Prime ideals in two-dimensional domains over the integers

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Abstract

Let $B$ be a finitely generated birational extension of $\mathbb{Z}[x]$, the ring of polynomials in one variable over the integers $\mathbb{Z}$. (That is, $B$ is a finitely generated extension of $\mathbb{Z}[x]$ contained in its quotient field $\mathbb{Q}(x)$.) Then $\text{Spec}(B)$ is order-isomorphic to $\text{Spec}(\mathbb{Z}[x])$. This affirms part of a conjecture of Wiegand (1986). © 1998 Elsevier Science B.V. All rights reserved.


1. Introduction

Let $R$ be a commutative Noetherian ring with 1. We consider the prime spectrum of $R$, $\text{Spec}(R)$, the set of prime ideals of $R$, as a partially ordered set under inclusion. (For Noetherian rings, the underlying partially ordered set determines the topology on the spectrum.) In the 1950s, Irving Kaplansky asked: "Which partially ordered sets arise as $\text{Spec}(R)$ for some Noetherian ring $R$?" [8]. This is still an unsolved problem. Some progress has been made towards a more modest goal: Describe those partially ordered sets arising from Noetherian two-dimensional domains which are related to polynomial rings. For example, Heinzer et al. [2–6] partially describe the prime spectra of finitely generated birational extensions of a polynomial ring over a semilocal one-dimensional domain and Wiegand [12,13] characterizes spectra of two-dimensional polynomial rings over the integers and over finite fields.

In particular, an interesting question arises: "For which Noetherian domains $R$ is $\text{Spec}(R) \cong \text{Spec}(\mathbb{Z}[x])$?". Wiegand shows in [12,13] that this is true if $R$ is a two-dimensional domain that is finitely generated as a $k$-algebra, where $k$ is an algebraic

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extension of a finite field. The spectrum of the polynomial ring in two variables over the rationals, however, is not order-isomorphic to Spec(\(\mathbb{Z}[x]\)). Ilic conjectures in [13]

**Conjecture 1.1.** Every two-dimensional domain \(A\) which is a finitely generated \(\mathbb{Z}\)-algebra has prime spectrum order-isomorphic to that of \(\mathbb{Z}[x]\).

He proves this in the case \(A = D[x]\), where \(D\) is an order in an algebraic number field. The conjecture is equivalent to the assertion that the spectrum of every two-dimensional domain which is a finitely generated \(\mathbb{Z}\)-algebra satisfies five specific axioms (the CZP Axioms 1.3) which hold for Spec(\(\mathbb{Z}[x]\)). Every pair of countable partially ordered sets satisfying these five axioms are isomorphic [12]. The fifth axiom is the most difficult to prove; it is trivial to verify the first four (see Lemma 1.6).

We prove here that Conjecture 1.1 holds for finitely generated birational extensions of \(\mathbb{Z}[x]\). Our main result, Theorem 3.5, is that if \(B = \mathbb{Z}[x][(g_1(x))/(f(x)), \ldots, (g_n(x))/(f(x))]\), where \(g_1(x), \ldots, g_n(x)\) and \(f(x)\) are polynomials in \(\mathbb{Z}[x]\) and \(f(x) \neq 0\), then Spec(\(B\)) is order-isomorphic to Spec(\(\mathbb{Z}[x]\)). We believe some generalization of our techniques might be used to establish Conjecture 1.1.

We begin by introducing the notation for partially ordered sets to be used throughout.

**Notation 1.2.** Let \(U\) be a partially ordered set satisfying the ascending and descending chain conditions. For \(u \in U\), the *height* of \(u\), \(ht(u)\), is the length of the longest chain to \(u\) from a minimal element in \(U\); the *dimension* of \(U\), \(\dim(U) = \max\{ht(u) \mid u \in U\}\).

For all \(u\) and \(v\) in \(U\), and for all subsets \(S\) and \(T\) of \(U\), define

\[
G_S(u) = \{y \in S \mid y > u\}, \quad G_S(u, v) = G_S(u) \cap G_S(v),
\]

\[
G_S(T) = \bigcap_{t \in T} G_S(t), \quad \text{and} \quad H_i(U) = \{u \in U \mid ht(u) = i\}.
\]

We write \(G(u), G(u, v), \) and \(G(T)\) for \(G_U(u), G_U(u, v), \) and \(G_U(T)\), respectively.

**The CZP axioms 1.3.** A partially ordered set \(U\) satisfies CZP provided:

(P1) \(U\) is countable with a unique minimal element.

(P2) \(U\) has dimension two.

(P3) For each element \(u\) of height one, \(G(u)\) is infinite.

(P4) For each pair \(u, v\) of distinct elements of height one, \(G(u, v)\) is finite.

(P5) Given finite subsets \(S\) and \(T\), with \(\emptyset \neq S \subseteq H_1(U)\) and \(T \subseteq H_2(U)\), there is a height-one element \(w\) in \(U\) such that \(w < t, \forall t \in T\), and whenever \(t' \in U\) is greater than both \(w\) and \(s\) for some \(s\) in \(S\), then \(t' \in T\).

