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# On Dominant Modules and Dominant Rings

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In [11] Kato introduced a notion of dominant modules: Let $A$ be a ring, $W_A$ a faithful, finitely generated projective module, and $B = \mathrm{End}(W_A)$ the endomorphism ring of $W_A$. Then he called $W_A$ dominant if ${}_BW$ is lower distinguished, i.e. contains a copy of each simple right $B$-module, and further obtained a categorical characterization on a dominant module [11] and a structure theorem for a ring having a dominant module [12]. Rutter [26] also obtained another characterization on a dominant module.

In this paper we shall cast its finite generation out of the definition of dominant modules; that is, a faithful projective module $W_A$ with $B = \mathrm{End}(W_A)$ is called a dominant module provided every simple factor module of ${}_BW$ is embedded into $S({}_BW)$, the socle of ${}_BW$ (and at the same time, in fact, each simple component of $S({}_BW)$ is isomorphic to a simple factor module of ${}_BW$). Our definition coincides with the original for the case where $W_A$ is finitely generated, because then ${}_BW$ is a generator and so is upper distinguished, i.e. every simple right $B$-module is isomorphic to a simple factor module of ${}_BW$. A ring $A$ will be called right (resp. left) dominant if there exists a dominant right (resp. left) $A$-module. In particular, in case $A$ has a finitely generated, dominant module, $A$ will be called a right (resp. left) dominant ring of finite type.[1]

The requirement to extend the definition of dominant modules has been motivated by the next:

Theorem 5.7. Let $A$ be a ring. Then $A$ is an endomorphism ring of a generator-cogenerator, say ${}_BW$, if and only if $A$ satisfies the next three conditions:

(i) $A = \underset{\sim}{Q}_l$, the maximal left quotient ring of $A$ itself.

(ii) $A$ is a right dominant ring of finite type.

(iii) A is a left $\aleph$-$\underset{\sim}{Q}F\,3$ ring. (Its definition will be stated later and of course $A$ is a left dominant ring.)

Moreover, $B$ has only finite many isomorphism classes of simple left $B$-modules if and only if $A$ becomes a left $\underset{\sim}{Q}F\,3$ ring in (iii) above mentioned.

[1] A right (resp. left) dominant ring differs from a dominant ring defined in [28, p. 226].



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This theorem is a generalization of a result of Ringel and Tachikawa [24] concerning the endomorphism ring of a linearly compact generator-cogenerator.

The purpose of this paper is to investigate not only dominant modules but also dominant rings in our sense. To do so, as a preliminary Section 1 is devoted to establish Theorem 1.4 concerning locally projective modules, which will play an important role on characterizing dominant modules, and which will be interesting by itself.

In Section 2, main results obtained by Kato [11, 12] and Rutter [26] will be extended to our case. In particular we shall establish two criteria on dominant modules: The first (Theorem 2.2) contains an extension of the characterization due to Rutter, which asserts that a projective module $W_A$ is dominant if and only if $\mathrm{Tr}(W_A)$, the trace ideal of $W_A$, is the smallest dense left ideal of $A$. The second (Theorem 2.4) contains an extension of the categorical characterization due to Kato, which asserts that a projective module $W_A$ with $B = \mathrm{End}(W_A)$ is dominant if and only if the functors $\mathrm{Hom}_B({}_BW_A, -)$ and ${}_BW_A \otimes -$ induce an equivalence $\mathscr{G}({}_BW) \sim \mathscr{D}(E({}_AA))$, where $E({}_AA)$ denotes the injective hull of ${}_AA$ and $\mathscr{G}({}_BW)$ (resp. $\mathscr{D}(E({}_AA))$) denotes the full subcategory consisting of all left $B$-modules generated by ${}_BW$ (resp. of all left $A$-modules with $E({}_AA)$-dominant dimension $\geqslant 2$).

In Section 3, we shall state an intrinsic characterization of right dominant rings (Theorem 3.1) and show that the property of rings to be right dominant is Morita-invariant (Proposition 3.3).

In Section 4, we shall treat a special but a useful right (resp. left) dominant ring: A ring $A$ will be called right pseudo-perfect provided there are pairwise non-isomorphic, local idempotents $\{e_\lambda \mid \lambda \in \Lambda\}$ of $A$ such that $[\sum_\lambda \oplus e_\lambda A]_A$ is dominant. Then it is remarkable that, in the above, $\sum_\lambda \oplus e_\lambda A$ is dominant if and only if $\sum_\lambda \oplus e_\lambda A$ is faithful and the distinct simple components of $S({}_AA)$ coincide with $\{Ae_\lambda/Je_\lambda \mid \lambda \in \Lambda\}$ up to multiplicity where $J = J(A)$, the Jacobson radical of $A$ (Theorem 4.1), and that $\sum_\lambda \oplus e_\lambda A$ is minimal dominant, i.e. is isomorphic to a direct summand of any dominant module and so is uniquely determined up to isomorphism (Proposition 4.3). A property of rings to be right pseudo-perfect is Morita-invariant (Proposition 4.5).

The class of right pseudo-perfect rings contains semi-perfect rings with essential left socle (and so right perfect rings) as well as right $QF$ 3 rings, and another example will be given by the endomorphism rings of upper distinguished cogenerators (Theorem 4.7).

In Section 5, we shall treat a more special right (resp. left) pseudo-perfect ring: A ring $A$ will be called right $\aleph$-$QF$ 3 if there exist pairwise orthogonal and pairwise non-isomorphic, local idempotents $\{e_\lambda \mid \lambda \in \Lambda\}$ of $A$ such that each $e_\lambda A$ $(\lambda \in \Lambda)$ is an injective module with a simple socle, and that $[\sum_\lambda e_\lambda A]_A$ is faithful. Similarly left $\aleph$-$QF$ 3 rings are defined. In case the cardinal of $\Lambda$ is finite, this is nothing else a right (resp. left) $QF$ 3 ring.

Then, in the above, $\sum_\lambda e_\lambda A$ is minimal dominant and the simple components of

$S({}_AA)$ as well as $S(A_A)$ are completely determined up to multiplicity (Theorem 5.1), and an analogue of Colby and Rutter [5] concerning right $QF$ 3 rings holds good (Proposition 5.4). Of course a property of rings to be right $\aleph$-$QF$ 3 is Morita-invariant. As an important example of right (resp. left) $\aleph$-$QF$ 3 rings we have the endomorphism ring of a generator-cogenerator (Theorem 5.7) as was stated before.

