On Riemann and Caputo fractional differences

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ABSTRACT

In this paper, we define left and right Caputo fractional sums and differences, study some of their properties and then relate them to Riemann–Liouville ones studied before by Miller K. S. and Ross B., Atıcı F.M. and Eloe P.W., Abdeljawad T. and Baleanu D., and a few others. Also, the discrete version of the $Q$-operator is used to relate the left and right Caputo fractional differences. A Caputo fractional difference equation is solved. The solution proposes discrete versions of Mittag-Leffler functions.

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1. Introduction and preliminaries about fractional calculus

Fractional calculus deals with the study of fractional order integrals and derivatives and their diverse applications [1–5]. Riemann–Liouville and Caputo are kinds of fractional derivatives. They all generalize the ordinary integral and differential operators. However, the fractional derivatives have fewer properties than the corresponding classical ones. As a result, it makes these derivatives very useful at describing the anomalous phenomena [6–11].

Very recently some solutions of equations containing left and right fractional derivatives were investigated [12–14] and some exact solutions were found [12,15]. The left and the right derivatives found interesting applications in fractional variational principles, fractional control theory as well as in fractional Lagrangian and Hamiltonian dynamics.

The discrete calculus is always preferred especially when computers are used to obtain the solutions of certain dynamic equations. The study of ordinary difference equations is widespread. However, the theory of fractional difference equations is still limited. In this article we continue the job done by the authors in [16–19] to develop the discrete fractional calculus. We define the right fractional sum and difference operators and obtain many of their properties. Then by using those properties we define the left and right Caputo fractional differences and relate them to the Riemann–Liouville ones. The discrete $Q$-operator is also defined and used to relate right and left Caputo fractional differences. The successive approximation is used to solve a fractional linear Caputo difference equation as well. Homogeneous and nonhomogenous linear fractional difference initial value problems are solved. Their solutions suggest discrete versions for the classical Mittag-Leffler functions. For the sake of comparison between our discrete results and the usual classical fractional integrals and derivatives, we refer the reader to [2,20,3] for more details.

For a function $f$ defined on an interval $J = [a, b]$, the Riemann–Liouville integrals $I_{a+}^\alpha f$ and $I_{b-}^\alpha f$ of order $\alpha \in \mathbb{C} (\text{Re}(\alpha) > 0)$ are, respectively, defined by [3]

$$
(I_{a+}^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)ds}{(t-s)^{1-\alpha}},
$$

and

$$
(I_{b-}^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(s)ds}{(s-t)^{1-\alpha}}.
$$

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Here $I^\alpha(\alpha)$ is the well-known Gamma function. The left and right Riemann–Liouville fractional derivatives $D_{a+}^\alpha f$ and $D_{b-}^\alpha f$ of order $\alpha \in \mathbb{C}$ $(\text{Re}(\alpha)) > 0$ are, respectively, defined by

$$
(D_{a+}^\alpha f)(t) = \left( \frac{d}{dt} \right)^n (I_{a+}^{n-\alpha} f)(t),
$$

(3)

and

$$
(D_{b-}^\alpha f)(t) = \left( -\frac{d}{dt} \right)^n (I_{b-}^{n-\alpha} f)(t),
$$

(4)

where $n = \lceil \text{Re}(\alpha) \rceil + 1$ and $\lceil \text{Re}(\alpha) \rceil$ is the greatest integer less than or equal to $\text{Re}(\alpha)$. While, left and right Caputo-fractional derivatives $^C D_{a+}^\alpha f$ and $^C D_{b-}^\alpha f$ are, respectively, defined by

$$
(^C D_{a+}^\alpha f)(t) = (I_{a+}^{n-\alpha} f^{(n)})(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{1+\alpha-n}} ds,
$$

(5)

and

$$
(^C D_{b-}^\alpha f)(t) = (-1)^n (I_{b-}^{n-\alpha} f^{(n)})(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \frac{f^{(n)}(s)}{(s-t)^{1+\alpha-n}} ds.
$$

(6)

**Property 1** ([3]). If $\text{Re}(\alpha) \geq 0$ and $\beta \in \mathbb{C}$ $(\text{Re}(\beta) > 0)$, then

$$
(I_{a+}^\beta (t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} (x-a)^{\beta+\alpha-1} \quad (\text{Re}(\alpha) > 0)
$$

(7)

$$
(D_{a+}^\alpha (t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} (x-a)^{\beta-\alpha-1} \quad (\text{Re}(\alpha) \geq 0)
$$

(8)

and

$$
(I_{b-}^\beta (b-t)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} (b-x)^{\beta+\alpha-1} \quad (\text{Re}(\alpha) > 0)
$$

