NORTH-HOLLAND

Generalized Quadrangles With a Regular Point and Association Schemes

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Dedicated to J. J. Seidel

Submitted by W. H. Haemers

ABSTRACT

There is a new method of constructing generalized quadrangles (GQs) which is based on covering of nets; all GQs with a regular point can be represented in this way. Here we first construct from a generalized quadrangle $\mathscr{Q}$ with a regular point a four-class association scheme $\mathscr{A}(\mathscr{Q})$ called in brief *geometric*. It is then natural to call pseudo-geometric any association scheme $\mathscr{A}$ with the same parameters as $\mathscr{A}(\mathscr{Q})$. We use eigenvalue techniques and the above method of construction to give a characterization of pseudo-geometric association schemes which are geometric.

* Research partially supported by EEC Human Capital and Mobility program, contract no. ERBCHBGCT920004.

LINEAR ALGEBRA AND ITS APPLICATIONS 226–228:87–104 (1995)

0024-3795/95/$9.50
SSDI 0024-3795(94)00305-W

1. INTRODUCTION

This paper finds its motivation in the well-known paper [B] of 1963 in which Bose introduced partial geometries in order to provide a setting and generalization for known characterization theorems for strongly regular graphs. Definitions and basic facts will be given mainly in Sections 2 and 3, and whenever it will be necessary to keep the paper as self-contained as possible (the reader is referred to [CvL], [C], [BI], [BCN], [H], [HP], and [PT] for further details). In this section, we would like to point out the contribution of J. J. Seidel, to whom the paper is dedicated, in our and in similar problems.

A strongly regular graph is called in [B] "geometric" if it is the point graph of a partial geometry. It is a pseudo-geometric graph if it has the same parameters as the point graph of a partial geometry. Many authors contributed to the fundamental problem, which dates back to Bose, to establish when a pseudo-geometric graph is geometric; in particular Seidel, with Cameron and Goethals (see [CGS] and also [H80]) proved that any pseudo-geometric $(s^2, s, 1)$-graph is geometric.

Association schemes with two associate classes are strongly regular graphs. Therefore it is natural to try to relate to a partial geometry also association schemes with more than two classes to be called "geometric," and then try to use association schemes with the same parameters (i.e., "pseudo-geometric") for constructing partial geometries.

In this paper, we consider a similar problem. Namely, given a generalized quadrangle $\mathcal{Q}$ of order (s, t) with a regular point (see Section 2 for the definition), we construct a four-class association scheme $\mathcal{A}(\mathcal{Q})$ which we define to be "*geometric*." A four-class association scheme $\mathcal{A}$ with the same parameters as $\mathcal{A}(\mathcal{Q})$ will be called "*pseudo-geometric*." The first purpose of the paper is to show how the eigenvalue techniques of Seidel and his school (see in particular [H]) can be used to characterize our pseudo-geometric association schemes which are geometric (for results in the same spirit see, for instance, [BH], [Pa], [HPa]).

In Section 2, we fix notation and we recall how a triangle-free partial plane which covers a net of order $(s - 1, t)$ can be completed to construct a $GQ(s, t)$ (see [Lö] and also [GO] for further construction methods of GQs using nets).

In Section 3, we construct our geometric association scheme $\mathcal{A}(\mathcal{Q})$ and we compute its parameters and eigenvalues.

In Sections 4 and 5, we find necessary conditions for reconstructing from a pseudo-geometric $\mathcal{A}$ a triangle-free partial plane which covers a net and thus, using the construction method in [Lö], we can reconstruct from $\mathcal{A}$ a generalized quadrangle $\mathcal{Q}[\mathcal{A}]$.

In Section 6, we prove that these conditions are also sufficient for $\mathscr{A}$ to be geometric.

The second purpose of this paper is to give an application of the above-mentioned result of [Lö]. We should note, however, that some of the arguments in Section 5 could also be given in terms of the quotient scheme (see [BCN, pp. 51–52]) of our pseudo-geometric $\mathscr{A} = (X; R_0, \dots, R_4)$. Actually, the graph (X, R_1) is a union of cliques and R_1 is an equivalence relation. Therefore the association scheme is imprimitive and we can form the quotient scheme. The corresponding partition on relations is $\{R_0, R_1\}$, $\{R_2, R_3\}$ $\{R_4\}$. The quotient scheme is a strongly regular graph with the same parameters of a net graph. We leave to the interested reader to give an approach in terms of quotient schemes.

2. DEFINITIONS AND BASIC FACTS

A *partial geometry* of *order* (s, t) and *parameter* α (briefly, $pG_\alpha(s, t)$) is an incidence structure with $s + 1$ points on a line, $t + 1$ lines on a point, two points on at most one line, such that for all antiflags (x, l) the antiflag number (i.e., the number of points in the line l collinear with the point x) is a constant $\alpha \neq 0$.

When $\alpha = 1$, a $pG_1(s, t)$ is a generalized quadrangle $GQ(s, t)$ of order (s, t) (see [PT]). When $\alpha = t$, a $pG_t(s, t)$ is a net $\mathscr{N}(s, t)$ of order (s, t). We warn the reader that with the standard terminology introduced by Bruck [Br], this is a net of order $s + 1$ and degree $t + 1$.

