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Implicit Function Theorems, Approximate Solvability of Nonlinear Equations, and Error Estimates

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1. INTRODUCTION

This paper deals with various implicit and inverse function theorems for nondifferentiable maps, with constructive homeomorphism results for nonlinear and semilinear not necessarily differentiable maps, and with error estimates of approximate solutions.

Let X and Y be Banach spaces and $T: X \to Y$ be continuously Fréchet differentiable at x_0 and satisfy the Lyusternik condition $T'(x_0)(X) = Y$, and thus right-invertible. Then the classical inverse function (open mapping) theorem of Graves [G] states that $T(x_0) \in \operatorname{Int} T(B_h(x_0))$ for all $h > 0$. Recently, there have been extensions of this result in many directions based on various iterative processes, topological degrees, and Ekeland's variational principle, depending on the structure and/or differentiability properties of T. For example, if T has either a strong Fréchet and a Hadamard derivative at x_0 and $T'(x_0)$ has an approximate right or other inverse, then some implicit function theorems, based on a generalized Newton–Kantorovich iterative process, have been obtained by Craven and Nashed and others (see [CN] and the references therein). In applications, many boundary value problems for differential equations or many control theory and optimization problems may not be locally linearized and may require results where the linear structure is not present. Motivated by this, a number of authors have obtained various generalization of the classical inverse and implicit function theorems to nondifferentiable maps having either some type of a multivalued derivative in a Banach space or a suitable variation in a complete metric space, which describes the infinites-

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imal behavior of a map at a given point. Various extensions, based on iterative processes, can be found in [E], [K], and [DMO] and the references therein. Inverse function theorems for set-valued maps having contingent or Clark's type of multivalued derivative, and the range in a finite- or infinite-dimensional space, have been proved by many authors and we refer to Aubin and Ekeland [AE] and Aubin and Frankowska [AF-1, AF-2] and the references therein. Recently, Frankowska [F] has obtained several first- and higher-order inverse mapping theorems for set-valued maps from a complete metric space to a Banach space by studying the corresponding open mapping principle using a variation of the map at a point and Ekeland's variational principle. Another type of implicit function theorem for compact maps, requiring a more special type of a multivalued derivative, has been obtained by Chow and Lasota [CL] using the Leray–Schauder degree theory.

New extensions of the classical implicit/inverse function theorem are given in Section 2. We prove some implicit and inverse function theorems for maps having a multivalued derivative at an initial solution x_0. The first few results involve pseudo A-proper and ϕ-condensing maps of the form $T(x, v) = Nx + M(x, v)$ and are based on the crucial new assumption that the isolated solution x_0 of $Nx = 0$ has a nontrivial index, i.e., the corresponding degree is nontrivial. This assumption is shown to hold for potential A-proper maps, as well as for some types N having a multivalued derivative. These results are proved using topological degree methods and extend considerably the work of Chow and Lasota [CL]. The last neighborhood open mapping–inverse function theorem involves nonlinear maps on closed subsets that have a special type of a multivalued derivative, and is proved by using an iterative process. It is an extension of Ehrmann's implicit function theorem [E] and of the open mapping theorem of Kachurovskii [K] (cf. also [DMO] for other results for nondifferentiable maps). It also extends an inverse function theorem of Aubin and Frankowska [AF] to nondifferentiable maps having an infinite-dimensional image space but defined on less general domains. However, we refer to [AF-2] for a constrained inverse function theorem for differentiable maps between two Banach spaces satisfying a transversality condition. The results of this section are applicable to boundary value problems for differential equations which may not be locally linearized (cf. [CL] for some such applications). They are also applicable to such BVP's in Banach spaces assuming some monotonicity or contractive-type condition on the nonlinear part, and to optimal control problems. As in [SS], Corollary 2.1 can be used to study various semilinear BVP's not in resonance involving nonlinearities depending also on the highest-order derivatives in such a way as to make the induced map A-proper.

Section 3 contains a basic approximation solvability result (Theorem 3.1) for nonlinear maps having a multivalued derivative and the error estimates for the approximate solutions. This result has many applications. For example, in [Mi-9], we have applied it to the constructive solvability of nonlinear Hammerstein (operator and integral) equations. It is also used extensively in the rest of the paper. In Section 4, we have established various constructive homeomorphism results for A-proper maps. First, we show that a continuous coercive and locally invertible A-proper map is a homeomorphism. Then, using this and Theorem 3.1, we show that a continuous locally injective A-proper map $T: X \to Y$ with closed range and which has a multivalued derivative $A(x)$ on X, with coercive finite-dimensional approximations, is a homeomorphism, the equation $Tx = f$ is approximation-solvable, and the corresponding error estimates hold. In particular, these assertions hold if T is a locally injective Fréchet differentiable A-proper map on X with closed range and the injective A-proper derivative $T'(x)$ on X. When $T = I - C$, with C a compact map, is coercive, continuously Fréchet differentiable and $T'(x)$ is injective on X, the homeomorphism assertion only for T has been proved by Krasnosel'skii and Zabreiko [KZ]. Applications to some special classes of nondifferentiable maps and to Fréchet differentiable asymptotically $\{B_1, B_2\}$-quasilinear maps are also given. The final result of the section asserts that the equation $Tx = f$ has the same finite or infinite number of solutions for each $f \in Y$, and each is obtained constructively. The nonconstructive part of the result is due to Ehrmann [E]. The results of Section 4 are applicable to BVP's for ordinary and partial differential equations with nonlinearities depending on the highest-order derivatives in such a way that the induced map is A-proper (see also [Mi-8]).

In Section 5, using Theorem 3.1, we prove a number of results dealing with the unique approximation solvability and error estimates for nonresonant semilinear equations $Ax - Nx = f$ in a Hilbert space, where A is a closed linear densely defined map with $\dim \ker(A) \leq \infty$ and N is a suitable nonlinear map such that $A - N$ is an A-proper map. For example, N can be a Lipschitz or a strongly monotone map or such that in a suitable reformulation the corresponding nonlinearity is contractive or monotone. These results are improvements of [Mi-6] and extend constructively some recent results of Fonda and Mawhin [FM] and its many special cases (Amann [A], Dancer [D], etc.) and of Ben-Naoum and Mawhin [BM]. They are applicable to BVP's for semilinear elliptic equations and periodic-BVP's for semilinear hyperbolic equations in several space variables. We refer to [Mi-10] for some such applications. Section 6 is devoted to constructive homeomorphism theorems and error estimates for approximation-stable A-proper maps.

2. IMPLICIT FUNCTION THEOREMS

Let $\{X_n\}$ and $\{Y_n\}$ be finite-dimensional subspaces of Banach spaces X and Y, respectively, such that $\dim X_n = \dim Y_n$ for each n and $\operatorname{dist}(x, X_n) \to 0$ as $n \to \infty$ for each $x \in X$. Let $P_n\colon X \to X_n$ and $Q_n\colon Y \to Y_n$ be linear projections onto X_n and Y_n, respectively, such that $P_n x \to x$ for each $x \in X$ and $\delta_0 = \sup\|Q_n\| < \infty$. Then $\Gamma = \{X_n, P_n; Y_n, Q_n\}$ is a projection scheme for (X, Y).

Let $T\colon D \subset X \to 2^Y$ be a multivalued map. We recall ([*Mi* − 1])

DEFINITION 2.1. T is said to be *approximation-proper* with respect to Γ (A-proper w.r.t. Γ, for short) if (i) $Q_n T\colon D \cap X_n \to 2^{Y_n}$ is upper semicontinuous (u.s.c. for short) for each n and (ii) whenever $\{x_{n_k} \in D \cap X_{n_k}\}$ is bounded and $\|Q_{n_k} y_{n_k} - Q_{n_k} f\| \to 0$ for some $y_{n_k} \in Tx_{n_k}$ and $f \in Y$, then a subsequence $x_{n_{k(i)}} \to x$ and $f \in Tx$. T is said to be *pseudo A-proper* w.r.t. Γ if in (ii) we do not require that a subsequence of $\{x_{n_k}\}$ converges to x for which $f \in Tx$.

For many examples of single-valued and multivalued A-proper and pseudo A-proper maps, we refer to [Mi-1−Mi-6]. For example, ball-condensing and, in particular, compact and k-contractive, perturbations of Fredholm maps of index zero, maps of type (S_+), sums of ball-condensing, and strongly monotone maps are all A-proper maps. Monotone-like maps and such perturbations of closed linear maps A with finite- or infinite-dimensional null space are pseudo A-proper maps.

A multivalued map $A\colon X \to 2^Y$ is said to be m-bounded if there is a positive constant m such that $\|y\| \le m\|x\|$ for all $x \in X$, $y \in Ax$. It is c-coercive if $\|u\| \ge c\|x\|$ for $x \in X$ and $u \in Ax$.

Next, we introduce a class of maps having a multivalued derivative.

DEFINITION 2.2. Let U be open in X and $T\colon \overline{U} \to Y$. A positively homogeneous map $A\colon X \to 2^Y$, with $A(x)$ convex and closed for each $x \in X$, is said to be a *multivalued derivative* of T at $x_0 \in U$ if there exists a map $R = R(x_0)\colon \overline{U} - x_0 \to 2^Y$ such that $R(x - x_0) = o(\|x - x_0\|)$, i.e., if $r(x - x_0)\colon \overline{U} - x_0 \to Y$ is a selection of $R(x - x_0)$: $r(x - x_0) \in R(x - x_0)$, then $\|r(x - x_0)\|/\|x - x_0\| \to 0$ as $x \to x_0$ and

$$Tx - Tx_0 \in A(x - x_0) + R(x - x_0) \qquad \text{for } x \text{ near } x_0.$$

The basic assumption in our first implicit function theorem is that a known initial solution has a nonvanishing degree.

THEOREM 2.1. *Let U be an open subset of X, $N\colon U \to Y$ be an A-proper map, and x_0 be an isolated solution to $Nx = 0$, $M\colon U \times B_r \to Y$ be continuous with $M(x, v) \to 0$ uniformly in x as $v \to 0$, and, for each fixed $v \in B_r$,*

$N + M(\cdot, v)$ be pseudo A-proper on each ball $\overline{B}_k(x_0) \subset U$. Suppose that either $\deg(Q_n N, B_\rho(x_0) \cap X_n, 0) \neq 0$ *for all large n and a small $\rho > 0$, or N has an u.s.c. A-proper homogeneous derivative $A\colon X \to 2^Y$ at x_0 such that $x = 0$ if $0 \in Ax$. Then there is an $r_0 \in (0, r]$ such that*

(a) *for every $v \in B_{r_0}$ there exists a solution $x_v \in B_\rho(x_0)$ of*

$$Nx + M(x, v) = 0. \tag{2.1}$$

(b) $\|x_v - x_0\| \to 0$ *as* $\|v\| \to 0$ *provided also A is c_1-coercive.*

LEMMA 2.1. *Let $A\colon X \to 2^Y$ be a positively homogeneous map. Then*

(a) *If A is u.s.c. and has closed and bounded values, then it is m-bounded.*

(b) *If A is A-proper w.r.t.* Γ *and $x = 0$ if $0 \in Ax$, then $Q_n A$ is c-coercive on X_n for all $n \geq n_0 \geq 1$ and some $c > 0$ independent of n.*

Proof. (a) If such an m does not exist, then there are $x_k \in X$ and $y_k \in Ax_k$ such that $\|y_k\| > k\|x_k\|$ for each $k > 0$. Since A is positively homogeneous, we have that

$$y_k/\big(k\|x_k\|\big) \in A\big(x_k/\big(k\|x_k\|\big)\big) \qquad \text{for all } k \geq 1.$$

But $x_k/(k\|x_k\|) \to 0$ and $\|y_k/(k\|x_k\|)\| > 1$, in contradiction to the upper semicontinuity of A at 0.

(b) This is Lemma 2.1 in [Mi-3]. ∎

Proof of Theorem 2.1. Suppose first that x_0 is an isolated solution and the above degree is nonzero. Then there is a small $\rho > 0$ such that $Nx \neq 0$ for all $x \in \overline{B}_\rho(x_0) \setminus \{x_0\}$. Arguing by contradiction and using the A-properness of N, it follows that there are a $\gamma > 0$ and an $n_0 \geq 1$ such that

$$\|Q_n Nx\| \geq \gamma \qquad \text{for all } x \in \partial B_\rho(x_0) \cap X_n,\ n \geq n_0. \tag{2.2}$$

Hence, since $M(x, v) \to 0$ uniformly in $x \in \overline{B}_r(x_0)$ as $v \to 0$, there is an $r_0 \leq r$ such that

$$\big\| Q_n Nx - tQ_n M(x, v) \big\| > 0 \qquad \text{for } x \in \partial B_\rho(x_0),\ v \in B_{r_0},\ t \in [0, 1].$$

Thus, for each $n \geq n_0$ and $v \in B_{r_0}$,

$$\deg\big(Q_n N - Q_n M(\cdot, v), B_\rho(x_0) \cap X_n, 0\big)$$
$$= \deg\big(Q_n N, B_\rho(x_0) \cap X_n, 0\big) \neq 0$$

and, consequently, $Q_n(Nx_n + M(x_n, v)) = 0$ for some $x_n \in B_\rho(x_0) \cap X_n$. Since $N + M(\cdot, v)$ is pseudo A-proper on $\overline{B}_\rho(x_0)$, it follows that there is an $x \in \overline{B}_\rho(x_0)$ such that $Nx + M(x, v) = 0$.