**Theorem 1.4** (Wiegand [13, Theorem 1]). Spec(\(\mathbb{Z}[x]\)) satisfies (P1)-(P5), and for every order \(D\) in an algebraic number field, Spec(\(D[x]\)) \(\cong\) Spec(\(\mathbb{Z}[x]\)). Every partially ordered set which satisfies (P1)-(P5) is order-isomorphic to Specc(\(\mathbb{Z}[x]\)).
Recall that "an order in an algebraic number field $k$" is an integral domain with quotient field $k$ which is a finitely generated $\mathbb{Z}$-module.

**Theorem 1.5** (Wiegand [14, Theorem 5]). Let $A$ be a two-dimensional domain finitely generated as a $k$-algebra where $k$ is an algebraic extension of a finite field. Then $\text{Spec}(A) \cong \text{Spec}(\mathbb{Z}[x])$.

**Lemma 1.6.** Let $R$ be a one-dimensional countable, Hilbert Noetherian domain, and let $C$ be a two-dimensional domain finitely generated as an $R$-algebra. Then $\text{Spec}(C)$ satisfies Axioms (P1)–(P4).

**Proof.** Note that $C$ has the form $(R[x_1,\ldots,x_n])/P_0$, where $x_1,\ldots,x_n$ are indeterminates and $P_0$ is a prime ideal of $R[x_1,\ldots,x_n]$ of height $n - 1$. Axioms (P1) and (P2) are trivial.

For (P3), we first prove that every maximal ideal of $C$ has height two. Since every maximal ideal of $R$ has height one, every maximal ideal of $R[x_1,\ldots,x_n]$ has height $n + 1$ by [9, p. 114, Exercise 31]. Therefore, every maximal ideal of $C$ has height two. If $Q$ is a height-one prime of $C$, then $Q$ is not maximal and so $C/Q$ has dimension one. Also $C/Q \cong (R[x_1,\ldots,x_n])/(Q')$ for some height-$n$ prime ideal $Q'$ of $R[x_1,\ldots,x_n]$ containing $P_0$. Clearly, $C/Q$ is a Hilbert ring; thus, $C/Q$ has infinitely many maximal ideals [9, Theorem 147]. Therefore (P3) holds.

For (P4), let $u \neq v \in \text{Spec}(C)$ with $\text{ht}(u) - \text{ht}(v) = 1$. Only finitely many maximal ideals contain both $u$ and $v$, since the zero-dimensional Noetherian ring $C/(u + v)$ has only finitely many minimal primes.

**Corollary 1.7.** Let $R$ be a two-dimensional domain finitely generated as a $\mathbb{Z}$-algebra. Then $\text{Spec}(R) \cong \text{Spec}(\mathbb{Z}[x]) \iff \text{Spec}(R)$ satisfies Axiom (P5) of (1.3).

Since Axiom (P5) of (1.3) is so important, we give a new definition related to that axiom.

**Definition 1.8.** Let $U$ be a partially ordered set of dimension $n$ and $0 < m < n$. A height-$(m,n)$-$U$-pair (or $(m,n)$-pair or $U$-pair, when $U$ or $(m,n)$ is understood) is a pair $(S,T)$ of finite subsets of $U$, where $T \subseteq H_m(U)$ and $\emptyset \neq S \subseteq H_n(U)$. If $(S,T)$ is such a pair and $w \in H_i(U)$, for $1 \leq i \leq n - 1$, satisfies $\bigcup_{s \in S} (G(w) \cap G(s)) \subseteq T \subset G(w)$, then $w$ is called a (height-$i$) radical element for $(S,T)$.

In other words, a radical element for a $U$-pair $(S,T)$ is an element $w$ of $U$ such that $w < t$, $\forall t \in T$, and whenever $t' \in U$ is greater than both $w$ and $s$ for some $s$ in $S$, then $t' \in T$, as in (P5).

Using this definition, (P5) above can be restated.

(P5) Every height-(1,2)-$U$-pair has a height-one radical element.

Since we are working primarily in two-dimensional partially ordered sets $U$, pairs are usually height-(1,2)-$U$-pairs and the "height-(1,2)" may be omitted.
The radical elements play an important role in defining isomorphisms between pairs of CZP sets. Say \( U = \{u_i\}_{i=0}^{\infty}, V = \{v_i\}_{i=0}^{\infty} \), are CZP sets with \( \text{ht}(u_0) = 0 = \text{ht}(v_0) \). If \( \phi_n : \{u_i\}_{i=0}^{n} \rightarrow V \) is a monomorphism preserving heights and order and \( H_1(\{u_i\}_{i=1}^{\infty}) \neq \emptyset \), the existence of radical elements enables \( \phi_n \) to be extended to a new height-one element \( w \) of \( U \) so that \( \phi(w) \in V \) is in the right place. (This is an oversimplification; the procedure is explained carefully in [12].)

Remarks 1.9. (1) If \( w \) is a radical element for a pair \((S, T)\) and \( |G(w)| = \infty \), then \( w \notin S \). This is because if \( w \in S \) and \( s = w \), then \( G(w) \cap G(s) = G(w) \) is not contained in the finite set \( T \). In particular, for a partially ordered set \( U \) satisfying (P3), a height-one radical element \( w \) for a \((1,2)\)-\( U \)-pair \((S, T)\) is never a member of \( S \).