Throughout this paper rings and modules will be assumed to be unitary, and for a right $A$-module $M$ we shall denote by $E(M_A)$, $J(M_A)$, $S(M_A)$, $\mathrm{Tr}(M_A)$, End $(M_A)$ and by Biend $(M_A)$ respectively the injective hull, the Jacobson radical, the socle, the trace ideal, the endomorphism ring and the biendomorphism ring (i.e. the double centralizer) of $M_A$. For subsets $X \subset A$ and $Y \subset M_A$ we shall denote by $r_X(Y)$ and $l_Y(X)$ respectively the right annihilator of $Y$ in $X$ and the left annihilator of $X$ in $Y$, i.e.

$$r_X(Y) = \{x \in X \mid Yx = 0\} \text{ and } l_Y(X) = \{y \in Y \mid yX = 0\}.$$

Similarly these notations will be used for a left $A$-module $M$. ${}_A\mathcal{M}$ (resp. $\mathcal{M}_A$) will always denote the category of all left (resp. right) $A$-modules.

## 1. Preliminary (locally projective modules)

Throughout this section, let $W_A$ be locally projective and $(A, B, W^*, W, (\ ,\ ), [\ ,\ ])$ the Morita context derived from $W_A$; i.e. $B = \mathrm{End}(W_A)$, ${}_AW^*{}_B = \mathrm{Hom}_A(W_A, A_A)$ and

$$(\ ,\ ): {}_AW^* \otimes_B W_A \to {}_AA_A, \quad \text{via} \quad f \otimes v \mapsto (f, v) = f(v)$$
$$[\ ,\ ]: {}_BW \otimes_A W^*{}_B \to {}_BB_B, \quad \text{via} \quad u \otimes f \mapsto [u, f]$$

defined by $[u, f]v = u(f, v)$, and so $g[u, f] = (g, u)f$ holds where $u, v \in W$ and $f, g \in W^*$.

For given subsets $U \subset W$ and $H \subset W^*$ let us denote respectively

$$(H, U) = \{\textstyle\sum (f_i, u_i) \text{ (finite sum)} \mid f_i \in H, u_i \in U \text{ for every } i\}$$

and

$$[U, H] = \{\textstyle\sum [u_i, f_i] \text{ (finite sum)} \mid u_i \in U, f_i \in H \text{ for every } i\}.$$

In particular set $T = (W^*, W)$ and $R = [W, W^*]$.

Following Zimmermann–Huisgen [35] we say that $W_A$ is locally projective (= universally torsionless [9]) provided, for each element $u \in W$, there exist $u_1, \dots, u_n \in W$ and $f_1, \dots, f_n \in W^*$ such that

$$u = \sum_{i=1}^{n} [u_i, f_i]u = \sum_{i=1}^{n} u_i(f_i, u).$$

Then in view of [35, Theorem 2.1] it is seen that $\mathrm{Tr}(W_A) = T = T^2$, $\mathrm{Tr}({}_BW) = R = R^2$, $W = WT$, $u \in Ru$ for each $u \in W$, $r \in Rr$ for each $r \in R$ (i.e. $[B/R]_B$ is flat), $TW^* = W^*R$ and that $W_A$ is flat. Furthermore the next will be well known.

LEMMA 1.1. *Under the above situation, denote respectively*

$$\mathscr{L}({}_BW) = \{{}_BU \mid {}_BU \subset {}_BW\}, \mathscr{L}({}_AA) = \{{}_AX \mid {}_AX \subset {}_AA,\ TX = X\},$$
$$\mathscr{L}({}_AW^*) = \{{}_AH \mid {}_AH \subset {}_AW^*,\ TH = H\} \text{ and } \mathscr{L}({}_BB) = \{{}_BY \mid {}_BY \subset {}_BR\}.$$

*Then the assertions below hold:*

(i) *There exists the following order-preserving bijection between* $\mathscr{L}({}_BW)$ *and* $\mathscr{L}({}_AA)$, *via*

$$U \mapsto (W^*, U), \qquad U \in \mathscr{L}({}_BW)$$
$$WX \leftarrow X, \qquad X \in \mathscr{L}({}_AA).$$ [35, *Theorem* 3.1]

(ii) *There exists the following order-preserving bijection between* $\mathscr{L}({}_AW^*)$ *and* $\mathscr{L}({}_BB)$, *via*

$$H \mapsto [W, H], \qquad H \in \mathscr{L}({}_AW^*)$$
$$W^*Y \leftarrow Y, \qquad Y \in \mathscr{L}({}_BB).$$

Proof is straightforward.

LEMMA 1.2. *Let $Y$ be any maximal left ideal of $B$ such that $R \not\subset Y$, and set $\mathscr{F}(Y) = \{{}_AK \mid TW^*Y \subset K \subsetneqq TW^*\}$. Then there exists a unique maximal member (denoted by $K(Y)$) in $\mathscr{F}(Y)$.*

*Proof.* For brevity set $N = B/Y$. Then by hypothesis $RN \neq 0$ and so $RN = N$ since ${}_BN$ is simple. As $[B/R]_B$ is flat, by [4] the next is pure exact:

$$0 \to R_B \to B_B \to [B/R]_B \to 0,$$

whence we have an exact sequence

$$0 \to R \otimes_B N \to B \otimes_B N \to [B/R] \otimes_B N \to 0,$$

and consequently $N \simeq R \otimes_B N \simeq R/RY$ since $[B/R] \otimes_B N = 0$. This shows $RY$ is a maximal (proper) member in $\mathscr{L}({}_BB)$. Noting $TW^*Y = W^*RY$, by Lemma 1.1 $TW^*Y$ is also a maximal (proper) member in $\mathscr{L}({}_AW^*)$.

