Clifford Theory for Semisimple $G$-Groups

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In this paper a Clifford theory for semisimple $G$-groups is developed, as a particular case of an abstract Clifford theory for $G$-functors.

Key Words: $G$-group; Clifford theory; $G$-functor.

1. INTRODUCTION

Let $G$ be a finite group and $H$ a subgroup of $G$. An $H$-group is a pair $(A, \psi)$ where $A$ is a group and $\psi : H \to \text{Aut}(A)$ is a group homomorphism. Then we say that $H$ acts on $A$ through $\psi$. If the action is understood, we shall often write that $A$ is an $H$-group.

If $(A, \psi)$ is an $H$-group, we can build the semidirect product $[A]H$, where the product is given by $(a, h)(a', h') = (aa'^{\psi(h^{-1})}, hh')$ for any $a, a' \in A$ and any $h, h' \in H$. Conversely, each semidirect product $[A]H$ determines an action of $H$ on $A$ given by the conjugation in $[A]H$.

If $(A, \psi)$ and $(B, \varphi)$ are $H$-groups, we define its direct product as the direct product of groups $A \times B$ with the action $(\psi, \varphi) : H \to \text{Aut}(A \times B)$ given by $(a, b)(\psi, \varphi)(h) = (a^{\psi(h)}, b^{\varphi(h)})$, for all $a \in A$, $b \in B$, and $h \in H$.

An $H$-group $(A, \psi)$ is said to be irreducible if there is no non-trivial proper normal subgroup $B$ in $A$ with $B^{\psi(h)} = B$ for all $h \in H$.

In this paper we are interested in semisimple $H$-groups, that is, $H$-groups which are a direct product of nonabelian irreducible $H$-groups.

An irreducible $H$-group $A$ can be regarded as a minimal normal subgroup of the semidirect product $[A]H$. Therefore, the following proposition holds (see [4, I 9.12]).

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PROPOSITION 1.1. Let \( H \leq G \). If \((A, \psi)\) is an irreducible nonabelian \( H\)-group and \( S \) is a minimal normal subgroup of \( A \), then \( S \) is simple and \( A = S^{\psi(h_1)} \times \cdots \times S^{\psi(h_m)} \) for any right transversal \( \{h_1, h_2, \ldots, h_m\} \) of \( N_H(S) \) in \( H \). Moreover, the minimal normal subgroups of \( A \) are precisely \( S^{\psi(h_1)}, \ldots, S^{\psi(h_m)} \).

Two \( H\)-sets \( X \) and \( Y \) are said to be \( H\)-isomorphic if there is a bijection \( f: X \to Y \) such that \( x^{h} = x^{f(h)} \) for all \( h \in H \). Recall that every transitive \( H\)-set is \( H\)-isomorphic to the \( H\)-set \( H/U \) of the right cosets of a subgroup \( U \) in \( H \) with the action given by right multiplication. Further the \( H\)-sets \( H/U \) and \( H/V \) are \( H\)-isomorphic if and only if \( U \) and \( V \) are \( H\)-conjugate subgroups (see [2]).

Remark 1.2. With the notation of the previous proposition, \( H \) permutes transitively the components \( S^{\psi(h)}, 1 \leq i \leq m \). Since

\[
S^{\psi(h_i)}S^{\psi(h_j)} = S^{\psi(h_i)}N_H(S)h_i \cdot h = N_H(S)h_j \quad (h \in H, 1 \leq i, j \leq m),
\]

the set \( \{S^{\psi(h_i)} \mid 1 \leq i \leq m\} \) is \( H\)-isomorphic to the \( H\)-set \( H/N_H(S) \).

DEFINITION 1.3. If \((A, \psi)\) is an irreducible \( H\)-group and \( S \) is a minimal normal subgroup of \( A \), we call the \( H\)-set associated with \((A, \psi)\) to the \( H\)-isomorphism class of \( H/N_H(S) \).

Remark 1.4. If we consider any other minimal normal subgroup \( T \) of \( A \) in the previous definition, then we obtain the \( H\)-set \( H/N_T(T) \) with \( T = S^{\psi(h)} \) for some \( h \in H \). Thus \( N_H(T) = N_T(S)h \) and so \( H/N_T(T) \) is \( H\)-isomorphic to \( H/N_H(S) \). Therefore, the \( H\)-set associated with an irreducible nonabelian \( H\)-group is well-defined.

If \((A, \psi)\) is a nonabelian \( H\)-group, a derivation or 1-cocycle from \( H \) to \( A \) is a map \( \delta: H \to A \) such that \((h_1h_2)^{\psi} = h_1^{\psi(h_2)}h_2^{\psi} \) for all \( h_1, h_2 \in H \) (see [8]).