(2) If \( S \subseteq S' \subseteq H_1(U) \) and \( T \subseteq T' \subseteq H_1(U) \), where \( j > i \), and \( t \neq s \), \( \forall t \in T' - T, s \in S \), then a radical element \( w \) for \((S', T')\) is also a radical element for \((S, T)\). To see this, note that \( T' \subseteq G(w) \Rightarrow T \subseteq G(w) \), and \( t \in G(w) \cap G(s) \) for some \( s \in S \Rightarrow t \in T' \) and \( t \notin T' - T \); thus \( t \in T \).

Thus, in particular, to show that a particular \( U \)-pair \((S, T)\) has a radical element, we can assume that, by expanding \( S \) if necessary, for every \( t \in T \), there exists an \( s \in S \) with \( s < t \).

(3) The rationale for calling the element \( w \) in (P5) a radical element is that, if \( U = \mathrm{Spec}(R) \) for some ring \( R \) and if \((S, T)\) is a \((1,2)\)-\( U \)-pair such that \( \forall t \in T \), there exists an \( s \in S \) with \( t > s \), then

\[
\text{If } I = \bigcap_{s \in S} s \text{ and } w \text{ is a radical element for } (S, T), \text{ then } \sqrt{I + w} = \bigcap_{m \in T} m.
\]

(4) If (P3) and (P5) hold and \((S, T)\) is a \( U \)-pair, then the number of radical elements for the pair is infinite. To see this, suppose that \( w \) is one such radical element; by Remark 1.9(1), \( w \notin S \). The pair \((S \cup \{w\}, T)\) has a radical element \( w' \notin S \cup \{w\} \) which is also a radical element of \((S, T)\). By continuing in this way we produce infinitely many radical elements for \((S, T)\).

2. Localizations and technical lemmas

We first study localizations of \( \mathbb{Z}[x] \).

Proposition 2.1. Let \( X \) be a partially ordered set and \( u_1, \ldots, u_n \) height-one elements of \( X \). Let \( U = X - \{u_1, \ldots, u_n\} \cup (\bigcup_{i=1}^{n} G_X(u_i)) \). If \( X \cong \mathrm{Spec}(\mathbb{Z}[x]) \), then \( U \cong \mathrm{Spec}(\mathbb{Z}[x]) \).

Proof. It is enough to check (P1)–(P5) for \( U \). Axiom (P1) is obvious. For (P2) and (P3), note that \( \mathrm{Spec}(\mathbb{Z}[x]) \) has infinitely many height-one elements, and thus \( X \) and \( U \) do. If \( u \) is a height-one element of \( U \), then \( G_U(u) = G_X(u) - (\bigcup_{i=1}^{n} G_X(u, u_i)) \).
Since $X$ satisfies (P3) and (P4), $|G_X(u)| = \infty$ and $|\bigcup_{i=1}^n G_X(u_i)| < \infty$. Thus, $|G_U(u)| = \infty$ and $\dim(U) = 2$. For (P4), $\forall u, v \in H_1(U)$, $G_U(u, v) \subseteq G_X(u, v)$; hence $G_U(u, v)$ is finite.

For (P5), every $(1, 2)$-$U$-pair $(S, T)$ is also an $X$-pair. By (P5) for $X$ and Remark 1.9(4), there are infinitely many height-one radical elements $w$ for $(S, T)$ as an $X$-pair such that $\bigcup_{s \in S} G_X(w, s) \subseteq T \subseteq G_X(w)$. Choose one such $w$ not in $\{u_1, \ldots, u_n\}$; then $w \in H_1(U)$ and

$$\bigcup_{s \in S} G_U(w, s) \subseteq \bigcup_{s \in S} G_X(w, s) \subseteq T \subseteq G_X(w) \cap U = G_U(w).$$

Thus, (P5) is satisfied. 

**Corollary 2.2.** Let $y$ be an indeterminate over $\mathbb{Z}[x]$ and let $f \in \mathbb{Z}[x]$ be a nonzero nonunit. Then

$$\Spec\left(\mathbb{Z}\left[\frac{y}{f y - 1}\right]\right) \cong \Spec\left(\mathbb{Z}\left[x, \frac{1}{f}\right]\right) \cong \Spec(\mathbb{Z}[x]).$$

**Proof.** Clearly $(\mathbb{Z}[x, y])/(f y - 1) \cong \mathbb{Z}[x, \frac{1}{f}]$. Let $u_1, \ldots, u_m$ be the height-one primes of $\mathbb{Z}[x]$ containing $f$. By Proposition 2.1, $\Spec(\mathbb{Z}[x][1/f]) \cong \Spec(\mathbb{Z}[x])$. 