Let now $K$ be any member in $\mathscr{F}(Y)$. Then, since $TW^*Y \subset TK \subset K \subsetneqq TW^*$ and $TK \in \mathscr{L}({}_AW^*)$, from the maximality of $TW^*Y$ it follows that $TW^*Y = TK$ and hence

$$[W, K] = [WT, K] = [W, TK] = [W, TW^*Y] = RY.$$

Therefore setting

$$K(Y) = \sum K \text{ where } K \text{ ranges over } \mathscr{F}(Y),$$

we have $[W, K(Y)] = \sum [W, K] = \sum RY = RY\,(\neq R)$, which implies that $K(Y)$ is the unique maximal member in $\mathscr{F}(Y)$. Thus the proof is completed.

Following Kato [13], for a given left ideal $I$ of $B$, left $B$-module $N$ is called $I$-flat if the functor $- \otimes_B N$ is exact on all short exact sequences (in $\mathscr{M}_B$) $0 \to Y' \to Y \to Y'' \to 0$ with $Y''I = 0$. Taking $R = I$ the next is obtained and the statement (iii) below is a portion of Kato and Ohtake [14].

LEMMA 1.3. *Under the same situation as Lemma* 1.1, *let $N$ be a left $B$-module with $RN = N$. Then,*

(i) *${}_BN$ is $R$-flat.*

(ii) *${}_ATW^* \otimes {}_BN \cong {}_AW^* \otimes {}_BN$ canonically.*

(iii) *${}_BW \otimes_A W^* \otimes_B N \cong {}_BN$ canonically.* [14]

*Proof.* To verify (i), let the sequence below:

$$0 \to Y' \xrightarrow{j} Y \to Y'' \to 0 \qquad \text{with} \quad Y''R = 0$$

be exact as right $B$-modules. Then we have the exact sequence

$$0 \to \operatorname{Ker}(j \otimes R) \to Y' \otimes {}_BR \to Y \otimes {}_BR \to Y'' \otimes {}_BR \to 0,$$

where obviously $Y'' \otimes {}_BR = 0$ and a routine calculation shows $\operatorname{Ker}(j \otimes R)R = 0$. Therefore we get further the exact sequence

$$0 = \operatorname{Ker}(j \otimes R) \otimes {}_BN \to Y' \otimes {}_BR \otimes {}_BN \to Y \otimes {}_BR \otimes {}_BN \to 0.$$

Since ${}_BR \otimes_B N \cong {}_BN$ in the same way as the former half of the proof of Lemma 1.2, we have $Y' \otimes {}_BN \cong Y \otimes {}_BN$ canonically, which means ${}_BN$ is $R$-flat since $Y'' \otimes {}_BN = 0$.

(ii) is a direct consequence of (i). For self-containedness we shall give the proof of (iii): Consider the canonical exact sequence

$$0 \to K \to W \otimes {}_AW^* \to R \to 0$$

where $K = \operatorname{Ker}[\ ,\ ]$ and so a routine calculation shows $KR = 0$. Then we get the exact sequence

$$0 = K \otimes {}_BN \to W \otimes {}_AW^* \otimes {}_BN \to R \otimes {}_BN \to 0,$$

whence we have ${}_BW \otimes {}_AW^* \otimes {}_BN \cong {}_BN$ canonically since ${}_BR \otimes {}_BN \cong {}_BN$ as was shown above. Thus the proof is completed.

Now we are in a position to prove the next:

THEOREM 1.4. *Let $W_A$ be a locally projective module with $B = \mathrm{End}(W_A)$, and set $T = \mathrm{Tr}(W_A)$, $R = \mathrm{Tr}({}_BW)$ and $W^* = \mathrm{Hom}_A(W_A, A_A)$. Denote by $\mathscr{S}({}_AA)$ and $\mathscr{S}({}_BB)$ respectively a family consisting of all isomorphism classes of simple left $A$-modules $M$ with $TM = M$, and of simple left $B$-modules $N$ with $RN = N$. Then there exists a bijection between $\mathscr{S}({}_AA)$ and $\mathscr{S}({}_BB)$, via*

$$\begin{aligned} {}_AM &\mapsto {}_BW \otimes {}_AM, & M \in \mathscr{S}({}_AA) \\ {}_AW^* \otimes {}_BN/J({}_AW^* \otimes {}_BN) &\leftarrow\!\dashv {}_BN, & N \in \mathscr{S}({}_BB). \end{aligned}$$

*Moreover, $\mathscr{S}({}_AA)$ and $\mathscr{S}({}_BB)$ respectively coincides with a family consisting of all isomorphism classes of simple factor modules of ${}_ATW^*$, and of simple factor modules of ${}_BW$.*

*Proof.*[2] At first assume ${}_BN$ is a simple module with $RN = N$. Then there exists a maximal left ideal $Y$ of $B$ such that $N \cong B/Y$ with $R \not\subset Y$, and hence by Lemma 1.3

$$W^* \otimes {}_BN \cong TW^* \otimes {}_BN \cong TW^*/TW^*Y.$$

Accordingly, in view of Lemma 1.2 ${}_AW^* \otimes {}_BN$ has a unique maximal (proper) submodule; that is, ${}_AW^* \otimes {}_BN/J({}_AW^* \otimes {}_BN) \cong TW^*/K(Y)$ is simple, and obviously $T[W^* \otimes {}_BN/J({}_AW^* \otimes {}_BN)] = W^* \otimes_B N/J({}_AW^* \otimes {}_B N)$, where $K(Y)$ means the same as in Lemma 1.2.

Furthermore from the exact sequence

$$0 \to K(Y)/TW^*Y \to TW^*/TW^*Y \to TW^*/K(Y) \to 0,$$

we get the next exact sequence

$$W \otimes_A [K(Y)/TW^*Y] \to W \otimes_A [TW^*/TW^*Y] \to W \otimes_A [TW^*/K(Y)] \to 0,$$

where $W \otimes_A [K(Y)/TW^*Y] = 0$ since $TK(Y) = TW^*Y$ by the proof of Lemma 1.2. Therefore, together with Lemma 1.3,

$${}_BN \cong {}_BW \otimes {}_AW^* \otimes {}_BN \cong {}_BW \otimes_A [{}_AW^* \otimes {}_BN/J({}_AW^* \otimes {}_BN)].$$

Next assume ${}_AM$ is a simple module with $TM = M$. Then there exists a maximal left ideal $X$ of $A$ such that $M \cong A/X$ with $T \not\subset X$, and hence $W \otimes {}_AM \cong W/WX$. Since $T \neq TX$ we have $W \neq WX$ by Lemma 1.1, i.e. $W \otimes {}_AM \neq 0$. Let further ${}_BL$ be a proper submodule of ${}_BW$ containing $WX$. Then, since $L = W(L : W)$ by Lemma 1.1 where $(L : W) = \{a \in A \mid Wa \subset L\}$, we have $(L : W) = X$ by the maximality of $X$, i.e. $L = WX$, which implies that ${}_BW \otimes {}_AX$ is simple. Evidently $R(W \otimes {}_AM) = W \otimes {}_AM$.

