Discrete Order Preserving Semigroups and Stability for Periodic Parabolic Differential Equations

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1. Introduction

It has been known for some time now that the dynamics of strongly order-preserving continuous-time semigroups is relatively uncomplicated. In fact Hirsch [9] has established under a compactness hypothesis a very general generic result for such systems stating that most orbits are quasi-convergent, that is, asymptotic to the set of equilibria, \( \omega(x) \subseteq E \). This abstract result has significant applications to the stability theory of autonomous parabolic equations of not necessarily selfadjoint type, which cannot be handled by the usual classical arguments involving Liapunov functions. In the present paper we are primarily interested in the global stability of solutions of parabolic equations in non-selfadjoint form with time periodic coefficients. The relevant mathematical structure then is that of discrete-time strongly order-preserving semigroups \( (S^n)_{n \in \mathbb{N}} \) on a Banach lattice. We now preview some of our main results in restricted generality beginning with an application that motivates the abstract hypotheses.

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Consider the equation

$$\frac{\partial u}{\partial t} + \mathcal{A} u = g(x, t, u) \quad \text{on} \quad \Omega \times (0, \infty)$$

$$\left. \frac{\partial u}{\partial n} \right|_{\partial \Omega} = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty) \quad (1)$$

$$u(x, 0) = u_0(x) \quad \text{on} \quad \Omega, \text{bounded domain in } \mathbb{R}^N.$$

Here $g$ is $T$-periodic in $t$, $\mathcal{A} := -A + \sum_{j=1}^{N} a_j(x) \partial_j$, $(\partial/\partial n)$ the Neumann boundary operator in the outward normal direction. Assume that there are constants $c_1, c_2 > c_1$ such that $g(x, t, c_1) > 0 > g(x, t, c_2)$ for all $x \in \bar{\Omega}$ and $t \in \mathbb{R}$. Then it can be shown that the order interval

$$[c_1, c_2] = \{ u_0 \in L^p(\Omega) / c_1 \leq u_0 \leq c_2 \}, \quad p > 1, \quad (2)$$

is positively invariant under the flow that (1) induces. Let $\lambda_1$ be the principal eigenvalue of the periodic parabolic problem ([5])

$$\mathcal{P} u = \lambda u, \quad \left. \frac{\partial u}{\partial n} \right|_{\partial \Omega} = 0$$

$$u(\cdot, 0) = u(\cdot, T),$$

$$\mathcal{P} := \frac{\partial}{\partial t} + \mathcal{A} - \bar{g}, \quad \bar{g}(x, t) = \max_{c_1 \leq \xi \leq c_2} \frac{\partial g}{\partial \xi}(x, t, \xi).$$

Our main hypothesis is that

$$\lambda_1 \geq 0. \quad (H)$$

Theorem 4, one of the major results of this work, states that under hypothesis $(H)$ any solution of (1) with initial conditions in $[c_1, c_2]$ converges, as $t \to +\infty$, to a periodic solution $w(t)$ of (1). Simple modifications of known examples show that $(H)$ is necessary in general for the stabilization of all orbits. The autonomous version of the theorem above was essentially established in Hirsch [7] as a consequence of his generic results. In our case we handle the periodic case as an application to an abstract theory for a class of discrete strongly order-preserving semigroups $(S^n)_{n \in \mathbb{N}}$ on a Banach lattice $X$ that enjoy an orbital stability property. In this class of maps $S$, orbits with compact closure are convergent (Theorem 1). Examples are strongly order-preserving nonexpansive maps (and so the results in [2] are subsummed here) and also strongly order-preserving maps with Liapunov stable orbits. This last type of more general maps is natural in the study of non-selfadjoint problems as the induced period map of Eq. (1)
under hypothesis (H) is not in general nonexpansive. Next we consider the robustness of the stabilization results under perturbations. Consider the variant of Eq. (1) that is obtained by replacing $g$ with $g(x, t, \xi) + h(x, t, \xi)$, where $h \to 0$ as $t \to +\infty$. More precisely, under the hypothesis

$$\int_0^{\infty} \left\| \sup_{c_1 < \xi < c_2} |h(\cdot, s, \xi)| \right\|_{L^p(\Omega)} ds < +\infty. \quad (3)$$

Theorem 6 states that any solution of the perturbed equation that stays eventually in $[c_1, c_2]$ converges, as $t \to +\infty$, to a periodic solution $w(t)$ of (1). If on the other hand we weaken (3) to read

$$\int_{nT}^{(n+1)T} \left\| \sup_{c_1 < \xi < c_2} |h(\cdot, s, \xi)| \right\|_{L^p(\Omega)} ds \to 0 \quad (4)$$
as $n \to +\infty$, we establish quasi-convergence (Theorem 5). Simple examples show that the quasi-convergence result is optimal under the hypothesis (4). Our theorems on the asymptotically periodic case are obtained as applications of abstract results on asymptotically autonomous discrete dynamical processes (Theorems 2, 3) that are of independent interest. By a passage to the limit $T \to 0$ in the period our results include, as a special case, assertions for continuous-time strongly order-preserving semigroups (cf. [7]). Important in our approach is the assumption of orbital stability of the considered semigroup in order to get definitive conclusions. Without this hypothesis only generic results can be expected in a general setting.

Our studies and efforts for identifying the appropriate class of maps have been guided by the work of Dafermos [6]. For some related work on discrete order-preserving systems we refer to Hirsch [8]. Our techniques differ significantly from those in [1], [2] and in particular we do not employ Liapunov operators.

This paper is structured as follows. In Part I we present the abstract results for the discrete order-preserving semigroups (Section 2) and for the asymptotically autonomous discrete dynamical processes (Section 3). In Part II we treat periodic and asymptotically periodic parabolic differential equations (Section 4).

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**PART I**

2. Discrete Strongly Order-preserving Semigroups

Let $X$ be a Banach lattice ($\leq$, $\|\cdot\|$) with the property

(X.1) $X$ has order-continuous norm (i.e., any increasing net which has a supremum is convergent in $X$).
This is equivalent to asking that $X$ is $\sigma$-order complete (i.e., the supremum of countable majorized subsets always exists) and has $\sigma$-order-continuous norm (i.e., any increasing sequence with a supremum is convergent), cf. [11, Def. II. 5.12 and Them. II, 5.10.]

Let further $D \subset X$ be a closed subset which satisfies

(D.1) $D$ is closed under the sup- and inf- operation;

(D.2) $D$ is order-connected in the sense that for each pair $x, y \in D, x < y$, there exists an ordered curve $(z_\lambda)_{0 \leq \lambda \leq 1}$ in $D$ connecting $x$ and $y$;

(D.3) $D$ is order-bounded from above (or below).