The nonabelian \( H\)-groups \((A, \psi)\) and \((B, \varphi)\) are said to be \( H\)-equivalent (see [5]) if there exist a group isomorphism \( f: A \to B \) and a derivation \( \delta: H \to B \) such that \( a^{\psi(h)f} = a^{\varphi(h)f} \) for all \( a \in A, h \in H \). If the derivation is trivial, the \( H\)-groups \((A, \psi)\) and \((B, \varphi)\) are said to be \( H\)-isomorphic. We denote \( [(A, \psi)] \) as the \( H\)-equivalence class of \((A, \psi)\).

If \( \delta: H \to B \) is simply a map such that \( a^{\psi(h)f} = a^{\varphi(h)f} \) for all \( a \in A, h \in H \), and the center of \( B \) is trivial, then it is easy to prove that \( \delta: H \to B \) is actually a derivation.

In this paper, we develop a Clifford theory for \( G\)-groups similar to the celebrated one for linear representations [1]. First we consider, for each \( H \leq G \), the semisimple \( H\)-groups, and define induction, restriction, and
conjugation on them. We then prove that these maps are compatible with the $H$-equivalence classes of $H$-groups.

With these ingredients, we construct the $G$-functor of the semisimple $G$-groups and prove that Clifford theory holds on it, as a particular case of the more general Clifford theory for $G$-functors that has been developed in [6].

We recall the definition of based $G$-functor, given in [6].

**Definition 1.5** (see Green [3]). Let $\mathcal{A} = (M, T, R, C, \mathcal{B})$ be a quintuple of families $M, T, R, C, \mathcal{B}$ of the following kinds:

- $M = (M_H)_{H \leq G}$ gives, for each subgroup $H$ of $G$, a free abelian semigroup $M_H$.
- $T = (T_H, K)$ and $R = (R_K, H)$ give, for each pair $(H, K)$ of subgroups of $G$ such that $H \leq K$, the respective homomorphisms of semigroups, $T_{H, K} : M_H \rightarrow M_K$ and $R_{K, H} : M_K \rightarrow M_H$.
- $C = (C_{H, g})$ gives, for each pair $(H, g)$ where $H$ is a subgroup and $g$ an element of $G$, the semigroup homomorphism $C_{H, g} : M_H \rightarrow M_{H^g}$, where $H^g = g^{-1}Hg$.
- $\mathcal{B} = (\mathcal{B}_H)_{H \leq G}$ gives, for each subgroup $H$ of $G$, the basis $\mathcal{B}_H$ of $M_H$.

We say that $\mathcal{A} = (M, T, R, C, \mathcal{B})$ is a **based $G$-functor** if these families of semigroups and maps satisfy the following axioms, where $D, H, K,$ and $L$ are subgroups of $G$ and $g, g'$ are elements of $G$:

(a) $T_{H, H} = \text{id}(M_H)$, $T_{H, K}T_{K, L} = T_{H, L}$ if $H \leq K \leq L$.

(b) $R_{H, H} = \text{id}(M_H)$, $R_{K, H}R_{H, D} = R_{K, D}$ if $D \leq H \leq K$.

(c) $C_{H, g}C_{H^g, K} = C_{H, gg'}$.

(d) $C_{H, h} = \text{id}(M_H)$ if $h \in H$.

(e) $T_{H, K}C_{K, g} = C_{H, g}T_{H^g, K}$.

(f) $R_{K, H}C_{H, g} = C_{K, g}R_{H^g, K}$.

(g) Mackey axiom. If $H \leq L$, $K \leq L$, and if $\Gamma$ is a transversal of the $(H, K)$-double cosets in $L$, then

$$T_{H, L}R_{L, K} = \sum_{g \in \Gamma} C_{H, g}R_{H^g, H^g \cap K}T_{H^g \cap K, K}.$$

Since $C_{H, g}$ is an isomorphism, $\beta C_{H, g} \in \mathcal{B}_{H^g}$ for any $\beta \in \mathcal{B}_H$.

For each $H \leq G$, we consider the symmetric bilinear form $\langle \cdot, \cdot \rangle : M_H \times M_H \rightarrow \mathbb{N}$ such that $\mathcal{B}_H$ is an orthonormal basis of $M_H$. 
2. THE $G$-FUNCTOR OF SEMISIMPLE $G$-GROUPS

In this section we define step by step a based $G$-functor $\mathcal{S} = (M, T, R, C, \mathcal{B})$.

For each $H \leq G$, $M_H$ is the abelian semigroup of all $H$-equivalence classes of semisimple $H$-groups, where the sum is given by the direct product of $H$-groups. The set $\mathcal{B}_H$ of all $H$-equivalence classes of irreducible nonabelian $H$-groups is the basis of $M_H$.