[2] By virtue of Lemma 1.3 our original proof was simplified.

Moreover, as was proved above, ${}_AW^* \otimes_B (W \otimes {}_AM)$ has a unique maximal (proper) submodule. Hence considering a composite of the canonical epimorphisms:

$$ {}_AW^* \otimes {}_BW \otimes {}_AM \to {}_AT \otimes {}_AM \to {}_AM, $$

we have ${}_AW^* \otimes {}_BW \otimes {}_AM/J({}_AW^* \otimes {}_BW \otimes {}_AM) \simeq {}_AM$. Thus we have shown $\mathscr{S}({}_AA) \approx \mathscr{S}({}_BB)$.

Finally, the last statement of the theorem easily follows from the fact that $T(TW^*) = TW^*$ and $RW = W$, and from the bijection just proved. Thus the proof is completed.

*Remark.* In case $W_A$ is finitely generated projective, $R = B$ and $W^* = TW^*$. Therefore Theorem 1.4 is an improvement of Rutter [25, Lemma 2.4].

COROLLARY 1.5. *Let $W_A$ be a locally projective module with $B = \mathrm{End}(W_A)$. Then $J({}_BW) \neq {}_BW$, and every simple factor module of a submodule of ${}_BW$ is isomorphic to a simple factor module of ${}_BW$.*

*Proof.* Let $L_i$ $(i = 1, 2)$ be submodules of ${}_BW$ such that $L_2/L_1$ is simple. (Conversely there always exist such submodules $L_1$ and $L_2$). Since $R(L_2/L_1) = L_2/L_1$ this is a direct consequence of Theorem 1.4.

*Remark.* The fact: $J({}_BW) \neq {}_BW$ was first obtained in [35, Corollary 3.3]. This guarantees $\mathscr{S}({}_AA) \neq \varnothing$ and $\mathscr{S}({}_BB) \neq \varnothing$ in Theorem 1.4.

## 2. DOMINANT MODULES

As was stated in the introduction, we shall extend the definition of dominant modules as follows: A faithful projective module $W_A$ with $B = \mathrm{End}(W_A)$ is called a dominant module provided every simple factor module of ${}_BW$ is embedded into $S({}_BW)$. In case $W_A$ is finitely generated this coincides with the original. In this section we shall establish two characterization of dominant modules (Theorems 2.2 and 2.4) and prove several properties of dominant modules, which contain the extensions of the results of Kato [11, 12, 13] and Rutter [26].

To begin with, recall that a left ideal $X$ of $A$ is dense in $A$ whenever ${}_AA$ is a rational extension of ${}_AX$ (Cf. [6]). The next provides the criteria on non-denseness of maximal left ideals of $A$.

LEMMA 2.1. *Let $X$ be a maximal left ideal of $A$. Then the following statements are equivalent.*

(a) *${}_AX$ is not dense in ${}_AA$.*

(b) *$r_A(X) \neq 0$, i.e. $l_A(r_A(X)) = X$.*

(c) *$A/X \subsetneq S({}_AA)$.*

Proof is trivial, noting the canonical isomorphism: $\mathrm{Hom}_A(A/X, A) \simeq r_A(X)$. Now we shall state the first criteria on dominant modules.

THEOREM 2.2. *Let $W_A$ be a projective module with $B = \mathrm{End}(W_A)$ and set respectively $T = \mathrm{Tr}(W_A)$ and $R = \mathrm{Tr}({}_BW)$. Then the following statements are equivalent.*

(a) *$W_A$ is dominant.*

(b) *$W_A$ is faithful, and $RN = N \Rightarrow N \subsetneq S({}_BW)$ for simple left $B$-modules $N$.*

(c) *$W_A$ is faithful, and $TM = M \Rightarrow M \subsetneq S({}_AA)$ for simple left $A$-modules $M$.*

(d) *$W_A$ is faithful, and $r_A(X) = 0 \Rightarrow W = WX$ for maximal left ideals $X$ of $A$.*

(e) *$T$ is the smallest dense left ideal of $A$.*

*Moreover, in the above statements* (b), (c) *and* (d), "$\Rightarrow$" *may be replaced by* "$\Leftrightarrow$".

*Proof.* First of all, since $W_A$ is projective $W = WT$ and $T = (W^*, W)$ and so it should be noted that

(*) $W_A$ is faithful $\Leftrightarrow$ $T_A$ is faithful $\Leftrightarrow$ ${}_AT$ is dense in ${}_AA$.

(a) $\Rightarrow$ (e): Assume $W_A$ is dominant. Then ${}_AT$ is dense in ${}_AA$ by (*). Hence it may suffice to show that $T$ is a minimal dense left ideal of $A$. To show this assume that there is a dense left ideal $D$ of $A$ such that $D \subsetneqq T$. Then $W \neq WD$ by Lemma 1.1 and so we can easily find $B$-submodules $L_1$ and $L_2$ such that $WD \subset L_1 \subset L_2$ and that $L_2/L_1$ is simple. Since $W_A$ is dominant, by Corollary 1.5 we have

$$\mathrm{Hom}_B(L_2/L_1, W) \neq 0.$$

On the other hand, setting $D_i = \{a \in A \mid Wa \subset L_i\}$ for $i = 1, 2$, obviously $D \subset D_1 \subset D_2$ and $L_i = WD_i$ by Lemma 1.1. Noting that $W_A$ is flat and so $L_2/L_1 \simeq W \otimes_A (D_2/D_1)$, we have

$$\mathrm{Hom}_B(L_2/L_1, W) \simeq \mathrm{Hom}_A(D_2/D_1, Q)$$

where $Q = \mathrm{Biend}(W_A)$. By [7, Proposition 4.6] however ${}_AQ$ is a rational extension of ${}_AA$ and so of ${}_AD$, and consequently $\mathrm{Hom}_A(D_2/D_1, Q) = 0$, a contradiction. Accordingly $T$ must be a minimal dense left ideal of $A$.