**Note 2.3.** It follows from Corollary 2.2 that if $x_1, \ldots, x_n$ are indeterminates over $\mathbb{Z}[x]$ and $f_1, \ldots, f_n$ are nonzero nonunits of $\mathbb{Z}[x]$, then

$$\Spec\left(\mathbb{Z}\left[x_1, \ldots, x_n\right]/(f_1x_1 - 1, \ldots, f_nx_n - 1)\right) \cong \Spec\left(\mathbb{Z}\left[x, \frac{1}{f_1}, \ldots, \frac{1}{f_n}\right]\right) \cong \Spec(\mathbb{Z}[x]).$$

**Notation 2.4.** Let $R$ be a Noetherian domain and $a, b \in R$. We say that $(a, b)$ is a generalized $R$-sequence if one of the following holds:

1. $(a, b)$ is an $R$-sequence, or
2. $a, b$ generate the unit ideal in $R$.

**Proposition 2.5.** Let $R$ be a Noetherian domain, $a, b$ a generalized $R$-sequence, and $y$ an indeterminate over $R$. Then $(a + by)$ is a prime ideal of $R[y]$.

**Proof.** See [9, p. 102, Exercise 3].

**Corollary 2.6.** Let $R$ be a unique factorization domain, $y$ an indeterminate over $R$, and $a$ and $b$ relatively prime elements of $R$; that is, the greatest common divisor of $a$ and $b$ is a unit. Then $(a + by)$ is a prime ideal of $R[y]$.

**Proof.** Use [9, p. 32], [9, p. 102, Exercise 5 and (2.5)].

**Lemma 2.7.** Let $U$ be a partially ordered set of dimension two satisfying (P3) and (P4) and containing infinitely many height-one elements. Suppose every $U$-pair $(S, T)$
with \(|T| = 1\) has a radical element. Then every \(U\)-pair of form \((S, \emptyset)\) has a radical element. Furthermore, in this case every \(U\)-pair \((S, T)\) with \(|T| \leq 1\) has infinitely many radical elements.

**Proof.** Note that \(H_2(U) \not\subseteq \bigcup_{s \in S} G(s)\). For, if \(H_2(U) \subseteq \bigcup_{s \in S} G(s)\) and \(p \in H_2(U) - S\); then \(G(p) \subseteq \bigcup_{s \in S} G(p, s)\) a finite set, contradicting (P3). Let \(t \in H_2(U) - \bigcup_{s \in S} G(s)\). A radical element \(w\) for the pair \((S, \{t\})\) is also a radical element for \((S, \emptyset)\) by Remark 1.9(2). For the last statement, let \((S, T)\) be a \(U\)-pair with \(|T| \leq 1\) and let \(w\) be a radical element for \((S, T)\). Then \(w \not\in S\) as in Remark 1.9(1). Furthermore, \((S \cup \{w\}, T)\) has a radical element \(w' \neq w\), which is another radical element for \((S, T)\). Continuing this process, we see that \((S, T)\) has infinitely many radical elements.

**Theorem 2.8.** Suppose that \(n > 0\), \(x_1, \ldots, x_n\) are indeterminates over \(\mathbb{Z}\) and \(R = \mathbb{Z}[x_1, \ldots, x_n]\). Then every \((n, n + 1)\)-Spec\((R)\)-pair \((S, T)\) has a height-one radical element.

**Proof.** Note that if \(n = 1\), this follows from Theorem 1.4. For \(n > 1\), let \(S = \{P_1, \ldots, P_r\}\). If \(T = \emptyset\), then choose \(x \in P_1 \cap \cdots \cap P_r\). Then \(1 - x \notin P_1 \cup \cdots \cup P_r\). Let \(Q\) be a prime ideal minimal over \(1 - x\); then \(Q\) is comaximal to \(P_1, \ldots, P_r\) and therefore, \(Q\) is a radical element. Thus, we may suppose that \(|T| \geq 1\). By Remark 1.9(2) we may assume that each element in \(T\) is above at least one element in \(S\). Let \(T = \{m_1, \ldots, m_d\}\), with \(d > 0\). Set \(I = P_1 \cap \cdots \cap P_r\), \(R' = R/I\). Then \(I\) is contained in each element of \(T\), and \(R'\) is a reduced one-dimensional Noetherian ring which is a finitely generated \(\mathbb{Z}\)-algebra. Thus, the Picard group of \(R'\) is finite (torsion) by [13, Finiteness Theorem]. By [7, Ch. VI, Proposition 1.13 and Corollary 1.17] and [13, Finiteness Theorem], \(R'\) has at most finitely many singular maximal ideals; therefore, every maximal ideal of \(R'\) is the radical of a principal ideal [13, Lemma 4]. Say \(m_i = \sqrt{I + (f)}\) where \(f_i \in R\) and set \(f = f_1 \cdots f_d\). Then \(\sqrt{I + (f)} = m_1 \cap \cdots \cap m_d\). Also \(f \notin \bigcup_{i=1}^r P_i\) since \(f \in P_i\) implies \(I + (f) \subseteq P_i\) and \(\sqrt{I + (f)} \subseteq P_i\), a contradiction. For each \(i\) choose \(g_i\) to be an irreducible element of \(P_i\) outside \(\bigcup_{i \neq j} P_j\) and let \(g = g_1 \cdots g_r \in I\). Then \(f\) and \(g\) do not have common irreducible factors and \((f, g) \neq (1)\), because \(f, g \in m_1\). By Proposition 2.5, \((f + gy)\) is a prime ideal of the polynomial ring \(R[y]\). By Hilbert's Irreducibility Theorem [10, Ch. VIII]) there exists a prime integer \(q \notin P_1, \ldots, P_r\) such that \(h = f + qy\) is an irreducible polynomial over \(\mathbb{Q}\), so irreducible over \(\mathbb{Z}\). Write \(w = hR\), a height-one prime of \(R\).