**Definition 2.1.** Let $H \leq K \leq G$ and let $\{x_i = 1, x_2, \ldots, x_n\}$ be a right transversal of $H$ in $K$. Consider the action of $K$ on the set $\{1, \ldots, n\}$ given by $i^k = j$ if $Hx_i k = Hx_j$ ($k \in K, i, j \in \{1, \ldots, n\}$). If $(A, \psi)$ is an $H$-group, we define its **induced $K$-group** $(A^K, \psi^K)$ as the direct product

$$A^K = A_1 \times \cdots \times A_n$$

of $n$ copies of the group $A$ with the action $\psi^K : K \to \text{Aut}(A_1 \times \cdots \times A_n)$ given by

$$(a_1, \ldots, a_n)^{\psi^K(g)} = (a_1^{\psi(x_1g^{-1})}, \ldots, a_n^{\psi(x_ng^{-1})}),$$

where $g \in K, i' = i g^{-1}$ and $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$.

Notice that the isomorphism class of $(A^K, \psi^K)$ does not depend on the choice of the transversal of $H$ in $K$. The next proposition is a direct consequence of the definition.

**Proposition 2.2.** Let $H \leq K \leq L \leq G$. If $(A, \psi)$ is an $H$-group, then the $L$-groups $A^L$ and $(A^K)^L$ are $L$-isomorphic.

**Notation 2.3.** Let $(A, \psi)$ be an $H$-group. If $U \leq H$, $B \leq A$, and $B^{\psi(u)} = B$ for all $u \in U$, we say that $B$ is $U$-invariant. The map $\psi|_U : U \to \text{Aut}(B)$ given by $\psi|_U(u) = \psi(u)|_B$, if $u \in U$, is a homomorphism. In this case we may consider the $U$-group $(B, \psi|_U)$.

**Proposition 2.4.** Let $H \leq K \leq G$ and let $(A, \psi)$ be an $H$-group.

(i) If $\{x_i = 1, x_2, \ldots, x_n\}$ is a right transversal of $H$ in $K$, then $A^K = A_1^{\psi(x_1)} \times \cdots \times A_n^{\psi(x_n)}$, where $A_i = A$ is $H$-invariant and $(A, \psi)$ is $H$-isomorphic to $(A_1, \psi^K|_H)$.

(ii) If $(C, \varphi)$ is an $H$-group which is $H$-equivalent to $(A, \psi)$, then the $K$-groups $A^K$ and $C^K$ are $K$-equivalent.

**Proof.** (i) It is a direct check.

(ii) Let $f : A \to C$ be a group isomorphism and $\delta : H \to C$ a derivation which give an $H$-equivalence between $A$ and $C$. Take a right transver-
sal \{x_1, \ldots, x_n\} of H in K. Set \(A_i = A\) and \(C_i = C\) for \(1 \leq i \leq n\), and define the maps

\[ f^K : A_1 \times \cdots \times A_n \to C_1 \times \cdots \times C_n, \quad \delta^K : K \to C_1 \times \cdots \times C_n \]

by

\[(a_1, \ldots, a_n)^{f^K} = (a'_1, \ldots, a'_n), \quad g^{\delta^K} = \left((x_{1i}^{-1}gx_i^{-1})^{\delta}, \ldots, (x_{ni}^{-1}gx_i^{-1})^{\delta}\right),\]

if \(a_i \in A, 1 \leq i \leq n, g \in K\). Then it is easy to check that the group isomorphism \(f^K\) and the map \(\delta^K\) give a K-equivalence.

Let \(H \leq K \leq G\). The assignment

\[ T_{H, K} : [(A, \psi)] \mapsto [(A^K, \psi^K)] \]

for each irreducible nonabelian H-group \((A, \psi)\) is consistent by Proposition 2.4 and affords a homomorphism \(T_{H, K} : M_H \to M_K\). This map satisfies axiom (a) by Proposition 2.2.

We quote next some results on induction.

**Remark 2.5.** Let \(F \leq H, (A, \psi)\) an irreducible H-group and B an F-invariant normal subgroup of A. We may not assert in general that A is H-isomorphic to the H-group induced of \(B, (\psi \mid _F)\), as it is shown in the following easy example.

**Example 2.6.** Let \(A = T_1 \times T_2 \times T_3\), where \(T_i = T\) is a nonabelian simple group. Let H be the symmetric group on \(1, 2, 3\) and F the subgroup of H generated by the transposition (12). \(A\) is an irreducible H-group for the action \(h : (t_1, t_2, t_3) \mapsto (t_{\psi}, t_{2\psi}, t_{3\psi}), \) where \(h \in H, t_i \in T, 1 \leq i \leq 3\). \(B = T_1 \times T_2\) is an irreducible F-invariant normal subgroup of A. However, \(B^H\) is not isomorphic to A.