Let $\mathscr{Q}$ be a $GQ(s, t)$. For a point x of $\mathscr{Q}$, we denote by $x^{\perp}$ the set of points of $\mathscr{Q}$ collinear with x. If S is a subset of the pointset of $\mathscr{Q}$, then

$$S^{\perp} := \bigcap_{x \in S} x^{\perp}.$$

A *point* ∞ of $\mathscr{Q}$ is said to be *regular* if for every x not collinear with ∞, we have $|\{x, \infty\}^{\perp\perp}| = t + 1$.

In all of this paper, the incidence structures considered will be *partial planes* (i.e., incidence structures of points and lines such that there is at most one line on two points or, equivalently, two lines meet in at most one point).

A morphism $\pi : X \to Y$ of two partial planes is said to be a *covering* if the following condition holds

(C) for every element $x \in X$ (x a point or a line), π maps x together with the elements incident with x bijectively to $\pi(x)$ and the elements incident with $\pi(x)$.

If Y is connected, the *index* of π is a well-defined number equal to cardinality of a preimage of an element of Y (see [C]).

Löwe proved in [Lö, 2.1] the following theorem.

THEOREM 2.1. *There exists a $GQ(s, t)$ with a regular point if and only if there exists a covering $\pi : X \to Y$ of index t where X is a triangle-free partial plane and $Y = \mathcal{N}(s-1, t)$ is a net.*

Let $\mathcal{N} = \mathcal{N}(s-1, t)$ be a net and let $\pi : X \to \mathcal{N}$ be a covering of index t where X is a triangle-free partial plane.

In the sequel we shall need the construction which yields a generalized quadrangle $\mathcal{Q}$ of order (s, t) from π.

Define *points* of $\mathcal{Q}$ as

(i) the points of X,
(ii) the symbols $[l]$, where l is a line of $\mathcal{N}$,
(iii) one new symbol ∞.

Lines are defined as

(a) the lines of X,
(b) the symbols $[\mathcal{L}]$, where $\mathcal{L}$ is a parallel class of lines in $\mathcal{N}$

Incidence is defined as follows: A point of type (i) is incident only with lines of type (a), and the incidence is induced from the old incidence relation on X. A point $[l]$ of type (ii) is incident with the t lines of type (a) of $\pi^{-1}(l)$ and with the line $[\mathcal{L}]$ of type (b), where $\mathcal{L}$ is the parallel class of l in the net $\mathcal{N}$.

The point ∞ is incident with no line of type (a) and all $t+1$ lines of type (b).

The generalized quadrangle $\mathcal{Q}$ of order (s, t) obtained from π as above has a regular point ∞ such that X is the partial plane obtained from $\mathcal{Q}$ by deleting ∞, all lines through ∞, and all points on those lines. Furthermore, all generalized quadrangles with a regular point can be constructed in this way ([Lö]).

3. THE GEOMETRIC ASSOCIATION SCHEME

Let $\mathcal{Q}$ be a generalized quadrangle of order (s, t) with $s > t > 2$ and regular point ∞. Let X be the subgeometry of $\mathcal{Q}$ induced by $\mathcal{Q}$ on the set of points of $\mathcal{Q}$ not collinear with ∞. For a point x of X, we denote by x_∞ the set

$$x_\infty = \{x, \infty\}^\perp. \tag{3.1}$$

Since ∞ is a regular point, we have $x_\infty^\perp = |\{x, \infty\}^{\perp\perp}| = t + 1$ and obviously $x_\infty^\perp \ni x, \infty$.

To define the geometric association scheme, we need the following lemma.

LEMMA 3.1. *With the above notation, if* x, $y \in X$, *then*

$$|x_\infty \cap y_\infty| \in \{0, 1, t + 1\}.$$

Proof. Since x is not collinear with ∞, on each of the $t + 1$ lines l_i $(i = 0, \dots, t)$ through ∞, we have exactly one point, say p_i, collinear with x. Thus $x_\infty = \{p_0, \dots, p_t\}$, and $|x_\infty| = t + 1$.

To prove the lemma, we show that $|x_\infty \cap y_\infty| \geqslant 2$ implies $x_\infty = y_\infty$. If $x = y$, obviously $x_\infty = y_\infty$. If $x \neq y$, let $x_\infty^\perp = \{\infty, x, x_2, \dots, x_t\}$. For each chosen $i = 0, \dots, t$, the lines $p_i\infty, p_i x, p_i x_2, \dots, p_i x_t$ are $t + 1$ different lines on p_i, hence they are *all* the lines through p_i. When $|x_\infty \cap y_\infty| \geqslant 2$, we may assume, without loss of generality, that $p_0 \neq p_1 \in y_\infty$. Thus there are i and j such that y is on $p_0 x_i$ and on $p_1 x_j$. It cannot be $y \neq x_i$, otherwise $\{p_1, y, x_i\}$ would be a triangle in $\mathscr{Q}$. Hence $y = x_i$, so $y \in \{x, \infty\}^{\perp\perp}$, which implies $x_\infty = y_\infty$. ∎

We consider on the points of X the five (symmetric) relations $R_0, \dots, R_4$ defined by

$$\begin{aligned}
\{x, y\} \in R_0 &\quad\Leftrightarrow\quad x = y, \\
\{x, y\} \in R_1 &\quad\Leftrightarrow\quad x \neq y \text{ and } x_\infty = y_\infty, \\
\{x, y\} \in R_2 &\quad\Leftrightarrow\quad x, y \text{ collinear}, \\
\{x, y\} \in R_3 &\quad\Leftrightarrow\quad x, y \text{ not collinear}, |x_\infty \cap y_\infty| = 1, \\
\{x, y\} \in R_4 &\quad\Leftrightarrow\quad x_\infty \cap y_\infty = \varnothing.
\end{aligned} \tag{3.2}$$

