinct divisors $Q_I{}^j = q^j_{i_1} + \cdots + q^j_{i_r}$ for $I = (i_1, ..., i_r)$ among the set $Q^j(p)$ not containing p; then to each such divisor there corresponds a unique one $P_I{}^j \equiv p'_1 + \cdots + p'_{N-r}$ such that

$$Q_I{}^j + P_I{}^j \in g_N{}^r.$$

This defines a correspondence[24]

$$C(p) = \sum_{\substack{I \subseteq [2,M] \\ 1 \leqslant j \leqslant \nu}} P_I{}^j + \binom{M-2}{r-1} \sum_{j=1}^{\nu} Q^j(p)$$

of type

$$C = C\left[(M-r)\phi(N-1, r-1, M) + \nu(M-1)\binom{M-2}{r-1}, \nu(N-r)\binom{M-1}{r} + \nu(M-1))\binom{M-2}{r-1}\right].$$

We now explain the index count: the latter index can be read off from the divisor $C(p)$. Conversely, given a point $p' \in C(p)$; then p' can appear either in the first or the second sum in $C(p)$. In the latter case, there are ν ways to add $(M-1)$ points to p' to make a divisor in $\gamma_M{}^1$, and thus multiplying $\nu(M-1)$ with $\binom{M-2}{r-1}$, the multiplicity with which the p' appears, we get the latter contribution to the first index of C. The other contribution is obtained by taking $p' \in \sum P_I{}^j$. Note that the divisors of $g_N{}^r$ containing p' form a g_{N-1}^{r-1}, and if we look at the $\mathscr{D} \in g_{N-1}^{r-1}$ having r points in $\gamma_M{}^1$, of which there are $\phi(N-1, r-1, M)$, then the remaining $(M-r)$ points of the $\gamma_M{}^1$ are precisely the candidates for p. This explains the indices of C. The next point to observe is that the value γ of C is zero; indeed, since the combinatorial identity $\binom{M-2}{r-1} \sum Q^j(p) = \sum Q_I{}^j$, implies

$$C(p) = \sum (P_I{}^j + Q_I{}^j),$$

and since the $P_I{}^j + Q_I{}^j(p) \in g_N{}^r$ are independent of p in $\mathrm{Jac}(X)$, we have that $\gamma = 0$. Hence by the Chasles-Cayley–Brill formula $\xi = n + n' + 2g\gamma$,

$$\xi = (M-r)\phi(N-1, r-1, M) + \nu(M-1)\binom{M-2}{r-1} + \nu(N-r)\binom{M-1}{r} + \nu(M-1)\binom{M-2}{r-1}.$$

The coincidences ξ can also be counted directly. We will have a coincidence whenever p coincides with $\sum Q^j(p)$ or $\sum P_I{}^j$. The first case occurs $d \times \binom{M-2}{r-1}$ times, where d is the coincidence of the $\gamma_M{}^1$. The second case occurs precisely if $p \in P_I{}^j$ for some j, I, but that happens precisely if the $r+1$ points $Q_I{}^j(p) + p$ are common to $\gamma_M{}^1$, $g_N{}^r$, i.e., $\phi(N, r, M)$ times, and of course all of the $r+1$ elements of $Q_I{}^j(p) + p$ give rise to such an occurrence, which thus occurs $(r+1)\phi(N, r, M)$ times. We thus conclude

$$\xi = \binom{M-2}{r-1} d + (r+1)\phi(N, r, M).$$

[24] We do not exhibit the polynomials defining this correspondence.

Upon equating the above expressions for ξ we have the recurrence relation

$$(M-r)\phi(N-1,r-1,M)+\nu(M-1)\binom{M-2}{r-1}+\nu(N-r)\binom{M-1}{r}$$
$$+\nu(M-1)\binom{M-2}{r-1}=\binom{M-2}{r-1}d+(r+1)\phi(N,r,M). \qquad (6)$$

We now proceed by induction. Suppose $\phi(N, s, M)$ $(s < r)$ has the desired form for all N; then a straightforward combinatorial computation shows that $\phi(N, r, M)$ also does, by substitution. To start the induction put $r = 1$ in (6) using $\phi(N-1, 0, M) = \nu(N-1)$, leading to the correct expression for $\phi(N, 1, M)$. This proves Lemma 3.