The assertion of Remark 2.5 holds if we add a new hypothesis:

**Proposition 2.7.** Let \(F \leq H \leq G, (A, \psi)\) an irreducible H-group, and B an F-invariant normal subgroup of A. Let \(\{h_1 = 1, h_2, \ldots, h_l\}\) be a right transversal of F in H. If the product of all \(B^{\psi(h_i)}\) is direct, then the H-group induced of \((B, \psi \mid _F)\) is H-isomorphic to \((A, \psi)\).

**Proof.** Since each \(B^{\psi(h_i)}\) is normal in A, so is \(B^{\psi(h_i)} \times \cdots \times B^{\psi(h_l)}\). Also it is H-invariant, because H permutes the components \(B^{\psi(h_i)}\) transitively. Since \(A\) is H-irreducible, \(A = B^{\psi(h_i)} \times \cdots \times B^{\psi(h_l)}\). Now the assertion follows from Proposition 2.4 (i). \(\square\)

As a particular case, by Proposition 1.1, we obtain a useful consequence.
Corollary 2.8. Let \((A, \psi)\) be an irreducible nonabelian \(H\)-group and let \(S\) be a minimal normal subgroup of \(A\). Then \((A, \psi)\) is \(H\)-isomorphic to the \(H\)-group induced of \((S, \psi|_U)\), where \(U = N_H(S)\).

Recall that, if \(U \leq H \leq K\), the \(K\)-set induced of \(H/U\) is isomorphic to \(K/U\) (see [2]).

Proposition 2.9. Let \(H \leq K \leq G\). If \((A, \psi)\) is an irreducible nonabelian \(H\)-group with associated \(H\)-set \(H/U\), then \((A^K, \psi^K)\) is an irreducible \(K\)-group and its associated \(K\)-set is \(K/U\).

Proof. If \(S\) is a minimal normal subgroup of \(A\) and \(N_H(S) = U\), then \((A, \psi)\) is the \(H\)-group induced of \((S, \psi|_U)\). Thus, \((A^K, \psi^K)\) is the \(K\)-group induced of \((S, \psi|_U)\). Now, it is easy to see that the permutation induced by \(K\) on the \(|K:U|\) copies of \(S\) is transitive. Therefore, \(A^K\) is irreducible and its associated \(K\)-set is \(K/U\). \(\blacksquare\)

We define now the restriction.

Definition 2.10. Let \(F \leq H \leq G\). If \((A, \psi)\) is an \(H\)-group, then the restricted action \(\psi_F : F \to \text{Aut}(A)\) defines an \(F\)-group which is called the \(F\)-group restricted of \((A, \psi)\) and denoted by \((A, \psi_F)\).

Proposition 2.11. Let \(F \leq H \leq G\) and let \((A, \psi)\) be an irreducible nonabelian \(H\)-group with associated \(H\)-set \(H/U\). Then:

(i) \((A, \psi_F)\) is the direct product of irreducible \(F\)-groups whose associated \(F\)-sets are precisely the transitive components of the \(F\)-set restricted of \(H/U\).

(ii) Let \((B, \varphi)\) be an irreducible nonabelian \(H\)-group which is \(H\)-equivalent to \((A, \psi)\) through the isomorphism \(f : A \to B\), say. Now, if \(C_1, \ldots, C_r\) are the \(F\)-irreducible constituents of \((A, \psi_F)\), then \(C_1^{(1)} , \ldots, C_r^{(1)}\) are the \(F\)-irreducible constituents of \((B, \varphi_F)\); further \(C_i^{(1)}\) is \(F\)-equivalent to \(C_i^{(1)}\) and \(N_H(C_i) = N_H(C_i^{(1)})\), \(1 \leq i \leq r\). In particular \((B, \varphi_F)\) and \((A, \psi_F)\) are \(F\)-equivalent.

Proof. (i) If \(S\) is a minimal normal subgroup of \(A\), then \(S\) is simple and \(A = S^{(\delta_1)} \times \cdots \times S^{(\delta_n)}\) for any right transversal \(\{x_1 = 1, x_2, \ldots, x_n\}\) of \(U = N_H(S)\) in \(H\).

Let \(\{x_1, \ldots, x_n\}\) be a transversal of the \((U, F)\)-double cosets in \(H\). Then \(A = \prod_{x_1} S^{\delta_1} \times \cdots \times \prod_{x_n} S^{\delta_n}\), where each \(\Gamma_j\) is a right transversal of \(F \cap U^{x_j}\) in \(F\) and each \(\prod_{x_j} S^{\delta_n}\) is an irreducible \(F\)-group.

