JOURNAL OF COMBINATORIAL THEORY, Series B 45, 345–355 (1988)

Paths in k-Edge-Connected Graphs

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Communicated by the Managing Editors

Received December 15, 1986

We prove (i) if G is a $2k$-edge-connected graph ($k \geqslant 2$), s, t are vertices, and f_1, f_2, g are edges with $f_i \neq g$ ($i = 1, 2$), then there exists a cycle C passing through f_1 and f_2 (a path P between s and t passing through f_1) but not passing through g such that $G - E(C)(G - E(P))$ is $(2k-2)$-edge-connected, where C and P are not necessarily simple and $E(C)$ is the set of edges of C. (ii) Every $3k$-edge-connected graph ($k \geqslant 1$) is weakly $(2k+1)$-linked and every $(3k-1)$-edge-connected graph ($k \geqslant 2$) is weakly $2k$-linked. © 1988 Academic Press, Inc.

1. INTRODUCTION

We consider finite undirected graphs possibly with multiple edges but without loops. Let G be a graph and let $V(G)$ and $E(G)$ be the set of vertices and edges of G, respectively. We allow repetition of vertices (but not edges) in a path and cycle. Mader [3] conjectured,

> If G is a k-edge-connected graph ($k \geqslant 4$) and s, t are vertices of G, then there exists a cycle C passing through s and t such that $G - E(C)$ is $(k-2)$-edge-connected,

and proved this conjecture (in fact Theorem 1(1)) for $k = 4$. We prove this when k is even.

THEOREM 1. *Suppose that $k \geqslant 4$ is an even integer, G is a k-edge-connected graph, $\{s, t\} \subset V(G)$, $\{f_1, f_2, g\} \subset E(G)$, and $f_i \neq g$ ($i = 1, 2$). Then*

(1) *There exists a cycle C passing through f_1 and f_2 but not passing through g such that $G - E(C)$ is $(k-2)$-edge-connected.*

(2) *There exists a path P between s and t passing through f_1 but not passing through g such that $G - E(P)$ is $(k-2)$-edge-connected.*

For odd k, the conjecture of Mader is still open, but the result of Theorem 1(1) does not always hold; Fig. 1 gives a counterexample.

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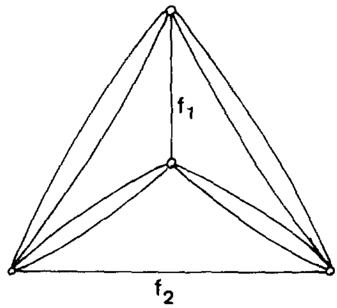

FIGURE 1

We call a graph G weakly k-linked, if for every k pairs of vertices (s_i, t_i), there exist edge-disjoint paths $P_1, ..., P_k$ such that P_i joins s_i and t_i $(1 \leqslant i \leqslant k)$. Let

$$g(k) := \min\{m \mid \text{if } G \text{ is } m\text{-edge-connected, then } G \text{ is weakly } k\text{-linked}\}.$$

Thomassen [6] conjectured

$$g(2k+1) = g(2k) = 2k+1 \qquad (k \geqslant 1).$$

The author [4] proved $g(3) = 3$ and Hirata, Kubota, and Saito [1] and Mader [3] proved $g(4) = 5$ and $g(k) \leqslant 2k-3$ $(k \geqslant 5)$. We prove

THEOREM 2. *$g(2k+1) \leqslant 3k$ $(k \geqslant 1)$ and $g(2k) \leqslant 3k-1$ $(k \geqslant 2)$.*

Notations and Definitions

$\lambda(G)$ denotes the edge-connectivity of G. Let $X, Y, \{x, y\} \subset V(G)$, $f \in E(G)$, and $X \cap Y = \emptyset$. We often denote $\{x\}$ by x. $V(f)$ denotes the set of end vertices of f. We denote by $\partial(X, Y; G)$ the set of edges with one end in X and the other in Y, and set $\partial(X; G) := \partial(X, V(G) - X; G)$, $e(X, Y; G) := |\partial(X, Y; G)|$, and $e(X; G) := |\partial(X, V(G) - X; G)|$. $\lambda(X, Y; G)$ denotes the maximal number of edge-disjoint paths between X and Y. We set $\bar{X} := V(G) - X$, $N(x; G) := \{a \in V(G) - x \mid e(a, x) > 0\}$, $N(X; G) := \bigcup_{x \in X} N(x:G)$, and $\Gamma(G, k) := \{Z \subset V(G) \mid \text{for each } a, b \in Z, \lambda(a, b; G) \geqslant k\}$. In all notations, we often omit G. G/X denotes the graph obtained from G by contracting X, and for $a \in X$, we denote the corresponding vertex in G/X by $\tilde{a}$. A path $P = P[x, y]$ denotes a path between x and y. For $a, b \in N(x)$ with $a \neq b$, $f \in \partial(x, a)$, and $g \in \partial(x, b)$, $G_x^{a,b}$ denotes the graph $(V(G), (E(G) \cup h) - \{f, g\})$, where h is a new edge between a and b and is called a lifting of G at x arising from the lifting of f and g at x. We call $G_x^{a,b}$ admissible if for each $y, z \in V(G) - x$ with $y \neq z$, $\lambda(y, z; G_x^{a,b}) = \lambda(y, z; G)$.