A typical example is the closed order-interval.

Moreover let $Z \subset X$ be a Banach space (norm $\| \cdot \|_Z$) continuously embedded in $X$, such that $\text{Int}_Z(X_+ \cap Z) \neq \emptyset$ ($X_+$ = positive cone in $X$).

We employ the standard notations of ordered Banach spaces

\[ x \leq y : \iff y - x \in X_+ \]
\[ x < y : \iff x \leq y, x \neq y \]
\[ x \ll y : \iff x, y \in Z, y - x \in \text{Int}_Z(X_+ \cap Z). \]

Finally let $S: D \to D$ be a mapping satisfying

(S.1) $S$ is strongly order-preserving: $S(D) \subset Z$, and $x < y \Rightarrow S(x) \ll S(y)$;

(S.2) $S: X \supset D \to Z$ is continuous;

(S.3) Each $x \in D$ is orbitally stable: for $\varepsilon > 0 \exists \delta = \delta(\varepsilon, x) > 0$ such that

\[ y^+(x) \subset U_\delta(y^+(x)), \quad \forall y \in U_\delta(x) \cap D. \]

Here $y^+(x) := \{ S^n(x) : n \in \mathbb{N} \}$ denotes the semiorbit of $x$, and $U_\varepsilon$ the $\varepsilon$-neighborhood (in $X$). In the following we will also make use of the notations:

\[ \omega(x) := \{ y \in D : \exists n \to \infty \text{ such that } S^m(x) \to y \text{ in } X \} \]
\[ = \omega\text{-limit set of } x; \]
\[ E := \{ y \in D : S(y) = y \} = \text{set of equilibria of } S. \]

The main result in this section is

**THEOREM 1.** Let all the assumptions above be satisfied, and let $x \in D$. Suppose $y^+(x)$ is relatively compact in $X$. Then

\[ \omega(x) = \{ q \}, \quad q \in E, \quad \text{and} \quad S^n(x) \to q \text{ in } Z. \]
Remark 2.1. Theorem 1 generalizes the related result in [2] in several ways. In particular the assumption on $S$ to be nonexpansive in $X$ is weakened to the orbital stability condition (S.3).

Proof of Theorem 1. Let $x \in D$ be such that $\gamma^+(x)$ is relatively compact in $X$. It is well-known that $\omega(x)$ is nonempty, compact, and invariant (i.e., $S(\omega(x)) = \omega(x)$) [10, Them. I. 5.2]. A sharpening is

**Lemma 2.2** (cf. Proposition 3.2 in [6]).

Let $x \in D$, $\gamma^+(x)$ relatively compact in $X$, and assume $y \in \omega(x)$ is orbitally stable. Then

$$\omega(y) \subset \omega(x) \subset \gamma^+(y).$$

**Proof.** It is clear that $\omega(y) \subset \omega(x)$. Let $z \in \omega(x)$. There exist sequences $(k_n)_{n \in \mathbb{N}}$, $(l_n)_{n \in \mathbb{N}}$, $k_n \to \infty$, $l_n \to \infty$, such that

$$y = \lim S^{k_n}(x), \quad z = \lim S^{l_n}(x).$$

Without loss of generality we may take

$$m_n := l_n - k_n \geq 0.$$

Since $S^{k_n}(x) \to y$, $y$ orbitally stable, it follows that there exist $(\mu_n)_{n \in \mathbb{N}}$, $\mu_n \geq 0$, such that

$$\| S^{l_n}(x) - S^{\mu_n}(y) \| = \| S^{m_n} S^{k_n}(x) - S^{\mu_n}(y) \| \to 0 \quad (2.1)$$

as $n \to \infty$. From

$$\| S^{\mu_n}(y) - z \| \leq \| S^{\mu_n}(y) - S^{l_n}(x) \| + \| S^{l_n}(x) - z \|, \quad (2.2)$$

using (2.1) and the definition of $z$ we obtain

$$\| S^{\mu_n}(y) - z \| \to 0 \quad (n \to \infty). \quad (2.3)$$

In the event $(\mu_n)_{n \in \mathbb{N}}$ is unbounded we obtain via (2.3) that $z \in \omega(y)$. If $\mu_n \leq c$, along a subsequence $\mu_n \to \mu$ and so

$$\mu \in S^\mu(y) \in \gamma^+(y).$$

An immediate consequence is

**Corollary 2.3.** If $x \in D$ is such that $\gamma^+(x)$ is relatively compact in $X$, and if $y \in \omega(x) \cap E$ is orbitally stable, then

$$\omega(x) = \{ y \}.$$
We assume in the following that $D$ is order-bounded from above. Let $M := \sup \omega(x)$, which exists by (D.3) and compactness of $\omega(x)$ and belongs to $D$ by (D.1). We distinguish between two cases.

**Case 1.** $M \in E$.
We observe that

$$\omega(x) = S(\omega(x)) \subseteq Z \quad \text{and} \quad M = S(M) \in Z.$$ 

**Lemma 2.4.** There exists $\xi \in \omega(x)$ with $\xi \notin M$.

**Proof.** Assume that $\xi \leq M$ for all $\xi \in \omega(x)$ and take $\varphi_0 \in \text{Int}_x(X_+ \cap Z)$ fixed. For each $\xi \in \omega(x)$ let $c(\xi) := \sup \{ c > 0 : c\varphi_0 \leq M \}$. Accepting for the time being that $c(\cdot) : \omega(x) \to \mathbb{R}$ is continuous we set $\tilde{c} := \min_{\omega(x)} c(\xi)$ which we note to be positive by the compactness of $\omega(x)$ in $Z$. It follows that

$$\xi \leq M - \tilde{c}\varphi_0 \quad \text{for all} \quad \xi \in \omega(x)$$

and so $M - \tilde{\varphi}_0$ is an upper bound for $\omega(x)$, contradicting the definition of $M$.