(ii) Since \(A = C_1 \times \cdots \times C_r\), then \(B = C_1^{(1)} \times \cdots \times C_r^{(1)}\). We must show that each \(C_i^{(1)}\) is \(F\)-irreducible. Let \(\beta : H \to B\) a derivation such that \(a^{\delta(h)} f = a f^{\delta(h)} \mu^\beta\) for all \(a \in A, \ h \in H\). If \(D \subset C_i^{(1)}\) with \(D\)-\(F\)-invariant, then \(D = E_f\) for some \(E \leq C_i\), and we have for all \(g \in F\),

\[
E f^{\delta(g)} = E_f \iff E f^{\delta(g) \mu^\beta} = E f^{\mu^\beta} \iff E f^{\delta(g) f} = E f^{f \mu^\beta} = E_f, \]

since \( E^j \leq B \) and \( g^\beta \in B \). Therefore, \( E^{g(\beta)} = E \) for all \( g \in F \). Since \( E \leq C_i \) and \( C_i \) is \( F \)-irreducible, either \( E = C_i \) or \( E = 1 \), and so \( D = C_i^j \) or \( D = 1 \), as desired.

Consider now the canonical epimorphism \( \pi_i : B \to C_i^j, \ 1 \leq i \leq n \). For any \( c \in C_i \) and \( g \in F \), we have \( c^{g(\beta)} = c^{(g(\beta))} = c^{g(\beta)g(\alpha)} \) since \( C_i^j \leq B \). Therefore, the isomorphism \( f \mid_{C_i} \) and the composition \( \beta \pi_i \) make \( C_i \) and \( C_i^j \) \( F \)-equivalent. Now it is easy to prove that \( N(h)(C_i) = N(h)(C_i^j) \).

Let \( H \leq K \leq G \) and \((B, \varphi)\) an irreducible nonabelian \( K \)-group. Then, by Proposition 2.11,

\[
R_{K, H} : [(B, \varphi)] \to [(B, \varphi_H)]
\]

is well defined and affords a homomorphism \( R_{K, H} : M_K \to M_H \). Obviously axiom (b) is satisfied.

We may now characterize the equivalence of \( H \)-groups in terms of their simple components.

**Theorem 2.12.** Let \((A, \psi)\) and \((B, \varphi)\) be irreducible nonabelian \( H \)-groups. Let \( S \) be a minimal normal subgroup of \( A \) and \( R \) a minimal normal subgroup of \( B \). Set \( U = N_H(S) \) and \( V = N_H(R) \). The following assertions are equivalent:

(i) The \( H \)-groups \((A, \psi)\) and \((B, \varphi)\) are \( H \)-equivalent.

(ii) \( U = V^h \) for some \( h \in H \) and the \( U \)-groups \((S, \psi \mid_U)\) and \((R^{\psi(h)}, \varphi \mid_U)\) are \( U \)-equivalent.

**Proof.** We can write \( A = S^{\psi(\alpha_1)} \times \cdots \cdots S^{\psi(\alpha_m)} \) and \( B = R^{\psi(x_1)} \times \cdots \times R^{\psi(x_n)} \), where \( \{\alpha_i\} = 1, \ldots, n \) is a right transversal of \( U = N_H(S) \) in \( H \) and \( \{x_1, \ldots, x_m\} \) is a right transversal of \( V = N_H(R) \) in \( H \). To see that (i) implies (ii), let \( f : A \to B \) be a group isomorphism and let \( \delta \) be a derivation such that \( a^{\psi(\alpha)h} = a^{(\psi(h))h^\delta} \) for all \( a \in A, h \in H \). Then \( (S^\psi(\alpha_i))h = (R^\psi(x_i))h \) for some \( j \in \{1, \ldots, m\} \).

Since \( S \) is an irreducible constituent of \((A, \psi_U)\), then \( N_H(S) = N_H(R^{\psi(x_i)}) \), by Proposition 2.11 (ii). Therefore \( U = V^{x_i} \) and the \( U \)-groups \((S, \psi \mid_U)\) and \((R^{\psi(x_i)}, \varphi \mid_U)\) are \( U \)-equivalent by the same proposition.

Conversely, suppose that \((S, \psi \mid_U)\) and \((R^{\psi(h)}, \varphi \mid_U)\) are \( U \)-equivalent. Then its induced \( H \)-groups are \( H \)-equivalent by Proposition 2.4 (ii).

**Remark 2.13.** Theorem 2.12 remains true if we put isomorphic in place of equivalent.

We now define \( G \)-conjugation.