Next, let A be odd. Then it suffices to show that $\deg(Q_n N, B_l(x_0) \cap X_n, 0) \neq 0$ for some $l \leq \rho$ and all large n. Let $c > 0$ and $n_0 \geq 1$ be as in Lemma 2.1 and $\delta_0 = \sup\|Q_n\|$. Let $\epsilon > 0$ be such that

$$\|y\|/\|x - x_0\| < c/(2\delta_0) \qquad \text{for all } \|x - x_0\| < \epsilon,\ y \in R(x - x_0).$$

Define $T(x) = N(x_0 + x)$ and let $l \leq \min\{\rho, \epsilon\}$. Define a homotopy H: $[0,1] \times \overline{B}_l \to Y$ by $H(t, x) = 1/(1+t)Tx - t/(1+t)T(-x)$. Then

$$Q_n H(t, x) \neq 0 \qquad \text{for } t \in [0,1],\ x \in \partial B_l \cap X_n,\ n \geq n_0. \tag{2.3}$$

If not, then there are an $n \geq n_0$, $x \in \partial B_l \cap X_n$, and $t \in [0,1]$ such that $Q_n H(t, x) = 0$, and therefore

$$1/(1+t)Q_n T(x) - t/(1+t)Q_n T(-x) = 0.$$

But $-T(-x) = -(N(x_0 - x) - Nx_0) \in Ax - R(-x)$, and therefore, by the convexity of Ax, we have that $0 \in Q_n Ax + Q_n R_1 x$, where $R_1 x = 1/(1 + t)R(x) - t/(1+t)R(-x)$. Hence, $-Q_n y \in Q_n Ax$ for some $y \in R_1(x)$, where $y = 1/(1+t)y_1 - t/(1+t)y_2$ with $y_1 \in Rx$ and $y_2 \in R(-x)$. By Lemma 2.1,

$$c\|x\| \leq \delta_0(\|y_1\| + \|y_2\|) < c\|x\|,$$

a contradiction. Thus, (2.3) holds and the Brouwer degree

$$\begin{aligned}\deg(Q_n N, B_l(x_0) \cap X_n, 0) &= \deg(Q_n T, B_l \cap X_n, 0)\\ &= \deg(Q_n H(1, \cdot), B_l \cap X_n, 0) \neq 0\end{aligned}$$

for each $n \geq n_0$ since $Q_n H(1, \cdot)$ is an odd map.

(b) Let x_v be a solution of Eq. (2.1) and $\|x_v - x\| < \epsilon$. The assumptions on M imply that we may assume that there is a monotone function $\delta(s) \geq 0$ such that $\delta(s) \to 0$ as $s \to 0$ and

$$\|M(x, v)\| < \delta(\|v\|) \qquad \text{for all } \|x - x_0\| < \epsilon.$$

Then, since we may assume that $c = c_1$ and

$$Nx_v - Nx_0 + M(x_v, v) \in A(x_v - x_0) + R(x_v - x_0) + M(x_v, v),$$

we get that, for some $u \in R(x_v - x_0)$,

$$c\|x_v - x_0\| \le \|u\| + \delta(\|v\|) \le c/(2\delta_0)\|x_v - x_0\| + \delta(\|v\|).$$

Hence, $\|x_v - x_0\| \le \delta(\|v\|)/(c - c/(2\delta_0))$, which implies (b). ∎

Remark 2.1. Analyzing the proof, we see that instead of x_0 being an isolated solution, it is enough to require that $Nx \neq 0$ for $x \in \partial B_\rho(x_0)$, or even that (2.2) holds, for some $\rho > 0$. If A is homogeneous, then Theorem 2.1(a) is valid without requiring that x_0 is an isolated solution. Indeed, one needs only to use the homotopy H with $Tx = N_1x + M_1x$, where $N_1x = N(x_0 + x)$ and $M_1x = M(x_0 + x, v)$ for $v \in B_r$. However, the c_1-coercivity of A in (b) implies that x_0 is an isolated solution.

To state a related result for ϕ-condensing maps, we recall that the *set measure of noncompactness* of a bounded set $D \subset X$ is defined as $\gamma(D) = \inf\{d > 0$: D has a finite covering by sets of diameter less than $d\}$. The *ball-measure of noncompactness* of D is defined as $\chi(D) = \inf\{r > 0 \mid D \subset \bigcup_{i=1}^n B_r(x_i),\ x \in X,\ n \in N\}$. Let ϕ denote either the set or the ball-measure of noncompactness. Then a map $T\colon D \subset X \to 2^X$ is said to be *ϕ-condensing* if $\phi(T(Q)) < \phi(Q)$ whenever $Q \subset D$ and $\phi(Q) \neq 0$.

THEOREM 2.2. *Let U be an open subset of X, $N\colon U \to X$ be a continuous and ϕ-condensing map with $Nx_0 = x_0$, and $M\colon U \times B_r \to X$ be continuous and ϕ-condensing with $M(x, v) \to 0$ uniformly in x as $v \to 0$. Suppose that either* $\deg(I - N, B_\rho(x_0), 0) \neq 0$ *for a small $\rho > 0$, or N has a homogeneous ϕ-condensing derivative A such that $x = 0$ if $x \in Ax$. Then there exist $r_0 \in (0, r]$ and $\rho > 0$ such that*

(a) *for every $v \in B_{r_0}$ there exists a solution $x_v \in B_\rho(x_0)$ of*

$$x = Nx + M(x, v). \tag{2.4}$$

(b) $\|x_v - x_0\| \to 0$ *as* $\|v\| \to 0$.

Proof. Since $x - Nx \neq 0$ for $\|x\| = \rho$, arguing by contradiction and using the ϕ-measure of noncompactness, we get a $\gamma > 0$ such that $\|x - Nx\| \ge \gamma$ for all $\|x\| = \rho$. This inequality also holds in the second case, since $I - A$ is c-coercive by Lemma 1.1 in [Mi-2] for some $c > 0$, and therefore x_0 is an isolated solution of $x - Nx = 0$. Let $\epsilon > 0$ and $\delta(s)$ be as in the proof of Theorem 2.1. Then, using the homotopy $H(t, x) = x - Nx - tM(x, v)$, we get that $\deg(I - N - M(\cdot, v), B_\rho(x_0), 0) \neq 0$ for each fixed $v \in \overline{B}_{r_0}$. This implies (a) under the degree assumption. If A is homogeneous, set $Tx = N(x_0 + x) - Nx_0$ and consider the homotopy $H(t, x) = x - 1/(1 + t)Tx - t/(1 + t)T(-x)$. Then, using the arguments similar to those in the proof of Theorem 2.1, we get that $\deg(I -$

$N, B_\rho(x_0), 0) \neq 0$ and (a) follows. The second part is proved as in Theorem 2.1. ∎

Remark 2.2. The condition $x \in Ax$ implies $x = 0$ replaces the condition that the Jacobian is not zero, or the invertibility of the derivative at 0 in the classical implicit function theorem.

Due to the generality of the maps involved, Theorems 2.1 and 2.2 are suitable, for example, for studying boundary value problems for ordinary differential equations in Banach spaces. When N and A are compact, Theorem 2.2 is due to Chow and Lasota [CL], where applications to BVP's for systems of ordinary differential equations are given.

The degree assumption in Theorems 2.1 and 2.2 holds if N is an odd map. Next, we shall show that it holds for gradient maps at an isolated critical point. We need the following result

THEOREM 2.3 (cf. [Mi-7]). *Let $U \subset X$ be a neighborhood of* 0, $f\colon \overline{U} \to R^1$ *be continuous and Gâteaux differentiable on U,* 0 *be its isolated critical point, and $f(0)$ be a local minimum at* 0. *Let*

(i) $f(0) < m(r) = \inf\{f(x) \mid x \in \partial B_r\}$ *for each* $0 < r \leq \rho$, *where* $f(0) < f(x)$ *for* $x \in B_\rho \setminus \{0\}$ *and some* $\rho > 0$.

(ii) $f'(x) \neq 0$ *for* $x \in \{x \in B_\rho,\ f(x) \geq k\}$ *for a suitable* $k > 0$.

If $N = f'\colon\ X \to X^*$ *is A-proper w.r.t.* $\Gamma = \{X_n, Y_n, Q_n\}$ *at* 0, *then* $\deg(Q_n N, B_\rho \cap X_n, 0\} \neq 0$ *for all large n.*

Remark 2.3. If f is C^1 on some $\overline{B}_R$ and satisfies the Palais–Smale (PS) condition, then condition (i) of Theorem 2.3 holds by Proposition 4 in Brezis and Nirenberg [BN] as well as condition (ii) if in addition f is bounded on $\overline{B}_\rho$. Conditions (i) and (ii) hold also if f is continuous in a Hilbert space and Gâteaux differentiable with f' being a bounded demicontinuous map of type (S_+) (i.e., $x_n \to x$ whenever $x_n \rightharpoonup x$ and $\limsup(f'(x_n), x_n - x) \leq 0$) (cf. [K]). It is well known that such maps are A-proper. We note also that (i) and (ii) hold if $N = f'$ is continuous and A-proper at 0 since such maps are proper on bounded and closed subsets. In particular, this is so if $f' = I - C$ with C compact.

Remark 2.4. The degree $\deg(Q_n N, B_\rho(x_0) \cap X_n, 0) \neq 0$ for all large n under other conditions. For example, if $f' = I - C$ with C compact and x_0 is an isolated critical point of f of mountain-pass type, then some sufficient conditions for $\deg(f', B_\rho, 0) = -1$, and therefore $\deg(Q_n f', B_\rho \cap X_n, 0) = -1$ for large n, have been given by Hofer [H].

Theorems 2.1 and 2.3 imply the following result.

THEOREM 2.4. *Let U and N be as in Theorem 2.3. Suppose that M: $U \times B_r \to X^*$ is continuous with $M(x, v) \to 0$ uniformly in x as $v \to 0$ and, for each $v \in B_r$, $N + M(\cdot, v)$ is pseudo A-proper w.r.t. Γ on each ball $\overline{B}_k \subset U$. Then there is an $r_0 \in (0, r]$ such that the conclusions of Theorem 2.1 hold with $x_0 = 0$.*

The next result does not require oddness of the multivalued derivative. A similar result holds also for ϕ-condensing maps, and includes a result in [SS] for compact maps.

COROLLARY 2.1. *Let L, K: $\overline{U} \subset X \to Y$ and F: $\overline{U} \to 2^Y$ be such that L is homogeneous, F is positively homogeneous with $F(x)$ starlike with respect to 0 for each $x \in \overline{U}$, $L + F$ is A-proper at 0 w.r.t. Γ, $x = 0$ if $0 \in Lx + Fx$, and either F is a multivalued derivative of K at 0 or $Kx \in Fx + Rx$ for all $\|x\|$ large with $|R| = \limsup_{\|x\| \to \infty} \|Rx\|/\|x\|$ sufficiently small. Suppose that M: $U \times B_r \to Y$ is continuous with $M(x, v) \to 0$ uniformly for x in bounded subsets as $v \to 0$ and, for each $v \in B_r$, $L + K + M(\cdot, v)$ is pseudo A-proper w.r.t. Γ on each ball $\overline{B}_k \subset U$. Then there are $\rho > 0$ and $r_0 \in (0, r]$ such that for each $v \in B_{r_0}$,*

(a) *$Lx + Kx + M(x, v) = 0$ has a solution $x_v \in B_\rho$.*

(b) *$\|x_v - x\| \to 0$ as $v \to 0$ provided $L + F$ is c_1-coercive.*

(c) *If $U = X$, $R \equiv 0$, and $M(x, v) = M_1 x$ for all v, with $|M_1|$ sufficiently small, then $L + K + M_1$ is onto.*

Proof. Since $x = 0$ if $0 \in Lx + Fx$, Lemma 2.1(b) implies that there are constants $c > 0$ and $n_0 \geq 1$ such that

$$\|Q_n Lx + Q_n y\| \geq c\|x\| \qquad \text{for } x \in X_n,\ y \in Fx,\ n \geq n_0. \tag{2.5}$$

Then, in the first case, there is a $\rho > 0$ sufficiently small such that

$$Q_n(L + tK)x \neq 0 \qquad \text{for } x \in \partial B_\rho \cap X_n,\ t \in [0, 1],\ n \geq n_0. \tag{2.6}$$

If not, then $0 \in Q_n(L + t_n F + t_n R)x_n$ for some $x_n \in X_n$, $x_n \to 0$, $x_n \neq 0$, and t_n. Set $u_n = x_n/\|x_n\|$. Then since Fx is starlike with respect to 0 for each x, it follows that $0 \in Q_n(L + F)u_n + t_n Q_n R(x_n)/\|x_n\|$ and $Q_n(Lu_n + y_n + t_n z_n/\|x_n\|) = 0$ for some $y_n \in Fu_n$ and $z_n \in Rx_n$. By (2.5), we get a contradiction

$$c \leq \|Q_n(Lu_n + y_n)\| \leq \|Q_n\|\,\|z_n\|/\|x_n\| \to 0 \qquad \text{as } n \to \infty.$$

Hence, (2.6) holds and $\deg(Q_n(L + K), B_\rho \cap X_n, 0) = \deg(Q_n L, B_\rho \cap X_n, 0) \neq 0$ for each $n \geq n_0$. In the second case, using similar arguments, we find a ρ large such that the last degree is again nonzero. Therefore, (a) and (b) follow from Theorem 2.1 and Remark 2.1 with $N = L + K$. For part (c) it is enough to observe that (2.6) holds for $L + tK + tM_1 - tf$ for each $f \in Y$, using a similar reasoning. ∎

For our next result, we introduce an Ω-neighborhood of a map T_0 with respect to $B_r(x_0)$. Here $D(S)$ is the domain of S.