Throughout the paper we shall make use of the following observation: If $a, b \in X \subset V(G)$, then $\lambda(a, b; G) \geqslant \lambda(a, b; G/\bar{X})$ and it is easy to give examples for which the inequality is strict. However, the inequality is an equality if $\bar{X}$ has a vertex z and a collection of $e(X)$ edge-disjoint paths from z to X.

2. Proof of Theorem 1

LEMMA 1 (Okamura [5]). *Suppose that $k \geqslant 4$, G is a 2-edge-connected graph, and $\{s, t\} \subset T \in \Gamma(G, k)$. Then*

(1) *If $a \in T - \{s, t\}$ and $e(a) < 2k$, then there exists a path $P[s, t]$ such that $a \notin V(P)$, $T \in \Gamma(G - E(P), k-2)$, and $\{s, t, a\} \in \Gamma(G - E(P), k-1)$.*

(2) *If $a \in V(G)$ and $\lambda(a, s) < k$, then there exists a path $P[s, t]$ such that $a \notin V(P)$, $T \in \Gamma(G - E(P), k-2)$, and $\lambda(a, s; G - E(P)) = \lambda(a, s; G)$.*

(3) *If $f_1, f_2 \in \partial(s)$, then there exists a cycle C such that $\{f_1, f_2\} \subset E(C)$ and $T \in \Gamma(G - E(C), k-2)$.*

LEMMA 2 (Mader [2]). *If G is a graph, $x \in V(G)$, $e(x) \geqslant 4$, $|N(x)| \geqslant 2$, and x is not a cut-vertex, then there exists an admissible lifting of G at x.*

LEMMA 3. *If $k \geqslant 3$ is an integer, G is a graph, $V(G) = W_1 \cup W_2$, $W_1 \cap W_2 = \emptyset$, $W_1 \in \Gamma(G, k)$, and each $x \in W_2$ has even degree, then we can obtain a k-edge-connected graph $G(W_1, k)$ from G such that $W_1 \subset V(G(W_1, k))$ by sequences of vertex-deletions and edge-liftings.*

Proof. We may assume G is connected. We can obtain $G(W_1, k)$ from G as follows.

Step 1. If $W_2 = \emptyset$, then let $G(W_1, k) := G$, and stop.

Step 2. If for an $x \in W_2$, $|N(x)| = 1$, then reset $G := G - x$ and go to Step 1.

Step 3. If for each $x \in W_2$, $e(x) = 2$, then let $G(W_1, k)$ be the k-edge-connected graph homeomorphic to G (that is at each $x \in W_2$, we lift $\partial(x)$ and then delete x), and stop. Otherwise let $x \in W_2$ and $e(x) \geqslant 4$.

Step 4. If x is a cut-vertex and each component of $G - x$ has a vertex of W_1, then reset $W_1 := W_1 \cup \{x\}$, $W_2 := W_2 - x$ (note that $W_1 \cup \{x\} \in \Gamma(G, k)$), and go to Step 1. If x is a cut-vertex and C is a component of $G - x$ such that $V(C) \cap W_1 = \emptyset$, then reset $G := G - V(C)$, and go to Step 1. Otherwise let G_x be an admissible lifting of G at x (see Lemma 2), reset $G := G_x$, and go to Step 1.

In what follows $G \to G(W_1, k)$ denotes this operation.

LEMMA 4. *Suppose that $k \geqslant 4$ is an even integer, G is a k-edge-connected graph, and for each $x \in V(G)$, $e(x) \leqslant k+1$. If $f \neq g \in E(G)$, then there exists a cycle C such that $f \in E(C)$, $g \notin E(C)$, and $\lambda(G-E(C)) \geqslant k-2$.*