(9)

$$
(D_{b-}^\alpha (b-t)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} (b-x)^{\beta-\alpha-1} \quad (\text{Re}(\alpha) \geq 0).
$$

(10)

In particular, if $\beta = 1$ and $\text{Re}(\alpha) \geq 0$, then the Riemann–Liouville fractional derivatives of a constant are, in general, not equal to zero:

$$
(D_{a+}^\alpha 1)(x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad (D_{b-}^\alpha 1)(x) = \frac{(b-x)^{-\alpha}}{\Gamma(1-\alpha)} \quad (0 < \text{Re}(\alpha) < 1).
$$

(11)

On the other hand, for $j = 1, 2, \ldots, \lceil \text{Re}(\alpha) \rceil + 1$,

$$
(D_{a+}^\alpha (t-a)^{\alpha-j})(x) = 0, \quad (D_{b-}^\alpha (b-x)^{\alpha-j})(x) = 0.
$$

(12)

**Property 2** ([3]). If $\text{Re}(\alpha) > 0$ and $\text{Re}(\beta) > 0$, then the equations

$$
(I_{a+}^\alpha I_{a+}^\beta f)(x) = (I_{a+}^{\alpha+\beta} f)(x), \quad (I_{b-}^\alpha I_{b-}^\beta f)(x) = (I_{b-}^{\alpha+\beta} f)(x)
$$

(13)

are satisfied at almost every point $x \in [a, b]$ for $f \in L_p(a, b), \ (1 \leq p \leq \infty)$. If $\alpha + \beta > 1$, then the relations in (13) hold at any point of $[a, b]$.

**Property 3** ([3]). The relation between Caputo fractional derivative and Riemann–Liouville fractional derivative is given by

$$
D_{a+}^\alpha y(t) = ^C D_{a+}^\alpha y(t) + \sum_{j=0}^{n-1} \frac{y^{(j)}(a)}{\Gamma(j+1-\alpha)} (t-a)^{j-\alpha}.
$$

(14)

**Property 4** ([3]). Let $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$ and let $n = \lceil \text{Re}(\alpha) \rceil + 1$. If $y(x) \in AC^n[a, b]$ or $y(x) \in C^n[a, b]$, then

$$
(I_{a+}^\alpha ^C D_{a+}^\alpha y)(x) = y(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!} (x-a)^k.
$$

(15)
and

\[(I_0^a \circ D_0^p \Delta y)(x) = y(x) - \sum_{k=0}^{n-1} \frac{(-1)^k y^{(k)}(b)}{k!} (b - x)^k.\]  \hspace{1cm} (16)

**Definition 1.** For \( \alpha, \beta, z \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), the classical Mittag-Leffler functions are defined by

\[E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.\]  \hspace{1cm} (17)

For \( \beta = 1 \), it is written that

\[E_\alpha(z) \equiv E_{\alpha, 1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.\]  \hspace{1cm} (18)

The classical Mittag-Leffler functions play an extremely important role in solving fractional linear dynamical systems. For example the solution to the Cauchy problem

\[(D_0^n + y)(x) = \lambda y(x) + f(x), \quad y(a) = b \quad (b \in \mathbb{R})\]  \hspace{1cm} (19)

given by

\[y(x) = bE_{\alpha} [\lambda(x - a)^\alpha] + \int_{a}^{x} (x - t)^{\alpha-1}E_{\alpha, \alpha}[\lambda(x - t)^\alpha]f(t)dt.\]  \hspace{1cm} (20)

This article is organized as follows: The first section is devoted to giving some preliminary results about fractional derivatives, integrals and fractional initial value problems. The discrete counterpart of these results have been recently discretized or are going to be discretized in this article. Some preliminary results about fractional sums and differences are given in Section 2. In Section 3, the discrete versions of left and right Caputo fractional differences are discussed. A nonhomogenous fractional linear initial value problem is solved, where the solution is expressed by a new introduced discrete type of Mittag-Leffler function. Finally, Section 4 is devoted to the discrete Q-operator and its role in transforming left fractional differences and sums to right ones or vise versa.

2. Preliminaries and essential results about fractional sums and differences

For a natural number \( n \), the fractional polynomial is defined by,

\[t^{(n)} = \prod_{j=0}^{n-1} (t - j) = \frac{\Gamma(t + 1)}{\Gamma(t + 1 - n)}\]  \hspace{1cm} (21)

where \( \Gamma \) denotes the special gamma function and the product is zero when \( t + 1 - j = 0 \) for some \( j \). More generally, for arbitrary \( \alpha \), define

\[t^{(\alpha)} = \frac{\Gamma(t + 1)}{\Gamma(t + 1 - \alpha)}\]  \hspace{1cm} (22)

where the convention is that division at the pole yields zero. Given that the forward and backward difference operators are defined by

\[\Delta f(t) = f(t + 1) - f(t), \quad \nabla f(t) = f(t) - f(t - 1)\]  \hspace{1cm} (23)

respectively, we define iteratively the operators \( \Delta^m = \Delta(\Delta^{m-1}) \) and \( \nabla^m = \nabla(\nabla^{m-1}) \), where \( m \) is a natural number.