LEMMA 4. *The coincidences of a correspondence* $C[n, n']$ *between* X *and* X' *satisfy*

$$\eta' \leqslant 2n'(n+g-1)$$

and

$$\eta \leqslant 2n(n'+g'-1).$$

Proof. A correspondence $C[n, n']$ gives rise to a $\gamma_n{}^1$ of index n'. Observe that

$$d = 2\ \#\{\text{divisors of order } n \text{ in } \gamma_n{}^1 \text{ with double points}\} = 2\eta'.$$

According to Lemma 3 and because of the nonnegativity of z, $z \equiv \#\{p = p_1 + \cdots + p_n \in \gamma_n{}^1$ such that $p + \cdots \in g^{n-1}_{n+g-1}\} = 2n'(n+g-1) - \eta' \geqslant 0$, from which Lemma 4 follows.

THEOREM 1. (*Castelnuovo*) *The following four statements are equivalent*:

(i) $z = 0$

(ii) $\gamma_n{}^1$ *is contained in a linear series* $g_n{}^l$

(iii) *the correspondence of* $\gamma_n{}^1$ *is given by one equation,*

(iv) *the homomorphism between* Jac(X) *and* Jac(X') *induced by the correspondence is trivial.*

Proof. (i) implies (ii). To begin with, let $z \geqslant 0$ be arbitrary; let $\gamma_n{}^1$ be given by the equations

$$f(x, y) = F_1(x, y, x', y') = F_2(x, y, x', y') = f'(x', y') = 0.$$

Take a generic g^{n-1}_{n+g-1} defined by

$$\psi \equiv \sum_{i=1}^{n} \lambda_i \psi_i = 0.$$

Eliminate y from f and F_1; this yields a polynomial

$$\theta_1(x, x', y') = 0.$$

Eliminate y from f and F_2:

$$\theta_2(x, x', y') = 0.$$

Given a point (x', y') on X', there are n solutions $(x_1(x', y'),..., x_n(x', y'))$ to these two equations (possibly after a birational transformation), i.e., n points on X. Eliminating y between the equation ψ and f yields a polynomial

$$\Theta(x, \lambda_1, \lambda_2, ..., \lambda_n) = 0.$$

Now we express the fact that the n roots $x_i(x', y')$ are also roots of $\Theta = 0$. This yields

$$\Theta(x_i(x', y'), \lambda_1, ..., \lambda_n) = 0, \qquad 1 \leqslant i \leqslant n, \text{ where } f'(x', y') = 0.$$

Note that this equation is parametrized by the g^{n-1}_{n+g-1}'s. Assume now $z = 0$, i.e., there is generically (in g^{n-1}_{n+g-1}) no solution to this system of equations in $\lambda_1, ..., \lambda_n$ for all (x', y') with $f(x', y') = 0$. But now we exhibit a special $\tilde{g}^{n-1}_{n+g-1}$ for which there is a solution; then for the ψ corresponding to the special choice of g^{n-1}_{n+g-1} there are an infinite number of solutions, by Bezout's theorem. To get a special ψ, take a group a of $\gamma_n{}^1$ and g other points $p_1, ..., p_g$ such that

$$a + \sum_1^g p_i \in g^n_{n+g}.$$

Pick p_1. Let $\tilde{g}^{n-1}_{n-g-1}$ be the fixed linear system[25] such that

$$\tilde{g}^{n-1}_{n+g-1} + p_1 \subseteq g^n_{n-g}.$$

But since

$$a + p_1 + \cdots \in g^n_{n+g},$$

we have that

$$a \in \tilde{g}^{n-1}_{n+g-1}$$

[25] Fixing one point in a linear system leads to a linear system of one fewer points and one fewer degrees of freedom. Indeed given a system defined by $\sum_0^n \lambda_i f_i(p) = g(\lambda, p) = 0$ and imposing the condition $g(\lambda, \tilde{p}) = 0$ enables us to eliminate, say, $\lambda_0 = -(f_0(\tilde{p}))^{-1} \sum_{i=1}^n \lambda_i f_i(\tilde{p})$, and thus substituting in the above yields a linear system of the form $\sum_{i=1}^n \lambda_i h_i = 0$.

and since $a \in \gamma_n{}^1$ we have indeed exhibited the right $\tilde{g}_{n+g-1}^{n-1}$ and thus

$$\gamma_n{}^1 + \cdots \subseteq \tilde{g}_{n+g-1}^{n-1}$$

and therefore

$$(\gamma_n{}^1 + p_1) + \cdots \subseteq \tilde{g}_{n+g-1}^{n-1} + p_1 \subseteq g_{n+g}^{n}\,.$$