DEFINITION 2.3. A map T is said to be in an $\Omega = (x_0, r, a, b)$-neighborhood of a map T_0 if $B_r(x_0) \subset D(T) \cap D(T_0)$ and $\Delta T = T - T_0$ satisfies

(a) $\|\Delta T x_0\| < a$.

(b) $\|\Delta Tx - \Delta Ty\| \le b\|x - y\|$ for all $x, y \in B_r(x_0)$.

Now, we give a neighborhood open mapping–inverse function theorem.

THEOREM 2.5. *Let $C \subset X$ be a closed convex subset, $x_0 \in C$, and T_0: $B_0 = B_{r_0}(x_0) \cap C \to Y$ with $T_0 x_0 = y_0$. Suppose that K: $X \to Y$ is a linear map with $K(C) = Y$ and, for some positive m and c with $mc^{-1} < 1$,*

(i) $\|T_0 x - T_0 y - K(x - y)\| \le m\|x - y\|$ *for* $x, y \in B_0$.

(ii) K^{-1}, *defined by* $K^{-1}(y) = \{x \in C \mid Kx = y\}$, *is c-Lipschitz, i.e.,*

$$K^{-1}(y_2) \subset K^{-1}(y_1) + c\|y_1 - y_2\|\overline{B}_1 \qquad \textit{for all } y_1, y_2 \in Y. \quad (2.7)$$

(iii) *Let, in addition, T be continuous if $N(K) = \ker K \neq \{0\}$.*

Let $\Omega = (x_0, r, a, b)$-neighborhood of T_0 with $0 < r \le r_0$ and $a, b \ge 0$ be such that $a = (c^{-1} - b - m)r > 0$. Then there is a $k > 0$ such that, for all $T \in \Omega$, $B_{h/k}(Tx) \subset T(B_h(x) \cap C)$ for each $x \in B_{r_0}(x_0)$ and $h \in [0, r_0)$. If $N(K) = \{0\}$ and $y_0 = 0$, then the equation $Tx = 0$ has a unique solution $x = x(T)$ which is continuous in T at $T = T_0$ in the sense that

$$\|x(T) - x_0\| \to 0 \qquad \textit{as } \|Tx_0\| \to 0. \quad (2.8)$$

Moreover, if $Tu_0 = v_0$, then T has a local inverse defined in a neighborhood of v_0 with the range in a neighborhood of u_0 and the corresponding solution $x(y)$ of $Tx = y$ is continuous in y, i.e., $x(y) \to u_0$ as $y \to v_0$.

Proof. Let $T \in \Omega$ with $r \le r_0$ and $\Delta T = T - T_0$. Then, for each $x, y \in B_r(x_0) \cap C$,

$$\begin{aligned}\|Ty - Tx - K(y - x)\| &\le \|\Delta Ty - \Delta Tx\| + \|T_0 y - T_0 x - K(y - x)\| \\ &\le (b + m)\|y - x\|. \end{aligned} \quad (2.9)$$

Now, $q = (b + m)c < 1$ and $p\rho_0 < r_0$ for $p = (1 - q)^{-1}c$ an some ρ_0. Then, for each $h \le r_0$, $h = sr_0$ and $k\rho_0 = r_0$ for some s and k. Moreover, since $p\rho_0 = pr_0/k < r_0$, we have that $ph/k = psr_0/k = p\rho_0 s < sr_0 = h$. Hence, for each $h \le r_0$, $p\rho < h$ with $\rho = h/k$. Let $x \in B_{r_0}(x_0) \cap C$ be fixed and $h \le r_0$. Let $y \in B_{h/k}(Tx)$ and define successive approximations as follows. Given $x_n \in B_h(x) \cap C$, there is an $\bar{x}_{n+1} \in C$ such that $K\bar{x}_{n+1} = Kx_n - (Tx_n - y)$ since $K(C) = Y$. By (ii), (2.7) holds for Kx_n and $K\bar{x}_{n+1}$. Hence, there are an $x_{n+1} \in K^{-1}(K\bar{x}_{n+1})$ and $w \in \overline{B}_1$ such that $x_n = x_{n+1} + c\|K\bar{x}_{n+1} - Kx_n\|w$. Then $Kx_{n+1} = Kx_n - (Tx_n - y)$ and

$$\|x_{n+1} - x_n\| \le c\|Kx_{n+1} - Kx_n\|. \tag{2.10}$$

Hence, starting with $x_1 = x \in B_h(x) \cap C$, we get

$$\begin{aligned} \|x_{n+1} - x_n\| &\le c\|Tx_n - y\| = c\|Tx_n - Tx_{n-1} - K(x_n - x_{n-1})\| \\ &\le (b+m)c\|x_n - x_{n-1}\| \end{aligned} \tag{2.11}$$

and

$$\|x_n - x_1\| \le \sum_{i=0}^{n-1} q^i \|x_2 - x_1\|,$$

with

$$\|x_2 - x_1\| \le c\|Tx_1 - y\| \le ch/k = c\rho.$$

Thus, by our choice of q and ρ, the sequence $\{x_n\} \subset B_{p\rho}(x)$ is Cauchy with the limit $\bar{x} \in \overline{B}_{p\rho}(x) \cap C \subset B_h(x) \cap C$. Let (iii) hold. Then, $Tx_n - y \to 0$ by (2.11) and the continuity of T implies that $y = T\bar{x} \in T(B_h(x) \cap C)$. Hence, we have shown that $B_{1/kh}(Tx) \subset T(B_h(x) \cap C) \subset TB_h(x)$ for each $x \in B_{r_0}(x_0)$ and each $h \in [0, r_0)$.

Next, let the inverse K^{-1} exist. Then $c = \|K^{-1}\|^{-1}$ and the map $V = K^{-1}(K - T)$ is l-contractive with $l = (b+m)\|K^{-1}\| < 1$ in $\overline{B}_r(x_0)$ and therefore $x_{n+1} = K^{-1}(Kx_n - Tx_n + y)$ implies that $T\bar{x} = y$. Next, let $y_0 = 0$. Since $T \in \Omega$, then $\|Vx_0 - x_0\| = \|K^{-1}Tx_0\| < (1-l)r$ and V maps $B_r(x_0)$ into itself. Hence, there exists a unique solution of $Vx = x$. Since the unique solvability of $Vx = x$ is equivalent to the unique solvability of $Tx = 0$, there is a unique solution $x(T)$ of $Tx = 0$ in $B_r(x_0)$ satisfying

$$\|x(T) - x_0\| \le (1-k)^{-1}\|K^{-1}Tx_0\| \le (1-k)^{-1}\|K^{-1}\|\,\|Tx_0\|.$$

Thus, (2.8) holds. Next, let $Tu_0 = v_0$ and $T_1 x = Tx - y$ for some $y \in B_\rho(v_0)$. Define $V_1 = K^{-1}(K - T_1)$ and note that $T_1 x = 0$ if and only if $V_1 x = x$. Then, for some $r > 0$, it is easy to see that, for $x \in \overline{B}_r(u_0)$,

$$\|V_1 x - u_0\| \le l\|x - u_0\| + \|K^{-1}\|\,\|v_0 - y\|.$$

Taking $\rho < \|K^{-1}\|^{-1}(1-l)r$, we see that V_1 maps $\overline{B}_r(u_0)$ into itself and is an l-contraction. Hence, the equation $Tx = y$ has a unique solution $x(y) \in \overline{B}_r(u_0)$ for each $y \in B_\rho(v_0)$. Moreover, $\|x(y) - u_0\| = \|K^{-1}(K - T_1)x - u_0\| \le l\|x - u_0\| + \|K^{-1}\| \|v_0 - y\|$ and therefore

$$\|x(y) - u_0\| \le (1-l)^{-1}\|K^{-1}\| \|v_0 - y\| \to 0 \qquad \text{as } y \to v_0. \quad \blacksquare$$

Remark 2.5. If K is continuous and $C \subset X$ is a closed convex cone with $K(C) = Y$, then K^{-1} is a Lipschitz set-valued map (cf. [AE]). If $N(K) = \{0\}$, $K(C) = Y$, and K^{-1} is continuous, then (2.7) holds. Moreover, if $C = X$, then each $T \in \Omega$ is an open map at each $x \in B_{r_0}(x_0)$, i.e., $B_{1/kh}(Tx) \subset TB_h(x)$ and, as noted in [DMO], this open mapping property is equivalent to the following distance estimate:

$$\|x - T^{-1}y\| \le k\|Tx - y\|$$

for all x in a neighborhood of x_0 and all y in a neighborhood of Tx_0 in Y. If T_0 has a strong Fréchet derivative $T_0'(x_0)$ at x_0, then we can take K to be any map near it in Theorem 2.5, or, if T_0 is as in Corollary 2.2 below, take $K = T_0'(x_0)$.

Remark 2.6. Even if T is defined on all of X, Theorem 2.5 does not imply the solvability of $Tx = y$ for each $y \in Y$. For example, consider the map $Tx = \arctan x$ for $x \in X = R^1$. Then all conditions of Theorem 2.5 hold but the equation $Tx = y$ is not solvable for all $y \in R^1$. For some additional conditions that imply $R(T) = Y$, see Theorem 4.4.

Recall that a map T is said to be weakly Gâteaux differentiable at x_0 if there is $T'(x_0) \in L(X,Y)$ such that

$$\left(t^{-1}[T(x_0 + th) - T(x_0)] - T'(x_0)h, y^*\right) \to 0$$
$$\text{as } t \to 0 \text{ for all } h \in X,\ y^* \in Y.$$

We have the following special case of Theorem 2.5.

COROLLARY 2.2. *Let $C \subset X$ be closed and convex, $x_0 \in C$, T_0: $B_0 = B_{r_0}(x_0) \cap C \to Y$ be continuous and weakly Gâteaux differentiable in a neighborhood of x_0, $T_0'(x_0)(C) = Y$, and $\|T_0'(x) - T_0'(x_0)\| \le m$ in B_0 for $m > 0$ sufficiently small. Then the conclusions of Theorem* 2.5 *hold true with $K = T_0'(x_0)$. Moreover, if $C = X$, then, for some $r_1 < r_2$, ρ_1, and l,*

$$\operatorname{dist}\left(T_0^{-1}(y_1) \cap B_{r_1}(x_0), T_0^{-1}(y_2) \cap B_{r_2}(x_0)\right)$$
$$\le l\|y_1 - y_2\| \qquad \textit{for } y_1, y_2 \in B_{\rho_1}(T_0x_0). \tag{2.12}$$

Proof. We shall first show that condition (i) holds with $K = T_0'(x_0)$ for each $x, y \in B_0$. Let $x, y \in B_0$ and $y^* \in Y^*$. Then

$$y^*(T_0 x - T_0 y - T_0'(x_0)(x - y))$$
$$= \int_0^1 \left([T_0'(y + t(x - y)) - T_0'(x_0)](x - y), y^* \right) dt$$

and choosing y^* with $\|y^*\| = 1$ such that the left-hand side is equal to $\|T_0 x - T_0 y - T_0'(x_0)(x - y)\|$, we obtain that

$$\|T_0 x - T_0 y - T_0'(x_0)(x - y)\| \le m\|x - y\| \qquad \text{for all } x, y \in B_0.$$

Since (ii) holds by Remark 2.5, then Theorem 2.5 is applicable.

It remains to show (2.12). Let $U \subset B_{r_0}(x_0)$ be a closed neighborhood of x_0. Since $T_0'(x_0)$ is surjective, there is a $c > 0$ such that, for all $y \in Y$, there is a solution u of the equation $T_0'(x_0)u = y$ satisfying $\|u\| \le c\|y\|$. Let $r > 0$ be such that $B_r(x_0) \subset U$. Then $y = T_0'(x)u + z$ for $z = (T_0'(x) - T_0'(x_0))y$ where $\|u\| \le c\|y\|$ and $\|z\| \le m\|y\|$. Hence, (2.12) holds as in the proof of Theorem 7.5.4 in [AE]. ∎

If $C = X$, then Theorem 2.5 generalizes a result of Ehrmann [E] when $N(K) = \{0\}$, and the open mapping theorem of Kachurovskii [K] when K is continuous and $N(K) \ne \{0\}$ (cf. also [DMO]). It also extends the inverse function theorem of Aubin and Frankowska [AF] to nondifferentiable maps with $\dim Y = \infty$ but defined on a less general domain. We refer to Aubin and Frankowska [AF] for a constrained inverse mapping theorem with $\dim Y = \infty$ involving a transversality condition. Corollary 2.2 extends an open mapping theorem of Browder [B] but without the estimate (2.12).