By Lemma 3.1, $R_1, \dots, R_4$ are a partition of the set of 2-elements subsets of X into four nonempty classes. It is long and tedious, but elementary, to verify the following two conditions:

(a) given $x \in X$, the number $n_i(x)$ of points $y \in X$ with $\{x, y\} \in R_i$ depends only on i, not on x (so we write this number as n_i);

(b) given $x, y \in X$ with $\{x, y\} \in R_k$, the number $p_{ij}^k(x, y)$ of points $z \in X$ with $\{x, z\} \in R_i$ and $\{y, z\} \in R_j$ depends only on i, j, k, not on x and y (so we write this number as p_{ij}^k).

Points x and y are called ith *associates* if $\{x, y\} \in R_i$, and $\mathscr{A}(\mathscr{Q}) = (X; R_0, \dots, R_4)$ is an *association scheme* with $m = 4$ associate classes, whose parameters n_i and $\{p_{ij}^k\}$ will be given in Theorem 3.2 below. For more information on association schemes, we refer the reader to [CvL, chapter 17]. We shall use as usual the notation A_i for the adjacency matrix of R_i, $P_i = (p_{ij}^k)$ for the corresponding intersection matrix, and P for the first eigenmatrix of the scheme.

THEOREM 3.2. *Let $\mathscr{Q}$ be a generalized quadrangle of order (s, t) with $s > t > 2$ and regular point ∞. Let X be the subgeometry of $\mathscr{Q}$ induced by $\mathscr{Q}$ on the set of points of $\mathscr{Q}$ not collinear with ∞. Then $\mathscr{A}(\mathscr{Q}) = (X; R_0, \dots, R_4)$, where the R_i's are defined by* (2), *is an association scheme with $m = 4$ associate classes with intersection matrices P_i given by*

$$P_0 = I_5,$$

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ t-1 & t-2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & t-1 & t-2 & 0 \\ 0 & 0 & 0 & 0 & t-1 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ p_{22}^0 & 0 & s-2 & t & t+1 \\ 0 & p_{23}^1 & (t-1)t & (t-2)t+s-2 & (t-1)(t+1) \\ 0 & 0 & (s-t)t & (s-t)t & p_{24}^4 \end{pmatrix},$$

where

$$p_{22}^0 = (s-1)(t+1) = p_{23}^1 = p_{32}^1,$$

$$p_{24}^4 = (-t+s-1)(t+1) = p_{42}^4.$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & t-1 & t-2 & 0 \\ 0 & p_{32}^1 & (t-1)t & (t-2)t+s-2 & (t-1)(t+1) \\ p_{33}^0 & p_{33}^1 & p_{33}^2 & p_{33}^3 & (t-1)^2(t+1) \\ 0 & 0 & (s-t)(t-1)t & (s-t)(t-1)t & p_{34}^4 \end{pmatrix}$$

where:

$$p_{33}^{0} = (s-1)(t-1)(t+1),$$

$$p_{33}^{1} = (s-1)(t-2)(t+1),$$

$$p_{33}^{2} = (t-1)((t-2)t+s-2)$$

$$p_{33}^{3} = (t-1)t+(t-2)^2t+(s-2)(t-2)$$

$$p_{34}^{4} = (-t+s-1)(t-1)(t+1) = p_{43}^{4},$$

$$P_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & t-1 \\ 0 & 0 & (s-t)t & (s-t)t & p_{42}^{4} \\ 0 & 0 & (s-t)(t-1)t & (s-t)(t-1)t & p_{43}^{4} \\ (s-1)(s-t)t & (s-1)(s-t)t & p_{44}^{2} & p_{44}^{3} & p_{44}^{4} \end{pmatrix},$$

where

$$p_{44}^{4} = t\big(t+(-t+s-1)^2-1\big),$$

$$p_{44}^{2} = p_{44}^{3} = t(-t+s-1)(s-t).$$

Furthermore, we have for the n_i's the following expressions:

$$n_0 = 1, \qquad n_1 = t-1, \qquad n_2 = (s-1)(t+1), \tag{3.3}$$

$$n_3 = (s-1)(t+1)(t-1), \qquad n_4 = t(s-t)(s-1). \tag{3.4}$$

Proof. As we already noted before, the proof is long and tedious but elementary; it can be done by simple counting arguments, using Lemma 3.1 and the definitions of GQ and of association scheme. We leave it to the interested reader. ■

We will call $\mathscr{A}(\mathscr{G})$, whose n_i's and intersection matrices are given in 3.2, the *geometric association scheme* (of the generalized quadrangle $\mathscr{G}$).

DEFINITION 3.3. A *pseudo-geometric association scheme of type* (s, t), $s > t > 2$, is a 4-class association scheme $\mathscr{A}$ with the same parameters as $\mathscr{A}(\mathscr{G})$ (see the statement of 3.2 for their expressions as functions of s, t).