**Claim.** \(w\) is a radical element for \((S, T)\).

First, since \(h \in I + (f)\), we have that \(w \subseteq I + (f) \subseteq \sqrt{I + (f)} = m_1 \cap \cdots \cap m_d\). Thus, \(w < m_i\) for each \(i = 1, \ldots, d\). On the other hand, if \(m\) is a maximal ideal of \(R\) such that \(m > w\) and \(m > P_i\) for some \(i \in \{1, \ldots, r\}\), then \((h, I) \subseteq m \Rightarrow f \in m \Rightarrow m \supseteq I + (f) \Rightarrow \sqrt{I + (f)} = m_1 \cap \cdots \cap m_d\). Thus, \(m \in \{m_1, \ldots, m_d\} = T\) and the claim holds. \(\square\)
Theorem 2.9. Let $R$ be a two-dimensional integral domain which is a finitely generated $\mathbb{Z}$-algebra. Then every $(1,2)$-$\text{Spec}(R)$-pair $(S, T)$ with $|T| \leq 1$ has infinitely many radical elements.

**Proof.** Write $R = A_n/P_0$, where $A_n = \mathbb{Z}[x_1, \ldots, x_n]$ is a polynomial ring over $\mathbb{Z}$ in $n$ indeterminates and $P_0$ is a height-$(n - 1)$ prime ideal of $A_n$. By Theorem 1.4 and Lemma 2.7 we may assume $n > 1$ and $|T| = 1$. □

Let $\pi$ be the projection $A_n \rightarrow R$ and $\psi$ the induced order-preserving isomorphism

$$\text{Spec}(R) \rightarrow \{Q \mid Q \in \text{Spec}(A_n), \ Q \subseteq P_0\}.$$ 

For convenience, let $\psi$ denote the image under $\psi$, i.e., $P_0^\psi = \psi(P)$, for $P \in \text{Spec}(R)$. Suppose $T = \{m\}$ and $S = \{P_1, \ldots, P_r\}$. Set $T' = \{m^\psi\}$, $S' = \{P_1^\psi, \ldots, P_r^\psi\}$ in $\text{Spec}(A_n)$; then $m^\psi, P_1^\psi, \ldots, P_r^\psi$ all contain $P_0$. Also $(S', T')$ is an $(n,n + 1)$-pair in $\text{Spec}(A_n)$. By Theorem 2.8, $(S', T')$ has a radical element $Q'$ of height one. If $Q' \subseteq P_0$, then $G(P_1^\psi, Q') = G(P_r^\psi) \not\subseteq T'$ since $G(P_1^\psi)$ is infinite; this contradicts $Q'$ being a radical element for $(S', T')$. Thus, $Q' \not\subseteq P_0$. Since $Q'$ is a height-one prime ideal of $A_n$, $Q'$ is principal; say $Q' = aA_n$. By applying the principal ideal theorem to $A_n/P_0$, we see that the ideal $P_0 + Q' = (P_0, a)A_n$ has height $n$. Also $P_0 + Q' \subset m^\psi$. Therefore, there exists a height-$n$ prime ideal $W'$ of $\mathbb{Z}[x_1, \ldots, x_n]$, such that $P_0 + Q' \subseteq W' < m^\psi$, and $Q' < W'$ since $n > 1$.

Claim. $W = \pi(W')$ is a radical element for $(S, T)$ in $R$.

First note that it has height one in $R$ and $W < m$ since $W^\psi - W' < m^\psi$. If $n$ is a maximal ideal of $R$, such that $n > W$ and $n > P_i$ for some $i$, then $n^\psi > W' > Q'$, $n^\psi > P_i^\psi$ implies $n^\psi \in T'$ since $Q'$ is a radical element for $(S', T')$. This implies $n^\psi = m^\psi$ and then $n = m \in T$. Thus, $W$ is a radical element for $(S, T)$. By Lemma 2.7, there are infinitely many radical elements for $(S, T)$. □

3. Birational extensions of $\mathbb{Z}[x]$

First, we note a general fact about birational extensions.

**Proposition 3.1.** Suppose that $C$ is an integral domain, $0 \neq c \subset C$ and $D$ is a birational extension such that $C \subseteq D \subseteq C[1/c]$. Let $V_D(c) = \{P \in \text{Spec}(D) \mid c \in P\}$, and $P \in \text{Spec}(D) - V_D(c)$. Then there exists a prime ideal $Q \in \text{Spec}(C) - V_C(c)$ such that $P = QC[1/c] \cap D$. Furthermore, for every $Q \in \text{Spec}(C) - V_C(c)$, $P = QC[1/c] \cap D$ is a prime ideal of $D - V_D(c)$ and $P \cap C = Q$. 