3. ERROR ESTIMATES FOR NONLINEAR OPERATOR EQUATIONS

In this section, we shall establish a constructive solvability and error estimates for the approximate solutions of nonlinear equations of the form

$$Tx = f, \qquad x \in X, f \in Y, \tag{3.1}$$

involving A-proper maps which have a multivalued derivative at a solution. Our basic result, announced in [Mi-4, Mi-6], is

THEOREM 3.1. *Let $T: \overline{U} \subset X \to Y$ be A-proper w.r.t. Γ and x_0 be a solution of Eq.* (3.1). *Suppose that A is an odd multivalued derivative of T at x_0 and there exist constants $c_0 > 0$ and $n_0 \ge 1$ such that*

$$\|Q_n u\| \ge c_0\|x\| \qquad \textit{for } x \in X_n, u \in Ax, n \ge n_0. \tag{3.2}$$

(a) *If x_0 is an isolated solution, then Eq.* (3.1) *is strongly approximation solvable in $B_r(x_0)$ for some $r > 0$ (i.e., $Q_nTx_n = Q_nf$ for some $x_n \in \overline{B}_r(x_0) \cap X_n$ and all large n and $x_n \to x_0$).*

(b) *If, in addition, A is c_1-coercive for some $c_1 > 0$, then x_0 is an isolated solution, the conclusion of* (a) *holds, and for any $\epsilon \in (0, c_0)$ approximate solutions $x_n \in \overline{B}_r(x_0) \cap X_n$ satisfy*

$$\|x_n - x_0\| \le (c_0 - \epsilon)^{-1}\|Tx_n - f\| \qquad \text{for } n \ge n_1 \ge n_0. \tag{3.3}$$

(c) *If x_0 is an isolated solution in $B_r(x_0)$, A is c_2-bounded for some c_2 and*

$$Tx - Ty \in A(x-y) + R(x-y) \qquad \text{whenever } x - y \in B_r, \tag{3.4}$$

and $\|r(x-y)\|/\|x-y\| \to 0$ as $x \to x_0$ and $y \to x_0$ for each selection function $r(x-y)$ of $R(x-y)$, then Eq. (3.1) *is uniquely approximation solvable in $B_r(x_0)$ and the unique solutions $x_n \in B_r(x_0) \cap X_n$ of $Q_nTx = Q_nf$ satisfy*

$$\|x_n - x_0\| \le k\|P_nx_0 - x_0\| \le c\,\mathrm{dist}(x_0, X_n), \tag{3.5}$$

where k depends on c_0, c_2, ϵ, and δ and $c = 2k\delta_1$, $\delta_1 = \sup\|P_n\|$.

Proof. (a) If $T_fx = Tx - f$, then T_f has the same properties as T and $T_fx_0 = 0$. Therefore, we may assume that $f = 0$. Let $r > 0$ be such that $\overline{B}_r(x_0) \subset U$ and

$$\frac{1}{1+t}\frac{\|y\|}{\|x\|} + \frac{t}{1+t}\frac{\|z\|}{\|x\|} < \frac{c_0}{\delta}$$
$$\text{for } t \in [0,1],\ \|x\| = r,\ y \in R(x),\ z \in R(-x).$$

Let $T_1x = T(x + x_0)$ for $\|x\| \le r$ and define a homotopy $H: [0,1] \times \overline{B}_r \to Y$ by $H(t,x) = 1/(1+t)T_1x - t/(1+t)T_1(-x)$. Then, as in the proof of Theorem 2.1, we have that

$$Q_nH(t,x) \ne 0 \qquad \text{for } t \in [0,1],\ x \in \partial B_r \cap X_n,\ n \ge n_0. \tag{3.6}$$

Thus, $\deg(Q_nT_1, B_r \cap X_n, 0) = \deg(Q_nH(1,\cdot), B_r \cap X_n, 0) \ne 0$ for each $n \ge n_0$ since $Q_nH(1,\cdot)$ is an odd map. Hence, $\deg(Q_nT, B_r(x_0) \cap X_n, 0) = \deg(Q_nT_1, B_r \cap X_n, 0) \ne 0$ for each $n \ge n_0$ and, consequently, $Q_nTx_n = 0$ for some $x_n \in B_r(x_0) \cap X_n$. Since T is A-proper and x_0 is an isolated solution, it follows easily that $x_n \to x_0$.

(b) Let us first show that x_0 is an isolated solution. Choose $r > 0$ such that $\|y\|/\|x\| < c_1$ for $0 < \|x\| \le r$ and $y \in R(x)$. Then, for each such x, there are $u \in Ax$ and $v \in Rx$ such that $T(x + x_0) - T(x_0) = u + v$ and

$$\|T(x + x_0) - f\| \ge \|u\| - \|v\| \ge \|x\|(c_1 - \|v\|/\|x\|) > 0.$$

Next, for any $x_n \in B_r(x_0) \cap X_n$ such that $Q_n T x_n = Q_n f$, we have that $Tx_n - Tx_0 \in A(x_n - x_0) + R(x_n - x_0)$ and therefore $Tx_n - f = u_n + v_n$ for some $u_n \in A(x_n - x_0)$ and $v_n \in R(x_n - x_0)$. Hence, for each large n, $c_1\|x_n - x_0\| \le \|Tx_n - f\| + \|v_n\| \le \|Tx_n - f\| + \epsilon\|x_n - x_0\|$ for any $\epsilon \in (0, c_1)$ and therefore (3.3) holds.

(c) By part (a), for each $n \ge n_1$ there is an $x_n \in B_r(x_0) \cap X_n$ such that $Q_n T x_n = Q_n f$ and $x_n \to x_0$. If the equation $Q_n T x = Q_n f$ had another solution $y_n \in B_r(x_0) \cap X_n$ for each $n \ge n_1$, then $0 \in Q_n A(y_n - x_n) + Q_n R(y_n - x_n)$, and therefore, for some $u_n \in R(y_n - x_n)$,

$$c_0 \le \delta\|u_n\|/\|x_n - y_n\| \to 0 \qquad \text{as } n \to \infty$$

in contradiction to $c_0 > 0$. Hence, the equation $Q_n T x = Q_n f$ is uniquely solvable in $B_r(x_0) \cap X_n$ for each $n \ge n_1$.

Now, let $\{x_n\}$ be the corresponding unique solutions and observe that $Q_n T P_n x_0 - Q_n T x_n \in Q_n A(P_n x_0 - x_n) + Q_n R(P_n x_0 - x_n)$ for each $n \ge n_1$. Choose $u_n \in A(P_n x_0 - x_n)$ and $v_n \in R(P_n x_0 - x_n)$ such that $Q_n T P_n x_0 - Q_n T x_n = u_n + v_n$. It follows from (3.2) that

$$\begin{aligned} c_0\|x_n - P_n x_0\| &\le \delta\|Tx_0 - TP_n x_0\| + \delta\|v_n\| \\ &\le \delta\|Tx_0 - TP_n x_0\| + \epsilon\|x_n - P_n x_0\| \end{aligned}$$

for any $\epsilon \in (0, c_0)$ and each $n \ge n_1$ large. Hence, for such n,

$$(c_0 - \epsilon)\|x_n - P_n x_0\| \le \delta\|Tx_0 - TP_n x_0\|.$$

But, since $Tx_0 - TP_n x_0 \in A(x_0 - P_n x_0) + R(x_0 - P_n x_0)$, it follows that

$$\|Tx_0 - TP_n x_0\| \le c_2\|x_0 - P_n x_0\| + \epsilon_1\|x_0 - P_n x_0\|$$

for any given $\epsilon_1 > 0$ and each $n \ge n_2 \ge n_1$ large. Combining the last two inequalities, it follows that, for each $n \ge n_2$,

$$\begin{aligned} \|x_n - x_0\| &\le \|x_n - P_n x_0\| + \|x_0 - P_n x_0\| \\ &\le \big(\delta(c_2 + \epsilon_1)/(c_0 - \epsilon) + 1\big)\|P_n x_0 - x_0\| = k\|P_n x_0 - x_0\|. \end{aligned}$$

Next, for each $z_n \in X_n$ and $x \in X$, we have that

$$\|P_n x - x\| = \|P_n(x - z_n) - (x - z_n)\| \le (1 + \|P_n\|)\|x - z_n\|.$$

Hence, for each $x \in X$,

$$\|P_n x - x\| \le (1 + \|P_n\|)\operatorname{dist}(x, X_n) \le 2\|P_n\|\operatorname{dist}(x, X_n)$$

and therefore

$$\|x_n - x_0\| \leq c \operatorname{dist}(x_0, X_n) \qquad \text{with } c = 2k\delta_1. \quad \blacksquare$$

Remark 3.1. Analyzing the above proof, we see that the oddness of A can be replaced by $\deg(Q_nT, B_r \cap X_n, 0) \neq 0$ for all large n (cf. also Theorem 2.3 and Corollary 2.1). Regarding (3.2), we refer to Lemma 2.1.

Inequality (3.5) shows that the problem of estimating the error $\|x_0 - x_n\|$ is reduced to a problem in approximation theory, i.e., to evaluate the distance $\operatorname{dist}(x_0, X_n) = \inf_{u_n \in X_n}\|x_0 - u_n\|$ between a vector $x_0 \in X$ and a subspace $X_n \subset X$. Often one is able to show that there exist constants $c(x_0) > 0$ and $\beta > 0$ such that the distance $\operatorname{dist}(x_0, X_n) \leq c(x_0)n^{-\beta}$ and therefore the following error estimate holds:

$$\|x_0 - x_n\| \leq c(x_0)n^{-\beta}. \tag{3.7}$$

In this case we say that the *order of convergence* is β. In applications there are numerous ways of constructing suitable subspaces $\{X_n\}$ which would lead to the order of convergence of approximate solutions and we refer to the books [Ci, SF] and so on. We note also that when $\{X_n\}$ are finite element subspaces of a Hilbert space, inequality (3.5) is an extension of Céa’s lemma [C] to nondifferentiable nonlinear maps. Theorem 3.1 with $T = I - C$, C-compact, contains a result of Schmitt [S].

When T is Fréchet differentiable, Theorem 3.1 reduces to the following result of the author [Mi-1], which extends a result of Krasnosel’skii [Kr] and Vainikko [V] when $T = I - C$ with C compact.

THEOREM 3.2 (cf. [Mi-1]). *Let $T\colon \overline{U} \subset X \to Y$ be A-proper w.r.t. Γ, x_0 be a solution of Eq.* (3.1), *and T be Fréchet differentiable at x_0 with $T'(x_0)$ A-proper w.r.t. Γ and injective. Then*

(a) *x_0 is an isolated solution, Eq.* (3.1) *is strongly approximation solvable in $B_r(x_0)$ for some $r > 0$, and* (3.3) *holds with $c_0 = \|(T'(x_0))^{-1}\|$.*

(b) *If, in addition, T is continuously Fréchet differentiable at x_0, then Eq.* (3.1) *is uniquely approximation solvable in $B_r(x_0)$ and the unique solutions $x_n \in B_r(x_0) \cap X_n$ of $Q_nTx = Q_nf$ satisfy* (3.5) *where k depends on c_0, $\|T'(x_0)\|$, ϵ, and δ and $c = 2k\delta_1$, $\delta_1 = \sup\|P_n\|$.*

Let us now give a version of Theorem 3.1 which is useful for proving error estimates of the form (3.7) in applications to differential equations. Let Z be a Banach space densely and continuously embedded in X and $\{X_n\}$ be finite-dimensional subspaces of Z such that $\operatorname{dist}(z, X_n) \to 0$ as $n \to \infty$ for each $z \in Z$. Then $\operatorname{dist}(x, X_n) \to 0$ as $n \to \infty$ for each $x \in X$ by the continuity of the embedding. If $P_n\colon X \to X_n$ are linear projections onto

X_n such that $\delta_1 = \sup\|P_n\| < \infty$, then $\Gamma = \{X_n, P_n, Y_n, Q_n\}$ is a projection scheme for (X, Y). We also assume that

(3.8) There are positive and monotonically decreasing functions $g_1(n)$ and $g_2(n)$ such that for each $z \in Z$ there exists a $u_n \in X_n$ such that

$$\|z - u_n\|_Z \le g_1(n) \qquad \text{and} \qquad \|z - u_n\|_X \le g_2(n).$$

(3.9) There is a positive monotonically increasing function $g_3(n)$ such that

$$\|u_n\|_Z \le g_3(n)\|u_n\|_X \qquad \text{for each } u_n \in X_n.$$

THEOREM 3.3. *Suppose that* (3.8) *and* (3.9) *and all conditions of Theorem* 3.1(c) *hold with* $\Gamma = \{X_n, P_n, Y_n, Q_n\}$ *as constructed above. Then Eq.* (3.1) *is uniquely approximation solvable in some ball* $B_r(x_0)$, *and if* $x_0 \in Z$ *the unique approximate solutions* $x_n \in B_r(x_0) \cap X_n$ *satisfy*

$$\|x_n - x_0\|_Z \le c \max\{g_1(n), g_2(n) \cdot g_3(n)\}. \tag{3.10}$$

Proof. By Theorem 3.1(c) we have that, for each large n,

$$\|x_n - x_0\| \le k\|P_n x_0 - x_0\| \le c \operatorname{dist}(x_0, X_n).$$

For $x_0 \in Z$ choose $u_n \in X_n$ such that (3.8) holds. Then

$$\begin{aligned}\|x_n - x_0\|_Z &\le g_1(n) + g_3(n)\|x_n - u_n\| \\ &\le g_1(n) + g_3(n)\big(k\|P_n x_0 - x_0\| + g_2(n)\big),\end{aligned}$$

which implies (3.10) for some constant c. ∎

In particular, if $g_i(h) = h^{r_i}$ for some $r_i < 0$, $i = 1, 2$, $r_3 > 0$, and the inequalities in (3.8) are replaced by

$$\|z - u_n\|_Z \le n^{r_1}\|z\|_Z \qquad \text{and} \qquad \|z - u_n\|_X \le n^{r_2}\|z\|_Z,$$

then the error estimate (3.10) becomes

$$\|x_n - x_0\|_Z \le c \max\{n^{r_1}, n^{r_2 + r_3}\}\|x_0\|_Z. \tag{3.11}$$

Estimates of the above type appear in approximations by the finite element method where each subspace X_n consists of splines (i.e., piecewise-polynomial functions) of fixed degree defined over a mesh (usually of triangles) laid out to approximately cover the spatial domain Ω of the problem. One of the principal assets of the finite element method is that, no matter how irregular the shape of the boundary $\partial\Omega$ of Ω, such meshes can be fitted very closely. A normalized mesh parameter h, $0 < h \le 1$, is assigned to each mesh so that the mesh is refined as $h \to 0$ and the dimension of X_n increases indefinitely. When $T = I - C$, with C compact and continuously Fréchet differentiable at x_0. Theorems 3.3 is due to Shaposhnikova [Sh].