Clearly, every geometric association scheme is pseudo-geometric. We note that for a pseudo-geometric association scheme,

$$s = p_{22}^2 + 2, \qquad t = p_{11}^0 + 1. \tag{3.5}$$

THEOREM 3.4. *The first eigenmatrix P of a pseudo-geometric association scheme $\mathscr{A}$ (i.e., the matrix with (i, j) entry the ith eigenvalue $p_j(i)$ of the matrix P_j given in the statement of* 3.2) *is*

$$P = \begin{pmatrix} 1 & t-1 & -t+s-1 & (1-t)(t-s+1) & t(t-s) \\ 1 & t-1 & -t-1 & (1-t)(t+1) & t^2 \\ 1 & -1 & -t-1 & t+1 & 0 \\ 1 & -1 & s-1 & 1-s & 0 \\ 1 & t-1 & (s-1)(t+1) & (s-1)(t-1)(t+1) & (1-s)t(t-s) \end{pmatrix}.$$

The multiplicities μ_i have the following expressions:

$$\mu_0 = (s-1)(t+1), \qquad \mu_1 = (s-1)(s-t),$$

$$\mu_2 = \frac{s^2(t-1)(s+1)}{s+t}, \tag{3.6}$$

$$\mu_3 = \frac{s^2(t-1)(t+1)}{s+t}, \qquad \mu_4 = 1. \tag{3.7}$$

Proof. The matrix P was obtained using MACSYMA. We were looking for a basis of $\mathbf{R}^5$ giving a simultaneous diagonalization of the matrices P_i. Since P_3 has five different eigenvalues, the corresponding eigenvectors $\mathbf{u}_0, \ldots, \mathbf{u}_4$ gave the required basis. Then, the entries of P were given by $\mathbf{u}_j P_i = p_i(j)\mathbf{u}_j$.

Using Theorem (17.12) of [CvL], we obtained for the multiplicities the above expressions. ∎

4. A PARTIAL PLANE FROM A PSEUDO-GEOMETRIC $\mathscr{A}$

Let $\mathscr{A} = (X; R_0, \ldots, R_4)$ be a pseudo-geometric association scheme of type (s, t) and let $\{p_{ij}^k\}$ be its parameters (see 3.2 and 3.3). We shall assume from now on that the maximal cliques with respect to R_2 (i.e., the maximal "complete subgraphs" of the graph R_2) have size $s = p_{22}^2 + 2$. It is not clear whether this condition can be dropped.

On the pointset X of $\mathscr{A} = (X; R_0, \ldots, R_4)$, we define an incidence structure $\mathscr{X}$ taking as set $\{\mathscr{L}\}$ of *lines* the set of the maximal "cliques with respect to R_2" (briefly, R_2-cliques) of $\mathscr{A}$. Incidence is containment.

THEOREM 4.1. *With the above notation, the following conditions hold:*

(a) *the incidence structure* $\mathscr{X} = (X, \{L\}, \in)$ *is a triangle-free partial plane*;

(b) *every point of* X *is on* $t + 1$ *lines.*

Proof. (a) Since $p_{22}^2 = s - 2$, a point which is second associate with two different points on a line must already be on that line.

(b) This follows from $n_2 = (s - 1)(t + 1)$, (a), and the assumption that every line contains s points. ■

The purpose of this section is to collect some information on the structure of $\mathscr{X}$. The main tool we shall use is eigenvalue interlacing (see [H80] or [H]) and mainly theorem 1.2.3 of [H80].

NOTATION 4.2. Let L be a line of the partial plane $\mathscr{X}$ or, equivalently, a maximal R_2-clique of the given pseudo-geometric association scheme $\mathscr{A} = (X; R_0, \ldots, R_4)$. We denote by Π_L the set

$$\Pi_L := \{x \in X \mid \exists p \in L \text{ such that } \{p, x\} \in R_1\}. \tag{4.1}$$

For $p \in L$, we denote by L_p the set

$$L_p := \{x \in \Pi_L \mid \{p, x\} \in R_1\}. \tag{4.2}$$

Let $T = T(L)$ be the pointset $L \cup \Pi_L$. This gives a partition $X = T \,\dot{\cup}\, (X - T)$ which yields a decomposition of each adjacency matrix A_i of the pseudo-geometric association scheme $\mathscr{A}$:

$$A_i = \begin{pmatrix} A_{iT} & C_i \\ C_i^T & A_{i(X-T)} \end{pmatrix}, \qquad i = 0, 1, \ldots, 4, \tag{4.3}$$

where C_i^T is the transpose of C_i.

We denote by B_i the corresponding matrix of the average row sums. We shall apply theorem 1.2.3 of [H80] to such a decomposition of the matrices A_i. First we collect some information on the set T.

LEMMA 4.3. *If $p, q \in L$, with $p \neq q$, then $L_p \cap L_q = \varnothing$. This implies*

$$|T| = st. \tag{4.4}$$

Proof. The first part of the statement is an easy consequence of $p_{11}^2 = 0$. For each of the s points of L, we have $n_1 = t - 1$ first associates. Thus if $p \in L$, then $|L_p| = n_1 = t - 1$. Now $\Pi_L = \dot{\bigcup}_{p \in L} L_p$, so that $|\Pi_L| = s(t - 1)$ and $|T| = |L| + |\Pi_L| = s + s(t - 1) = st$. ■

LEMMA 4.4. *Let $p \in L$. Then for $y \in \Pi_L$, we have either $\{p, y\} \in R_1$ or $\{p, y\} \in R_3$.*