Proof. Consider the following diagram:

\[
\begin{array}{ccc}
\text{Spec}(D) - V_D(c) & \xrightarrow{\psi} & \text{Spec}(D[1/c]) \\
\uparrow & & \uparrow \\
\text{Spec}(C) - V_C(c) & \xrightarrow{\sigma} & \text{Spec}(C[1/c])
\end{array}
\]

The isomorphism $\psi$ from Spec$(D) - V_D(c)$ to Spec$(D[1/c])$ is defined by $\psi(Q) = QC[1/c]$. The inverse of $\psi$ is given by $\psi^{-1}(Q') = Q' \cap D$. Moreover, there is a similar isomorphism $\sigma$ from Spec$(C[1/c])$ to Spec$(C) - V_C(c)$. Now the isomorphism $\varphi : \text{Spec}(C) - V_C(c) \to \text{Spec}(D) - V_D(c)$ is $\psi^{-1} \circ \sigma$; equivalently, $\varphi$ is given by $\varphi(P) = PC[1/c] \cap D$. \(\square\)

Setting and notation 3.2. We consider birational extensions of $A = \mathbb{Z}[x]$ of form $B = \mathbb{Z}[x, g_1/f, \ldots, g_d/f]$, where $f$ and the $g_i$ are polynomials in $\mathbb{Z}[x]$ and $f \neq 0$. Then

\[A = \mathbb{Z}[x] \subseteq B = \mathbb{Z}\left[x, \frac{g_1}{f}, \ldots, \frac{g_d}{f}\right] \subseteq \mathbb{Z}\left[x, \frac{1}{f}\right] = B_f = A_f.
\]

Corollary 3.3. Let $A$, $B$, and $B_f$ be as in Setting 3.2. If $P \in \text{Spec}(B) - V_B(f)$ has height one, then there exists an irreducible element $h \in \mathbb{Z}[x]$ such that $P = hA_f \cap B$. Furthermore, for every irreducible element $h \in \mathbb{Z}[x]$ which is relatively prime to $f$ in $\mathbb{Z}[x]$, $P = hA_f \cap B$ is a prime ideal of $B$.

Proof. This is clear by Proposition 3.1, because every height-one prime ideal of the unique factorization domain $\mathbb{Z}[x]$ is generated by an irreducible element. \(\square\)

The following lemma is a consequence of the Artin–Rees Lemma.

Lemma 3.4. Let $R$ and $C$ be Noetherian integral domains, with $h$ a nonzero element of $C$ and $I$ a nonzero ideal of $C$. Then there exists a positive integer $e$ so that

1. For every $d \in C$ and every $k > e$, $hd \in I^k \Rightarrow d \in I^{k-e}$.
2. For every $d \in C$, $x \in I^{e+1}$ and $k \geq 0$,

\[(h + x)d \in I^k \iff hd \in I^k \iff hd \in I^k \text{ and } xd \in I^{k+1}.
\]

3. Let $f$ be a nonzero element of $R$ and $g_1, \ldots, g_r \in R$. Suppose that $C = R[g_1/f, \ldots, g_r/f]$, a birational extension of $R$, and that $I = fC$. Then, for every $d \in R$, every $x \in f^e + C$ and every $k \geq 0$,

\[(h + x)\frac{d}{f^k} \in C \iff \frac{hd}{f^k} \in C \iff \left(\frac{hd}{f^k} \in C \text{ and } \frac{zd}{f^k} \in fC\right).
\]
Proof. Let $e$ be as in the Artin–Rees Lemma [11, p. 59] so that for all $n > e$,
\[ I^n \cap hC = I^{n-e}(I^e \cap hC). \]
For (1) write $hd = ha_1 d_1 + \cdots + ha_r d_r$, where each $ha_i \in C$, each $a_i \in I^e \cap hC$ and each $d_i \in I^{k-e}$. Then $d = a_1 d_1 + \cdots + a_r d_r \in I^{k-e}$.

For the $\Rightarrow$ direction of the first equivalence of (2), let $(h + x)d \in I^k$. Then $hd \in xdC + I^k$. If $k \leq e$, then $hd \in I^{e+1} + I^k \subseteq I^k$. Suppose that $k > e$, that $hd \notin I^k$ and that $n < k$ is the largest power of $I$ so that $d \in I^n$. Then
\[ hd \in xdC + I^k \subseteq I^{e+1}I^n + I^k. \]
Then $hd \notin I^k \Rightarrow e + 1 + n < k$, and $hd \in I^{e+1} + I^k$. Now $e + n + 1 > e \Rightarrow d \in I^{n+1}$, by part (1), contradicting the maximality of $n$. Therefore, $hd \in I^k$.

For the $\Rightarrow$ direction of the second equivalence of (2), suppose that $hd \in I^k$. If $k \leq e$, then $x \in I^{e+1} \subseteq I^{k+1}$. If $k > e$, then $d \in I^{k-e}$, by (1). Thus, $xd \in I^{k+1}$.