4. CONSTRUCTIVE HOMEOMORPHISM THEOREMS

The existence of a local inverse of T has been studied in Section 2. In this section, we shall give some (constructive) homeomorphism theorems for nonlinear maps.

We say that T satisfies condition $(+)$ if whenever $Tx_n \to f$ in Y, then $\{x_n\}$ is bounded in X. It relates to the closedness of $R(T)$ as follows:

PROPOSITION 4.1. *Let T: $X \to Y$ be continuous.*

(a) *If T is A-proper and condition $(+)$ holds, then $R(T)$ is closed in Y.*

(b) *If $R(T)$ is closed and T is an open map, then condition $(+)$ holds.*

Proof. (a) Let $y_k \in T(X)$ be such that $y_k \to y$ in Y and $x_k \in X$ such that $Tx_k = y_k$. Then there is an $r > 0$ such that $\{x_k\} \subset \overline{B}_r$ by condition $(+)$. Since T restricted to $\overline{B}$ is proper, it follows that $\{x_n\}$ is precompact and therefore some subsequence $x_{n_k} \to x$. Since T is continuous, we have $Tx = y$.

(b) Assume that $\{x_n\} \subset X$ is such that $Tx_n \to f$. Let $x \in X$ be such that $Tx = f$. For $r > 0$, $TB_r(x)$ is open and contains f and therefore $Tx_n \in TB_r(x)$ for all large n. Moreover, $x_n \in B_r(x)$ for all large n since T is an open map. Hence, $\{x_n\}$ is bounded and condition $(+)$ holds. ∎

THEOREM 4.1. *Let T: $X \to Y$ be continuous, A-proper w.r.t. Γ, satisfy condition $(+)$, and be locally invertible on X. Then T is a homeomorphism onto Y.*

Proof. We know that $R(T)$ is closed by Proposition 4.1. Since T is locally invertible, each point of $T(X)$ possesses a neighborhood consisting of points of $T(X)$. Hence, $R(T)$ is open and therefore $T(X) = Y$.

It remains to show that T is injective. First, we shall show that $T^{-1}(y)$ is a finite set for each $y \in Y$. Suppose that $S = T^{-1}(y)$ is infinite for some $y \in Y$. Then any sequence $\{x_n\} \subset S$ is bounded by condition $(+)$ and, since T is a proper map when restricted to a bounded set, there is a subsequence converging to some z with $Tz = y$. Hence, each neighborhood of z contains a solution of $Tx = y$ in contradiction to the local invertibility of T. Next, let $x_1 \neq x_2$ and $Tx_1 = Tx_2 = y$ and $I = [y, 0]$ be a segment in Y. Let $t \in [0, 1]$ be fixed. Since $S(y)$ is finite and T is locally invertible, there is an $\epsilon_t > 0$ such that T is invertible on $B_t = B(ty, \epsilon_t)$ whatever the preimage of ty is fixed. Let $\epsilon_t^* < \epsilon_t$. Then the family $\{B(ty, \epsilon_t^*) \mid t \in [0,1]\}$ is an open cover for the compact set I. Hence, there are $t_1, \dots, t_k$ in $[0,1]$ such that $\{B_i = B(t_i y, \epsilon_{t_i}^*) \mid 1 \le i \le k\}$ covers I.

Next, we shall construct two continuous curves γ_1 and γ_2 with the initial points at x_1 and x_2 and the end points in $T^{-1}(0)$ and having no common points. We may assume that $y \in B_1$. Since T is locally invertible on B_1, a part of I is in a one-to-one correspondence with an arc of the curve with the initial point at x_1. Repeat the process for $B_2, \dots, B_k$. Since T is locally invertible on all larger spheres $B(t_i y, \epsilon_{t_i})$, we get a continuous curve γ_1 starting at x_1 and ending at a point of $T^{-1}(0)$ such that $T(\gamma_1) = I$. Similarly, we construct the above-mentioned continuous curve γ_2 with $T(\gamma_2) = I$. These curves have no points in common. If not, let z be a common point, and, for simplicity, we may assume that $z = 0$. Using the local invertibility of T, we see that the two curves coincide in a part lying in a neighborhood of 0, and therefore $\gamma_1 = \gamma_2$, a contradiction.

Now, the segment $J = [x_1, x_2]$ induces a closed curve C passing through y. Consider the central homothety and let C_t be the image of C at $t \in [0,1]$. Clearly, each C_t is a closed curve. Using the compactness of each C_t and the above reasoning, we can construct a continuous curve C_t' in X for each t with $T(C_t') = C_t$ and having the end points on γ_1 and γ_2. Since T is locally invertible, there is a neighborhood U of 0 where T is bijective. Let t be sufficiently small such that $C_t \subset U$. Then the corresponding curve C_t' is closed, in contradiction to it being open. Hence, T is injective and therefore it is a homeomorphism. ∎

THEOREM 4.2 (cf. [Mi-4]). *Let $T\colon X \to Y$ be continuous, locally injective, A-proper w.r.t. Γ for (X, Y), and satisfy condition (t). Suppose that T has an odd multivalued derivative $A(x_0)\colon X \to 2^Y$ at each $x_0 \in X$ and there exist an $n_0 = n_0(x_0) \ge 1$ and $c_0 = c_0(x_0) > 0$ such that*

$$\|Q_n u\| \ge c_0 \|x\| \qquad \text{for } x \in X_n,\ u \in A(x_0)x,\ n \ge n_0. \tag{4.1}$$

Assume that T and each $A(x_0)$ satisfy (3.4) *and $A(x_0)$ is $c_2(x_0)$-bounded for some $c_2(x_0) > 0$. Then T is onto Y and Eq.* (3.1) *is uniquely approximation solvable for each $f \in Y$ and* (3.5) *holds. If, in addition, $A(x_0)$ is c_1-coercive for some $c_1 = c_1(x_0) > 0$, then the estimate* (3.3) *holds.*

Proof. In view of Theorems 3.1 and 4.1 and the closedness of $R(T)$, it suffices to show that $R(T)$ is open in Y and that T is locally invertible. Let $x_0 \in X$ be fixed and $\epsilon > 0$ such that T is injective on $\bar{B}_\epsilon(x_0)$. We need to show that $T(B_\epsilon(x_0))$ is open in Y. Define a map $T_1\colon \bar{B}_\epsilon \to Y$ by $T_1(y) = T(x) - T(x_0)$, where $y = x - x_0$ with $x \in \bar{B}_\epsilon(x_0)$. Then $T_1(0) = 0$ and $T(B_\epsilon(x_0)) = T_1(B_\epsilon) + T(x_0)$ is open if such is $T_1(B_\epsilon)$. In view of the invariance of domain theorem for A-proper maps (cf. [Mi-3]), it suffices to show that, for each $y_0 \in B_\epsilon$ and some small $r_0 > 0$ such that $\overline{B}_{r_0}(y_0) \subset B_\epsilon$, we have that $\deg(Q_nT_1 - Q_nT_1(y_0), B_{r_0}(y_0) \cap X_n, 0) \neq 0$ for each $n \geq n_0$.

Let $n \geq n_0$ be fixed and define a map T_n: $\overline{B}_n = \overline{B}_{r_0} \cap X_n \to Y_n$ by $T_n(u) = Q_nT_1(y) - Q_nT_1(y_0)$, where $u = y - y_0$ with $y \in B_{r_0}(y_0) \cap X_n$.

Let H_n: $[0,1] \times \overline{B}_n \to Y_n$ be given by

$$\begin{aligned} H_n(t,u) &= T_n\left(\frac{u}{1+t}\right) - T_n\left(\frac{-tu}{1+t}\right) \\ &= Q_nT_1\left(\frac{y+ty_0}{1+t}\right) - Q_nT_1\left(2y_0 - \frac{y_0+ty}{1+t}\right). \end{aligned}$$

Then there exists an $n_1 \geq n_0$ such that $H_n(t,u) \neq 0$ for all $u \in \partial B_n$, $t \in [0,1]$, and $n \geq n_1$. If not, there exist $t_{n_k} \in [0,1]$, $t_{n_k} \to t$, and $u_{n_k} \in \partial B_{n_k}$ such that $H_{n_k}(t_{n_k}, u_{n_k}) = 0$ for each k. Therefore, by (3.4), $0 \in Q_{n_k}A(x_0)(y_{n_k} - y_0) + Q_{n_k}R(x_0;\ y_{n_k} - y_0) = Q_{n_k}A(x_0)(u_{n_k}) + Q_{n_k}R(x_0; u_{n_k})$, or, setting $v_{n_k} = u_{n_k}/\|u_{n_k}\|$,

$$0 \in Q_{n_k}A(x_0)(v_{n_k}) + Q_{n_k}R(x_0; u_{n_k})/\|u_{n_k}\| \qquad \text{for each } k.$$

Let $w_{n_k} \in A(x_0)(v_{n_k})$ and $z_{n_k} \in R(x_0; u_{n_k})$ be such that $Q_{n_k}w_{n_k} + Q_{n_k}z_{n_k}/\|u_{n_k}\| = 0$. Since r_0 is sufficiently small we arrive at the contradiction

$$c_0 = c_0\|v_{n_k}\| \leq \|Q_{n_k}w_{n_k}\| \leq \delta\|z_{n_k}\|/\|u_{n_k}\| < c_0.$$

Hence, $H_n(t,u) \neq 0$ on $[0,1] \times \partial B_n$ for $n \geq n_1$ and, consequently,

$$\begin{aligned} &\deg\big(Q_nT_1 - Q_nT_1(y_0), B_{r_0}(y_0) \cap X_n, 0\big) \\ &\qquad = \deg\big(H_n(1,\cdot), B_{r_0}(y_0) \cap X_n, 0\big) \neq 0 \end{aligned}$$

for each $n \geq n_1$ since $H_n(1,\cdot)$ is an odd map on $\overline{B}_n$.

Finally, we claim that T is locally invertible. Since T is locally injective, for each $x_0 \in X$, there is an $\epsilon > 0$ such that T: $B_\epsilon(x_0) \to Y$ is injective. Let $Tx_0 = f_0$. As shown above, $TB_\epsilon(x_0)$ is open and therefore there is a $\delta > 0$ such that $B_\delta(f_0) \subset TB_\epsilon(x_0)$. Hence, for each $f \in B_\delta(f_0)$, there is a unique $x \in B_\epsilon(x_0)$ such that $Tx = f$, i.e., T is locally invertible. ∎

Next, we shall look at T having a uniform multivalued derivative A in the sense that $Tx - Ty \in A(x-y)$ whenever $x - y \in U$.

THEOREM 4.3. (a) *Let* $T: X \to Y$ *be* A*-proper w.r.t.* Γ *satisfy condition* $(+)$. *Suppose that* $U \subset X$ *is a neighborhood of* 0, $A: U \to 2^Y$ *satisfies* (3.2) *on* $U \cap X_n$, *and*

$$Tx - Ty \in A(x-y) \qquad \text{whenever } x - y \in U. \tag{4.2}$$

Then $T(X) = Y$.

(b) *If, in addition,* $U = X$, A *is positively homogeneous, and* A *is* c_1*-coercive and* c_2*-bounded for some constants* $c_1, c_2 > 0$, *then* T *is a homeomorphism and Eq.* (3.1) *is uniquely approximation solvable for each* f *in* Y *and the estimates* (3.3) *and* (3.5) *hold.*

Proof. We have that $T(X) = Y$ and $Q_nT: X_n \to Y_n$ is a homeomorphism for each large n by Theorem 2.5 in [Mi-3]. Moreover, if $0 \in Ax$, then $x = 0$ by the c_1-coercivity and T is locally injective. Indeed, let $x \in X$ be fixed and $\epsilon > 0$ such that $\overline{B}_{4\epsilon} \subset U$. Then, if for some $x_1, x_2 \in \overline{B}_{2\epsilon}(x)$ we have that $Tx_1 = Tx_2$, then, by condition (4.2), $0 = Tx_1 - Tx_2 \in A(y_1 - y_2)$ for some $y_1 - y_2 \in U$ since $x_1, x_2 \in \overline{B}_{2\epsilon}(x) \subset U(x) = U + x$ and, consequently, $x_1 - x_1 = y_1 - y_2$ for some $y_1, y_2 \in U$ with $\|y_1 - y_2\| \leq 4\epsilon$. Hence, $y_1 = y_2$ and therefore $x_1 = x_2$. The conclusions in (b) now follow from Theorems 4.2 and 3.1 (b) and (c) and Remark 3.1. ∎

THEOREM 4.4. (a) *Let* $T: X \to Y$ *be continuous,* A*-proper w.r.t.* Γ, *and have the closed range. Suppose that for each* $u \in X$ *there exist a ball* $B_r(u) \subset X$, *a linear map* $K: X \to Y$ *with* $K(X) = Y$, *and positive constants* m *and* c *such that* $mc < 1$ *and*

(i) $\|Tx - Ty - K(x-y)\| \leq m\|x - y\|$ *for* $x, y \in \overline{B}_r(u)$.