Proof. Let q be the point of L which is first associate of $y \in \Pi_L$. If $q = p$, then $\{p, y\} \in R_1$. If $q \neq p$, then q and p, which are both on L, are second associates, by definition of the lines in $\mathscr{L}$. Let p and y be k-th associates. Since $p_{12}^k = 0$ for $k \neq 3$, while $p_{12}^3 = 1$, the only possibility is $k = 3$. Therefore, when $q \neq p$ we have $\{p, y\} \in R_3$. This completes the proof. ■

THEOREM 4.5. *We have*

(i) *the set T is a disjoint union of s R_1-cliques of size t, namely,*

$$T = \dot{\bigcup_{p \in L}} \left(L_p \cup \{p\}\right). \tag{4.5}$$

(ii) *If $x, y \in T$, then $\{x, y\} \notin R_4$.*

(iii) *For every $x \in T$, there are exactly $s - 1$ points z of T with $\{x, z\} \in R_2$.*

Proof. (i) The proof of Lemma 4.3 shows that $\Pi_L = \dot{\bigcup}_{p \in L} L_p$ and $|L_p| = n_1 = t - 1$. Since $L \cap \Pi_L = \emptyset$, we have (4.5). It remains to prove that $L_p \cup \{p\}$ is a clique with respect to R_1. since $p_{11}^k \neq 0$ only for $k = 0, 1$, two different points of L_p are necessarily first associates. This, by definition of L_p, completes the proof of (i).

(ii) By Lemma 4.4, we only have to prove the statement when x and y are in Π_L. Let p and q be points of L first associates with x and y, respectively. We proved in (i) that if $p = q$, then x and y are first associates. If $p \neq q$, we obtain from Lemma 4.4 that p and y are third associates, hence there is at least the point p among the points which are first associates of x and third associates of y. Since $p_{13}^k \neq 0$ only for $k = 2, 3$, then x and y are either second or third associates and the proof of (ii) is complete.

(iii) If x is on L, then there are exactly $s - 1$ further points on L and by Lemma 4.4 we have the statement.

If x is not on L, there is a unique point p on L which is first associate of x. Let $q \neq p$ be another point of L. Then every point y of L_q is a third associate of p (see 4.4) and since $p_{12}^3 = 1$, there exists precisely one $p(y) \in L_p$ such that $\{p, p(y)\} \in R_1$ and $\{p(y), y\} \in R_2$. From $p_{12}^2 = 0$, we infer that for $y \neq y'$, also $p(y) \neq p(y')$. Hence the map $L_q \to L_p$, $y \mapsto p(y)$ is injective. Because $|L_p| = |L_q|$, it is also surjective. Therefore there exists a unique $z \in L_q$ with $p(z) = x$. There are $s - 1$ possible choices of q, whence (iii). ■

LEMMA 4.6. *The matrices B_j, $j \neq 2$, of the average row sums are*

$$B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} t-1 & 0 \\ 0 & t-1 \end{pmatrix}, \tag{4.6}$$

$$B_3 = \begin{pmatrix} (t-1)(s-1) & t(s-1)(t-1) \\ t(t-1) & (t-1)[t(s-2)+(s-1)] \end{pmatrix}, \tag{4.7}$$

$$B_4 = \begin{pmatrix} 0 & t(s-t)(s-1) \\ t(s-t) & t(s-t)(s-2) \end{pmatrix}. \tag{4.8}$$

Furthermore, the interlacing of the eigenvalues of B_j and A_j for $j = 0, 1, 3, 4$ is tight. Therefore, the four submatrices of A_j have constant row sums, so that the B_j's are actually the matrices of the row sums.

Proof. The statement is trivial for A_0 and B_0.

It follows from $n_1 = t - 1$ and from the construction of T that B_1 is given by (6). Now looking at the matrix P of Theorem 3.4, we see that the eigenvalues of A_1 are $t - 1$ and -1 and the first eigenvalue has multiplicity greater than 1, thus the interlacing is tight also for B_1 and A_1.

By Theorem 4.5, a point x of T has $t - 1$ first associates in T, $s - 1$ second associates in T, and there are no fourth associates of x which are in T. If σ is the number of points in T which are third associates of x, then by (4.4), we get $|T| = st = 1 + (s - 1) + (t - 1) + \sigma$. This yields the first entry of B_3, namely,

$$\sigma = (s - 1)(t - 1). \tag{4.9}$$

Every row of the submatrix $(A_{3T}\ C_3)$ of A_3 contains $n_3 = st(s - 1)(t - 1)^2$ entries equal to 1. Since exactly $(t - 1)(s - 1)$ are in A_{3T}, we have that the matrix C_3 (which has st rows and $s^2t - st = st(s - 1)$ columns) has constant row sum $n_3 - (t - 1)(s - 1) = t(s - 1)(t - 1)$. Thus the sum of all entries of C_3 is exactly $st^2(s - 1)(t - 1)$, so that dividing by the number $st(s - 1)$ of columns, we obtain, for the average column sum of C_3, the expression $t(t - 1)$. Since this is the average row sum of C_3^T, and A_3 has constant row sum n_3, we have that $n_3 - t(t - 1) = (t - 1)[t(s - 2) + (s - 1)]$ is the average row sum of $A_{3(X-T)}$. Therefore, the matrix B_3 is given by (7). Now the eigenvalues of B_3 are $(s - 1)(t - 1)(t + 1)$ and $(t - 1)(s - t - 1)$. From the matrix P of Theorem 3.4, we see that A_3 has largest eigenvalue $(s - 1)(t - 1)(t + 1)$ with multiplicity 1 and second largest eigenvalue $(t - 1)(s - t - 1)$. Hence the interlacing is tight by definition.