To establish continuity for $c(\cdot)$ we proceed as follows. Let $\xi_n \to \xi$ and assume that $c(\xi_n) \leq c(\xi) - \delta_0$, $\delta_0 > 0$. By definition of $c(\cdot)$, $\xi + (c(\xi) - \delta_0/2) \varphi_0 \ll M$ or equivalently

$$\left(c(\xi) - \frac{3\delta_0}{4}\right) \varphi_0 \ll M - \xi - \frac{\delta_0}{4} \varphi_0.$$

Taking $n$ large we may assume that $\xi_n - \xi \leq (\delta_0/4) \varphi_0$ and so

$$\left(c(\xi) - \frac{3\delta_0}{4}\right) \varphi_0 \ll M - \xi_n + \left(\xi_n - \xi - \frac{\delta_0}{4} \varphi_0\right) \ll M - \xi_n.$$

Therefore

$$\xi_n + \left(c(\xi) - \frac{3\delta_0}{4}\right) \varphi_0 \ll M,$$

from which we conclude $c(\xi_n) \leq c(\xi) - 3\delta_0/4$, $n$ large, a contradiction, and so lower semicontinuity of $c(\cdot)$ is established. To establish upper semicontinuity take $\xi_n \to \xi$ and assume that $c(\xi_n) \geq c(\xi) + \delta_0$, $\delta_0 > 0$. By the definition of $c(\cdot)$ once more

$$\xi_n + \left(c(\xi_n) - \frac{\delta_0}{2}\right) \varphi_0 \ll M,$$
and so for $n$ large enough

$$\xi + \left( c(\xi_n) - \frac{\delta_0}{2} \right) \varphi_0 \leq M + \xi - \xi_n \leq M + \frac{\delta_0}{4} \varphi_0.$$ 

Therefore

$$\xi + \left( c(\xi_n) - \frac{3\delta_0}{4} \right) \varphi_0 \leq M,$$

from which it follows that $c(\xi) \geq c(\xi_n) - \frac{3\delta_0}{4}$, a contradiction. 

We now establish that $\omega(x) = \{M\}$ and so conclude the proof of Theorem 1 in the first case. Indeed, if $\xi = M$, then $M \in \omega(x) \cap E$ and hence $\omega(x) = \{M\}$ by Corollary 2.3. If however $\xi < M$, by the invariance of $\omega(x)$ there exists $\xi \in \omega(x)$ with $\xi = S(\xi)$. Now either

$$\xi = M \Rightarrow \xi = S(\xi) = S(M) = M: \text{ contradiction,}$$

or

$$\xi < M \Rightarrow \xi = S(\xi) \leq S(M) = M: \text{ contradiction.}$$

Case II. $M \notin E$.

Since $\omega(x) \leq M$, we have $\omega(x) = S(\omega(x)) \leq S(M)$ and hence, by the definition of $M$, $M \leq S(M)$. Thus $M < S(M)$ in this case. Iterating this inequality, and making use of the order-boundedness of $D$ from above, we infer by (X.1) that $S^n(M) \nrightarrow q(n \to \infty)$ in $X$, with $q \in E$. Hence

$$S^n(M) \nrightarrow q \quad \text{in} \quad Z, \text{ by (S.2).}$$

Lemma 2.5. For each $y \in D$ with $y \leq M$, it holds that

$$S^n(y) \nrightarrow q \quad \text{in} \quad Z, \quad \text{as} \quad n \to \infty.$$ 

Proof.

The proof is divided in three steps:

(i) Since $M < S(M) \leq S(q) = q$, there exists $\delta > 0$ such that for $u \in D$,

$$u \leq q, \quad \| q - u \| < \delta \Rightarrow S^n(u) \nrightarrow q \quad \text{in} \quad Z.$$ 

In fact, for sufficiently small $\delta > 0$ we have $M \leq u \leq q$. Then

$$q = S^n(M) \leq S^n(u) \leq S^n(q) = q,$$

i.e., $S^n(u) \nrightarrow q$ in $X$ and thus in $Z$ by (S.2).
Let now \( y \in D, y \leq M \), and denote by \((z_i)_{0 \leq i \leq 1}\) the ordered curve in \( D \) with \( z_0 = M, z_1 = y \), guaranteed by (D.2). Set
\[
A := \{ \lambda \in [0, 1] : S^n(z_\lambda) \to q \}.
\]
Note that \( 0 \in A \).

(ii) \( A \) is open in \([0, 1]\). Indeed, let \( \lambda_0 \in A \). Then there exists \( n_0 \in \mathbb{N} : \| q - S^{n_0}(z_{\lambda_0}) \|_Z < \delta/2 \) (with \( \delta > 0 \) from step (i)), and by continuity of \( S^{n_0} \), there exists \( \varepsilon > 0 : \| S^{n_0}(z_\lambda) - S^{n_0}(z_{\lambda_0}) \|_Z < \delta/2, \forall \lambda \in [0, 1], |\lambda - \lambda_0| < \varepsilon \).
Hence, by step (i) with \( u := S^{n_0}(z_\lambda) \), it follows that \( S^n(z_\lambda) \to q \) \((n \to \infty)\) for such \( \lambda \).

(iii) \( A \) is closed in \([0, 1]\). Indeed let \( \lambda_0 \in \overline{A} \). In the event \( q \notin \omega(z_{\lambda_0}) \) it follows by Corollary 2.3 that \( \omega(z_{\lambda_0}) = \{ q \} \) and so \( \lambda_0 \in A \). On the other hand if \( q \notin \omega(z_{\lambda_0}) \) and so \( q \notin \gamma^+(z_{\lambda_0}) \), it follows that \( \operatorname{dist}_X(q, \gamma^+(z_{\lambda_0})) = \sigma > 0 \). By orbital stability of \( z_{\lambda_0} \) there exists \( \hat{\lambda} \in A \) such that
\[
S^n(z_{\hat{\lambda}}) \in U_{\sigma/2}(\gamma^+(z_{\lambda_0})), \quad \forall n \in \mathbb{N}.
\]
Since \( S^n(z_{\hat{\lambda}}) \to q \) as \( n \to \infty \) we conclude that
\[
\operatorname{dist}_X(q, \gamma^+(z_{\lambda_0})) \leq \frac{\sigma}{2},
\]
contradicting the definition of \( \sigma \). Therefore \( q \in \omega(z_{\lambda_0}) \) is the only possibility and so \( A \) is closed.

Thus \( A = [0, 1], \) and the assertion of Lemma 2.5 follows.

Now we have the situation \( \omega(x) \leq M \leq q \). Let \( y \in \omega(x) \). By Lemma 2.5, \( S^n(y) \to q \) in \( Z \). However, \( S^n(y) \in \omega(x) \) for all \( n \in \mathbb{N} : \) contradiction. Thus Case II: \( M \notin E \) is impossible.

The proof of Theorem 1 is complete.

Remark 2.6. Theorem 1 admits variants which differ slightly in the assumptions on \( X, D, Z, \) and \( S \). We close this section by presenting a version which seems to be particularly useful. Assume

(X.1*) \( X \) has the property that increasing, norm-bounded sequences converge. This property, which is stronger than (X.1), holds in particular if \( X \) is weakly sequentially complete, hence if \( X \) is reflexive. Nonreflexive Banach lattices enjoying (X.1*) are, e.g., the AL-spaces cf. [10, p. 92, Example 7.]