Here are some of the properties of the above factorial function.

**Lemma 2 ([17]).** Assume the following factorial functions are well defined.

(i) \( \Delta t^{(\alpha)} = \alpha t^{(\alpha-1)} \),
(ii) \( (t - \mu) t^{(\mu)} = t^{(\mu+1)} \), where \( \mu \in \mathbb{R} \),
(iii) \( \mu^{(\mu)} = \Gamma(\mu + 1) \),
(iv) If \( t \leq r \), then \( t^{(\alpha)} \leq r^{(\alpha)} \) for any \( \alpha > r \).
(v) If \( 0 < \alpha < 1 \), then \( t^{(\alpha)} \geq (t^{(1)})^\alpha \).
(vi) \( t^{(\alpha + \beta)} = (t - \beta)^{(\alpha)} t^{(\beta)} \).
For any \( f \) defined on \( N \), we point to the endpoints up to which we takethe fractional sum or difference. However, one has to note that if \( \alpha > 0 \) and \( \sigma(s) = s + 1 \). Then, the \( \alpha \)-th fractional sum of \( f \) is defined (as done in [16] and used in [17–19]) by
\[
\Delta^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-1} (t - \sigma(s))^{(\alpha-1)} f(s).
\]

Note that \( \Delta^{-\alpha} \) maps functions defined on \( N \) to functions defined on \( N_{\alpha} \). Also note that
\[(i) \quad u(t) = \Delta^{-\alpha}f(t), \quad t \in N, \quad u(a + j - 1) = 0, \quad j = 1, 2, \ldots, n \]
\[(ii) \quad \text{the Cauchy function } \frac{(t-\sigma(s))^{(\alpha-1)}}{(n-1)!} \text{ vanishes at } s = t - (n - 1), \ldots, t - 1.\]

As used to be done in usual fractional calculus, the Riemann left and right fractional differences are to be, respectively, defined by
\[
\Delta^\nu f(t) = \Delta^{\nu+1} \Delta^{-n\nu} f(t) \quad \text{and} \quad \nabla^\nu f(t) = (-1)^n \nabla^{\nu+1} \nabla^{-n\nu} f(t)
\]
where \( n = [\alpha] + 1 \). It is clear that the fractional left difference operator \( \Delta^\nu \) maps functions defined on \( N \) to functions defined on \( N_{\alpha+n} \), while the fractional right difference operator \( \nabla^\nu \) maps functions defined on \( b \) to functions defined on \( b-(n-\alpha)N \).

Throughout this article, for simplicity we write \( \Delta^\nu \) and \( \nabla^\nu \) in place of \( \Delta^\nu_a \) and \( \nabla^\nu_b \), respectively, where \( \alpha \in \mathbb{R} \). Otherwise, we point to the end points up to which we take the fractional sum or difference. However, one has to note that if \( \alpha = n \in \mathbb{N} \), then
\[
\Delta^\nu f(t) = \Delta^\nu f(t) \quad \text{and} \quad \nabla^\nu f(t) = (-1)^n \nabla^\nu f(t).
\]

The \( v \)-th left fractional sum behaves well in composition. In fact, Theorem 2.2 in [17] states.

**Lemma 3.** Let \( f \) be a real-valued function, and let \( \mu, \nu > 0 \). Then, for all \( t \) such \( t = a + \mu + \nu, \mod(1) \), we have
\[
\Delta^{-\nu}[\Delta^{-\mu} f(t)] = \Delta^{-\nu+\mu} f(t) = \Delta^{-\mu}[\Delta^{\nu} f(t)].
\]

Theorem 2.1 and Remark 2.2 in [18] are to be summarized in the following lemma.

**Lemma 4.** For any \( \nu \in \mathbb{R} \), we have
\[
\Delta^{-\nu} \Delta^\nu f(t) = \Delta \Delta^{-\nu} f(t) - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)} f(a)
\]
where \( f \) is defined on \( N \).

Following inductively, Theorem 2.2 in [18] states.

**Lemma 5.** For any \( \alpha \in \mathbb{R} \) and any positive integer \( p \), the following equality holds:
\[
\Delta^{-\alpha} \Delta^p f(t) = \Delta^p \Delta^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{(\alpha-p+k)}}{\Gamma(\alpha + k - p + 1)} \Delta^k f(a)
\]
where \( f \) is defined on \( N \).
Parallel with Lemma 5 above, the authors in [19] proved.