For (3), for every $b \in C$, $k \geq 0$,
\[ \frac{b}{f^k} \in C \iff b \in f^kC = I^k \quad \text{and} \quad \frac{b}{f^k} \in fC \iff b \in f^{k+1}C = I^{k+1}. \]
Thus, using (2),
\[ \frac{(h + x)d}{f^k} \in C \iff (h + x)d \in I^k \iff hd \in I^k \iff \frac{hd}{f^k} \in C. \]
Also,
\[ (h + x)d \in I^k \iff hd \in I^k \text{ and } xd \in I^{k+1} \iff \frac{hd}{f^k} \in C \text{ and } \frac{xd}{f^k} \in fC. \]

Note. If $H$ is a finite set of nonzero elements of a Noetherian integral domain $C$, then there exists a positive integer $e$ that works as in Lemma 3.4 for every $h \in H$.

**Theorem 3.5.** Let $n$ be a positive integer and let $B = \mathbb{Z}[x][((g_1(x))/(f(x))), \ldots, (g_n(x))/(f(x))]$, where the $g_i(x)$ and $f(x)$ are polynomials in $\mathbb{Z}[x]$ and $f \neq 0$. Then $\text{Spec}(B)$ is order-isomorphic to $\text{Spec}(\mathbb{Z}[x])$.

Proof. We need check only Axiom 1.3 (P5) for $\text{Spec}(B)$ by Corollary 1.7. Let $(S, T)$ be a $(1,2)$-$\text{Spec}(B)$-pair; we seek a radical element for $(S, T)$. By Remark 1.9(2) we may assume

(1) Every height-one prime ideal of $B$ containing $f$ is in $S$.

(2) Each element of $T$ contains at least one element of $S$.

If $|T| \leq 1$, we are done by Theorem 2.9, so we assume that $|T| > 1$. Let $A \in T$. We assume by induction that $(S, T - \{A\})$ and $(S, \{A\})$ have radical elements $P_1$ and $P_2$, respectively. Then $P_1 \neq P_2$, because otherwise $A \supseteq P_1$, $A \supseteq Q$, for some $Q \in S$ by (2), and the radical property of $P_1$ for $(S, T - \{A\})$ would imply $A \in T - \{A\}$. Let
\{Q_1, \ldots, Q_m\} = S - V_p(f)\), the height-one prime ideals of \(B\) which are in \(S\) but do not contain \(f\).

Now \(f \notin P_1 \cup P_2\), since \(S\) contains all height-one primes containing \(f\) and neither \(P_1\) nor \(P_2\) is in \(S\) by Remark 1.9(1). Thus by Corollary 3.3, \(P_1 = h_1A_f \cap B\), \(P_2 = h_2A_f \cap B\), and, for \(1 \leq j \leq m\), \(Q_j = b_jA_f \cap B\), where \(h_1, h_2\) and the \(b_j\) are pairwise relatively prime irreducible elements of \(\mathbb{Z}[x]\), all relatively prime to \(f\). Let \(e\) be as in Lemma 3.4 with \(I := fB, B := C\) and \(h := h_1h_2\). Note that \(h = h_1h_2, v = \prod_{j=1}^m b_jf^{2e}\) form a generalized \(\mathbb{Z}[x]\)-sequence because they have no common irreducible factors.

By Corollary 2.6, if \(y\) is another indeterminate over \(\mathbb{Z}[x]\), \((h + vy)\) is a prime ideal of \(\mathbb{Z}[x][y]\). Apply Hilbert’s Irreducibility Theorem [10, Ch. VIII] to get a prime integer \(p\) outside every element of \(S \cup \{P_1, P_2\}\) such that \(h + pv\) is an irreducible polynomial of \(\mathbb{Z}[x]\). Thus, \(W' := (h + pv)\mathbb{Z}[x]\) is a height-one prime ideal of \(\mathbb{Z}[x]\). Note that \(f \notin hA\), but \(f \notin hA\) since \(f\) is not in \((h_1)\) or \((h_2)\). So \(f \notin W'\). By Corollary 3.3 \(W := (h + pv)A_f \cap B\) is a prime ideal of \(B\). We show that \(W\) is a radical element for \((S, T)\).

**Claim 1.** If \(\mathcal{M} \subseteq T\), then \(W \subseteq \mathcal{M}\).

**Proof.** Let \(z \in W\). Then \(z = (h + pv)\frac{d}{f^k} \in B\), for some \(d \in A\) and \(k \geq 0\). By the choice of \(e\), we have that \(hd/f^k \in B\) and \(pvdf/f^k \in fB\) from Lemma 3.4. Thus,

\[
\frac{hd}{f^k} = \frac{h_1h_2d}{f^k} \in (h_1A_f \cap B) \cap (h_2A_f \cap B) = P_1 \cap P_2 \subseteq \mathcal{M},
\]

and

\[
\frac{pvdf}{f^k} = p\prod_{j=1}^m b_jf^{2e}d \in (b_jA_f \cap B) \cap fB = \bigcap Q_j \cap fB \subseteq \mathcal{M},
\]

since \(P_1\) or \(P_2\) is contained in \(\mathcal{M}\), and some \(Q_j\) or \(fB\) is contained in \(\mathcal{M}\). Therefore, \(z \in \mathcal{M}\). \(\square\)

**Claim 2.** For each maximal ideal \(\mathcal{M}\) in \(B\), if \(W \subseteq \mathcal{M}\) and \(Q \subseteq \mathcal{M}\), for some \(Q \in S\), then \(P_1\) or \(P_2 \subseteq \mathcal{M}\).