Proof. It is easy to see that G is 2-connected. We proceed by induction on $|E(G)|$. If there is an $x \in V(f) \cap V(g)$, then for any $h \in \partial(x) - \{f, g\}$, by Lemma 1(3) G has a cycle C such that $\{f, h\} \subset E(C)$ and $\lambda(G-E(C)) \geqslant k-2$. Then $g \notin E(C)$. Thus let $V(f) \cap V(g) = \emptyset$. Let $V(g) = \{x, y\}$. If $e(x) = k$, then let G_x be an admissible lifting at x (see Lemma 2) and let $G_1 := G_x(V(G)-x, k)$ (see Lemma 3). In G_1 $f \neq g$, and by induction G_1 has a cycle C such that $f \in E(C)$, $g \notin E(C)$, and $\lambda(G_1 - E(C)) \geqslant k-2$. Let C_1 be the corresponding cycle in G and let C_2 be the simple subcycle of C_1 containing f, then C_2 is a required cycle. Thus let $e(x) = e(y) = k+1$. If $\lambda(G-g) \geqslant k$, then the result holds in $G-g$. Thus for some $X \subset V(G) - y$, $x \in X$ and $e(X) = k$. Let $V(f) = \{a, b\}$, we may let $X \cap V(f) = \emptyset$ or $\{a\}$. If $X \cap V(f) = \emptyset$, then G/X has a required cycle C. If $\tilde{x} \notin V(C)$, then C is a required cycle for G. If $\tilde{x} \in V(C)$, then let $\partial(\tilde{x}; G/X) \cap E(C) = \{h_1, h_2\}$. By Lemma 1(3) $G/\bar{X}$ has a cycle C_1 such that $\{h_1, h_2\} \subset E(C_1)$ and $\lambda(G/\bar{X} - E(C_1)) \geqslant k-2$. Then $g \notin E(C_1)$. By combining C and C_1 in G we have a required cycle. Therefore $X \cap V(f) = \{a\}$. Let $h \in \partial(X) - \{f, g\}$, $G_1 := G/X$, and $G_2 := G/\bar{X}$. By Lemma 1(3) for $i = 1, 2$, G_i has a cycle C_i such that $\{f, h\} \subset E(C_i)$ and $\lambda(G_i - E(C_i)) \geqslant k-2$. By combining C_1 and C_2 in G we have a required cycle.

The proof of Lemma 5 will be given later.

LEMMA 5. *Suppose that $k \geqslant 4$ is an even integer and G is a graph. If*

(i) *$V(G) = \{u\} \cup A \cup W_1 \cup W_2$ (disjoint union), $A \neq \emptyset$, and either $W_2 = \emptyset$ or $W_2 = \{b\}$ and b has even degree,*

(ii) *$V(G) - W_2 \in \Gamma(G, k-2)$, and for each $X \subset W_1 \cup W_2$ with $X \cap W_1 \neq \emptyset$, $e(X) \geqslant k$,*

(iii) *for each $x \in W_1$, $e(x) \leqslant k+1$, and for each $x \in V(G) - W_1$, $e(x) \leqslant k-1$.*

(iv) *$f \in \partial(u)$, $g \in E(G) - f$, and $\{f, g\} \neq \partial(b)$,*

then for some $a \in A$, there exists a path $P[u, a]$ such that $f \in E(P)$, $g \notin E(P)$, and $V(G) - W_2 \in \Gamma(G * P, k-2)$. Here $G * P$ denotes the graph $(V(G), (E(G) \cup h) - E(P))$ and h is a new edge between u and a.

Proof of Theorem 1

First we prove that Theorem 1(1) implies Theorem 1(2). Let h be a new edge between s and t and let $G_1 := (V(G), E(G) \cup h)$. Then by (1), G_1 has a cycle C such that $\{h, f_1\} \subset E(C)$, $g \notin E(C)$, and $\lambda(G_1 - E(C)) \geqslant k-2$. $C - h$ is a required path of G.

Now we prove Theorem 1(1) by induction on $|E(G)|$. If G is not 2-connected, then we can deduce the result by using induction on some blocks. Thus we may assume that G is 2-connected. If for an $x \in V(G)$, $e(x) \geqslant k+2$, then by Lemma 2 for some $a, b \in N(x)$ with $a \neq b$, $G_x^{a,b} =: G_1$ is admissible. If $f_i \neq g$ $(i=1, 2)$ in G_1, then by induction the result holds in G_1. Thus let $\partial(a, x; G) = \{f_1\}$ and $\partial(b, x; G) = \{g\}$. $|N(x; G_1)| \geqslant 2$, otherwise for each $h_1, h_2 \in \partial(x; G_1) - f_2$, the result holds in $G - \{h_1, h_2\}$. Thus for some $y, z \in N(x; G_1)$ with $y \neq z$, $(G_1)_x^{y,z}$ is admissible, $f_i \neq g$ $(i=1, 2)$ in $G_x^{y,z}$, and the result holds in $G_x^{y,z}$. Thus we may assume that for each $x \in V(G)$, $e(x) \leqslant k+1$. Let

$$F := \{h \in E(G) \mid \text{there is a cycle } C \text{ such that } \{h, f_1\} \subset E(C),$$
$$g \notin E(P), \text{ and } \lambda(G - E(C)) \geqslant k-2\}.$$