Theorem 4.5 implies that the (1, 1) entry of B_4 is 0. Since A_4 has constant row sum n_4, the (1, 2) entry of B_4 is $n_4 = t(s - t)(s - 1)$. As in the previous paragraph, this implies that multiplying by $st/st(s - 1) = 1/(s - 1)$, we obtain the average row sum $t(s - t)$ of C_4^T, which yields $n_4 - t(s - t) = t(s - t)(s - 2)$ as average row sum of $A_{4(X-T)}$. Now the eigenvalues of B_4 are $(s - 1)t(s - t)$ and $t(t - s)$. From the matrix P of Theorem 3.4, we see that A_4 has largest eigenvalue $(s - 1)t(s - t)$ and smallest eigenvalue $t(t - s)$. Hence the interlacing is tight by definition.

By (ii) in theorem 1.2.3 of [H80], we have that the four submatrices of A_j have constant row sums given by the corresponding elements of B_j for $j = 0, 1, 3, 4$. ■

LEMMA 4.7. *We have*

$$B_2 = \begin{pmatrix} s - 1 & t(s - 1) \\ t & (s - 1)(t + 1) - t \end{pmatrix}. \tag{4.10}$$

Furthermore, the four submatrices of A_2 *have constant row sums given by the corresponding elements of* B_2.

Proof. It is an easy consequence of Lemma 4.6, since

$$J = A_0 + A_1 + A_2 + A_3 + A_4. \tag{4.11}$$

Obviously, the fact that B_2 is given by (4.10) could be proved directly as before, but since the interlacing of the eigenvalues is not tight, to deduce that the four submatrices of A_2 have constant row sums, we need to use Lemma 4.6 and (4.10). ∎

We conclude this section by applying these results to give information on the set $X - T$.

THEOREM 4.8. *For every point* y *of* $X - T$, *the following conditions hold*:

(i) *there are exactly* t *points of* T *which are second associates with* y,
(ii) *if every line through* y *meets* T *in at most one point, then there exists exactly one line on* y *which does not meet* T.

Proof. The submatrix C_2^T of the adjacency matrix A_2 has constant row sums t, by Lemma 4.7; therefore, for every point y of $X - T$, we have exactly t points of T which are second associates with y and we get (i).

If every line through y meets T in at most one point, then by Theorem 4.1(b), there is exactly one line through y which does not meet T, proving (ii). ∎

As in the proof of Theorem 4.8(i), other incidence properties of the sets T and $X - T$ can be deduced easily from the fact that the B_i's are actually the matrices of "constant" row sums of the matrices A_i.

5. A SECOND INCIDENCE STRUCTURE FROM $\mathscr{A}$

Let $\mathscr{A} = (X; R_0, \ldots, R_4)$ be a pseudo-geometric association scheme of type (s, t), $s > t > 2$, and thus with parameter p_{ij}^k given by the (k, j) entry of the matrix P_i of Theorem 3.2. Let $t := p_{11}^0 + 1$.

As in Section 4, we shall assume that the maximal R_2-cliques have size $s = p_{22}^2 + 2$.

DEFINITION 5.1. We define an incidence structure $\mathcal{N} = \mathcal{N}[\mathcal{A}]$ *as follows. Every* $x \in X$ *gives a point* $\bar{x}$ *of* $\mathcal{N}$, *namely, the t-subset of the elements of X which are zero or first associates of* x:

$$\bar{x} := \{y \in X \mid \{x, y\} \in R_1\} \cup \{x\}. \tag{5.1}$$

A *line* $\overline{L}$ of $\mathcal{N}$ is obtained from a line L of the partial plane $\mathcal{X} = (X, \{L\}, \in)$ defined at the beginning of Section 4, by

$$\overline{L} := T(L) = L \cup \Pi_L, \tag{5.2}$$

where Π_L is defined in (4.1).

By abuse of notation, we shall denote by $\bar{x}$ and $\overline{L}$ the corresponding subsets of X; the meaning will be clear from the context.

The point $\bar{x}$ of $\mathcal{N}$ is *incident* with the line $\overline{L}$ (and we write $\bar{x} \mathrel{\mathrm{I}} \overline{L}$) if the subset $\bar{x}$ is contained in the subset $\overline{L} = T(L)$. Note that by Theorem 4.5 (i), either $\bar{x} \subset \overline{L}$ or $\bar{x} \cap \overline{L} = \varnothing$.

Clearly, the map $\pi : \mathcal{X} \to \mathcal{N}$, defined by

$$x \mapsto \bar{x}, \qquad L \mapsto \overline{L},$$

sends flags of $\mathcal{X}$ in flags of $\mathcal{N}$ and thus preserves incidence.

LEMMA 5.2. *If* $\mathcal{N}$ *is a partial plane, then every* $T(L)$ *is a disjoint union of lines of* $\mathcal{X}$, *and no other line of* $\mathcal{X}$ *has more than one point in* $T(L)$.