**Definition 2.14.** If \( H \leq G, g \in G \), and \((A, \psi)\) is an \( H \)-group, we write \((A, \psi^g)\) for the \( H^g \)-group \( A \) with the action \( \psi^g : H^g \to \text{Aut}(A) \)
given by $\psi^g(h^g) = \psi(h)$, for all $h \in H$. We call $(A, \psi)$ the conjugate of $(A, \psi)$ by $g$.

Notice that if $(A, \psi)$ is irreducible, then $(A, \psi^g)$ is also irreducible for all $g \in G$. Moreover, if $g \in H$, then $(A, \psi^g)$ is $H$-isomorphic to $(A, \psi)$.

Let $H \leq G$, $g \in G$, and $(A, \psi)$ be an irreducible nonabelian $H$-group. Then

$$ C_{H,g} : [(A, \psi)] \mapsto [(A, \psi^g)] $$

is clearly well defined and affords a homomorphism $C_{H,g} : M_H \to M_{H^g}$. It is immediate that axioms (e), (d), and (f) are satisfied. To prove that axiom (e) is satisfied it suffices to observe the following. If $(x_1, x_2, \ldots, x_n)$ is a right transversal of $H$ in $K$ and $g \in G$, then $(x_1^g, x_2^g, \ldots, x_n^g)$ is a right transversal of $H^g$ in $K^g$, and $H^g x_i^g k^g = H^g x_j^g$ if and only if $H x_i k = H x_j$ for all $k \in K$.

We state without proof the next result.

PROPOSITION 2.15. Let $(A, \psi)$ be an $H$-group, $U \leq H$, and $h \in H$. If $B \trianglelefteq A$ is $U$-invariant, then:

(i) $(B^{\psi(h)}, \psi|_U)$ is an $U^h$-group which is $U^h$-isomorphic to $(B, (\psi|_U)^h)$.

(ii) The $H$-group induced of $(B, \psi|_U)$ is $H$-isomorphic to the $H$-group induced of $(B^{\psi(h)}, \psi|_U)$.

It remains to prove that axiom (g) is also satisfied to conclude that $\mathbb{S}$ is a based $G$-functor.

THEOREM 2.16 (Mackey Axiom). Let $H$, $K \leq L \leq G$ and let $(A, \psi)$ be an irreducible nonabelian $H$-group. Then the $K$-restricted of $A^L$ is $K$-isomorphic to $\prod_{g \in \Gamma} (A^G, (\psi |_{H^g \cap K})^k)$, where $\Gamma$ is a right transversal of the $(H, K)$-double cosets in $L$.

Proof. Let $(x_1, \ldots, x_n)$ be a right transversal of $H$ in $L$. We know that $A^L = A^{\psi(x_1)}_1 \times \cdots \times A^{\psi(x_n)}_n$, where $A_i = A$. If $x_1, \ldots, x_n$ is a full set of representatives of the $(H, K)$-double cosets of $L$, then

$$ A^L = \prod_{g \in \Gamma_1} A^{\psi(x_1,g)}_1 \times \cdots \times \prod_{g \in \Gamma_n} A^{\psi(x_n,g)}_n, $$

where each $\Gamma_j$ is a right transversal of $K \cap H^{x_i}$ in $K$. Now, by Proposition 2.7, each $\prod_{g \in \Gamma_j} A^{\psi(x_i,g)}_i$ is the $K$-group induced of $(A^{\psi(x_i)}_i, \psi^L |_{K \cap H^{x_i}})$, where $(A^{\psi(x_i)}_i, \psi^L |_{H^{x_i}})$ is $H^{x_i}$-isomorphic to the conjugate of $(A_i, \psi^k |_H)$ by $x_i$, and so $H^{x_i}$-isomorphic to the conjugate of $(A, \psi^k |_H)$ by $x_i$. \[\square\]
3. CLIFFORD THEORY ON THE G-FUNCTOR OF SEMISIMPLE G-GROUPS

It is shown in [6] that Clifford theory holds on a based $G$-functor $(M, T, R, C, \mathcal{B})$ if and only if the following property is satisfied:

(P) If $H \leq K \leq G$ and $\alpha \in \mathcal{B}_K$, then for any $\beta \in \mathcal{B}_H$ such that $\langle \alpha R_{K,H}, \beta \rangle \neq 0$, there exist $S, H \leq S \leq K$, and $\gamma \in \mathcal{B}_S$ such that $\langle \gamma T_{S,K}, \alpha \rangle \neq 0$ and $\gamma R_{S,H} = n \beta$ for some $n \in \mathbb{N}$.