Assume $F \neq E(G) - g$. By Lemma 4 $F \neq \varnothing$. For some $h \in F$ and $f \in E(G) - F - g$, h and f have a common end vertex, say u. Let C be a cycle such that $\{h, f_1\} \subset E(C)$, $g \notin E(C)$, and $\lambda(G - E(P_1)) \geqslant k-2$. Let $A := V(C) - u$, $W_1 := V(G) - V(C)$, and $W_2 = \varnothing$. Then by Lemma 5 for some $a \in A$, $G - E(C)$ has a path $P[u, a]$ such that $f \in E(P)$, $g \notin E(P)$, and $(G - E(C)) * P$ is $(k-2)$-edge-connected. In C there are two disjoint paths joining u and a. Let P_1 be one of them containing f_1, and let $C_1 := P \cup P_1$, then $\lambda(G - E(C_1)) \geqslant k-2$. Thus $F = E(G) - g$.

To prove Lemma 5 we need some lemmas.

Lemma 6. *Suppose that $k \geqslant 2$ is an even integer, G is a graph, $V(G) = W_1 \cup W_2$, $W_1 \cap W_2 = \varnothing$, and each $v \in W_2$ has even degree. Then*

(1) *If $W_1 \in \Gamma(G, k)$, $x \in V(G)$, and $e(x)$ is odd, then for some $y \in W_1$, $\lambda(x, y) \geqslant k+1$.*

(2) *If $X \subset V(G)$, $e(X) \leqslant k+1$, $x \in X \cap W_1$, $y \in \bar{X} \cap W_1$, $(X \cap W_1) \cup \{\tilde{y}\} \in \Gamma(G/\bar{X}, k)$, and $(\bar{X} \cap W_1) \cup \{\tilde{x}\} \in \Gamma(G/X, k)$, then $W_1 \in \Gamma(G, k)$.*

Proof. (1) Since $x \notin W_2$, $e(x) \geqslant k+1$. If for some $X \subset V(G)$ with $x \in X$, $e(X) = k$, then choose X with this property such that $|X|$ is minimal, if not, then let $X := V(G)$. For some $y \in X - x$, $e(y)$ is odd. Then $y \in W_1$, $e(y) \geqslant k+1$, and $\lambda(y, x) \geqslant k+1$.

(2) Clearly we may let $e(X) = k+1$. By (1) for some $a \in X$, $\lambda(a, \bar{X}) = k+1$ and for some $b \in \bar{X}$, $\lambda(b, X) = k+1$, and so $W_1 \in \Gamma(G, k)$.

Lemma 7. *Suppose that G is a graph and X, $Y \subset V(G)$. Then*

(1)

$$e(X-Y) + e(Y-X) = e(X) + e(Y) - 2e(X \cap Y, \overline{X \cup Y}),$$

$$e(X \cap Y) + e(X \cup Y) = e(X) + e(Y) - 2e(X-Y, Y-X).$$

(2) *If* $\lambda(G)\geqslant k$, $X-Y$, $Y-X$, $X\cap Y$, *and* $\overline{X\cup Y}$ *are not empty and* $e(X)=e(Y)=k$, *then* k *is even and* $e(X-Y)=e(X\cap Y)=k$.

Proof. (1) Simple counting.

(2) By (1) $e(X-Y)=e(X\cap Y)=k$. Thus $k=e(X)\equiv e(X-Y)+e(X\cap Y)\equiv 0 \pmod 2$.

Proof of Lemma 5

We proceed by induction on $|E(G)|$. Let $\mathscr{P}(G,f,A,W_1)$ be the set of required paths of G and assume $\mathscr{P}(G,f,A,W_1)=\varnothing$. Note that $e(b)\geqslant 4$ if $W_2\neq\varnothing$. Let $V(f)=\{u,v\}$. Then $v\notin A$.

(2.1) *If* $X\subset W_1\cup W_2$ *and* $|X|\geqslant 2$, *then* $e(X)\geqslant k+2$.

Proof. Assume $e(X)\leqslant k+1$. Let $x\in X$ and let $P[u,a]\in\mathscr{P}(G/X,f,A,(W_1-X)\cup\tilde{x})$ (g might not be in $E(G/X)$). If $\tilde{x}\notin V(P)$, then we may let $e(X)=k+1$ and by Lemma 6(1) for some $y\in X$, $\lambda(y,\bar{X})=k+1$ and for some $z\in\bar{X}-W_2$, $\lambda(z,X;G*P)\geqslant k-1$, and so $V(G)-W_2\in\Gamma(G*P,k-2)$. Thus let $\tilde{x}\in V(P)$ and $h_1,h_2\in\partial(\tilde{x};G/X)\cap E(P)$. If $X\subset W_1$, then let $G_1:=G/\bar{X}$, and if $b\in X$, then for $Y:=V(G/\bar{X})-b$, let $G_1:=(G/\bar{X})(Y,k)$ (see Lemma 3). By induction for each graph H such that $|E(H)|<|E(G)|$, Lemma 5 holds, and so in G_1 Theorem 1(1) holds (see the proof of Theorem 1). Thus G_1 has a cycle C such that $\{h_1,h_2\}\subset E(C)$, $g\notin E(C)$, and $\lambda(G_1-E(C))\geqslant k-2$. Let C_1 be the corresponding cycle in $G/\bar{X}$. Let $P_1[u,a]$ be the path in G obtained by combining P and C_1. Then by Lemma 6 P_1 is a required path.