Hence by symmetry

$$(\gamma_n{}^1 + p_i) + \cdots \subseteq g_{n+g}^{n}\,, \qquad 1 \leqslant i \leqslant g. \tag{7}$$

Let $b \in \gamma_n{}^1$; then for some unique divisor $c = c(b)$

$$b + c \in g_{n+g}^{n}\,.$$

But, by (7), $p_i \in c$; we conclude $c = \sum_{i=1}^{g} p_i$. Hence

$$\gamma_n{}^1 + \sum_1^g p_i \subseteq g_{n+g}^{n}\,.$$

Therefore $\gamma_n{}^1$ is contained in a linear system g_n^{1+k}, $k \geqslant 0$. To establish the converse ((ii) $\Rightarrow$ (i)), assume $\gamma_n{}^1 \subseteq g_n^{1+k}$, which is complete; we need to show that $\gamma_n{}^1$ and an arbitrary (complete) g_{n+g-1}^{n-1} have nothing in common. Now, assume to the contrary, that $z > 0$; lat a be that divisor, i.e., for some points $p_1, \ldots, p_{g-1}$

$$a + \sum_1^{g-1} p_i \in g_{n+g-1}^{n-1}\,, \qquad a \in \gamma_n{}^1.$$

Fixing $\sum_1^{g-1} p_i$ determines[26] a unique complete $\tilde{g}_n{}^h$ such that

$$\tilde{g}_n{}^h + \sum_1^{g-1} p_i \subseteq g_{n+g-1}^{n-1}$$

which clearly also contains a; therefore

$$a \in \tilde{g}_n{}^h \cap g_n^{1+k}$$

and hence they coincide;[27] thus

$$\gamma_n{}^1 \subseteq g_n^{1+k} = \tilde{g}_n{}^h \subseteq g_{n+g-1}^{n-1}\,,$$

[26] See footnote 25.

[27] Because the divisors of both linear systems are equivalent to a in Jac(X) and both systems are complete by assumption.

which is absurd, because every generic linear system g_{g+n-1}^{n-1} would have something in common.

We now show that (ii) implies (iii); so assume $\gamma_n{}^1 \subsetneq g_n{}^l$. Embed the curves X and X' in projective spaces $(\mathbb{P}^l)^*$ and $\mathbb{P}^l$, $l \geqslant 2$, respectively, as follows: if $\psi = (\psi_0, ..., \psi_l)$ are the defining meromorphic functions for $g_n{}^l$, define the map

$$p \in X \to \psi(p) \in (\mathbb{P}^l)^*$$

and a hyperplane $\{x \mid (y, x) = \sum_0^l y_i x_i = 0\}$ intersects X in the points $\psi(p_1), ..., \psi(p_n)$, where $(y, \psi(p_j)) = 0$ $(1 \leqslant j \leqslant n)$; therefore we can identify the hyperplane with a divisor in $g_n{}^l$. Also note that y may be thought of as an element of $\mathbb{P}^l$. Let C: $X \to X'$ be the correspondence associated with $\gamma_n{}^1$; if $p' \in X'$; then $C^{-1}(p') \in g_n{}^l \simeq \mathbb{P}^l$; therefore X' can be embedded in $\mathbb{P}^l$. Let X and X' be given by

$$X: f(x) = 0, \qquad X': f'(x') = 0.$$

Then the correspondence is given by

$$\sum_{i=0}^{l} x_i y_i = 0,$$

which produces the one equation defining the correspondence, establishing (iii). Now assume (iii): the correspondence is given by one equation, besides the curve equations

$$f(x, y) = 0, \qquad f'(x', y') = 0,$$
$$R(x, y, x', y') = \sum_{i,i} x^i y^j \lambda_{ij}(x', y') = 0,$$

where λ_{ij} are also polynomials in (x', y'). Note that for different values of (x', y') the divisors of zeros of R are equivalent in $\mathrm{Jac}(X)$. This implies that the map defined by

$$p' \in X', \qquad f(p') = \int^{C(p')} \omega \in \mathrm{Jac}(X),$$

is constant. Think of X' as being embedded in $\mathrm{Jac}(X')$; then any map f, as defined above, extends by linearity to a unique homomorphism

$$h: \mathrm{Jac}(X') \to \mathrm{Jac}(X) \qquad \text{with} \quad h\,|_{X'} = f,$$

possibly after a new choice of origin. The fact that $f =$ constant implies that $h \equiv 0$ on $\mathrm{Jac}(X)$, establishing (iv). That (iv) implies (iii) can be done by retracing one's steps. This ends the proof of Theorem 1.

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