(ii) K^{-1} *is a multivalued* c*-Lipschitz map.*

Then T *is surjective, i.e.,* $T(X) = Y$.

(b) *Let, in addition,* T *satisfy condition* (t). K^{-1} *exist,* $\delta mc < 1$, *and*

(iii) $\|Q_nKx\| \geq c^{-1}\|x\|$ *on* X_n *for all large* n.

Then T *is a homeomorphism and, for each* $f \in Y$, *Eq.* (3.1) *is strongly approximation solvable and the estimate* (3.3) *holds. If each* K *is also continuous, then Eq.* (3.1) *is uniquely approximation solvable and* (3.5) *holds.*

Proof. (a) T is an open map at each $x \in X$ by Theorem 2.5 and therefore $R(T)$ is open. Hence, $R(T) = Y$ since $R(T)$ is closed.

(b) T is locally invertible by Remark 2.5 and is therefore a homeomorphism by Theorem 4.1. Let $f_0 \in Y$ be fixed, x_0 be the solution of $Tx = f_0$, and r such that conditions (i) and (iii) hold with u replaced by x_0.

Then the map A defined by $Ax = \{y \mid \|y - Kx\| \le m\|x\|\}$ for $x \in B_r(x_0)$ is a multivalued derivative of T at x_0 since, by condition (i),

$$Tx - Ty \in A(x - y) = \{z \mid \|z - K(x - y)\| \le m\|x - y\|\} \quad \text{for } x, y \in B_r(x_0).$$

Next, we claim that (3.2) holds. Indeed, if $x \in \overline{B}_r \cap X_n$ and $y \in Ax$, then

$$\|Q_n y\| \ge \|Q_n Kx\| - \delta\|y - Kx\| \ge (c^{-1} - \delta m)\|x\|,$$

and therefore (3.2) holds by the homogeneity of A. Moreover, for $y \in Ax$ with $\|x\| \le r$,

$$\|y\| \ge \|Kx\| - \|y - Kx\| \ge \|Kx\| - m\|x\| \ge (c^{-1} - m)\|x\|.$$

Hence, the first conclusion in (b) follows from Theorem 3.1(a) and (b). Next, suppose that each K is continuous. Then, for each $y \in Ax$ with $x \in \overline{B}_r$,

$$\|y\| \le \|y - Kx\| + \|Kx\| \le m\|x\| + \|K\|\,\|x\| = (m + \|K\|)\|x\|.$$

Hence, the second conclusion follows from Theorem 3.1(c). ▮

THEOREM 4.5. (a) *Let $U \subset X$ be a neighborhood of* 0 *and A: $U \to 2^Y$, with $A(x)$ compact, be u.s.c., ϕ-condensing, and $x = 0$ if $x \in Ax$. Suppose that N: $X \to Y$ is continuous and $Nx - Ny \in A(x - y)$ whenever $x - y \in U$. Then $I - N$ is bijective.*

(b) *If, in addition, A and N are ball-condensing on X and A is positively homogeneous, then the equation $x - Nx = f$ is uniquely approximation solvable w.r.t. $\Gamma = \{X_n, P_n\}$ for each $f \in X$ and the approximate solutions $x_n \in X_n$ of $x - P_n Nx = P_n f$ satisfy* (3.3) *and* (3.5).

Proof. $I - N$ is bijective by Corollary 1.4 in [Mi-2]. Moreover, $I - N$ is c_1-coercive by Lemma 1.1 in [Mi-2], for some $c_1 > 0$, while $I - A$ satisfies (3.2) by Lemma 2.1 in [Mi-3] and is c_2-coercive by Lemma 2.1. Finally, $I - P_n N$: $X_n \to X_n$ is a homeomorphism for each $n \ge n_0$ by Theorem 2.5 in [Mi-3]. Next, we claim that for each $f \in X$ there are an $r > 0$ and $n_0 \ge 1$ such that $\deg(I - P_n N, B_r \cap X_n, P_n f) = \deg(I - N, B_r, f) \ne 0$ for each $n \ge n_0$. Indeed, for a given $f \in X$, select an $r > 0$ such that $f \in (I - N)(B_r)$. Since $I - N$ is a homeomorphism, $\deg(I - N, B_r, f) \ne 0$. Then the homotopy $H(t, x) = tP_n Nx + (1 - t)Nx$ on $[0,1] \times \overline{B}_r$ is admissible and $\deg(I - N, B_r, f) = \deg(I - P_n N, B_r, P_n f) = \deg(I - P_n N, B_r \cap X_n, P_n f)$. Hence, the claim is valid and $x - P_n Nx = P_n f$ is solvable in $B_r \cap X_n$. Thus, the conclusions of the theorem follow from Theorem 3.1 and Remark 3.1. ▮

When A and N are compact maps, Theorem 4.5(a) is due to Lasota and Opial [LO]. For an application of this theorem to linear boundary value problems for ordinary differential equations, we refer to [Mi-8].

In the case of differentiable maps, we have

THEOREM 4.6. (a) *Let T: $X \to Y$ be Fréchet differentiable, A-proper w.r.t. Γ, and have the closed range in Y. Then, if $T'(x)$ is injective and A-proper w.r.t. Γ for each $x \in X$, Eq.* (3.1) *is strongly approximation solvable in a neighborhood $B_r(x_0)$ of each of its solution x_0 for $f \in Y$ and* (3.3) *holds.*

(b) *If, in addition, T is continuously Fréchet differentiable in X, then T is a homeomorphism and Eq.* (3.1) *is uniquely approximation solvable for each $f \in Y$ and* (3.5) *holds.*

Proof. Let $x_0 \in X$ be fixed. Since $T'(x_0)$ is an A-proper injection, there are a $c_0 = c_0(x_0) > 0$ and $n_0 \geq 1$ such that, for each $n \geq n_0$,

$$\|Q_n T'(x_0)x\| \geq c_0\|x\| \qquad \text{for } x \in X_n.$$

Since $T'(x_0)$ is a bijection, by Krasnosel'skii and Zabreiko's result [KZ], T is surjective. Hence, the conclusion follows from Theorem 3.2. If T is continuously Fréchet differentiable, then the assertions of (b) follow by Theorem 4.2. ∎

Remark 4.1. If $T = I - C$, C compact and continuously Fréchet differentiable, the homeomorphism assertion only is due to Krasnosel'skii and Zabreiko [KZ].

Next, we shall give an application of Theorem 3.1 to asymptotically $\{B_1, B_2\}$-quasilinear maps of the form $T = I - N$ in a Hilbert space H. Let B_1, B_2: $H \to H$ be self-adjoint maps and write $B_1 \leq B_2$ if $(B_1x, x) \leq (B_2x, x)$ for all $x \in H$. Let $\sigma(B_i)$ be the spectrum of B_i, $i = 1, 2$, $1 \notin \sigma(B_1) \cup \sigma(B_2)$, $\sigma(B_1) \cap (1, \infty) = \{\lambda_1, \dots, \lambda_k\}$ and $\sigma(B_2) \cap (1, \infty) = \{\mu_1, \dots, \mu_l\}$, where the λ_i's and μ_j's are the eigenvalues of B_1 and B_2, respectively, of finite multiplicities. Suppose that the sum of the multiplicities of the λ_i's is equal to the sum of the multiplicities of the μ_j's. Then we say that B_1 and B_2 form a regular pair.

Following Krasnosel'skii and Zabreiko [KZ], a (nonlinear) map A: $H \to H$ is said to be $\{B_1, B_2\}$-quasilinear on a set $M \subset H$ if for each $x \in M$ there exists a linear self-adjoint map C: $H \to H$ such that $B_1 \leq C \leq B_2$ and $Cx = Ax$. A map N: $H \to H$ is said to be asymptotically

$\{B_1, B_2\}$-quasilinear if there is a $\{B_1, B_2\}$-quasilinear outside some ball map A such that

$$|N - A| = \limsup_{\|x\| \to \infty} \frac{\|Nx - Ax\|}{\|x\|} < \infty.$$

For such maps we have

THEOREM 4.7. (a) *Let* $N\colon N \to H$ *be Fréchet differentiable and such that* $N'(x)$ *is self-adjoint,* $B_1 \le N'(x) \le B_2$ *for some regular pair* $\{B_1, B_2\}$, *and* $I - N'(x)$ *is* A*-proper w.r.t.* $\Gamma = \{X_n, P_n\}$ *for each* $x \in H$. *Suppose that* $H_t = I - tN - (1-t)B_0$ *is* A*-proper w.r.t.* Γ_0 *for each* $t \in [0,1]$ *and some self-adjoint map* B_0 *with* $B_1 \le B_0 \le B_2$. *Then, for each* $f \in H$, *the equation* $Tx = x - Nx = f$ *is strongly approximation solvable in a neighborhood* $B_r(x_0)$ *of each of its solutions* x_0 *and the estimate* (3.2) *holds.*

(b) *If, in addition,* N *is continuously Fréchet differentiable in* H, *then* $I - N$ *is a homeomorphism and the equation* $x - Nx = f$ *is uniquely approximation solvable for each* $f \in H$ *and the estimate* (3.5) *holds.*

Proof. For each $x, y, h \in H$ and some $t \in (0,1)$, we have that $(Nx - Ny, h) = (N'(y + t(x-y))(x-y), h)$, and therefore

$$\begin{aligned}\|Nx - Ny\| &\le \sup_{0 \le t \le 1} \|N'(y + t(x-y))\|\,\|x - y\| \\ &\le \max\{\|B_1\|, \|B_2\|\}\|x - y\|.\end{aligned}$$

Hence, N is bounded and, consequently, H_t is an A-proper homotopy w.r.t. Γ. Since $Nx = C(x)x + N(0)$, where $C(x) = \int_0^1 N'(tx)\,dt$, it follows that N is asymptotically $\{B_1, B_2\}$-quasilinear with $|N - A| = 0$ and $Ax = C(x)x$. Thus, $I - N$ is surjective by Theorem 3.4 in [Mi-4]. Moreover, since 1 is not an eigenvalue of $N'(x)$ for each $x \in H$ as shown in [Mi-4], the conclusions of the theorem follow from Theorems 3.1 and 4.2 as above. ∎

THEOREM 4.8. (a) *Let* $T\colon U \subset X \to Y$ *be such that for each* $u \in U$ *there is a ball* $B_r(u) \subset U$, *a linear map* K, *and a constant* m *such that*

(i) $\|Tx - Ty - K(x-y)\| \le m\|x - y\|$ *for* $x, y \in \overline{B}_r(u)$.

(ii) K *has a bounded inverse* K^{-1} *on* $TB_r(u)$ *and* $(\|K^{-1}\|^{-1} - m)r \ge c > 0$ *for some constant* c *independent of* $u \in U$.

Then T *is surjective and locally invertible.*

(b) *Let, in addition,* T *be* A*-proper w.r.t.* Γ, *and, for all large* n,

(iii) $\|Q_n Kx\| \ge c_0\|x\|$ *for* $x \in X_n$ *and some* c_0 *with* $\delta m c_0 < 1$.

Then, for each $f \in Y$, *Eq.* (3.1) *is strongly approximation solvable in a neighborhood of each its solution* x_0 *and the estimate* (3.3) *holds. Moreover, if each* K *is also continuous, then Eq.* (3.1) *is uniquely approximation solvable in a neighborhood of each its solution* x_0 *and* (3.5) *holds.*

Proof. The assertions in part (a) follow by Theorem 4.1 in [E], while those in part (b) can be proved as in the proof of Theorem 4.4(b). ∎

Remark 4.2. In view of Theorem 4.1(a) in [E], the condition $(\|K^{-1}\|^{-1} - m)r \geq c > 0$ can be replaced by: for each $R > 0$ there exists a constant $c = c(R) > 0$ such that $(\|K^{-1}\|^{-1} - m)r \geq c$ for $\|u\| \leq R$, and $\|Tx\| \to \infty$ as $\|x\| \to \infty$ with $x \in U$.

Remark 4.3. Taking $X = Y = R^1$ and $Tx = \tan x$, it can be shown that all conditions of Theorem 4.8 are satisfied (see [E]). Hence, Theorem 4.8 does not ensure that T has an inverse defined on all of Y.

When T is also continuous, we have the following result dealing with the invertibility of T and the number of solutions of $Tx = f$.

THEOREM 4.9. *Let all conditions of Theorem* 4.8 *hold and* T *be continuous on* U. *Then there exists a finite or infinite number* Λ *of open connected domains* $U_\lambda \subset U$ *such that*

(i) $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ *and the sets* U_λ *are mutually disjoint.*

(ii) *For each* $\lambda \in \Lambda$, *the restriction* T_λ *of* T *to* U_λ *is a homeomorphism of* U_λ *onto* Y.

(iii) *If* $U = X$, *then* T *is a homeomorphism of* X *onto* Y.

(iv) *If, in addition,* T *is an* A*-proper map w.r.t.* Γ *and* (iii) *of Theorem* 4.8 *holds, then the conclusion of Theorem* 4.8(b) *hold.*

Proof. Assertions (i)–(iii) are Theorem 6.1 in [E] and (iv) follows from Theorem 4.8. ∎

5. APPLICATIONS TO SEMILINEAR EQUATIONS

In this section, we shall consider semilinear maps of the form $T = A - N$ with A linear and not necessarily continuous. We have the following constructive inverse function theorem.