(2.2) *If* $X\subset V(G)$, $|X|\geqslant 2$, *and* $|\bar{X}|\geqslant 2$, *then* $e(X)\geqslant k$.

Proof. Assume $e(X)=k-2$ or $k-1$ and $v\notin X$. If $u\in X$, then by (ii) $A-X\neq\varnothing$ and $\varnothing\neq\mathscr{P}(G/X,f,A-X,W_1-X)\subset\mathscr{P}(G,f,A,W_1)$ by induction and by Lemma 6. If $u\notin X$, then $A\cap X\neq\varnothing$. Let $x\in X$ and $P_1[u,a]\in\mathscr{P}(G/X,f,(A-X)\cup\tilde{x},W_1-X)$. If $a\neq\tilde{x}$, then P_1 is a required path, thus let $a=\tilde{x}$ and $h\in\partial(\tilde{x};G/X)\cap E(P_1)$. Let $P_2\in\mathscr{P}(G/\bar{X},h,A\cap X,W_1\cap X)$. By combining P_1 and P_2 in G we can get a required path.

(2.3) *Let* $x=b$ *if* $W_2\neq\varnothing$ *and* $x\in W_1$ *if* $W_2=\varnothing$. *If* $|N(x)|=n$ *and* $N(x)=\{y_1,\dots,y_n\}$, *then* $n\geqslant 2$ *and* $V(G)-x\in\Gamma(G_x^{y_1,y_i},k-2)$ $(2\leqslant i\leqslant n)$. *Moreover if* $e(x,y_1)=1$, *then* $n\geqslant 3$.

Proof. If $n=1$, then $\lambda(G-x)\geqslant k-2$ and $e(y_1;G-x)\leqslant k+1-4$, a contradiction. Thus $n\geqslant 2$. By (2.2) for each $2\leqslant i\leqslant n$, $V(G)-x\in\Gamma(G_x^{y_1,y_i},k-2)$. Assume $e(x,y_n)=1$ and $n=2$. Then $e(\{x,y_1\})\leqslant e(y_1)-2$, contrary to (ii) or (iii).

(2.4) *$W_2=\emptyset$ and for each $x\in W_1$, $e(x)=k+1$.*

Proof. Let $x=b$ if $W_2\neq\emptyset$, and let $x\in W_1$ and $e(x)=k$ if $W_2=\emptyset$. By (2.3) we can obtain a lifting G_x of G at x such that $V(G)-x\in\Gamma(G_x, k-2)$, $f\neq g$ in G_x, and $\{f, g\}\neq\partial(x; G_x)$. By (2.1) for each $X\subset W_1\cup W_2$ with $X\neq\{x\}$, $e(X; G_x)\geqslant k$, and so there is a $P\in\mathscr{P}(G_x, f, A, W_1-x)$. If $x=b$, then $P\in\mathscr{P}(G, f, A, W_1)$. If $x\in W_1$, then let P_1 be the corresponding path in G. If P_1 is not simple, then let P_2 be the simple subpath of P_1 between u and a, then $P_2\in\mathscr{P}(G, f, A, W_1)$.

Since $v\in W_1$, if for some $x\in W_1$ there is an $h\in\partial(v, x)$, then by (2.1), (2.2), and (2.4) the result holds in $G-h$. Thus $\{u\}\neq N(v)\subset\{u\}\cup A$. By (2.3) for some $a\in N(v)-u$, there is a lifting $G_v^{u,a}$ such that $\lambda(G_v^{u,a})\geqslant k-2$ and $f\neq g$ in $G_v^{u,a}$.

3. Proof of Theorem 2

The proof of Lemma 8 will be given later.