(e) $\Rightarrow$ (d): Trivial by virtue of Lemma 2.1.

(d) $\Rightarrow$ (c): Assume (d). For any simple left $A$-module $M$ with $TM = M$, there is a maximal left ideal $X$ of $A$ such that $M \simeq A/X$ with $T \not\subset X$, and hence $W \neq WX$ by Lemma 1.1. Therefore $X$ must be a non-dense left ideal of $A$ by hypothesis, and so $M \subsetneq S({}_AA)$ by Lemma 2.1.

(c) $\Rightarrow$ (b): Assume (c). Let $N$ be a simple left $B$-module with $RN = N$. Then by Theorem 1.4 there exists a simple left $A$-module $M$ such that $TM = M$

and ${}_BN \cong {}_BW \otimes {}_AM$. By hypothesis, however, ${}_AM \subsetneq {}_AA$ and so ${}_BW \otimes {}_AM \subsetneq {}_BW \otimes {}_AA$, i.e. ${}_BN \subsetneq {}_BW$ since $W_A$ is flat.

(b) ⇒ (a): Trivial by virtue of Theorem 1.4. Thus we have shown the equivalence of (a)–(e). As for the last statement of the theorem, we have only to show that the inverse implication "⇐" is necessarily valid. In fact, the inverse implications at (b), (c) and (d) easily follow respectively from Corollary 1.5 together with Theorem 1.4, from Lemma 2.1 together with (*) above mentioned, and from Lemmas 1.1 and 2.1 together with (*). Thus the proof of the theorem is completed.

*Remark.* In case $W_A$ is finitely generated projective, Rutter obtained (a) ⇔ (e) [26, Theorem 1.4].

Following Kato [13], for a right ideal $I$ of $A$, we shall call a left $A$-module $M$ $I$-injective if the functor $\mathrm{Hom}_A(-, {}_AM)$ is exact on all short exact sequences (in ${}_A\mathscr{M}$) $0 \to X' \to X \to X'' \to 0$ with $IX'' = 0$. Evidently ${}_AM$ is $I$-injective if and only if $\mathrm{Ext}_A{}^1(X, M) = 0$ for every left $A$-module $X$ with $IX = 0$.

Turn our attention to a hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ in ${}_A\mathscr{M}$. Let $\mathbb{F}$ be its associated (left) Gabriel topology (Cf. Stenström [27, 28]). Then recall that a left $A$-module $M$ is $\mathbb{F}$-closed (resp. $\mathbb{F}$-injective) if the canonical homomorphism:

$$M \to \mathrm{Hom}_A(X, M) \text{ via } m \mapsto [x \mapsto xm] \qquad (m \in M, x \in X)$$

is an isomorphism (resp. an epimorphism) for every $X \in \mathbb{F}$. It is well known that (Cf. [8], [17], [27])

$${}_AM \text{ is } \mathbb{F}\text{-injective} \Leftrightarrow \mathrm{Ext}_A{}^1(X, M) = 0 \text{ for every } X \in \mathscr{T}$$

and that

$${}_AM \text{ is } \mathbb{F}\text{-closed} \Leftrightarrow {}_AM \text{ is } \mathbb{F}\text{-injective and } M \in \mathscr{F}.$$

Then we have the next:

LEMMA 2.3. *Let $I$ be an idempotent two-sided ideal of $A$, and set respectively*

$$\mathscr{T} = \{X \in {}_A\mathscr{M} \mid IX = 0\}, \qquad \mathscr{F} = \{X \in {}_A\mathscr{M} \mid r_X(I) = 0\}$$

*and $\mathbb{F} = \{{}_AX \subset A \mid I \subset X\}$. Then $(\mathscr{T}, \mathscr{F})$ is a hereditary torsion theory with $\mathbb{F}$ as its associated Gabriel topology, and further a left $A$-module $M$ is $\mathbb{F}$-closed if and only if ${}_AM$ is $I$-injective and $r_M(I) = 0$.*

Proof is trivial.

Following Tachikawa [30], for an injective module ${}_AE$, a left $A$-module $M$ is said to be $E$-dom. dim $M \geqslant n$ if there exists an exact sequence

$$0 \to M \to E_1 \to \cdots \to E_n$$

where each $E_i$ $(i = 1,\ldots, n)$ is a direct product of copies of $E$.

Now we can state the second criteria on dominant modules.

THEOREM 2.4. *Let $W_A$ be a projective module with $B = \mathrm{End}(W_A)$, and set $T = \mathrm{Tr}(W_A)$ and $R = \mathrm{Tr}({}_BW)$ rrspectively. Then the following statements are equivalent.*

(a) *$W_A$ is dominant.*

(b) *$\mathscr{D}(E({}_AA)) = {}_T\mathscr{L}$, where $\mathscr{D}(E({}_AA))$ and ${}_T\mathscr{L}$ denote respectively the full subcategory of ${}_A\mathscr{M}$ such as:*

$$\mathscr{D}(E({}_AA)) = \{{}_AM \mid E({}_AA)\text{-dom. dim } M \geqslant 2\}$$

*and*

$${}_T\mathscr{L} = \{{}_AM \mid {}_AM \textit{ is T-injective and } r_M(T) = 0\}.$$

(c) *The functors $\mathrm{Hom}_B({}_BW_A, -)$: ${}_B\mathscr{M} \to {}_A\mathscr{M}$ and ${}_BW_A \otimes -$: ${}_A\mathscr{M} \to {}_B\mathscr{M}$ induce an equivalence*

$$\mathscr{G}({}_BW) \sim \mathscr{D}(E({}_AA)),$$

*where $\mathscr{G}({}_BW)$ denotes the full subcategory of ${}_B\mathscr{M}$ generated by ${}_BW$, i.e. $\mathscr{G}({}_BW) = \{{}_BN \mid RN = N\}$.*

*Proof.* At first we shall consider two familiar (left) Gabriel topologies on $A$:

$$\mathbb{D} = \{X \mid X \text{ is a dense left ideal of } A\}$$

and

$$\mathbb{P} = \{X \mid X \text{ is a left ideal of } A \text{ containing } T\}.$$

Denote by $\mathscr{G}_{\mathbb{D}}$ (resp. $\mathscr{G}_{\mathbb{P}}$) the Giraud subcategory of ${}_A\mathscr{M}$ consisting of all $\mathbb{D}$-closed (resp. $\mathbb{P}$-closed) left $A$-modules. Then by [17] or [21] (Cf. [27, p. 44]) we have $\mathscr{G}_{\mathbb{D}} = \mathscr{D}(E({}_AA))$ and by Lemma 2.3 $\mathscr{G}_{\mathbb{P}} = {}_T\mathscr{L}$.