Further assume that \( D \subset X \) is a closed subset satisfying (D.1)–(D.2) (but which may be unbounded in \( X \)).

On the mapping \( S : D \to D \) conditions (S.1)–(S.2) are imposed as well as the following sharpening of (S.3),
(S.3*) Each \( x \in D \) is stable: for \( \varepsilon > 0 \) \( \exists \delta = \delta(\varepsilon, x) > 0 \) such that
\[
S^n(y) \in U_\varepsilon(S^n(x)), \quad \forall y \in U_\delta(x) \cap D.
\]

Under these assumptions Theorem 1 holds again.

Proof. We only note the points which differ from the previous arguments. The existence of \( M := \sup \omega(x) \) follows by compactness of \( \omega(x) \) and hypothesis (X.1). Case I, \( M \in E \), remains unchanged. In Case II, \( M \not\in E \), we obtain again that \( S^n(M) \) is increasing in \( D \). In order to prove that the sequence \( (S^n(M))_{n \in \mathbb{N}} \) is norm-bounded in \( X \)—and hence convergent in \( X \) by (X.1*)—we fix \( y \in \omega(x) \) and proceed as in the proof of Lemma 2.5:

Let \((z_k)_{0 \leq k \leq 1}\) be an ordered curve in \( D \) connecting \( y \) and \( M \) : \( z_0 = y \), \( z_1 = M \), and let \( A^* := \{ \lambda \in [0, 1] : (S^n(z_{\lambda})))_{n \in \mathbb{N}} \) is bounded in \( X \} \). Note that \( 0 \in A^* \). It follows immediately from (S.3*) (already from (S.3)) that \( A^* \) is open in \([0, 1]\). To show that \( A^* \) is also closed, let \( \lambda_0 \in A^* \), and assume that \( \lambda_0 \not\in A^* \). Then there is a subsequence \( (z_{\lambda_k})_{k \in \mathbb{N}} \) such that \( \|S^n(z_{\lambda_k})\| \to \infty \) \((k \to \infty)\). Let \( \lambda \in A^* \) such that \( \|z_{\lambda} - z_{\lambda_0}\| < \delta = \delta(1, z_{\lambda_0}), \delta \) given by (S.3*). Since \((S^n(z_{\lambda}))_{k \in \mathbb{N}} \) is bounded in \( X \), we arrive at a contradiction to (S.3*).

Thus \( A^* = [0, 1] \), and \( (S^n(M)) \) is norm-bounded, hence convergent also in this case. The rest of the proof of Theorem 1 remains unchanged.

3. Asymptotically Autonomous Discrete Dynamical Processes

Let \((S_n)_{n \in \mathbb{N}} \) be a sequence of mappings \( S_n : D_n \to D_{n+1} \) which are continuous in the \( X \)-topology. For \( n \geq 1 \) set
\[
T_n := S_{n-1} \circ \cdots \circ S_0 : D_0 \to D_n
\]
and take \( T_0 := \text{Id}_{D_0} \). Then \((T_n)_{n \in \mathbb{N}} \) is a discrete dynamical process. The semiorbit of \( x \in D_0 \) is now denoted by \( \hat{\gamma}^+(x) := \{T_n(x) : n \in \mathbb{N}\} \). We assume that \((T_n)\) is asymptotic to a strongly order-preserving, orbitally stable discrete semigroup \((S^n)_{n \in \mathbb{N}} \): suppose there exists a closed subset \( D \subseteq X \) with \( \forall D : D_n \forall n \in \mathbb{N} \), satisfying (D.1)–(D.3), and a mapping \( S : D \to D \cap \Omega \) satisfying (S.1)–(S.3), such that \( S_n \to S (n \to \infty) \) along relatively compact trajectories:

(P.1) If \( \hat{\gamma}^+(x) \) is relatively compact in \( X \), then
\[
\|S_n \circ T_n(x) - S \circ T_n(x)\| \to 0 \quad (n \to \infty).
\]

Let \( \hat{\omega}(x) := \{y \in D : \exists n_k \to \infty \text{ such that } T_{n_k}(x) \to y \text{ in } X \} \) be the \( \omega \)-limit set of \( x \in D_0 \) relative to \((T_n)\), but keep
\[
E := \{y \in D : S(y) = y\}.
\]
THEOREM 2. Under the above hypotheses, if $x \in D_0$ is such that $\tilde{\gamma}^+(x)$ is relatively compact in $X$, then $\tilde{\omega}(x) \subseteq E$.

Hence $x$ is quasi-convergent. We shall see in terms of a simple example in Section 4 that we cannot expect convergence, unless $S_n$ converges to $S$ sufficiently fast.

(P.1') For each relatively compact $\tilde{\gamma}^+(x)$ there exists a sequence $(d_n)_{n \in \mathbb{N}}$ of positive numbers with $\sum_{n=0}^{\infty} d_n < \infty$, such that $\forall n \in \mathbb{N}$

$$\|S_n \circ T_n(x) - S \circ T_n(x)\| \leq d_n.$$ 

We further substitute (S.3) by the stronger stability assumption

(S.3') For each compact set $K \subset D$ there exists a constant $C > 0$ such that

$$\|S^n(x) - S^n(y)\| \leq C \|x - y\|, \quad \forall x, y \in K, \forall n \in \mathbb{N}.$$ 

THEOREM 3. In the situation of Theorem 2, replace conditions (P1) and (S.3) by (P.1') and (S.3'), respectively. Let $x \in D_0$ with $\tilde{\gamma}^+(x)$ relatively compact in $X$. Then the orbit is convergent:

$$\tilde{\omega}(x) = \{q\}, q \in E.$$ 

Consider the discrete semigroup $(S^n)_{n \in \mathbb{N}}$. A set

$$\gamma(y) = \{y_n : n \in \mathbb{Z}, y_0 = y\}$$

is called an entire orbit of the semigroup through $y \in D$ if $y_{n+1} = S(y_n)$ for all $n \in \mathbb{Z}$. Set

$$\alpha(y) := \{z \in D : \exists n_k \rightarrow -\infty \text{ such that } y_{n_k} \rightarrow z \text{ in } X\}.$$ 

LEMMA 3.1 (cf. Propositions 2.2 and 3.2 in [6]).