Lemma 8. *Suppose that $k\geqslant 4$ and $n\geqslant 2$ are integers, G is a 2-connected graph, $V(G)=T\cup W_1\cup W_2$ (disjoint union), $T=\{s_1, ..., s_n, t_1, ..., t_n\}$, $|T|=2n$, $T\cup W_1\in\Gamma(G, k)$, $|W_2|\leqslant 2$, for each $x\in W_2$, $e(x)\leqslant k-1$ is even, and*

(i) *if k is odd, then for each $x\in T\cup W_1$, $e(x)=k$,*

(ii) *if k is even, then $e(s_i)=e(t_i)=k$ $(1\leqslant i\leqslant n)$ and for each $x\in W_1$, $e(x)=k$ or $k+1$.*

Then there is a subgraph $G^*\subset G$ such that

(a) *for some $1\leqslant i<j\leqslant n$, $G-E(G^*)$ has edge-disjoint paths $P_1[s_i, t_i]$ and $P_2[s_j, t_j]$,*

(b) *$V(G^*)=K_1\cup K_2$ and $K_1\cap K_2=\emptyset$,*

(c) *$T-\{s_i, t_i, s_j, t_j\}\subset K_1\in\Gamma(G^*, k-3)$,*

(d) *for each $x\in K_2$, $e(x; G^*)$ is even.*

Proof of Theorem 2

Let $\alpha=0$ or 1, $m\geqslant 1$ be an integer, $k:=3m-\alpha\geqslant 3$, and $n:=2m+1-\alpha$. Assume that G is a k-edge-connected graph and $\{s_1, ..., s_n, t_1, ..., t_n\}:=T$ are vertices of G (not necessarily distinct). We prove that there are edge-disjoint paths $P_1, ..., P_n$ such that P_i joins s_i and t_i $(1\leqslant i\leqslant n)$. We may assume that G is 2-connected. If k is odd and $e(x)=:d>k$ for some $x\in V(G)$, then we replace x by d vertices of degree k (Fig. 2 gives an

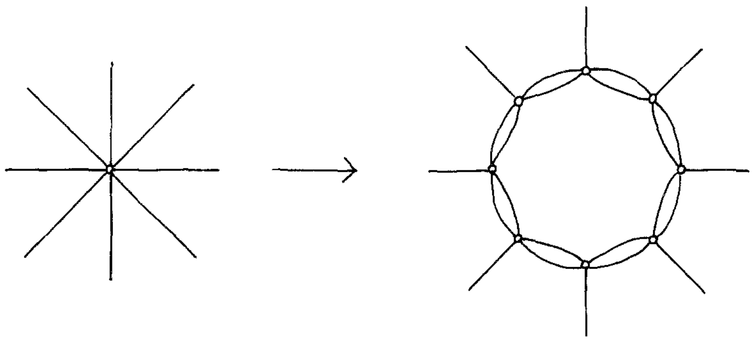

FIGURE 2

example with $d = 8$ and $k = 5$) and assign x on one of the new vertices, producing a new graph G_1. If the result holds in G_1, then it also holds for G. Thus we may assume

(3.1) *If k is odd, then G is k-regular.*

If k is even and $e(s_1)$ is odd, then we replace s_1 by two vertices of degree $e(s_1)-1$ and $e(s_1)$, and assign s_1 on the new vertex of degree $e(s_1)-1$, producing a new graph G_1 (see Fig. 3). If the result holds in G_1,then it also holds for G. Thus we may assume that $e(s_i)$ and $e(t_i)$ $(1 \leqslant i \leqslant n)$ are even. We proceed by induction on $|E(G)|$. If k is even, $x \in V(G)$, and $e(x) \geqslant k+2$, then an admissible lifting G_x of G at x is k-edge-connected and the result holds in G_x. Thus

(3.2) *If k is even, then $e(s_i) = e(t_i) = k$ $(1 \leqslant i \leqslant n)$ and for each $x \in V(G)$, $e(x) = k$ or $k+1$.*

By [4, 1, 3] $g(3) = 3$ and $g(4) = 5$, and so we may let $k \geqslant 6$, $n \geqslant 5$, and $m \geqslant 2$. If $s_1 = s_2$, then by Theorem 1 there is a path $P[t_1, t_2]$ such that $s_1 \in V(P)$ and $\lambda(G - E(P)) \geqslant k-3$. By induction $G - E(P)$ has edge-disjoint

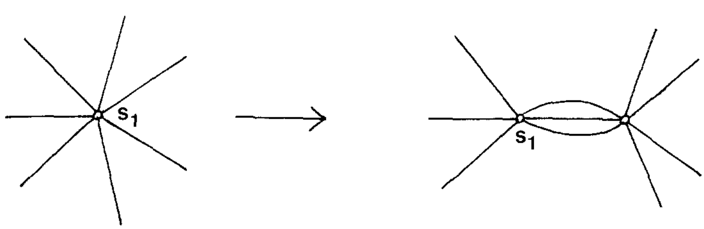

FIGURE 3

paths $P_3[s_3, t_3], ..., P_n[s_n, t_n]$. Thus let $|T| = 2n$. By Lemma 8 there is a subgraph $G^* \subset G$ such that (a), (b), (c), and (d) hold. By Lemma 3 $G^*(K_1, k-3)$ is $(k-3)$-edge-connected, and by induction $G^*(K_1, k-3)$ has $(n-2)$ edge-disjoint paths joining (s_l, t_l) $(1 \leqslant l \leqslant n,\ l \neq i, j)$. Thus the result holds in G.