**Lemma 6 ([19]).** For any \( \alpha > 0 \), the following equality holds:
\[
\nabla^{-\alpha} \nabla_b f(t) = \nabla_b \nabla^{-\alpha} f(t) - \frac{(b - t)^{(\alpha - 1)}}{\Gamma(\alpha)} f(b)
\]  
(35)

where \( f \) is defined on \( \mathbb{N} \).

**Remark 7 ([19]).** Let \( \alpha > 0 \) and \( n = [\alpha] + 1 \). Then, with the help of Lemma 6 we can have
\[
\nabla_b \nabla^\alpha f(t) = \nabla_b \nabla_b^n (\nabla^{-n\alpha} f(t)) = \nabla_b^n (\nabla^{-n\alpha} f(t))
\]  
(36)
or
\[
\nabla_b \nabla^\alpha f(t) = \nabla_b^n \left[ \nabla^{-n\alpha} \nabla_b f(t) + \frac{(b - t)^{(n - \alpha - 1)}}{\Gamma(n - \alpha)} f(b) \right].
\]  
(37)

Then, using the identity
\[
\nabla_b^n (b - t)^{(n - \alpha - 1)} = \frac{(b - t)^{(-\alpha - 1)}}{\Gamma(-\alpha)}
\]  
(38)
we infer that (35) is valid for any real \( \alpha \).

Using Lemma 6 and Remark 7 the authors in [19] arrived at the following.

**Theorem 8 ([19]).** For any real number \( \alpha \) and any positive integer \( p \), the following equality holds:
\[
\nabla^{-\alpha} \nabla_b^p f(t) = \nabla_b^p \nabla^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(b - t)^{(\alpha - p - k)}}{\Gamma(\alpha + k - p + 1)} \nabla_b^k f(b)
\]  
(39)

where \( f \) is defined on \( \mathbb{N} \) and we remind that \( \nabla_b^k f(t) = (-1)^k \nabla_b^k f(t) \).

In order to prove the commutative property for the right fractional sums, the following power rule obtained in [19] is needed. Here we shall give its proof in detail as the proof in [19] was sketched shortly.

**Lemma 9.** Let \( \alpha > 0 \), \( \mu > 0 \). Then,
\[
\nabla_{b-\alpha} (b - t)^{(\mu)} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \alpha + 1)} (b - t)^{(\mu + \alpha)}.
\]  
(40)

**Proof.** The proof can be achieved by checking that both sides of the identity (40) verify the difference equation
\[
((b - (\mu + \alpha)) - t + 1) \nabla_b g(t) = (\mu + \alpha) g(t), \quad g(b - (\mu + \alpha)) = \Gamma(\mu + 1).
\]  
(41)

Let \( l(t) = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \alpha + 1)} (b - t)^{(\mu + \alpha)} \) and \( h(t) = \nabla_{b-\alpha} (b - t)^{(\mu)} \).

First, consider \( l(t) \) and apply (iii) of Lemma 2 to see that
\[
l(b - (\mu + \alpha)) = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \alpha + 1)} (\mu + \alpha)^{(\mu + \alpha)} = \Gamma(\mu + 1).
\]

Also apply (25) and a similar identity to (ii) of Lemma 2 to see that
\[
(b - (\mu + \alpha) - t + 1) \nabla_b l(t) = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \alpha + 1)} (b - (\mu + \alpha) - t + 1)(\mu + \alpha)(b - t)^{(\mu + \alpha - 1)} = (\mu + \alpha)l(t).
\]

Second, consider \( h(t) \). Noting that
\[
h(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b-\alpha} (\rho(s) - t)^{(\alpha - 1)} (b - s)^{(\mu)},
\]
we see that
\[
h(b - (\mu + \alpha)) = \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b-\alpha} (\rho(s) - b + \mu + \alpha)^{(\alpha - 1)} (b - s)^{(\mu)} = \frac{1}{\Gamma(\alpha)} (\alpha - 1)^{(\alpha - 1)} (\mu)^{(\mu)} = \Gamma(\mu + 1).
\]
Finally, we show that \( h(t) \) satisfies the desired difference equation. Using (ii) of Lemma 2, adding and subtracting \( \mu \) and adding and subtracting \( b \), we see that

\[
h(t) = \frac{(b - (\mu + \alpha) - t + 1)}{\Gamma(\alpha)} \sum_{s=t+1}^{b-\mu} (\rho(s) - t)^{(\alpha-2)}(b - s)^{(\mu)} - \frac{1}{\Gamma(\alpha)} \sum_{s=t+1}^{b-\mu} (b - s - \mu)(\rho(s) - t)^{(\alpha-2)}(b - s)^{(\mu)}.
\]