**Proof.** If \(P_1 \notin \mathcal{M}\), choose \(d_1 \in \mathbb{Z}[x]\) so that \(h_1d_1/f^k \in P_1 - \mathcal{M}\). Suppose \(d_2 \in \mathbb{Z}[x]\) is such that \(h fds/f^m \in P_2\). Then \((h_1d_1/f^k)(h_2d_2/f^m) \in B\), so by Lemma 3.4(3) \((pv)(f^{k+m})d_1d_2 = p\prod b_jf^{2e}(d_1d_2)/(f^{m+k}) \in fB\). Now

\[
z = \frac{h_1d_1}{f^k} \frac{h_2d_2}{f^m} + p\prod b_jf^{2e} \frac{d_1d_2}{f^{m+k}} = (h + pv)\frac{d_1d_2}{f^{m+k}} \in W \subseteq \mathcal{M}.
\]

Also

\[
pv\frac{d_1d_2}{f^{m+k}} = p\prod b_jf^{2e} \frac{d_1d_2}{f^{m+k}} \in fB \cap \bigcap Q_j \subseteq \mathcal{M},
\]
since \( Q \subseteq \mathcal{H} \) for some \( Q \in S \). Thus, \((h_1h_2d_1d_2)/(f^{k+m}) \in \mathcal{H} \). But \( h_1d_1/f^k \in B - \mathcal{H} \). Thus \( h_2d_2/f^m \in \mathcal{H} \). We conclude that \( P_2 \subseteq \mathcal{H} \), as desired. \(\square\)

**Completion of proof.** By Claim 1, \( T \subseteq G(W) \). By Claim 2, whenever \( W \subseteq \mathcal{H} \) and \( Q \subseteq \mathcal{H} \), for some \( Q \in S \), we have that \( P_1 \) or \( P_2 \) is contained in \( \mathcal{H} \). Since \( P_1 \) and \( P_2 \) are radical elements, this implies that \( \mathcal{H} \subseteq T \). Thus, \( G(W) \cap G(Q) \subseteq T \). Now Axiom 1.3 (P5) holds, so \( \text{Spec}(B) \cong \text{Spec}(\mathbb{Z}[x]) \). \(\square\)

As an application of Theorem 3.5 we now extend some results of [5] concerning the spectra of certain two-dimensional birational extensions of polynomial rings over semilocal one-dimensional domains. The crucial Axiom (P6) there is related to the Axiom (P5) of (1.3) above:

**Definition 3.6** (Axiom P6). A two-dimensional partially ordered set \( U \) satisfies Axiom (P6) provided that for every nonempty finite subset \( T \) of \( H_2(U) \), the “exactly less than” set \( L_e(T) \) is infinite, where

\[
L_e(T) := \{ u \mid u \in H_1(U), G(u) = T \}.
\]

If Axiom (P6) holds for the spectrum of a birational extension \( C \) of the type studied in [5], then the structure of \( j\)-Spec\((C) \) is sufficient to completely determine Spec\((C) \). (Here \( j\)-Spec\((C) = \{ P \mid P \in \text{Spec}(C), P = \bigcap \text{maximal ideals} \} \).) The \( j \)-spectra for \( C \) (depending on \( f \) and \( g \)) is determined in [5], so the major missing piece is to see when Axiom (P6) holds.

Theorem 4.5 in [5] states that certain birational extensions satisfy Axiom (P6): 

**Theorem 3.7** (Heinzer [5, Theorem 4.5]). Let \( R \) be a semilocal domain of dimension one with maximal ideals \( m_1, \ldots, m_r \). Assume \((f, g)\) is an \( R[x] \)-sequence and \( f(x) \in R[x] - (\bigcup_{i=1}^r m_i R[x]) \). Suppose there exists a one-dimensional Noetherian domain \( D \subseteq R \) such that \( R \) is a localization of \( D \) and \( \text{Spec}(D[x, g/f]) \) satisfies Axiom (P5) of (1.3). Then \( \text{Spec}(R[x, g/f]) \) satisfies Axiom (P6), and is uniquely determined by \( j\)-Spec\((R[x, g/f]) \).

**Corollary 3.8.** Let \( R \) be a localization of \( \mathbb{Z} \) with only finitely many maximal ideals, \( m_1, \ldots, m_r \). Let \( f, g \) be a \( \mathbb{Z}[x] \)-sequence such that \( f \notin \bigcup_{i=1}^r m_i R[x] \). Then \( \text{Spec}(R[x, g/f]) \) satisfies Axiom (P6), and therefore, is uniquely determined by \( j\)-Spec\((R[x, g/f]) \).

**Proof.** By Theorem 3.5, \( \text{Spec}(\mathbb{Z}[x, g/f]) \) satisfies (P5). Now apply Theorem 3.7.

Note that in [5], the authors do not include rings satisfying the hypotheses of Corollary 3.8 among those for which Axiom (P6) hold. Therefore, this result is a slight extension of their work.
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References