**Proposition 3.1.** Let $F \leq H \leq G$ and $(A, \varphi)$ an irreducible nonabelian $H$-group. Let $B$ be an $F$-invariant normal subgroup of $A$ and suppose that $(B, \varphi \mid_F)$ is irreducible. Set $S = N_H(B)$. Then $F \leq S$, $B$ is $S$-invariant, the $F$-group restricted of $(B, \varphi \mid_S)$ is $(B, \varphi \mid_F)$ and $(A, \varphi)$ is $H$-isomorphic to the $H$-group induced of $(B, \varphi \mid_S)$. In particular property (P) holds on the $G$-functor $\mathcal{S}$.

**Proof.** By Proposition 2.7, it suffices to show that $A = B^{\varphi(x_1)} \times \cdots \times B^{\varphi(x_k)}$ for a right transversal $(x_1, \ldots, x_k)$ of $S$ in $H$. Since each $B^{\varphi(x_i)}$ is a normal subgroup of $A$ which is $F$-irreducible, then, for each $1 \leq j \leq n$, $B^{\varphi(x_j)} \cap \prod_{i \neq j} B^{\varphi(x_i)}$ must be either trivial or equal to $B^{\varphi(x_j)}$.

Suppose that the latter is true for some $j$. Then $B^{\varphi(x_j)} \leq \prod_{i \neq j} B^{\varphi(x_i)}$ and so $A = \prod_{i \neq j} B^{\varphi(x_i)}$. But, for $i \neq j$, $B^{\varphi(x_i)} \cap B^{\varphi(x_j)} = 1$ and so $B^{\varphi(x_j)} \leq C_A(B^{\varphi(x_i)})$. Therefore, $B^{\varphi(x_j)} \leq Z(A)$, a contradiction because $Z(A) = 1$.

**Definition 3.2.** Let $F \leq H \leq G$ and $(B, \varphi)$ an irreducible nonabelian $F$-group. We call the *inertia subgroup* of $(B, \varphi)$ in $H$ to the subgroup

$$\{ h \in H \mid (B, \varphi^h) \text{ is } F\text{-equivalent to } (B, \varphi) \}.$$ 

As a consequence of [6] and by Proposition 3.1 we obtain the version of the first Clifford’s theorem for the based $G$-functor $\mathcal{S}$.

**Theorem 3.3.** Let $F \leq H \leq G$ and let $(A, \varphi)$ be a nonabelian irreducible $H$-group. Let $B$ be an $F$-invariant normal subgroup of $A$ such that $(B, \varphi \mid_F)$ is irreducible and let $Y$ be the inertia subgroup of $(B, \varphi \mid_F)$ in $H$. Then there exists $e \in \mathbb{N}$ such that

$$[(A, \varphi)]_{R_{H,F}} = e\left( [(B, \varphi \mid_F)]^{g_1} + \cdots + [(B, \varphi \mid_F)]^{g_n} \right),$$

where $(g_1, g_2, \ldots, g_n)$ is any right transversal of $Y$ in $H$ and $[(B, \varphi \mid_F)]^{g_i} = [(B, \varphi \mid_F)]C_{F,g_i}$. 


We prove now that this based $G$-functor satisfies a finiteness condition, called property (F) in [6]:

(F) For each $\delta \in \mathcal{B}_1$, there exist only finitely many elements $\alpha \in \mathcal{B}_G$ with $\langle \alpha R_{G,1}, \delta \rangle \neq 0$.

**Theorem 3.4.** The based $G$-functor $\mathbb{S}$ satisfies property (F).

**Proof.** It is clear that the equivalence classes of nonabelian irreducible 1-groups are precisely the isomorphism classes of nonabelian simple groups with the trivial action. Now, if $(A, \psi)$ is an irreducible nonabelian $G$-group such that $\langle [(A, \psi)]R_{G,1}, [(S, 1)] \rangle \neq 0$, then $A$ must be a direct product of copies of the simple group $S$.

Moreover, if $U = N_G(S)$, then the $G$-equivalence class of $A$ is determined by the $U$-equivalence class of $(S, \psi|_U)$ by Theorem 2.12.

Since $\{\phi : U \to \text{Aut} S \mid U \leq G, \phi \text{ is a homomorphism}\}$ is a finite set, then there are only finitely many $G$-equivalence classes of irreducible nonabelian $G$-groups $(A, \psi)$ in whose restriction $(S, 1)$ occurs.

Therefore, by [6, 2.12], the following definition is consistent.

**Definition 3.5.** Let $H \leq K \leq G$. If $(B, \varphi)$ is an irreducible nonabelian $H$-group, we define the $K$-elevated of $[(B, \varphi)]$ by

$$[(B, \varphi)]_{E H, K} = \sum_{[(A, \psi) \in [K_H \times K]} \langle [(A, \psi)]_R K, [((B, \varphi))_1]\rangle [(A, \psi)],$$

where the sum ranges over a full set of representatives of $K$-equivalence classes of irreducible nonabelian $K$-groups. This definition can be extended to the semigroup $M_H$ and so we obtain a new family of semigroup homomorphisms $E_{H, K} : M_H \to M_K$ such that

$$\langle \alpha R_{K, H}, \beta \rangle = \langle \alpha, \beta E_{H, K} \rangle$$

for any $\alpha \in M_K$ and $\beta \in M_H$.