The rest of verification is direct and similar to that in Lemma 2.3 in [17] □

Actually, formula (40) is the analogous of

\[
\Delta^{\alpha+\mu}_{a+\mu} (t-a)^{(\mu)} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \alpha + 1)} (t-a)^{(\mu+\alpha)}.
\] (42)

**Theorem 10** ([19]). Let \( \alpha > 0, \mu > 0 \). Then, for all \( t \) such that \( t \equiv b - (\mu + \alpha) \) (mod 1), we have

\[
\nabla^{-\nu}[\nabla^{-\mu} f(t)] = \nabla^{-\nu} [\nabla^{-\mu} f(t)] = \nabla^{-\nu} f(t)
\]

where \( f \) is defined on \( bN \).

As a consequence of Lemma 5, Lemma 3, (27) and that \( \Delta^{-(n-\alpha)} f(a + n - \alpha - 1) = 0 \), the following result can be directly stated.

**Proposition 11** ([19]). For \( \alpha > 0 \), and \( f \) defined in a suitable domain \( N_a \), we have for \( t \in N_a + n \subset N_a \)

\[
\Delta^{\alpha}_{a+\mu} \Delta^{\alpha}_{-\mu} f(t) = f(t),
\]

and

\[
\Delta^{\alpha}_{a+n-a-1} \Delta^{\alpha} f(t) = f(t), \quad \text{when} \quad \alpha \not\in \mathbb{N},
\]

\[
\Delta^{\alpha} \Delta^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{(k)}}{k!} \Delta^{k} f(a), \quad \text{when} \quad \alpha = n \in \mathbb{N}.
\] (46)

Similarly, with the help of Lemma 6, Theorem 10, (29) and that \( \nabla^{-\nu} f(b - (n - \alpha) + 1) = 0 \), for the right sums and differences the following is obtained.

**Proposition 12** ([19]). For \( \alpha > 0 \), and \( f \) defined in a suitable domain \( N_a \), we have for \( t \in b-a N \subset b N \)

\[
\nabla^{\alpha}_{b-a} \nabla^{\alpha} f(t) = f(t),
\]

and

\[
\nabla^{\alpha}_{b-(n-a)+1} \nabla^{\alpha} f(t) = f(t), \quad \text{when} \quad \alpha \not\in \mathbb{N},
\]

\[
\nabla^{\alpha} \nabla^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{(k)}}{k!} \nabla^{k} f(b), \quad \text{when} \quad \alpha = n \in \mathbb{N}.
\] (49)

3. **Caputo fractional difference**

Analogous to the usual fractional calculus we can define

**Definition 13.** Let \( \alpha > 0, \alpha \not\in \mathbb{N} \). Then, the \( \alpha \)-order Caputo left fractional and right fractional differences of a function \( f \) defined on \( N_a \) and \( bN \), respectively, are defined by

\[
\Delta^{\alpha}_c f(t) \triangleq \Delta^{-(n-\alpha)} \Delta^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \sum_{s=t}^{t-(n-\alpha)} (t - \sigma(s))^{(n-\alpha-1)} \Delta^{\alpha} f(s)
\]

(50)

and

\[
\nabla^{\alpha}_c f(t) \triangleq \nabla^{-(n-\alpha)} \nabla^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \sum_{s=t+(n-\alpha)}^{b} (\rho(s) - t)^{(n-\alpha-1)} \nabla^{\alpha} f(s)
\]

(51)

where \( n = [\alpha] + 1 \).

If \( \alpha = n \in \mathbb{N} \), then

\[
\Delta^{\alpha}_c f(t) \triangleq \Delta^{\alpha} f(t) \quad \text{and} \quad \nabla^{\alpha}_c f(t) \triangleq \nabla^{\alpha} f(t).
\]
Also, it is clear that \( \Delta^\alpha \) maps functions defined on \( N_n \) to functions defined on \( N_{n+(n-\alpha)} \), and that \( \nabla^\alpha \) maps functions defined on \( bN \) to functions defined on \( b_{-n-\alpha}N \).

If, in Lemma 5 and Theorem 8 we replace \( \alpha \) by \( n-\alpha \) and \( p \) by \( n \), where \( n = [\alpha] + 1 \). Then, we can relate the Riemann and Caputo fractional left and right differences. Namely, we state.