Let $y \in D$, $\gamma(y)$ a relatively compact (in $X$) entire orbit of $(S^n)_{n \in \mathbb{N}}$ through $y$, and assume all elements of $\gamma(y)$ are orbitally stable. Then

$$\omega(y) = \alpha(y) = \overline{\gamma(y)}.$$ 

Proof. By the compactness of $\overline{\gamma(y)}$, $\alpha(y) \neq \emptyset$. Next we show that $\alpha(y)$ has no nontrivial closed positively invariant subsets and therefore conclude that if $A \subset \alpha(y)$, $A$ closed positively invariant, $A \neq \emptyset$, then $A = \alpha(y)$. We proceed by contradiction. So assume $\bar{n} \in \mathbb{Z}$ is such that $y_{\bar{n}} \notin A$. Take

$$\epsilon := \text{dist}_X(y_{\bar{n}}, A) > 0,$$

and let $z \in A \subset \alpha(y)$ and $n_k \rightarrow -\infty$ such that $y_{n_k} \rightarrow z$. By the orbital stability of $z$ there exists $\delta > 0$ such that for $u \in B_\delta(z)$, $S^n(u) \in U_{\epsilon/2}(\gamma^+(z))$, $\forall n \in \mathbb{N}$. 

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Now choose $k_0 < \bar{n}$ such that $\| y_{n_0} - z \| < \delta$. Therefore $S^n(y_{n_0}) \in U_{\varepsilon/2}(\gamma^+(z))$ and since $\gamma^+(z) \subset A$,

$$S^n(y_{n_0}) \subset U_{\varepsilon/2}(A), \quad \forall n \in \mathbb{N}.$$ 

Set $n := \bar{n} - n_0$ to conclude that $y_n \in U_{\varepsilon/2}(A)$, contradicting the definition of $\varepsilon$. Consequently $\alpha(y)$ is a minimal closed positively invariant set and $\gamma(y) \subset \alpha(y)$. Now $\omega(y) \subset \gamma(y) \subset \alpha(y)$, and since $A := \omega(y)$ is closed and positively invariant, we obtain $\omega(y) = \alpha(y) = \gamma(y)$.

**Proof of Theorem 2.**

Let $y \in \tilde{\omega}(x)$. By compactness and a diagonal process we obtain an entire orbit $\gamma(y)$ of $(S^n)_{n \in \mathbb{N}}$ through $y$, with $\gamma(y) \subset \tilde{\gamma}^+(x)$. Now Lemma 3.1 implies that $\gamma(y) = \omega(y)$. But $\omega(y) = \{ q \}$, $q \in E$, by Theorem 1. Hence $y \in \gamma(y) = \{ q \} \subset E$. Since this holds for any $y \in \tilde{\omega}(x)$, the assertion of Theorem 2 follows.

**Proof of Theorem 3.**

Let $y \in \tilde{\omega}(x)$. By Theorem 2, $y \in E$. Therefore there exists $k \to \infty$ such that $T_N(x) \to y$. Let $K$ in condition (S.3') be the set

$$\tilde{\gamma}^+ (x) \cup S(\tilde{\gamma}^+(x)).$$

We show that $T_N(x) \to y$ as $n \to \infty$ hence $\tilde{\omega}(x) = \{ y \}$. For $n \in \mathbb{N}$ we have

$$\| T_{n+m}(x) - y \|$$

$$\leq \| S_{n+m-1} \circ T_{n+m-1}(x) - S \circ T_{n+m-1}(x) \| + \| S \circ S_{n+m-2} \circ T_{n+m-2}(x) - S \circ S \circ T_{n+m-2}(x) \|$$

$$+ \| S^2 \circ S_{n+m-3} \circ T_{n+m-3}(x) - S^2 \circ S \circ T_{n+m-3}(x) \|$$

$$+ \cdots + \| S^m \circ T_{n}(x) - S^m(y) \| \leq M \left( \sum_{n-k_0}^{\infty} d_n \| T_{n}(x) - y \| \right)$$

(by (P.1'), (S.3')) $\leq \varepsilon \quad \forall k > k_0 = k_0(\varepsilon), \forall m \in \mathbb{N}.$

The proof of Theorem 3 is complete.

We close this section with a few remarks.

**Remark 3.2.** In practice $T_n$ is generated from a nonautonomous evolution process. Such a process is defined as a map

$$T: \mathbb{R} \times X \times \mathbb{R}^+ \to X$$
satisfying

(i) \( T'(x, 0) = x, t \in \mathbb{R}, x \in X \)

(ii) \( T'(x, \sigma + \tau) = T'^{t+\sigma}(T'(x, \sigma), \tau), t \in \mathbb{R}, x \in X, \sigma, \tau \in \mathbb{R}^+ \).

\( T'(x, \tau) \) stands for the value of the solution at time \( t + \tau \) with initial condition \( x \) at time \( t \). The restriction on \( T \) on the integers gives rise to a discrete dynamical process:

\[ S_m(x) := T^m(x, 1) \] and so by (ii) \( T^m(x, n) = S_{n+m-1} \circ \cdots \circ S_m \), we get \( T_n \) by setting \( T_n(x) = T^0(x, n) \).

**Remark 3.3.** In attempting to extend (P.1) by replacing the limit \( S \) with a set of limit points one is faced with the unpleasant fact that the class of orbitally stable maps is not closed under composition. To see this let \( V = \mathbb{R}^3 \) and

\[
A = \begin{pmatrix}
0 & 0 & 0 \\
0 & -\lambda & 0 \\
0 & 1 & -\lambda
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1/2 \\
0 & 0 & 0
\end{pmatrix},
\]

and define

\[ S_1(x) := e^{At}x, \quad S_2(x) := Bx. \]

One then can verify that for the choice of \( \lambda = e^{-2/2} \) the linear map \( S := S_2 \circ S_1 \) is not orbitally stable by showing that \( \| S^n(e_2) \| \to \infty \), for

\[ e_2 = \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \]

although \( \| e^{At} \| \leq M \), and \( \| B \| \leq 1/2 \).

**Remark 3.4.** Obvious variants of Theorems 2 and 3, based on the alternative hypotheses on \( X, D, S \) mentioned in Remark 2.6, hold. Their proofs are identical.