THEOREM 5.1 (cf. [Mi-6]). *Let* $A: D(A) \subset X \to Y$ *be a closed linear densely defined map and* $C: X \to Y$ *be linear and such that* $A - C: D(A) \subset X \to Y$ *is a bijection and* $d = \|(A - C)^{-1}\|^{-1}$. *Suppose that* $N: X \to Y$ *is nonlinear and continuous.*

(a) *Let, for some* $k \in (0, d)$,

$$\|(N - C)x - (N - C)y\| \leq k\|x - y\| \qquad \text{for all } x, y \in X. \tag{5.1}$$

Then Eq. (3.1) *is uniquely solvable for each* $f \in Y$ *and the solution is the limit of the iterative process*

$$Ax_n - Cx_n = Nx_{n-1} - Cx_{n-1} + f. \tag{5.2}$$

(b) *Equation* (3.1) *is uniquely approximation solvable w.r.t.* $\Gamma = \{X_n, P_n, Y_n, Q_n\}$ *with* $Q_n(A - C)x = (A - C)x$ *on* Y_n *and* $\delta = \max\|Q_n\| = 1$ *for each* $f \in Y$ *and the approximate solutions* $\{x_n \in X_n\}$ *satisfy*

$$\|x_n - x\| \le c\|(A + N)x_n - f\| \qquad \text{for some } c \text{ and all large } n. \tag{5.3}$$

If A is defined on all of X, then the approximate solutions also satisfy (3.5).

(c) *If* $k = d$, *X is uniformly convex,* $\delta = 1$, *and*

$$\|Nx - Cx\| \le a\|x\| + b \qquad \text{for some } a < k,\ b > 0, \tag{5.4}$$

then Eq. (3.1) *is solvable for each* $f \in X$.

Let us now discuss some special cases in the Hilbert space H setting. For $c \in \sigma(A) \cap (-\infty, 0]$, define $d_c^- = \operatorname{dist}(c, \sigma(A) \cap (-\infty, c))$. The following result with $c = 0$ was proved by the author [Mi-6 Proposition 2.7].

THEOREM 5.2. *Let* A: $D(A) \subset H \to H$ *be a self-adjoint map and* N: $H \to H$ *satisfy*

(i) $(Nx - Ny, x - y) \ge \alpha\|x - y\|^2$ *for all* $x, y \in H$.

(ii) $\|Nx - Ny\| \le \beta\|x - y\|$ *for all* $x, y \in H$.

(a) *If* (i) *and* (ii) *hold and* $\beta^2 < \alpha d_c^- + c(d_c^- - c - 2\alpha)$ *for some* $c \le 0$, *then Eq.* (3.1) *is uniquely solvable and the solution is the limit of the iterative process* (5.2). *Moreover, Eq.* (3.1) *is uniquely approximation solvable w.r.t.* $\Gamma = \{H_n, P_n\}$ *with* $\delta = \max\|P_n\| = 1$ *for each* $f \in H$ *and* (5.3) *holds. If A is defined on all of H, then the approximation solutions also satisfy* (3.5).

(b) *If* $\beta^2 \le \alpha d_c^- + c(d_c^- - c - 2\alpha)$ *and, for some* $a < \lambda = c - d_c^-/2$ *and* $b > 0$,

$$\|Nx - \lambda x\| \le a\|x\| + b \qquad \text{for all } x \in H,$$

then Eq. (3.1) *is solvable for each* $f \in H$.

Proof. We follow the arguments of Proposition 2.7 in [Mi-6]. Let $\lambda = c - d_c^-/2$ and $Cx = \lambda x$. Then $\lambda \notin \sigma(A)$ and $d = \operatorname{dist}(\lambda, \sigma(A)) > 0$ with $d = \|(A - \lambda I)^{-1}\|^{-1}$. Using conditions (i) and (ii), we get

$$\|Nx + \lambda x - (Ny + \lambda y)\| \le (\beta^2 + \lambda^2 + 2\alpha\lambda)^{1/2}\|x - y\|.$$

By our choice of λ and the condition on β, we get

$$\beta^2 + \lambda^2 + 2\alpha\lambda = \beta^2 + \alpha d_c^- + c(d_c^- - c - 2\alpha) + (d_c^-/2)^2$$
$$< (d_c^-/2)^2 = d^2.$$

Hence, the conclusions follow from Theorem 5.1. ∎

Remark 5.1. Since $\alpha \leq \beta$, the conditions imposed on α and β require that they belong to $(|c|, |c| + d_c^-)$. Hence, $c \leq 0$ is chosen so that this fact holds.

Remark 5.2. Theorem 5.2 extends a result of Smiley [Sm] in various ways, whose proof is based on the Liapunov–Schmidt alternative method, and the obtained error estimate is of a different type.

THEOREM 5.3. *Let $A: D(A) \subset H \to H$ be self-adjoint, $N: H \to H$ be a gradient map, and C, $B^{\pm}: H \to H$ be self-adjoint maps such that*

(i) $(B^-(x-y), x-y) \leq (Nx - Ny, x-y) \leq (B^+(x-y), x-y)$ *for all $x, y \in H$.*

(ii) $\|B^{\pm} - C\| \leq d = \min\{|\lambda| \,|\, \lambda \in \sigma(A - C)\}$.

(a) *If the inequality is strict in* (ii), *then Eq.* (3.1) *is uniquely solvable and the solution is the limit of the iterative process* (5.2). *Moreover, Eq.* (3.1) *is uniquely approximation solvable w.r.t.* $\Gamma = \{H_n, P_n\}$ *with* $\max\|P_n\| = 1$ *for H for each $f \in H$ and the approximate solutions satisfy* (5.3). *If A is defined on all of H, then the approximate solutions also satisfy* (3.5).

(b) *If, in addition, there are $0 < a < d$ and $b \geq 0$ such that*

$$\|Nx - Cx\| \leq a\|x\| + b \qquad \text{for all } x \in H,$$

then Eq. (3.1) *is solvable for each $f \in H$.*

Proof. Since C is a gradient of the functional $x \to (Cx, x)/2$, $N - C$ is a gradient map and

$$-\|B^- - C\|\,\|x-y\|^2$$
$$\leq ((B^- - C)(x-y), x-y), ((B^+ - C)(x-y), x-y)$$
$$\leq \|B^+ - C\|\,\|x-y\|^2.$$

Hence, by Lemma 1 in [M],

$$\|(N - C)x - (N - C)y\| \leq k\|x - y\| \qquad \text{for all } x, y \in H,$$

where $k = \max(\|B^- - C\|, \|B^+ - C\|)$. Since $d = \|(A - C)^{-1}\|^{-1}$, the conclusions follow from Theorem 5.1. ∎

Remark 5.3. If $B^- = \alpha I$ and $B^+ = \beta I$ and $[\alpha, \beta]$ is contained in the resolvent set $\rho(A)$ of A, then we can take $C = \lambda I$ for some $\lambda \in (\alpha, \beta)$ in part (a) of Theorem 5.3 (cf. [Mi-6]). In this case the unique solvability of Eq. (3.1) was proved by Amann [A] and a different proof of it was given in Mawhin [M]. Part (b) allows a bigger value of β as the following result shows. For $c \in \sigma(A) \cap (0, \infty)$, define $d_c^+ = \operatorname{dist}(c, \sigma(A) \cap (c, \infty))$.

THEOREM 5.4. *Let A: $D(A) \subset H \to H$ be self-adjoint, N: $H \to H$ be a gradient map, and $\alpha, \beta \in R$ be such that*

$$\alpha \|x - y\|^2 \le (Nx - Ny, x - y) \le \beta \|x - y\|^2 \qquad \text{for } x, y \in H.$$

(a) *If either $c \in \sigma(A) \cap (-\infty, 0]$ and $-c < \alpha \le \beta < -c + d_c^-$, or $c \in \sigma(A) \cap (0, \infty)$ and $-c - d_c^+ < \alpha \le \beta < -c$, then Eq.* (3.1) *is uniquely solvable and the solution is the limit of the iterative process* (5.2). *Moreover, Eq.* (3.1) *is uniquely approximation solvable w.r.t. $\Gamma = \{H_n, P_n\}$ with* $\max\|P_n\| = 1$ *for each $f \in H$ and* (5.3) *holds. If A is defined on all of H, then the approximate solutions also satisfy* (3.5).

(b) *If the conditions in* (a) *hold with each "$<$" sign replaced by "$\le$" and, for some $a < \lambda$ with $\lambda = c - d_c^-/2$ if $c \le 0$ and $\lambda = c + d_c^+/2$ if $c > 0$, and $b > 0$,*

$$\|Nx - \lambda x\| \le a\|x\| + b \qquad \text{for all } x \in H,$$

then Eq. (3.1) *is solvable for each $f \in H$.*

Proof. As above, we have that

$$\|Nx + \lambda x - Ny - \lambda y\| \le \max(|\alpha + \lambda|, |\beta + \lambda|)\|x - y\|.$$

By our choice of λ as given in (b), we conclude that $|\alpha + \lambda| \le d = \operatorname{dist}(\lambda, \sigma(A)) = d_c^{\pm}/2$ and $|\beta + \lambda| \le d$ with the inequalities being strict in part (a). Hence, Theorem 5.1 is applicable. ∎

When N is also Gâteaux differentiable, Theorem 5.4 was proved in [Mi-6]. Without the constructive solvability assertions and the error estimates, it is due to Ben-Naoum and Mawhin [BM] when $c = 0$.

The next result deals with conditions which imply the contractivity property of the nonlinear map in a suitable reformulation of Eq. (3.1).

THEOREM 5.5. *Let A: $D(A) \subset H \to H$ be self-adjoint, N: $H \to H$ be a gradient map, and $C, B^{\pm}$: $H \to H$ be self-adjoint maps such that*

(i) *$N - B^-$ and $B^+ - N$ are monotone*

and either one of the following conditions holds:

(ii) $H = H^- \oplus H^+$ *for some closed subspaces* $H^\pm$ *and the projections* $P^\pm\colon H \to H^\pm$ *are such that* $P^\pm(D(A)) \subset D(A)$ *and, for some* $\gamma > 0$,

$$((A - B^-)x, x) \le -\gamma\|x\|^2, \qquad x \in D(A) \cap H^-, \tag{5.5}$$

$$((A - B^+)x, x) \ge \gamma\|x\|^2, \qquad x \in D(A) \cap H^+. \tag{5.6}$$

(iii) $((A - B^-)x, x) < 0$ *for* $x \in D(A) \cap H^-$ *and* $((A - B^+)x, x) > 0$ *for* $x \in D(A) \cap H^+$

and either $A - (1-t)B^- - tB^+$ *has a closed range or* A *has a compact resolvent*.

(iv) $A - (1-t)B^- - tB^+$ *has a bounded inverse for each* $t \in [0,1]$.

Then, for each $f \in H$, *Eq.* (3.1) *is uniquely solvable,* (5.2) *holds, and, if* (ii) *holds, it is uniquely approximation solvable w.r.t.* $\Gamma = \{P_n, H_n\}$ *with* $P_n Ax = Ax$ *on* H_n *and the approximate solutions satisfy* (5.3). *If* A *is defined on all of* H, *then the approximation solutions also satisfy* (3.5).

The following lemma from [Mi-6] is needed for the proof.

LEMMA 5.1. *Let condition* (ii) *of Theorem* 5.5 *hold. Then there are* $\epsilon > 0$ *and* $c > 0$ *such that, for any self-adjoint maps* $B_1, B_2, C \in L(H)$ *with* $B^- \le B_1$, $B_2 \le B^+$, *and* $B_1 - \epsilon I \le C \le B_2 + \epsilon I$, *we have that*

$$\|Ax - Cx\| \ge c\|x\| \qquad \text{for all } x \in D(A). \tag{5.7}$$

Proof of Theorem 5.5. Lemma 5.1 implies that $A - C$ has a continuous inverse since it is self-adjoint and has a closed range. Hence, condition (ii) with $C = (1-t)B^- + tB^+$ implies (iv). Moreover, condition (iii) also implies (iv). To see this, it is enough to show that $A - (1-t)B^- - tB^+$ is one-to-one. If not, then there is an $x \ne 0$ such that $Ax - (1-t)B^-x - tB^+x = 0$. Then $x = x_1 + x_2 \in H^- + H^+$ and, by the symmetry of the operators,

$$\begin{aligned} 0 &= \left(Ax - (1-t)B^-x - tB^+x, x_2 - x_1\right) \\ &= \left(Ax_2 + Ax_1 - (1-t)\left(B^-x_2 + B^-x_1\right) - t\left(B^+x_2 + B^+x_1\right), x_2 - x_1\right) \\ &= \left(\left(A - (1-t)B^- - tB^+\right)x_2, x_2\right) - \left(\left(A - (1-t)B^- + tB^+\right)x_1, x_1\right) \\ &\ge \left(\left(A - B^+\right)x_2, x_2\right) - \left(\left(A - B^-\right)x_1, x_1\right) > 0, \end{aligned}$$

a contradiction. Now, if (iv) holds, then Eq. (3.1) is uniquely solvable for each $f \in H$ by a result of Fonda and Mawhin [FM]. When (ii) holds, $A - N$ is known to be A-proper and $\gamma/2$-strongly K-monotone and therefore the second assertion follows. ∎

Remark 5.4. Theorem 5.5(ii) gives a constructive proof of a result of Tersian [T] and part (iv) is due to Fonda and Mawhin [FM]. Our proof of the unique solvability in (ii) is new. This result also extends many other earlier ones (Amann [A], Dancer [D], etc.). For various applications to ordinary, elliptic, and hyperbolic equations, we refer to the above-cited works and to [Mi-10].