Since there is the bijection between (left) Gabriel topologies on $A$ and equivalence classes of Giraud subcategories of ${}_A\mathscr{M}$ (Cf. [8], [27]), by Theorem 2.2 we have readily (a) $\Leftrightarrow \mathbb{D} = \mathbb{P} \Leftrightarrow$ (b).

Next, since $W_A$ is projective, $RW = W$ and $[B/R]_B$ is flat, and so by Kato [13, Theorem 6.2] ($\mathrm{Hom}_B({}_BW_A, -)$, ${}_BW_A \otimes -$) induces an equivalence $\mathscr{G}({}_BW) \sim {}_T\mathscr{L}$. Hence we have (b) $\Leftrightarrow$ (c). Thus the proof is completed.

*Remark.* In case $W_A$ is (faithful) finitely generated projective, $\mathscr{G}({}_BW) = {}_B\mathscr{M}$ and Kato obtained (a) $\Leftrightarrow$ (c) [11, Theorem 1] and observed (a) $\Leftrightarrow$ (b) (Cf. [13, Corollary 7.3]).[3]

[3] Recently I have received a preprint [22] from K. Nishida. He also has obtained $(a') \Rightarrow (c)$ independently. Here $(a')$ implies the case where $W_A$ is faithful, locally projective and where every simple factor module of ${}_BW$ is embedded into $S({}_BW)$. (However, replacing $(a)$ by $(a')$, Theorem 2.2 is valid and so is Theorem 2.4 for a locally projective module $W_A$.)

From Theorem 2.2 several properties on dominant modules will be deduced. The next is an extension of Kato [10, Corollary 5].

COROLLARY 2.5. *Let* $W_A$ *be a dominant module with* $Q = \mathrm{Biend}(W_A)$. *Then* $Q$ *is the maximal left quotient ring of* $A$.

Proof is the same as in [26, Corollary 1.7] by virtue of Theorem 2.2.

For left (or right) $A$-modules $L_1$ and $L_2$, define $L_1 \sim^w L_2$ (resp. $L_1 \sim L_2$) if each of $L_1$ and $L_2$ is isomorphic to a direct summand of a direct sum (resp. a finite direct sum) of copies of the other. Then the next is an extension of Rutter [26, Corollary 1.6].

COROLLARY 2.6. *Let* $W_A$ *be a dominant module and* $V_A$ *a given module. Then* $V_A$ *is dominant if and only if* $V_A \sim^w W_A$.

Proof is the same as in [26, Corollary 1.6] by virtue of Theorem 2.2.

The following also is an extension of Kato [12, Remark 2].

COROLLARY 2.7. *Let* $W_A$ *be a dominant module. Then* ${}_A[E(S({}_AA))]$ *is faithful.*

*Proof.* In the first of all, it should be noted that ${}_A[E(S({}_AA))]$ is faithful if and only if, for any non-zero element $a$ of $A$, there exist subideals $L_i$ $(i = 1, 2)$ of $Aa$ such that $L_2/L_1 \subsetneq S({}_AA)$.[4]

Let $a$ be a non-zero element of $A$. Since $W_A$ is faithful we can find an element $v$ in $W$ with $va \neq 0$. Taking a simple factor module of $Bva$ where $B = \mathrm{End}(W_A)$, there is a maximal left ideal $Y$ of $B$ such that $Bva/Yva \cong B/Y$ and $\mathrm{Tr}({}_BW) = R \not\subset Y$, and then it holds that

$$(K(Y), va) \neq (TW^*, va)$$

where $T = \mathrm{Tr}(W_A)$, $W^* = \mathrm{Hom}_A(W_A, A_A)$ and $K(Y)$ denotes the same as in Lemma 1.2. Because, if $(K(Y), va) = (TW^*, va)$ then we have $(TW^*Y, va) = (TW^*, va)$ since $TK(Y) = TW^*Y$ by Lemma 1.2, whence

$$Yva = RYva = W(TW^*Y, va) = W(TW^*, va) = Rva = Bva,$$

a contradiction.

Therefore noting that $(TW^*, v)\, a/(K(Y), v)a \cong TW^*/K(Y)$ is simple, we have $(TW^*, v)\, a/(K(Y), v)a \subsetneq S({}_AA)$ by Theorem 2.2. Thus the proof is completed.

Finally, for a projective module $W_A$ with a dual basis $\{u_\lambda, f_\lambda \mid \lambda \in \Lambda\}$, we shall call $\sum_{\lambda\in\Lambda} f_\lambda(W)$ the right pretrace ideal of $W_A$ associated with $\{u_\lambda, f_\lambda \mid \lambda \in \Lambda\}$. This depends on the choice of its dual basis. The next is a slight extension of

[4] Without any restriction this equivalence is valid, which was first observed by Y. Iwanaga (Cf. [12, Remark 2]).

Morita [20, Theorem 2.2] and of Faith [7, Proposition 4.6], which asserts that the Morita's version on retainment of faithful projectivity depends rather on its right (or left) pretrace ideal than on its trace ideal.

LEMMA 2.8. *Let $W_A$ be a faithful projective module with $Q = \mathrm{Biend}(W_A)$ and with a right pretrace ideal $T_0$, and set $A_0 = \mathbb{Z}1_A + T_0$ where $\mathbb{Z}$ denotes the ring of rational integers. Then $T_0 = T_0^2 = T_0Q$ and, for any subring $C$ of $Q$ containing $A_0$, $W_C$ becomes a faithful projective module with $CT_0 = \mathrm{Tr}(W_C)$ and with $Q = \mathrm{Biend}(W_C)$. Moreover $Q_C$ is torsionless and ${}_CQ$ is a rational extension of $CT_0$.*

Proof is trivial (Cf. Faith [7, Proposition 4.6]).