**Part II**

4. **Periodic and Asymptotically Periodic Parabolic Initial-boundary Value Problems**

Consider the initial value problem

\[
\begin{align*}
\left\{ \begin{array}{ll}
\partial_t u + \mathcal{A}(t) u = g(x, t, u) & \text{on } \Omega \times (0, \infty) \\
\mathcal{B}u = 0 & \text{on } \partial \Omega \times (0, \infty) \\
u(0) = u_0 & \text{in } \Omega.
\end{array} \right.
\]

**IVP**
Here

\[ \mathcal{A}(t) := - \sum_{j,k=1}^{N} a_{jk}(x, t) \partial_j \partial_k + \sum_{j=1}^{N} a_j(x, t) \partial_j + a_0(x, t) \]

is linear and uniformly elliptic on the bounded smooth domain \( \Omega \subseteq \mathbb{R}^N \), with Hölder-continuous and \( T \)-periodic coefficient functions (\( T > 0 \) given). \( \mathcal{B} \) is a linear boundary operator (Dirichlet, Neumann, or regular oblique derivative type), and \( g \) a smooth function on \( \Omega \times \mathbb{R} \times \mathbb{R} \). More precisely we take the coefficients of the liner operator in \( C^{\mu, \mu/2}(\bar{\Omega} \times \mathbb{R}) \), \( 0 < \mu < 1 \), and \( g: (x, t, \xi) \to g(x, t, \xi) \) continuous, \( g(\cdot, \cdot, \xi) \in C^{\mu, \mu/2}(\bar{\Omega} \times \mathbb{R}) \) uniformly for \( \xi \) in bounded intervals and identical hypotheses for \( \partial g/\partial \xi \) and \( \partial \Omega \in C^{2, \mu} \).

(a) We first assume \( g \) to be \( T \)-periodic in \( t \). The smooth function \( u \in C^{1,0}(\Omega \times (0, T]) \cap C^{1,0}(\bar{\Omega} \times [0, T]) \) is called a subsolution if

\[ \begin{cases} \partial_t u + \mathcal{A}(t) u \leq g(x, t, u) & \text{in } \Omega \times (0, T] \\ \mathcal{B} u \leq 0 & \text{on } \partial \Omega \times (0, T] \end{cases} \]

a supersolution is defined with obvious modifications. We assume the existence of an ordered pair \( u < \bar{u} \) of periodic sub- and supersolutions. Set \( u_0 := u(0), \bar{u}_0 := \bar{u}(0) \), as well as \( X := L^p(\Omega)(N < p < \infty) \) and \( D := \{ [u_0, \bar{u}_0]_X := \{ u_0 \in X: u_0 \leq u_0 \leq u_0 \leq \bar{u}_0 \} \). Define \( S \) by: \( u_0 \in D \to S(u_0) := u(T) \in X \), where \( u(t) \) is the solution of the (IVP). By the parabolic maximum principle, \( S \) is well defined and maps \( D \) into itself. The smoothing property of analytic semigroups implies that \( S(D) \subset Z \), where \( Z \) is either \( C^0(\bar{\Omega}) \) (Dirichlet) or \( C^1(\bar{\Omega}) \). By the strong maximum principle, \( S \) is strongly order-preserving.

Define the \( T \)-periodic function \( \tilde{g} \) by

\[ \tilde{g}(x, t, \xi) := \max_{\nu(x, t) \leq \xi \leq \bar{\nu}(x, t)} \frac{\partial g}{\partial \xi}(x, t, \xi). \]

It is shown in [5] that the periodic eigenvalue problem

\[ \begin{cases} \partial_t w + \mathcal{A}(t) w - \tilde{g}(t) w = \lambda w & \text{in } \Omega \times \mathbb{R} \\ \mathcal{B} w = 0 & \text{on } \partial \Omega \times \mathbb{R} \\ w(t) = w(t + T) & \text{in } \Omega, \forall t \in \mathbb{R} \end{cases} \]

has a real principal eigenvalue \( \lambda_1 \), characterized as the unique eigenvalue having an eigenfunction which is positive in \( \Omega \times \mathbb{R} \). Our crucial hypothesis is

\[ \lambda_1 > 0. \]
**Lemma 4.1.** Hypothesis (H) implies the existence of a constant \( C \) such that

\[
\| S^n(u_0) - S^n(v_0) \|_X \leq C \| u_0 - v_0 \|_X, \quad \forall u_0, v_0 \in D, \forall n \in \mathbb{N}.
\]

Thus the stronger stability condition (S.3') holds in \( D \). A straightforward application of Theorem 1 then gives

**Theorem 4.** Under hypothesis (H), for each \( u_0 \in D \) the solution \( u(t) \) of the (IVP) exists \( \forall t \geq 0 \) and converges in \( C^1(\bar{\Omega}) \) to a periodic solution of the (IVP),

\[
\| u(t) - w(t) \|_{C^1(\bar{\Omega})} \to 0 \quad (t \to \infty).
\]

**Proof of Lemma 4.1.**

Let

\[ A(t) := \text{sectorial operator in } X \text{ induced by } (\mathcal{A}(t), \mathcal{B}), \]

\[ \tilde{A}(t) := \text{sectorial operator in } X \text{ induced by } (\mathcal{A}(t) - \mathcal{g}(t), \mathcal{B}), \]

\[ \tilde{A}(t) = A(t) - \mathcal{g}(t). \]

\( A(t), \tilde{A}(t) \) are \( T \)-periodic, \( g(t, \cdot): X \supset D \to X \) is the Nemytski operator induced by \( g(\cdot, t, \cdot) \). Let \( u_0, v_0 \in D, u(t), v(t) \) the associated global solutions of (IVP).

\[
\frac{d}{dt} (u - v)(t) = -A(t)(u - v)(t) + g(t, u(t)) - g(t, v(t))
\]

\[
= -\tilde{A}(t)(u - v)(t) + (g_{\xi}(t) - \mathcal{g}(t))(u - v)(t),
\]

where

\[ g_{\xi}(t) := \int_0^1 \frac{\partial \mathcal{g}}{\partial \xi}(t, v(t) + s(u(t) - v(t))) \, ds. \]

By the variation of constants formula

\[
(u - v)(t) = \mathcal{U}(t, 0)(u_0 - v_0) + \int_0^t \mathcal{U}(t, s)(g_{\xi}(s) - \mathcal{g}(s))(u - v)(s) \, ds, \quad (4.1)
\]

where

\[ \mathcal{U}(t, s) \text{ is the evolution operator in } X \text{ associated to } \tilde{A}(t). \]

We split the proof in two steps.