COROLLARY 5.1. *Let A: $D(A) \subset H \to H$ be self-adjoint and N: $H \to H$ be a gradient map such that, for some self-adjoint maps $B^{\pm}$: $H \to H$,*

(i) *$N - B^-$ and $B^+ - N$ are monotone.*

(ii) *$\beta^+ = \sum_{i=1}^m \lambda_i^{\pm} P_i^{\pm}$ commute with A, where $P_i^{\pm}$: $H \to \ker(B^{\pm} - \lambda_i^{\pm})$ are orthogonal projections with $P_i^- = P_i^+$ for $1 \le i \le m$, $\lambda_1^{\pm} \le \cdots \le \lambda_m^{\pm}$, and $\lambda_i^{\pm}$ are pairwise distinct.*

(iii) *$\bigcup_{i=1}^m [\lambda_i^-, \lambda_i^+] \subset \rho(A)$—the resolvent set of A.*

Then Eq. (3.1) *is uniquely approximation solvable w.r.t.* Γ *for each $f \in H$ and* (5.2) *and* (5.3) *hold. If A is defined on all of H, then the approximate solutions also satisfy* (3.5).

Proof. By Lemma 2.4 in [Mi-6], there are orthogonal subspaces $H^{\pm}$ such that $H = H^- \oplus H^+$ and conditions (5.5) and (5.6) hold. Hence, the result follows from Theorem 5.5. ∎

When $B^{\pm}$ are not of the form (ii), we need to assume more on the linear part A.

COROLLARY 5.2. *Let A: $D(A) \subset H \to H$ be self-adjoint, N: $H \to H$ be a gradient map, and $C_1, C_2, B^{\pm}$ be continuous self-adjoint maps such that $C_1 \le B^-$, $B^+ \le C_2$, and*

(i) *$N - B^-$ and $B^+ - N$ are monotone maps.*

(ii) *The spectrum $\sigma(A)$ is countable, consists of eigenvalues, and the corresponding eigenvectors form a complete orthonormal system in H.*

(iii) *There are two consecutive finite multiplicity eigenvalues $\lambda_k < \lambda_{k+1}$ of A such that*

$$\lambda_k \|x\|^2 < (C_1 x, x) \le (C_2 x, x) < \lambda_{k+1} \|x\|^2 \qquad \text{for } x \in H \setminus \{0\}.$$

Then Eq. (3.1) *is uniquely approximation solvable w.r.t.* Γ *for each $f \in H$ and* (5.2) *and* (5.3) *hold. If A is defined on all of H, then the approximate solutions also satisfy* (3.5).

Proof. Let H^- (resp. H^+) be the subspaces of H spanned by the eigenvectors of A corresponding to the eigenvalues $\lambda_i \le \lambda_k$ (resp. $\lambda_i \ge \lambda_{k+1}$). By Lemma 2.5 in [Mi-6], there is a $\gamma > 0$ such that (5.5) and (5.6) hold. Hence, the conclusion follows from Theorem 5.5. ∎

Remark 5.5. If λ_k (resp. λ_{k+1}) is of infinite multiplicity, then Corollary 5.2 is still valid if we assume in (iii)

$$(\lambda_k + \epsilon)\|x\|^2 \leq (C_1 x, x) \left(\text{resp.}\ (C_2 x, x) \leq \left(\lambda_{k+1} - \epsilon\|x\|^2\right)\right.$$

$$\text{for } 0 \neq x \in H.$$

Regarding (5.7), we also have the following useful result.

LEMMA 5.2. *Let* A: $D(A) \subset H \to H$ *be self-adjoint. Then* (5.7) *holds for each continuous self-adjoint map* C *with* $B_1 \leq C \leq B_2$ *if there is an* $a > 0$ *such that either*

(i) $0 < a < \min\{|\lambda| \,|\, \lambda \in \sigma(A - C)\}$, *or*

(ii) *each* C *commutes with* A *and* $\operatorname{dist}(\sigma(A), \sigma(C)) > a$.

For a discussion of the case when $H^- \oplus H^+ \neq H$, we refer to [A, Mi-6].

6. CONSTRUCTIVE HOMEOMORPHISM THEOREMS FOR A-STABLE MAPS

In this section, we continue our study of Eq. (3.1) with T neither differentiable nor having a multivalued derivative. We shall show that similar error estimates hold provided that T is locally Lipschitz and approximation stable, i.e., an inequality of type (6.1) below holds. We say that T: $X \to Y$ is locally p-Lipschitz for some $p > 0$ if for each $x \in X$ there are positive numbers r and M (depending on x) such that

$$\|Ty - Tz\| \leq M\|y - z\|^p \qquad \text{for all } y, z \in \overline{B}_r(x).$$

Define the function

$$m(q) = \begin{cases} 1, & 0 < q \leq 1, \\ 2^{q-1}, & q > 1. \end{cases}$$

For such maps, the following result was announced in [Mi-4].

THEOREM 6.1 (cf. [Mi-4]). *Let* T: $X \to Y$ *be surjective and locally* p-*Lipschitz and there are a function* c: $R^+ \to R^+$ *and numbers* $q > 0$ *and* $n_0 \geq 1$ *such that* $c(r)r^q \to \infty$ *as* $r \to \infty$ *and, for each* $r > 0$,

$$\|Q_n Tx - Q_n Ty\| \geq c(r)\|x - y\|^q \qquad \textit{for } x, y \in \overline{B}_r \cap X_n,\ n \geq n_0. \quad (6.1)$$

Then T is a homeomorphism and, for each $f \in Y$, Eq. (3.1) *is uniquely approximation solvable w.r.t. Γ, the approximate solutions $x_n \in \overline{B}_r \cap X_n$ for some r, and, for $n \geq n_0$,*

$$\|x_n - x_0\|^q \leq k_1 \|P_n x_0 - x_0\|^p + k_2 \|P_n x_0 - x_0\|^q$$
$$\leq c_1 d(x_0, X_n)^p + c_2 d(x_0, X_n)^q, \tag{6.2}$$

where d is the distance, the constants k_1 and k_2 depend on r, δ, q, and x_0, and $c_i = 2k_i\delta_1$, $i = 1, 2$, $\delta_1 = \sup\|P_n\|$.

Proof. Let $r > 0$ and $x, y \in \overline{B}_r$. Then, since Γ is projectionally complete, there are $x_n, y_n \in X_n$ with $\|x_n\| = \|x\|$ and $\|y_n\| = \|y\|$ and $x_n \to x$, $y_n \to y$. Moreover, for each $n \geq n_0$,

$$\|Q_n T x_n - Q_n T y_n\| \geq c(r)\|x_n - y_n\|^q,$$

and passing to the limit we obtain that

$$\|Tx - Ty\| \geq c(r)\|x - y\|^q, \qquad x, y \in \overline{B}_r. \tag{6.3}$$

In particular, for $\|x\| = r$ we get

$$\|Tx\| \geq \|Tx - T0\| - \|T0\| \geq c(r)\|x\|^q - \|T0\|$$

and therefore $\|Tx\| \to \infty$ as $\|x\| \to \infty$. Next, let $y_0 \in Y$ be fixed, $Tx_0 = y_0$, r, R, and $\epsilon > 0$ such that $B_\epsilon(x_0) \subset T^{-1}(B_R(y_0)) \subset B_r$. Then, if $u \in B_R(y_0)$, $u = Tx$ for some $x \in B_r(x_0)$ and

$$\|T^{-1}u - T^{-1}y_0\| \leq c(r)^{-1/q}\|u - y_0\|^{1/q},$$

which implies the continuity of T^{-1} at y_0. Hence, T is a homeomorphism.

Let us now prove the second part of the theorem. Since $Q_n T\colon X_n \to Y_n$ is continuous and injective by (6.1), $Q_n T(X_n)$ is open in Y_n by the Brouwer invariance of domain theorem. Moreover, $Q_n T(X_n)$ is closed in Y_n for, if $Q_n T x_k \to y$ for some $\{x_k\} \subset X_n$, then $\{x_k\}$ is a Cauchy sequence and therefore $Q_n T x_k \to Q_n T x$ for some $x \in X_n$. Hence, $Q_n T\colon X_n \to Y_n$ is bijective for each $n \geq n_0$. Next, let $f \in Y$ be fixed and x_0 and $x_n \in X_n$ be the unique solutions of Eq. (3.1) and $Q_n T x = Q_n f$ with $n \geq n_0$, respectively. Set $\|x_n\| = r_n$. Then $\|Q_n f\| = \|Q_n T x_n\| \geq c(r_n) r_n^q - \|Q_n T0\|$ and, consequently, $\{x_n\}$ is bounded. Let $r > 0$ be such that x_0 and $\{x_n\}$ are contained in $\overline{B}_r$. Set $\delta = \sup\|Q_n\|$. Then, for all large n,

$$c(r)\|x_n - P_n x_0\|^q \leq \|Q_n T x_n - Q_n T P_n x_0\|$$
$$\leq \delta\|Tx_0 - TP_n x_0\| \leq \delta M(x_0)\|x_0 - P_n x_0\|^p$$

and

$$\begin{aligned}\|x_n - x_0\|^q &\le \left(\|x_n - P_n x_0\| + \|P_n x_0 - x_0\|\right)^q \\ &\le m(q)\left(\|x_n - P_n x_0\|^q + \|P_n x_0 - x_0\|^q\right) \\ &\le m(q)\left(\delta M(x_0) c^{-1}(r)\|P_n x_0 - x_0\|^p + \|P_n x_0 - x_0\|^q\right).\end{aligned}$$

Hence, (6.2) holds as in Theorem 3.1 ∎

Remark 6.1. If, for example, $p = q = 1$ and $c(r) \equiv$ constant in Theorem 6.1, then we have that $\|Tx - Ty\| \ge c\|x - y\|$ for all $x, y \in X$ and one easily sees that the approximate solutions also satisfy (3.3).

COROLLARY 6.1. *Let $T\colon X \to Y$ be surjective, weakly Gâteaux differentiable on X, and satisfy* (6.1). *Suppose that for each $x \in X$ there are positive constants r and M (depending only on x) such that $\|T'(y)\| \le M$ for all $y \in \overline{B}_r(x)$. Then the conclusions of Theorem* 6.1 *hold with $p = 1$.*

Proof. It suffices to show that T is locally Lipschitz on X. But this follows easily by the mean value theorem. ∎

Strengthening condition (6.1) to the strong-monotonicity condition for $T\colon X \to X^*$, we shall now derive a simpler formula for the rate of convergence of approximate solutions. A similar result has been proven earlier by Ciarlet, Schultz, and Varga [CSV] using different arguments, where one can also find a number of applications to quasilinear elliptic partial differential equations.

THEOREM 6.2 (cf. [Mi-4]). *Let $T\colon X \to X^*$ be surjective and locally Lipschitz and, for some $1 < q > p$ and $c(r)$ with $c(r)r^q \to \infty$ as $r \to \infty$ and $r > 0$,*

$$(Tx - Ty, x - y) \ge c(r)\|x - y\|^q \qquad \text{for } x, y \in \overline{B}_r. \tag{6.4}$$

Then T is a homeomorphism and, for each $f \in Y$, Eq. (3.1) *is uniquely approximation solvable w.r.t. $\Gamma = \{X_n, P_n; Y_n = R(P_n^*), P_n^*\}$, the approximate solutions $x_n \in \overline{B}_r \cap X_n$ for some r and for each n*

$$\|x_n - x_0\| \le k\|P_n x_0 - x_0\|^{1/(q-p)} \le c \operatorname{dist}(x_0, X_n)^{1/(q-p)},$$

where k depends on $M(x_0)$ and $c(r)$, and $c = 2k\delta_1$, $\delta_1 = \sup\|P_n\|$.

Proof. It is easy to see that (6.4) implies (6.1) with $c(r)$ replaced by $\delta^{-1}cr$, where $\delta = \sup\|P_n^*\|$. Hence, as in the proof of Theorem 6.1, we see that T is a homeomorphism and $P_n^* T\colon X_n \to Y_n$ is bijective for each n. Moreover, for each $f \in X^*$ fixed, the solution x_0 of $Tx = f$ and the approximate solutions x_n of $P_n^* Tx = P_n^* f$ belong to a ball $\overline{B}_r$ for some $r > 0$. Hence,

$$\begin{aligned} c(r)\|x_n - x_0\|^q &\le (Tx_n - Tx_0, x_n - x_0) \\ &= (Tx_n - Tx_0, P_n x_0 - x_0) \\ &\quad + (Tx_n - Tx_0, x_n - P_n x_0) \\ &= (Tx_n - Tx_0, P_n x_0 - x_0) \\ &\le \|Tx_n - Tx_0\| \, \|P_n x_0 - x_0\| \\ &\le M(x_0)\|x_n - x_0\|^p \|P_n x_0 - x_0\|. \end{aligned}$$

Set $k = (c^{-1}(r)M(x_0))^{1/(q-p)}$. Then, for each n, we have that

$$\|x_n - x_0\| \le k\|P_n x_0 - x_0\|^{1/(q-p)} \le c \operatorname{dist}(x_0, X_n)^{1/(q-p)}. \quad \blacksquare$$

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