Applying this to dominant modules we obtain the following.

PROPOSITION 2.9. *Let $W_A$ be a dominant module with a right pretrace ideal $T_0$ and $Q$ the maximal left quotient ring of $A$, and set $A_0 = \mathbb{Z}1_A + T_0$. Then, for any subring $C$ of $Q$ containing $A_0$, $W_C$ is a dominant module with $CT_0 = \mathrm{Tr}(W_C)$ and with $Q = \mathrm{Biend}(W_C)$, and hence $CT_0$ is the smallest dense left ideal of $C$, $Q$ is a maximal left quotient ring of $C$ and $Q_C$ is torsionless.*

Proof is trivial in view of Lemma 2.8 together with Corollary 2.5 and Theorem 2.2.

## 3. DOMINANT RINGS

As was stated in the introduction, a ring $A$ is said to be right (resp. left) dominant if there exists a dominant right (resp. left) $A$-module. In particular $A$ is called a right (resp. left) dominant ring of finite type if there exists a finitely generated, dominant right (resp. left) $A$-module.

As for dominant rings it seems to the author that until hitherto there is no intrinsic characterization (Cf. [12, Theorem 1]). But using (e) in Theorem 2.2 it is readily obtained.

THEOREM 3.1. *A ring $A$ is right dominant if and only if $A$ has the smallest dense left ideal $T$ and there exist elements $t_{\lambda\mu}$ in $T$, $(\lambda, \mu) \in \Lambda \times \Lambda$, with an index set $\Lambda$, satisfying the next three conditions:*

(i) *For each $\mu \in \Lambda$, $t_{\lambda\mu} = 0$ for almost all $\lambda \in \Lambda$.*

(ii) $\sum_{\mu\in\Lambda} t_{\lambda\mu}t_{\mu\nu} = t_{\lambda\nu}$ *for every* $(\lambda, \nu) \in \Lambda \times \Lambda$.

(iii) $T = \sum_{\lambda,\mu\in\Lambda} At_{\lambda\mu}A$.

*In particular, $A$ is right dominant of finite type if and only if there exists a finite index set $\Lambda$ in the above.*

*Proof.* Assume $A$ is right dominant. Then there is a dominant module $W_A$ with a dual basis $\{u_\lambda, f_\lambda \mid \lambda \in \Lambda\}$. Setting

$$t_{\lambda\mu} = f_\lambda(u_\mu) \qquad \text{for} \quad (\lambda, \mu) \in \Lambda \times \Lambda,$$

by a routine calculation it is readily seen that $t_{\lambda\mu}$, $(\lambda, \mu) \in \Lambda \times \Lambda$, satisfies (i), (ii) and (iii) in the theorem. Also $T = \mathrm{Tr}(W_A)$ is the smallest dense left ideal of $A$ by Theorem 2.2.

Conversely, assume that there exist the smallest dense left ideal $T$ of $A$ and the elements $t_{\lambda\mu}$, $(\lambda, \mu) \in \Lambda \times \Lambda$, in $T$ satisfying (i), (ii) and (iii) in the theorem. Set now

$$B = \mathrm{End}(A_A^{(\Lambda)})$$

where $A^{(\Lambda)}$ denotes the direct sum of $\Lambda$ copies of $A$. As the free module $A_A^{(\Lambda)}$ may be regarded as the right $A$-module consisting of all column-finite $\Lambda \times 1$ matrices over $A$, $B$ is the column-finite $\Lambda \times \Lambda$ matrix ring over $A$. Further set

$$e = (t_{\lambda\mu})_{\lambda,\mu}.$$

Then clearly $e \in B$ and hence we may set

$$W_A = e \cdot A_A^{(\Lambda)}.$$

It now suffices to show that $W_A$ is dominant. Denote by $u_\lambda$ and $f_\lambda$ respectively the $\lambda$th column vector of $e$ and the restriction (to $W$) of the $\lambda$th canonical projection: $A^{(\Lambda)} \to A$. Then, using a routine matrix calculation, from (i) and (ii) it follows that $f_\lambda(u) = 0$ for almost all $\lambda \in \Lambda$ and $u = \sum_\lambda u_\lambda f_\lambda(u)$ for each $u \in W$; that is, $W_A$ is a projective module with the dual basis $\{u_\lambda, f_\lambda \mid \lambda \in \Lambda\}$. Furthermore $f_\lambda(u_\mu) = t_{\lambda\mu}$ and so by (iii) we have easily $\mathrm{Tr}(W_A) = T$, which is the smallest dense left ideal of $A$ by hypothesis. Accordingly $W_A$ is dominant by Theorem 2.2. Thus the proof is completed.

The next is a direct consequence of Proposition 2.9.

COROLLARY 3.2. *Let $A$ be a right dominant ring, $T$ the smallest dense left ideal of $A$, and $t_{\lambda\mu}$, $(\lambda, \mu) \in \Lambda \times \Lambda$, the elements of $T$ satisfying* (i), (ii) *and* (iii) *in Theorem* 3.1. *Further let $Q$ be the maximal left quotient ring of $A$, and set respectively $T_0 = \sum_{\lambda,\mu\in\Lambda} t_{\lambda\mu}A$ and $A_0 = \mathbb{Z}1_A + T_0$. Then any subring $C$ of $Q$ containing $A_0$ is a right dominant ring, and further $CT_0$ is the smallest dense left ideal of $C$, $Q$ is the maximal left quotient ring of $C$ and $Q_C$ is torsionless.*

PROPOSITION 3.3. *A property of a ring to be right* (*or left*) *dominant is Morita-invariant.*

*Proof.* Assume $\mathscr{M}_A \sim \mathscr{M}_B$, via

$$X \mapsto F(X) = X \otimes_A U_B, \qquad X \in \mathscr{M}_A$$
$$G(Y) = Y \otimes_B V_A \leftarrow Y, \qquad Y \in \mathscr{M}_B,$$

where ${}_AU$ is a finitely generated, projective generator with $B = \mathrm{End}({}_AU)$ and ${}_BV_A = \mathrm{Hom}_A({}_AU_B, {}_AA)$. Further assume $A$ is right dominant; that is, there exists a dominant module $W_A$ with $C = \mathrm{End}(W_A)$. Then it will be shown that $F(W)_B$ is dominant.