**Step I.** \( u_0 \geq v_0 \).
By the parabolic maximum principle \((u - v)(t) \geq 0\). Using this and the inequality \(g(t) - \bar{g}(t) \leq 0\) we obtain from (4.1)

\[
(u - v)(nT) \leq \bar{U}(nT, 0)(u_0 - v_0).
\] (4.2)

Periodicity gives

\[
\bar{U}(nT, 0) = \bar{U}(T, 0)^n.
\]

Set \(\bar{U} := \bar{U}(T, 0)\). It is shown in [5] that \(\bar{U}\), in \(C^1(\Omega)\) or \(C^1(\bar{\Omega})\), is a compact strongly positive operator. By the Krein–Rutman Theorem

\[
y_1 := \text{spr}(\bar{U}) > 0
\]

is a simple eigenvalue of \(\bar{U}\), the only one on the circle

\[
\{z \in C : |z| = \text{spr}(\bar{U})\}.
\]

It is established in [5] that \(\lambda_1 = -(1/T) \log y_1\) and so hypothesis (H) implies \(y_1 \leq 1\). By the canonical decomposition of the spectrum

\[
\sigma(\bar{U}) = \sigma_1 \cup \sigma_2, \quad \sigma_1 = \{y_1\}, \quad \sigma_2 \subset \{|y| \leq 1 - \delta\}, \quad \delta > 0.
\]

Letting \(X = X_1 \oplus X_2\), \(\bar{U} = \bar{U}_1 \oplus \bar{U}_2\) we obtain that \(\|\bar{U}_1\|_{L(X_1)} \leq 1\) and

\[
1 - \delta \geq \text{spr}(U_2) = \lim \|U_2^n\|_{L(X_2)}^{1/n}
\]

and so

\[
\|\bar{U}_2^n\|_{L(X_2)} < 1 \quad \text{for} \quad n \geq n_0.
\]

It follows that

\[
\|\bar{U}^n\|_{L(X)} \leq M, \quad n \in \mathbb{N}.
\]

Using this information in (4.2) we obtain

\[
\|(u - v)(nT)\|_X \leq M\|u_0 - v_0\|_X, \quad n \in \mathbb{N}.
\]

**Step II.** \(u_0, v_0\) general.

Set

\[
w_0 := \inf f(u_0, v_0), \\
\bar{w}_0 := \sup (u_0, v_0)
\]

and let \(w(t), \bar{w}(t)\) be the associated solution of the (IVP). By the maximum principle we obtain

\[
w(t) \leq u(t), \quad v(t) \leq \bar{w}(t).
\]
Therefore using properties of the $L^p$-norm and Step I we obtain
\[
\|(u - v)(nT)\|_X \leq \|((\hat{w} - w)(nT))\|_X \\
\leq M\|\hat{w}_0 - w_0\|_X \\
= M\|u_0 - v_0\|_X.
\]

From (4.3) it follows that
\[
\|S^n(u_0) - S^n(v_0)\|_X \leq M\|u_0 - v_0\|_X, \quad \forall n \in \mathbb{N}, u_0, v_0 \in D.
\]

**Proof of Theorem 4.** By Theorem 1, $S^n(u_0) \to q$ in $X$ (and so in $Z$), $S(q) = q$. Set $w(t) :=$ solution of (IVP) with $w(0) = q$. It follows that $w$ is $T$-periodic and
\[
\|u(t) - w(t)\|_Z \leq C\|u([t] - 1)T - w([t] - 1)T\|_X \quad \text{(by [3, Lemma 3.5])}
\]
\[
eq C\|S^{[t]}(u_0) - q\|_X \to 0 \quad (t \to \infty).
\]

The proof of the Theorem is complete.  

**Remark 4.2.** We note that the constant $C$ in Lemma 4.1 may be greater than 1 and so a complete stabilization theory for the (IVP) under hypothesis (H) cannot be obtained within the class of nonexpansive semigroups. To see this we will consider the autonomous case, which is subsumed in Theorem 4 above by taking $T = (1/n)$ $(n \geq 1)$ and so conclude, by continuity, that the solution converges towards an equilibrium of the (IVP). In this case $\lambda_1$ is the principal eigenvalue of the elliptic operator $A - \bar{g}$. We take the linear equation $(g = 0)$
\[
u_x - u_{xx} - u_x = 0 \quad \text{in} \quad (0, \pi) \times (0, \infty)
\]
\[
u_x(0, t) = u_x(\pi, t) = 0
\]
\[
u(x, 0) = u_0(x).
\]

In this case $\mathcal{A}u = -u_{xx} - u_x$ and $\text{Re} \sigma(A) \geq 0$, $0$ being a simple eigenvalue. Standard linear theory implies that $A$ generates a $C^0$ semigroup $S(t)$ on $X = L^2(0, \pi)$. Now nonexpansiveness of $S$ in $L^2$ would imply, via the Lumer–Phillips Theorem, that $A$ is accretive, \[
\int_0^\pi A\varphi \cdot \varphi \, dx \geq 0, \forall \varphi \in D(A) = \{u \in X: u'' \in X, u'(0) = u'\pi) = 0\},
\]
which is clearly false.

**Remark 4.3.** The hypothesis (H) in general cannot be relaxed without affecting the conclusion of Theorem 4. We will show this by considering the autonomous case and constructing an example of a linear elliptic operator $A_\varepsilon$ on the unit disc, with homogeneous Dirichlet conditions, which has principal eigenvalue equal to $-\varepsilon (\varepsilon > 0)$ and such that the associated
parabolic problem has periodic solutions. We refer the reader to Hirsch [7] for a very related construction due to G. Auchmuty. First take $-A$ on $\Omega$, the unit disc, with homogeneous Dirichlet conditions and choose a real eigenvalue $c$ with complex eigenfunction $\omega = h(r) e^{im\theta}$, $m$ integer $\neq 0$, $x = (r, \theta)$. Next note that the operator

$$\tilde{A}u := -Au - \frac{\partial u}{\partial \theta} + cu, \quad u|_{S\Omega} = 0$$

has $im$ as an eigenvalue. Let $\lambda$ be its principal eigenvalue which, by the Krein–Rutman Theorem, has to be negative, and define $A_x u := (\epsilon/|\lambda|) \tilde{A}u$. Note that $\text{Re} \sigma(A_x) \geq -\epsilon$. Finally we observe that any linear combination of the Real and Imaginary parts of $h(r) e^{i(\epsilon/|\lambda|)m(\theta + t)}$ solves the equation

$$\begin{align*}
\frac{\partial u}{\partial t} - A_x u &= 0 \\
u|_{S\Omega} &= 0.
\end{align*}$$

(b) We now turn to the asymptotically periodic initial-boundary value problem, which we denote by (IBP): we still assume $\mathcal{A}(t)$ to be $T$-periodic in $t$ but admit nonlinearities $\tilde{g}$ having a decomposition

$$\tilde{g}(x, t, \xi) = g(x, t, \xi) + h(x, t, \xi),$$

where $g$ is $T$-periodic in $t$ and $h(\cdot, t, \cdot) \to 0$ as $t \to \infty$ (in a sense to be made precise). The initial-value problem with the periodic nonlinearity $g$ we still denote by (IVP).

We intend to apply Theorems 2 and 3 and define the mappings $S_n(n \in \mathbb{N})$ by

$$u_0 \in X \to S_n(u_0) = \tilde{u}((n + 1) T),$$

where $\tilde{u}$ is the solution of the (IBP) but with initial condition $\tilde{u}(nT) = u_0$. The limit operator $S$ is still the period map for the ($T$-periodic) (IVP).