Proof. Let M be a line of $\mathcal{X}$ meeting $T(L)$ in two different points p and q. The $\bar{p}$ and $\bar{q}$ are incident with both $\overline{L}$ and $\overline{M}$, and since $\mathcal{N}$ is a partial plane, we get that either $\overline{L} = \overline{M}$ or $\bar{p} = \bar{q}$. Since p and q are in M, they are second associates, hence $\bar{p} \neq \bar{q}$ so that the only possibility remains $\overline{L} = \overline{M}$, in particular $M \subset L \cup \Pi_L$. If p and q are on L, then $L = M$. Otherwise, $M \cap L = \varnothing$ and $M \subset \Pi_L$. It follows from Theorem 4.5(iii) that two different lines of $\mathcal{X}$ contained in Π_L are skew lines and that Π_L is covered by lines. ∎

THEOREM 5.3. $\mathcal{N}$ *is a net of order* $(s-1, t)$ *if and only if* $\mathcal{N}$ *is a partial plane.*

Proof. By definition, a net is a partial plane.

Assume that $\mathcal{N}$ is a partial plane. First we compute its parameters.

Theorem 4.5(i) implies that every line $\bar{L}$ of $\mathcal{N}$ has s points. Let $\bar{x}$ be a point of $\mathcal{N}$ and let $L_0, \ldots, L_t$ be the lines of $\mathcal{X}$ through the point x. We claim that $\bar{L}_0, \ldots, \bar{L}_t$ are precisely the lines in $\mathcal{N}$ through $\bar{x}$.

First, if $\bar{L}_i = \bar{L}_j$, then $L_i \subset L_j \cup \Pi_{L_j}$. Since x is on L_i and L_j, we have $L_i = L_j$ (see Lemma 4.4 and the proof of Lemma 5.2). Thus the lines $\bar{L}_0, \ldots, \bar{L}_t$ are pairwise different.

Let $\bar{L}$ be a line on $\bar{x}$. By Lemma 5.2, x is on a line M which is contained in $L \cup \Pi_L$. Hence there exists an i, $0 \leqslant i \leqslant t$, such that $M = L_i$, which implies $\bar{L} = \bar{L}_i$. Therefore $\bar{L}_0, \ldots, \bar{L}_t$ are all the lines through $\bar{x}$.

It remains to prove that for any antiflag $(\bar{x}, \bar{L})$, the antiflag number is t. It follows from Lemma 5.2 and Theorem 4.8(ii) that t of the lines on x, say $L_1, \ldots, L_t$, meet $\bar{L}$ in exactly one point, while the last line L_0 does not meet $\bar{L}$. Then $\bar{L}_1, \ldots, \bar{L}_t$ intersect $\bar{L}$, and since $\mathcal{N}$ is a partial plane, the intersection points $\bar{x}_i = \bar{L}_i \cap \bar{L}$ are pairwise different. We claim that $\bar{L}_0$ does not meet $\bar{L}$.

Every point of L_0 is exterior to $\bar{L}$. By Lemma 4.7, every point of L_0 is second associate with t points of $\bar{L}$. Since $p_{22}^2 = s - 2$ and $|L_0| = s$ by the assumption on the size of maximal R_2-cliques, this yields a decomposition of $L \cup \Pi_L$ into pairwise disjoint sets of size t; so for each $x \in L \cup \Pi_L$, there exists a point on L_0 which is second associate with x. By Lemma 4.4, Π_{L_0} consists of points which are never second associates with points on L_0, hence $\Pi_{L_0} \cap (L \cup \Pi_L) = \varnothing$. This implies $\bar{L} \cap \bar{L}_0 = \varnothing$, which completes the proof. ∎

We can now use Theorem 2.1 to reconstruct from a pseudo-geometric association scheme a generalized quadrangle.

THEOREM 5.4. *Let $A = (X; R_0, \ldots, R_4)$ be a pseuologeometric association scheme with parameters $\{p_{ij}^k\}$ (i.e., p_{ij}^k is given by the (k, j) entry of the matrix P_i of Theorem 3.2). Suppose that maximal R_2-cliques have size $s = p_{22}^2 + 2$. Consider the incidence structures $\mathcal{X} = (X, \{L\}, \in)$ and $\mathcal{N}$, where $\{L\}$ is the set of maximal R_2-cliques and $\mathcal{N}$ is defined in Definition 5.1. Let $\pi : \mathcal{X} \to \mathcal{N}$ be the natural projection map. If $\mathcal{N}$ is a partial plane, then the following hold:*

(i) *π is a covering of index $t = p_{11}^0 + 1$,*
(ii) *$\mathcal{X}$ is a triangle-free partial plane,*
(iii) *$\mathcal{N}$ is a net or order $(s - 1, t)$,*
(iv) *there exists a $GQ(s, t)$ with a regular point.*

Proof. (i) follows from the proof of Theorem 5.3 (ii) is Lemma 4.1(a). (iii) is Theorem 5.3. Now (iv) follows from (i)–(iii) using Theorem 2.1. For the actual construction of the GQ, see Section 2.

6. THE CHARACTERIZATION THEOREM

Given a pseudo-geometric association scheme $\mathscr{A} = (X; R_0, \ldots, R_4)$ with parameters $\{p_{ij}^k\}$, let $s := p_{22}^2 + 2$ and $t := p_{11}^0 + 1$. In Sections 4 and 5, we have associated with $\mathscr{A}$ two incidence structures $\mathscr{X} = \mathscr{X}[\mathscr{A}]$ and $\mathscr{N} = \mathscr{N}[\mathscr{A}]$ and we have given conditions under which the natural projection $\pi : \mathscr{X} \to \mathscr{N}$ yields a generalized quadrangle $\mathscr{Q} = \mathscr{Q}[\mathscr{A}]$ of order (s, t) with a regular point ∞.