Proof of Lemma 8

Suppose that G satisfies the hypothesis of Lemma 8, but the result does not hold. Choose G with this property such that $|E(G)|$ is minimal.

(3.3) $W_2 = \varnothing$.

Proof. Assume $x \in W_2$. Then $e(x) \geqslant 4$. By Lemma 2 we have an admissible lifting G_x of G at x. The result holds in G_x and so in G.

Case 1. k is odd.

(3.4) *If* $x, y \in W_1$ *and* $f \in \partial(x, y)$, *then* $V(G) - \{x, y\} \notin \Gamma(G - f, k)$.

Proof. Otherwise the result holds in $G - f$ with $V(G - f) = T \cup (W_1 - \{x, y\}) \cup \{x, y\}$.

(3.5) *If* $a, b \in T$, *then* $e(a, b) = 0$.

Proof. If $f \in \partial(s_1, t_1)$, then by Lemma 1(1) G has a path $P[s_2, t_2]$ such that $s_1 \notin V(P)$ and $\lambda(G - E(P)) \geqslant k - 2$. $G^* := G - E(P) - f$ is a required graph. If $f \in \partial(s_1, s_2)$, then by Theorem 1 $G - f$ has a path $P[t_1, t_2]$ such that $s_1 \in V(P)$ and $\lambda(G - f - E(P)) \geqslant k - 3$.

(3.6) *If* $x \in W_1$, $f \in \partial(x, s_1)$, *and* $h \in \partial(x, t_1)$, *then* $V(G) - \{x, s_1\} \notin \Gamma(G - f, k)$.

Proof. Assume $V(G) - \{x, s_1\} \in \Gamma(G - f, k)$. Then $V(G) - x \in \Gamma(G - \{f, h\}, k - 1)$. By Lemma 1(2) $G - \{f, h\}$ has a path $P[s_2, t_2]$ such that $x \notin V(P)$ and $\lambda(G - \{f, h\} - E(P)) \geqslant k - 3$.

(3.7) *If* $x \in W_1$ *and* $f_i \in \partial(x, s_i)$ $(i = 1, 2)$, *then* $V(G) - \{x, s_1, s_2\} \notin \Gamma(G - \{f_1, f_2\}, k)$.

Proof. Assume $V(G) - \{x, s_1, s_2\} \in \Gamma(G - \{f_1, f_2\}, k)$. Let $y \in N(x) - \{s_1, s_2\}$, $h \in \partial(x, y)$, and $G_1 := G - \{f_1, f_2, h\}$. Then $V(G) - x \in \Gamma(G_1, k - 1)$ and $e(x; G_1)$ is even. By Lemma 3 and Theorem 1 G_1 has a path $P[t_1, t_2]$ such that $s_1 \in V(P)$ and $V(G_1) - x \in \Gamma(G_1 - E(P), k - 3)$. Let $G^* := G - E(P) - \{f_1, f_2, h\}$.

(3.8) *If* $X \subset V(G)$, $|X| \geqslant 2$, *and* $|\bar{X}| \geqslant 2$, *then* $e(X) \geqslant k + 1$.

Proof. Assume $e(X)=k$ and for each $Y \subsetneq X$ with $|Y| \geqslant 2$, $e(Y) \geqslant k+1$. Let $u \in \bar{X}$ and $G_1 := G/\bar{X}$. If $Z \subset V(G)$, $|Z| \geqslant 2$, $|\bar{Z}| \geqslant 2$, and $X \cap Z \neq \emptyset \neq X - Z$, then $e(Z) \geqslant k+1$. For if not, then $Z - X \neq \emptyset \neq \overline{X \cup Z}$, contrary to Lemma 7(2). Thus if $x, y \in X$ and $f \in \partial(x, y)$, then $V(G) - \{x, y\} \in \Gamma(G-f, k)$, and so by (3.4) $N(X \cap W_1 : G_1) \subset T \cup \{\tilde{u}\}$. By (3.5) $N(X \cap T; G_1) \subset W_1 \cup \{\tilde{u}\}$. Thus G_1 is a bipartite graph with the partition $(X \cap T, (X \cap W_1) \cup \{\tilde{u}\})$ or $((X \cap T) \cup \{\tilde{u}\}, X \cap W_1)$ (note that G_1 is k-regular and k is odd) and $|V(G_1)| \geqslant 6$. Thus for some $x \in X \cap W_1$, $e(x, \tilde{u}; G_1) = e(x, \bar{X}; G) < (k-1)/2$. Then by Lemma 2 for some $a, b \in X \cap T$, $G_x^{a,b}$ is admissible, contrary to (3.6) or (3.7).