**Theorem 14.** For any \( \alpha > 0 \), we have

\[
\Delta^\alpha f(t) = \Delta^\alpha f(t) - \sum_{k=0}^{n-1} \frac{t-a}{k!(k+1)} \Delta^k f(a)
\]

and

\[
\nabla^\alpha f(t) = \nabla^\alpha f(t) - \sum_{k=0}^{n-1} \frac{b-t}{k!(k+1)} \nabla^k f(b).
\]

In particular, when \( 0 < \alpha < 1 \), we have

\[
\Delta^\alpha f(t) = \Delta^\alpha f(t) - \frac{(t-a)^{(-\alpha)}}{\Gamma(1-\alpha)} f(a)
\]

and

\[
\nabla^\alpha f(t) = \nabla^\alpha f(t) - \frac{(b-t)^{(-\alpha)}}{\Gamma(1-\alpha)} f(b).
\]

One can note that the Riemann and Caputo fractional differences, for \( 0 < \alpha < 1 \), coincide when \( f \) vanishes at the end points.

The following identity is useful to transform Caputo fractional difference equations into fractional summations.

**Proposition 15.** Assume \( \alpha > 0 \) and \( f \) is defined on suitable domains \( N_n \) and \( bN \). Then

\[
\Delta^{-\alpha}_{a+(n-\alpha)} \Delta^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{(k)}}{k!} \Delta^k f(a)
\]

and

\[
\nabla^{-\alpha}_{b-(n-\alpha)} \nabla^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{(k)}}{k!} \nabla^k f(b).
\]

In particular, if \( 0 < \alpha \leq 1 \) then

\[
\Delta^{-\alpha}_{a+(n-\alpha)} \Delta^\alpha f(t) = f(t) - f(a) \quad \text{and} \quad \nabla^{-\alpha}_{b-(n-\alpha)} \nabla^\alpha f(t) = f(t) - f(b).
\]

**Proof.** The proof of (56) is followed by applying the definition and then using Lemma 3 and Proposition 11 (46). While the proof of (57) is followed by applying the definition and then using Theorem 10 and Proposition 12 (49). □

Using the identity (37) and the identity

\[
\Delta^n \frac{(t-a)^{(n-\alpha-1)}}{\Gamma(n-\alpha)} = \frac{(t-a)^{(n-\alpha-1)}}{\Gamma(\alpha)}
\]

we can compute the Riemann and Caputo differences of certain functions. For example, for \( \beta > n = [\alpha] + 1 \), we have

\[
\Delta^\alpha (t-a)^{(\beta-1)} = \Delta^\alpha (t-a)^{(\beta-1)} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{(\beta-\alpha-1)}
\]

and

\[
\nabla^\alpha (b-t)^{(\beta-1)} = \nabla^\alpha (b-t)^{(\beta-1)} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (b-t)^{(\beta-\alpha-1)}.
\]

However,

\[
\Delta^\alpha 1 = \nabla^\alpha 1 = 0
\]

while

\[
\Delta^\alpha 1 = \frac{(t-a)^{(-\alpha)}}{\Gamma(1-\alpha)}, \quad \nabla^\alpha 1 = \frac{(b-t)^{(-\alpha)}}{\Gamma(1-\alpha)}.
\]

Example 17. Let $0 < \alpha \leq 1$, $a = \alpha - 1$ and consider the left Caputo fractional difference equation
\[
\Delta^\alpha_0 y(t) = \lambda y(t + \alpha - 1), \quad y(a) = a_0, \quad t \in N_0.
\] (64)
Note that the solution $y(t)$, if exists, is defined on $N_0$ and hence $\Delta^\alpha_0 y(t)$ becomes defined on $N_{a+1(1-\alpha)} = N_0$. Thus, if we apply $\Delta^{\alpha \sigma}$ on the Eq. (64) then we see that
\[
y(t) = a_0 + \frac{\lambda}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t - \sigma(s))^{(\alpha-1)} y(s + \alpha - 1).
\]
To obtain an explicit clear solution, we apply the method of successive approximation. Set $y_0(t) = a_0$ and
\[
y_m(t) = a_0 + \lambda \Delta^{\alpha \sigma} y_{m-1}(t + \alpha - 1), \quad m = 1, 2, 3, \ldots.
\]
For $m = 1$, we have by the power formula (42)
\[
y_1(t) = a_0 \left[ 1 + \frac{\lambda t^{(\alpha)}}{\Gamma(\alpha + 1)} \right].
\]
For $m = 2$, we also see that
\[
y_2(t) = a_0 + \lambda \Delta^{\alpha \sigma} \left[ a_0 + \frac{(t + \alpha - 1)^{(\alpha)}}{\Gamma(\alpha + 1)} \right] = a_0 \left[ 1 + \frac{\lambda t^{(\alpha)}}{\Gamma(\alpha + 1)} + \frac{\lambda^2 (t + \alpha - 1)^{(2\alpha)}}{\Gamma(2\alpha + 1)} \right].
\]
If we proceed inductively and let $m \to \infty$ we obtain the solution
\[
y(t) = a_0 \left[ 1 + \sum_{k=1}^{\infty} \frac{\lambda^k (t + (k - 1)(\alpha - 1))^{(k\alpha)}}{\Gamma(k\alpha + 1)} \right].
\]
If we set $\alpha = 1$ we come to the conclusion that $(\lambda + 1)^t = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} t^k$ on the time scale $Z$.