In order to guarantee that these mappings are well-defined we assume:

(i) There exist $\bar{u} < \tilde{u}$, sub- and supersolutions of the (IVP) as in part (a) above, and set again $D := [\bar{u}_0, \tilde{u}_0]_X$.

(ii) There exist $\tilde{u}_0 < \bar{u}_0$ in $D$ such that the associated solutions $\tilde{g}(t) < \bar{g}(t)$ of the (IVP) exist $\forall t \geq 0$ and are bounded in $L^\infty(\Omega \times (0, \infty))$, and such that $\forall n \in \mathbb{N}$,

$$D_n := [\tilde{g}(nT), \bar{g}(nT)]_X \subset D.$$
Let \( \tilde{c} := \max \left\{ \| \tilde{u} \|_{L^\infty(\Omega \times [0, \infty))}, \| \tilde{u} \|_{L^2(\Omega \times [0, \infty))}, \| u \|_{L^\infty(\Omega \times [0, T])}, \| \tilde{u} \|_{L^\infty(\Omega \times [0, T])} \right\} < \infty \). To ensure that \( S_n \to S(n \to \infty) \) along the trajectory of \( u_0 \in D_0 \), we suppose

\[
\int_{nT}^{(n+1)T} \| \sup_{|\xi| \leq \tilde{c}} |h(\cdot, s, \xi)| \|_{L^p(\Omega)} \, ds \to 0 \quad (n \to \infty). \tag{4.4}
\]

We now can state

**Theorem 5.** If condition (H) holds for the nonlinearity \( g \), and if (4.4) holds, for any \( u_0 \in D_0 \) the solution of the \((\text{IVP})\) is quasi-convergent:

\[ \tilde{\omega}(u) \subset E = \{ w \in D : w \text{ is } T\text{-periodic point of the } (\text{IVP}) \}. \]

**Remark 4.4.** The following simple example (suggested by Ball [4]) shows that we need not have convergence under the above conditions. Consider the

\[
\begin{cases}
\partial_t u - \Delta u = \cos t + h(t) & \text{in } \Omega \times (0, \infty) \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega \times (0, \infty) \\
u(0) = 0 & \text{on } \Omega,
\end{cases}
\]

(\(\text{IVP}\))

where \( h: \mathbb{R} \to \mathbb{R} \) is such that \( h(t) \to 0(t \to \infty) \), but

\[
\limsup_{s \to 0} \int_{0}^{s} h(s) \, ds = 1, \quad \liminf_{s \to 0} \int_{0}^{s} h(s) \, ds = -1.
\]

Here \( T = 2\pi \), and the solution is given by \( u(t) = \sin t + \int_{0}^{t} h(s) \, ds \). Thus \( \tilde{\omega}(0) = \{-1, 1\} \).

In order to obtain convergence, we strengthen condition (4.4) to

\[
\int_{0}^{\infty} \| \sup_{|\xi| \leq \tilde{c}} |h(\cdot, s, \xi)| \|_{L^p(\Omega)} \, ds < \infty. \tag{4.5}
\]

Then we can state the stronger

**Theorem 6.** Under hypothesis (4.5), for any \( u_0 \in D_0 \) the solution \( u(t) \) of the \((\text{IVP})\) converges to a periodic solution \( w(t) \) of the \((\text{IVP})\) as \( t \to \infty \).

**Proof of Theorems 5 and 6.** Take \( u_0 \in D_n, n \in \mathbb{N} \), and set \( u(t), \tilde{u}(t) \) to be the solutions of the \((\text{IVP})\) and \((\text{IVP})\), respectively, with the initial condition \( u(nT) = \tilde{u}(nT) = u_0 \). For \( t \geq nT \) we have
\[ \ddot{u}(t) = U(t, nT) u_0 + \int_{nT}^t U(t, s) \tilde{g}(s, \ddot{u}(s)) \, ds \]

\[ u(t) = U(t, nT) u_0 + \int_{nT}^t U(t, s) g(s, u(s)) \, ds, \]

where \( U(t, s) \) is the evolution operator associated with \( A(t) \). Subtracting,

\[ (\ddot{u} - u)(t) = \int_{nT}^t U(t, s) [ \tilde{g}(s, \ddot{u}(s)) - g(s, u(s)) ] \, ds. \]

Periodicity implies

\[ U(t, s) = U(t + T, s + T). \]

Therefore

\[ \| U(t, s) \|_{L^\infty(X)} \leq M, \quad nT \leq s \leq t \leq (n + 1) T, \quad \forall n \in \mathbb{N}. \]

For \( nT \leq t \leq (n + 1) T \),

\[ \|(\ddot{u} - u)(t)\|_X \leq M \int_{nT}^t \| \tilde{g}(s, \ddot{u}(s)) - g(s, u(s)) \|_X \, ds + M \int_{nT}^t \| g(s, \ddot{u}(s)) - g(s, u(s)) \|_X \, ds \]

\[ \leq M \int_{nT}^t \| h(s, \ddot{u}(s)) \|_X \, ds + MC \int_{nT}^t \| (\ddot{u} - u)(s) \|_X \, ds, \]

where we used that \( g \) is Lipschitz in \( \xi \), \( |\xi| \leq \bar{c} \). Therefore

\[ \|(\ddot{u} - u)(t)\|_X \leq Md_n + MC \int_{nT}^t \|(\ddot{u} - u)(s)\|_X \, ds \]

for all \( t \) in \([nT, (n + 1) T]\), all \( n \in \mathbb{N} \),

\[ d_n := \int_{nT}^{(n + 1)T} \| \sup_{|\xi| \leq \bar{c}} |h(\cdot, s, \xi)| \|_X \, ds. \]

Applying Gronwall's inequality we obtain

\[ \|(\ddot{u} - u)((n + 1) T)\|_X \leq Md_n \exp \left( \int_{nT}^{(n + 1)T} MC \, ds \right) = Me^{MC(n + 1)T} d_n. \]
So,
\[ \| (S_n - S)(u_0) \| _X \leq C' d_n, \quad \forall u_0 \in D. \]

Theorems 2 and 3 render now correspondingly Theorems 5 and 6. The proof is complete.

**Remark 4.5.** We can alternatively base our application on the hypothesis mentioned in Remarks 2.6 and 3.4. This has the advantage that we do not have to localize our considerations: we may take \( D = X = L^p(\Omega)(N < p < \infty) \) and do not need any pair of ordered sub- and supersolutions. On the other hand, we need to impose global conditions on \( g \) and \( \tilde{g} \) (in \( \xi \in \mathbb{R} \)).

**REFERENCES**