By (3.3), (3.4), (3.5), and (3.8) G is a bipartite graph with the partition (T, W_1). Let $x \in W_1$. By Lemma 2 for some $a, b \in N(x) \subset T$, $G_x^{a,b}$ is admissible, contrary to (3.6) or (3.7).

Case 2. k is even.

(3.9) *If* $x \in W_1$, *then* $e(x) = k+1$.

Proof. Assume $e(x)=k$. By Lemma 2 there is an admissible lifting G_x of G at x. The result holds in G_x with $V(G_x) = T \cup (W_1 - x) \cup \{x\}$, and it also holds for G.

(3.10) *If* $x, y \in W_1$ *and* $f \in \partial(x, y)$, *then* $\lambda(G-f) \leqslant k-1$.

(3.11) *If* $a, b \in T$, *then* $e(a, b) = 0$.

Proof. We can prove $e(s_i, t_i) = 0$ $(1 \leqslant i \leqslant n)$ in the same way as (3.5). If $g \in \partial(s_1, s_2)$, then let $f \in \partial(s_1) - g$. By Theorem 1 G has a path $P[t_1, t_2]$ such that $f \in E(P)$, $g \notin E(P)$, and $\lambda(G - E(P)) \geqslant k-2$. Let $G^* := G - E(P) - g$.

(3.12) *If* $x_1, x_2 \in W_1$, $a_1, a_2 \in T$, $f_i \in \partial(x_i, a_1)$ $(i=1,2)$, *and* $g \in \partial(x_2, a_2)$, *then* $V(G) - a_1 \notin \Gamma(G - \{f_1, f_2\}, k)$.

Proof. Assume $V(G) - a_1 \in \Gamma(G - \{f_1, f_2\}, k)$. If $a_1 = s_1$ and $a_2 = t_1$, then by Lemma 1 $G - - \{f_1, f_2\}$ has a path $P[s_2, t_2]$ such that $t_1 \notin V(P)$ and $V(G) - s_1 \in \Gamma(G - \{f_1, f_2\} - E(P), k-2)$. Let $G^* := G - E(P) - \{f_1, f_2, g\}$. If $a_1 = s_1$ and $a_2 = s_2$, then let $h \in \partial(a_2) - g$. By Theorem 1 and Lemma 3 $G - \{f_1, f_2\}$ has a path $P[t_1, t_2]$ such that $h \in E(P)$, $g \notin E(P)$, and $V(G) - s_1 \in \Gamma(G - \{f_1, f_2\} - E(P), k-2)$. Let $G^* := G - E(P) - \{f_1, f_2, g\}$.

(3.13) *If* $X \subset V(G)$, $|X| \geqslant 2$, *and* $|\bar{X}| \geqslant 2$, *then* $e(X) \geqslant k+1$.

Proof. Assume $e(X) = k$ and for each $Y \subsetneq X$ with $|Y| \geqslant 2$, $e(Y) \geqslant k+1$.

Let $u \in \bar{X}$. If $x, y \in X$ and $f \in \partial(x, y)$, then $V(G) - \{x, y\} \in \Gamma(G - f, k)$. For if not, then for some $Z \subset V(G) - x$, $y \in X$, $e(Z) = k$, $|Z| \geqslant 2$, and $|\bar{Z}| \geqslant 2$, and so $Z - X \neq \varnothing \neq \overline{X \cup Z}$ and by Lemma 7(2) $e(X - Z) = e(X \cap Z) = k$. Thus $|X| = 2$ and $e(x) = e(y) = k$. Then by (3.9) $\{x, y\} \subset T$, contrary to (3.11). Thus by (3.10) $N(X \cap W_1; G/\bar{X}) \subset T \cup \{\tilde{u}\}$, and by (3.11) $X \cap W_1 \neq \varnothing \neq X \cap T$. By (3.9) $|X \cap W_1| \geqslant 2$, and so $|X \cap T| \geqslant 2$. Let $a \in X \cap T$. Since $e(a, \bar{X}) < k/2$ (otherwise $e(X - a) = k$), by Lemma 2 for some $x, y \in N(a) \cap X$, $G_a^{x,y}$ is admissible. By (3.11) $\{x, y\} \subset W_1$. Let $f_1 \in \partial(a, x)$ and $f_2 \in \partial(a, y)$, then $V(G) - a \in \Gamma(G - \{f_1, f_2\}, k)$. Let $b \in ((N(x) \cup N(y)) \cap X) - a$, then $b \in T$, contrary to (3.12).

By (3.3), (3.10), (3.11), and (3.13) G is a bipartite graph with the partition (T, W_1). Let $a \in T$. By Lemma 2 for some $x, y \in N(a) \subset W_1$, $G_a^{x,y}$ is admissible and we can deduce a contradiction (see the proof of (3.13)).

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