From $\mathscr{Q} = \mathscr{Q}[\mathscr{A}]$, we can construct an association scheme $\mathscr{A}(\mathscr{Q}) = \mathscr{A}^* = (X, R_0^*, \ldots, R_4^*)$ as in Section 3. In this section, we will show that $\mathscr{A}^* = \mathscr{A}(\mathscr{Q}[\mathscr{A}]) = \mathscr{A}$ and thus we shall complete the proof of the following characterization theorem.

THEOREM 6.1. *A pseudo-geometric association scheme* $\mathscr{A} = (X; R_0, \ldots, R_4)$ *is geometric if and only if*

(I) *the maximal* R_2*-cliques of* $\mathscr{A}$ *have size* $s = p_{22}^2 + 2$,
(II) *the structure* $\mathscr{N}[\mathscr{A}]$ *defined in Definition* 5.1 *is a partial plane.*

Clearly (see Section 3) the subgeometry of $\mathscr{Q}[\mathscr{A}]$ induced on the set of points not collinear with ∞ is the triangle-free partial plane $\mathscr{X} = \mathscr{X}[\mathscr{A}] = (X, \{L\}, \in)$ of Section 4.

LEMMA 6.2. *If* $x \in X$, *then in the generalized quadrangle* $\mathscr{Q}[\mathscr{A}]$,

$$x_\infty = \{[\bar{L}] \mid x \in T(L)\}. \tag{6.1}$$

Proof. The points $[\bar{L}]$ of $\mathscr{Q}[\mathscr{A}]$, $\bar{L}$ a line of $\mathscr{N}$, are all collinear with ∞ (see Section 2). Now x is collinear with $[\bar{L}]$ if and only if there exists a line M in X through x such that $\pi(M) = \bar{L}$, that is, $\bar{M} = \bar{L}$. Hence $x \in T(L)$.

Suppose now that $x \in T(L)$. By Lemma 5.2, there exists a line M on x with $M \subset T(L)$. Then $\bar{M} = \bar{L}$ and so x and $[\bar{L}]$ are collinear in $\mathscr{Q}[\mathscr{A}]$. ■

THEOREM 6.3. *The relations* R_i *and* R_i^* *defined on X are equal for all* $i = 0, \ldots, 4$. *In particular,* $\mathscr{A} = \mathscr{A}^*$ *and* $\mathscr{A}$ *is geometric.*

Proof. Trivially $R_0 = R_0^*$. Next we show that $R_i^* \subseteq R_i$ for all i.

If $\{x, y\} \in R_1^*$, then $x_\infty = y_\infty$. Therefore by Lemma 6.2, for every line L on x, the point y is in $T(L)$. Since $t > 2$, we deduce from Lemma 4.1(b) that there are at least three lines on x. Let M be a line on x that does not meet y. By Lemma 4.4, $\{x, y\}$ is either in R_3 or in R_1. If $\{x, y\}$ would be in R_3, since $p_{12}^3 = 1$, there would exist exactly one z such that $\{x, z\} \in R_2$ and $\{z, y\} \in R_1$. Let L be a line on x but not on z or on y. Since $y \in L \cup \Pi_L$, there must be a point $p \neq x$ on L such that $\{p, y\} \in R_1$. This yields the contradiction $p = z \in L$. Therefore $\{x, y\} \in R_1$.

If $\{x, y\} \in R_2^*$, then x and y are collinear in $\mathscr{X}$, therefore $\{x, y\} \in R_2$.

If $\{x, y\} \in R_3^*$, then x and y are not collinear in $\mathscr{X}$, hence $\{x, y\} \notin R_2$. It follows from $|x_\infty \cap y_\infty| = 1$ and Lemma 6.2 that there exists a line L in $\mathscr{X}$ such that $x, y \in T(L)$. This implies $\{x, y\} \notin R_4$, by Theorem 4.5(ii). Since $t > 2$, there also exists a line M in $\mathscr{X}$ such that $x \in T(M) = M \cup \Pi_M$ but $y \notin T(M)$. By definition of Π_M, we deduce $\{x, y\} \notin R_1$. Therefore $\{x, y\} \in R_3$.

If $\{x, y\} \in R_4^*$, then $x_\infty \cap y_\infty = \varnothing$ and so $\{x, y\} \notin R_1 \cup R_2$. If $\{x, y\}$ would be in R_3, then $p_{12}^3 = 1$ would imply the existence of a line L through x such that y is in relation R_1 to one point of L; thus y would be in $T(L)$. Because of Lemma 6.2, this contradicts $x_\infty \cap y_\infty = \varnothing$. Therefore $\{x, y\} \in R_4$.

Thus $R_i^* \subseteq R_i$ for all $i = 1, \ldots, 4$. Since $X \times X$ is finite and since $\{R_i\}$ and $\{R_i^*\}$ are partitions of $X \times X$, we get the desired result. ∎

It would be interesting to know if one can weaken condition (II) or, at least, replace it by a condition closer to the parameters of $\mathscr{A}$. So far, research in this direction seems to indicate that this is not possible.

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Received 24 May 1994; final manuscript accepted 31 October 1994