**Remark 3.6.** The definition above is independent of the chosen representative of the $H$-equivalence class of $(B, \varphi)$. Moreover, Proposition 2.4 (i) asserts that the coefficient of $[(B, \varphi)]T_{H, K}$ in $[(B, \varphi)]E_{H, K}$ is different from zero.

We obtain finally, by [6, 2.10 and 2.15], the second Clifford’s theorem for the based $G$-functor $\mathbb{S}$. 
Theorem 3.7. Let $F \leq H \leq G$. Suppose that $(B, \varphi)$ is an irreducible nonabelian $F$-group and let $Y$ be the inertia subgroup of $(B, \varphi)$ in $H$. If we write

$$\mathcal{A} = \{[(D, \phi)] | (D, \phi) \text{ irreducible } Y\text{-group}, \quad \langle[(D, \phi)]_{R_{Y, F}}, [(B, \varphi)]\rangle \neq 0\}$$

$$\mathcal{C} = \{[(A, \psi)] | (A, \psi) \text{ irreducible } H\text{-group}, \quad \langle[(A, \psi)]_{R_{H, F}}, [(B, \varphi)]\rangle \neq 0\}$$

then

(i) $[(D, \phi)]T_{Y, H} \in \mathcal{C}$ for all $[(D, \phi)] \in \mathcal{A}$.

(ii) If $(D, \phi)$ is an irreducible nonabelian $Y$-group, then

$$\langle[(D, \phi)]_{R_{Y, F}}, [(B, \varphi)]\rangle = \langle[(D, \phi)]T_{Y, H} R_{H, F}, [(B, \varphi)]\rangle.$$

(iii) If $[(D, \phi)] \in \mathcal{A}$ and $[(D, \phi)]T_{Y, H} = [(A, \psi)]$, then $[(D, \phi)]$ is the unique element of $\mathcal{A}$ which is a component of $[(A, \psi)]_{R_{H, Y}}$.

(iv) The map $\mathcal{A} \rightarrow \mathcal{C}$ given by $[(D, \phi)] \mapsto [(D, \phi)]T_{Y, H}$ is a well-defined bijection.

(v) $[(D, \phi)]E_{Y, H} = [(D, \phi)]T_{Y, H}$ for all $[(D, \phi)] \in \mathcal{A}$.

Remark 3.8. Let $[S]$ be the isomorphism class of the nonabelian simple group $S$. For each $H \leq G$, set

$$\mathcal{B}_H^{[S]} = \{[(A, \psi)] \in \mathcal{B}_H | A \text{ is a direct product of copies of } S\}.$$

Let $M_H^{[S]}$ be the subsemigroup of $M_H$ generated by $\mathcal{B}_H^{[S]}$. Then we obtain a based $G$-subfunctor $\mathfrak{S}^{[S]} = (\mathfrak{S}^{[S]}, T, R, C, \mathcal{B}^{[S]})$ of $\mathfrak{S}$ and, by [6, 2.9], the based $G$-functor $\mathfrak{S}$ is the coproduct of all based $G$-subfunctors $\mathfrak{S}^{[S]}$, $[S]$ ranging over all the isomorphism classes of nonabelian simple groups.

Remark 3.9. If we take $M_H$ as the set of all the $H$-equivalence classes of $H$-groups and the basis $\mathcal{B}_H$ of all $H$-equivalence classes of nonabelian indecomposable $H$-groups, the corresponding functor would have not satisfied property (P). For instance, let $G = \langle g_1 \rangle \times \langle g_2 \rangle \cong C_2 \times C_2$, $H = \langle g_1 \rangle$, and $A = \langle \tau, \sigma, \rho; \tau^2 = 1 = \sigma^3 = \rho^2, \sigma^2 = \sigma^2, \tau^2 = \tau, \sigma^\rho = \sigma \rangle \cong \Sigma_3 \times C_2$.

Then $A$ is an indecomposable $G$-group with the action given by $\tau g_2 = \tau \rho$, $\sigma g_2 = \sigma^2$, $\rho g_2 = \rho$, and $g_1$ acting trivially on $A$, although $A$ is not irreducible. In this case, the $H$-group restricted of $A$ is $\langle \tau, \sigma \rangle \times \langle \rho \rangle$, which is not a direct product of $G$-conjugated $H$-groups and so Clifford’s theorems do not hold.
REFERENCES