Definition 18 (Discrete Mittag-Leffler). For $\lambda \in \mathbb{R}$ and $\alpha, \beta, z \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, the discrete (like) Mittag-Leffler functions are defined by
\[
E_{(\alpha, \beta)}(\lambda, z) = \sum_{k=0}^{\infty} \frac{\lambda^k (z + (k - 1)(\alpha - 1))^{(k\alpha)}}{\Gamma(k\alpha + 1)}.
\] (65)
For $\beta = 1$, it is written that
\[
E_{(\alpha)}(\lambda, z) = \sum_{k=0}^{\infty} \frac{\lambda^k (z + (k - 1)(\alpha - 1))^{(k\alpha)}}{\Gamma(k\alpha + 1)}.
\] (66)

The following specific semigroup property is useful to proceed in our main results.

Lemma 19. For $\alpha > 0$ and $k = 1, 2, 3, \ldots$, we have
\[
\Delta^{\alpha \sigma} (\Delta^{\alpha\sigma} f)(t + k(\alpha - 1)) = (\Delta^{\alpha\sigma} f)(t + k(\alpha - 1)).
\]

Proof. We prove for $k = 1$ and the rest follows inductively. Hence we show that
\[
\Delta^{\alpha \sigma} (\Delta^{\alpha \sigma} f)(t + \alpha - 1) = (\Delta^{\alpha \sigma} f)(t + \alpha - 1).
\]
To this end,
\[
\Delta^{\alpha \sigma} (\Delta^{\alpha \sigma} f)(t + \alpha - 1) = \frac{1}{\Gamma(\alpha)} \sum_{r=0}^{t-\alpha} \sum_{s=0}^{r-1} (t - \sigma(r))^{(\alpha-1)} (r + \alpha - 1 - \sigma(s))^{(\alpha-1)}.
\]
If we change the order of summation we reach at
\[
\Delta^{\alpha \sigma} (\Delta^{\alpha \sigma} f)(t + \alpha - 1) = \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} f(s) \frac{1}{\Gamma(\alpha)} \sum_{r=s+1}^{t-\alpha} (t - \sigma(r))^{(\alpha-1)} (r + \alpha - 1 - \sigma(s))^{(\alpha-1)}.
\] (67)
That is
\[ \Delta_0^{-\alpha} (\Delta_0^{-\alpha} f)(t + \alpha - 1) = \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha-1} f(s) \Delta_+^{\alpha s} (t - (s + 2 - \alpha))^{(\alpha - 1)}. \] (68)

Finally, by (42) we obtain
\[ \Delta_0^{-\alpha} (\Delta_0^{-\alpha} f)(t + \alpha - 1) = \frac{1}{\Gamma(2\alpha)} \sum_{s=0}^{t-\alpha-1} (t + \alpha - 1 - \sigma(s))^{(2\alpha - 1)} f(s). \]

Which is exactly equal to \((\Delta_0^{-2\alpha} f)(t + \alpha - 1). \) □

**Example 20.** Let \(0 < \alpha \leq 1, a = \alpha - 1\) and consider the left Caputo nonhomogeneous fractional difference equation
\[ \Delta_c^\alpha y(t) = \lambda y(t + \alpha - 1) + f(t), \quad y(a) = a_0, \quad t \in N_0. \] (69)

Note that the solution \(y(t)\), if exists, is defined on \(N_a\) and hence \(\Delta_c^\alpha y(t)\) becomes defined on \(N_{a+(1-\alpha)} = N_0\). Thus, if we apply \(\Delta_0^{-\alpha}\) on the Eq. (69) then by (58) we see that
\[ y(t) = a_0 + \frac{\lambda}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t - \sigma(s))^{(\alpha - 1)} y(s + \alpha - 1) + \Delta_0^{-\alpha} f(t). \]

To obtain an explicit clear solution, we apply the method of successive approximation. Set \(y_0(t) = a_0\) and
\[ y_m(t) = a_0 + \Delta_0^{-\alpha} [\lambda y_{m-1}(t + \alpha - 1) + f(t)], \quad m = 1, 2, 3, \ldots. \]

For \(m = 1\), we have by the power formula (42)
\[ y_1(t) = a_0 \left[ 1 + \frac{\lambda t^{(\alpha)}}{\Gamma(\alpha + 1)} \right] + \Delta_0^{-\alpha} f(t). \]

For \(m = 2\), we also see with the help of Lemma 19 that
\[ y_2(t) = a_0 + \lambda \Delta_0^{-\alpha} \left[ a_0 + \frac{(t + \alpha - 1)^{(\alpha)}}{\Gamma(\alpha + 1)} \right] + \Delta_0^{-\alpha} f(t) + \lambda \Delta_0^{-2\alpha} f(t + \alpha - 1) \]
\[ = a_0 \left[ 1 + \frac{\lambda t^{(\alpha)}}{\Gamma(\alpha + 1)} + \frac{\lambda^2 (t + \alpha - 1)^{(2\alpha)}}{\Gamma(2\alpha + 1)} \right] + \Delta_0^{-\alpha} f(t) + \lambda \Delta_0^{-2\alpha} f(t + \alpha - 1). \]

If we proceed inductively, making use of Lemma 19 and letting \(m \to \infty\), we obtain the solution
\[ y(t) = a_0 \left[ 1 + \sum_{k=1}^{\infty} \lambda^k (t + (k - 1)(\alpha - 1))^{(k\alpha)} \frac{1}{\Gamma(k\alpha + 1)} \right] + \sum_{k=1}^{\infty} \lambda^{k-1} (\Delta_0^{-k\alpha} f)(t + (k - 1)(\alpha - 1)). \]

Then,
\[ y(t) = a_0 E_{(\alpha)}(\lambda, t) + \sum_{k=0}^{\infty} \lambda^k \frac{1}{\Gamma(ak + \alpha)} \sum_{s=0}^{t-\alpha} (t + k(\alpha - 1) - \sigma(s))^{(k\alpha + \alpha - 1)} f(s). \] (70)

Since division over bally yields zero then
\[ y(t) = a_0 E_{(\alpha)}(\lambda, t) + \sum_{k=0}^{\infty} \lambda^k \frac{1}{\Gamma(ak + \alpha)} \sum_{s=0}^{t-\alpha} (t + k(\alpha - 1) - \sigma(s))^{(k\alpha + \alpha - 1)} f(s). \] (71)

Interchanging the order of sum in (71) and making use of Lemma 2(vi), then we conclude that
\[ y(t) = a_0 E_{(\alpha)}(\lambda, t) + \sum_{k=0}^{\infty} \sum_{s=0}^{t-\alpha} \lambda^k \frac{(t - \sigma(s) + (k - 1)(\alpha - 1))^{(k\alpha)}}{\Gamma(ak + \alpha)} (t - \sigma(s) + k(\alpha - 1))^{(\alpha - 1)} f(s). \] (72)

That is
\[ y(t) = a_0 E_{(\alpha)}(\lambda, t) + \sum_{s=0}^{t-\alpha} E_{(\alpha,\alpha)}(\lambda, t - \sigma(s)) f(s). \] (73)
4. The Q-operator and fractional sums and differences

Recall that if \( f(s) \) is defined on \( N_a \cap bN \) and \( a \equiv b \pmod{1} \) then \( (Qf)(s) = f(a + b - s) \). It is clear that \( Q^2f(t) = f(t) \). If we use the change of variables \( u = a + b - s \) then it is easy to see that

\[
\Delta^{-\alpha}Qf(t) = Q\nabla^{-\alpha}f(t),
\]

and hence

\[
\Delta^\alpha_Qf(t) = Q(\nabla^\alpha f)(t).
\]

Therefore, the Q-operator agrees with its continuous counterpart when applied to left and right fractional Riemann integrals and the Caputo derivatives. More generally, this discrete version of the Q-operator can be used to transform the discrete delay-type fractional functional difference dynamic equations to advanced ones. For details on the continuous counterparts see [13].

Example 21. Let \( 0 < \alpha \leq 1, a = \alpha - 1 \) and \( b \) such that \( a \equiv b \pmod{1} \). Let \( y(t) \) be defined on \( N_a \cap bN \). Consider the following Caputo right fractional difference equation

\[
\nabla^\alpha_Q y(t) = \lambda y(2a + b - t), \quad (Qy)(b) = a_0.
\]

If we apply the Q operator on the Caputo fractional difference equation (74), then we obtain the left Caputo fractional difference equation (64) that has been solved in Example 17.

Remark 22. (1) Throughout the article we followed the delta time-scale-calculus to define our fractional differences. We have used nabla to stand for the right fractional differences just to simplify the notation.

(2) It is of interest to generalize the previously obtained results in discrete fractional calculus to the time scale \( h\mathbb{Z} \) and more general time scales. For recently obtained results on the time scale \( h\mathbb{Z} \) we refer to [21].

(3) It is of interest to follow the nabla calculus approach to obtain similar results in discrete fractional calculus. For recent work in this direction we refer to [22].

References