As is readily seen, the equivalence between categories of modules preserves both faithfulness and projectives, and so $F(W)_B$ is faithful and projective. Next we note

$$\mathrm{End}(F(W)_B) = \mathrm{End}(W_A) = C,$$

and since ${}_AA \sim {}_AU$ we have ${}_CW = {}_CW \otimes {}_AA \sim {}_CW \otimes {}_AU = {}_CF(W)$, whence

$$S({}_CW) \sim S({}_CF(W)).$$

Moreover we want to show that $\mathrm{Tr}({}_CW) = \mathrm{Tr}({}_CF(W))$. Since ${}_AU$ is finitely generated projective we have

$$\mathrm{Hom}_A({}_AU, {}_AA) \otimes_A \mathrm{Hom}_C({}_CW_A, {}_CC) \cong \mathrm{Hom}_C({}_CW \otimes {}_AU, {}_CC)$$

via

$$\varphi \otimes \psi \mapsto [w \otimes u \mapsto \psi(w\varphi(u))],$$

where $\varphi \in \mathrm{Hom}_A({}_AU, {}_AA)$, $\psi \in \mathrm{Hom}_C({}_CW, {}_CC)$, $w \in W$ and $u \in U$. Thereby

$$\mathrm{Tr}({}_CF(W)) \subset \mathrm{Tr}({}_CW).$$

On the other hand, as ${}_AU \otimes {}_BV_A \cong {}_AA_A$ and so ${}_CW \otimes {}_AU \otimes {}_BV \cong {}_CW$, applying the above argument to ${}_CF(W)$ instead of ${}_CW$ we have similarly $\mathrm{Tr}({}_CW) = \mathrm{Tr}({}_CF(W) \otimes {}_BV) \subset \mathrm{Tr}({}_CF(W))$. Consequently $\mathrm{Tr}({}_CW) = \mathrm{Tr}({}_CF(W))$, which we shall denote by $R$.

Take now any simple module ${}_CN$ with $RN = N$. Then, since $W_A$ is dominant, ${}_CN \subsetneq S({}_CW)$ by Theorem 2.2 and hence ${}_CN \subsetneq S({}_CF(W))$ because $S({}_CW) \sim S({}_CF(W))$. Therefore $F(W)_B$ is dominant by Theorem 2.2, and so $B$ is a right dominant ring. Thus the proof is completed.

## 4. Pseudo-perfect rings

Recall that an idempotent $e$ of a ring $A$ is said to be local provided $eAe$ is a local ring; that is, $eAe/eJe$ is a division ring where $J = J(A)$, the Jacobson radical of $A$. As is well known, $e$ is local $\Leftrightarrow Ae/Je$ is simple $\Leftrightarrow eA/eJ$ is simple.

Two idempotents $e$ and $f$ in $A$ are said to be isomorphic to each other provided $eA \cong fA$. As is well known, $eA \cong fA \Leftrightarrow Ae \cong Af \Leftrightarrow Ae/Je \cong Af/Jf \Leftrightarrow eA/eJ \cong fA/fJ$.

Now, in this section we shall treat a special but a useful right (or left) dominant ring: A ring $A$ is defined to be right pseudo-perfect if there are pairwise non-isomorphic, local idempotents $\{e_\lambda \mid \lambda \in \Lambda\}$ of $A$ such that $[\sum_{\lambda\in\Lambda} \oplus e_\lambda A]_A$ is dominant. Similarly left pseudo-perfect rings will be defined.

Such a dominant module is characterized by the next.

THEOREM 4.1. *Let $A$ be a ring and $\{e_\lambda \mid \lambda \in \Lambda\}$ pairwise non-isomorphic, local idempotents of $A$. Then the next statements are equivalent.*

(a) $[\sum_\lambda \oplus e_\lambda A]_A$ *is dominant.*

(b) $[\sum_\lambda \oplus e_\lambda A]_A$ *is faithful, and* $Ae_\lambda/Je_\lambda \subsetneq S({}_AA)$ *for every* $\lambda \in \Lambda$, *i.e.* $\sum_{\lambda\in\Lambda} \oplus Ae_\lambda/Je_\lambda \sim^w S({}_AA)$.

(c) ${}_A[E(S({}_AA))]$ *is faithful, and* $\sum_{\lambda\in\Lambda} \oplus Ae_\lambda/Je_\lambda \sim^w S({}_AA)$.

*Proof.* (a) $\Leftrightarrow$ (b): At first set $W_A = \sum_\lambda \oplus e_\lambda A$ and assume $W_A$ is faithful. Then $W_A$ is a faithful projective module with $\mathrm{Tr}(W_A) = \sum_\lambda Ae_\lambda A$. Let ${}_AM$ be any simple module with $(\sum_\lambda Ae_\lambda A)M = M$. Then $e_\lambda M \neq 0$ for some $\lambda \in \Lambda$ and consequently $M \cong Ae_\lambda/Je_\lambda$. On the other hand, acturally setting $M = Ae_\lambda/Je_\lambda$ for a given $\lambda \in \Lambda$, we have $M = (\sum_\lambda Ae_\lambda A)M$ since $e_\lambda Ae_\mu \subset J$ for $\lambda \neq \mu$. Therefore by virtue of Theorem 2.2 we have obtained the equivalence (a) $\Leftrightarrow$ (b).

(a) $\Rightarrow$ (c): Trivial by Corollary 2.7 and by the implication (a) $\Rightarrow$ (b) proved above.

(c) $\Rightarrow$ (b): We have only to prove that $W_A = \sum_\lambda \oplus e_\lambda A$ is faithful. To do so, let $a$ be any non-zero element of $A$. Then there exist subideals $L_i$ $(i = 1, 2)$ of $Aa$ such that $L_2/L_1 \subsetneq S({}_AA)$, because ${}_A[E(S({}_AA))]$ is faithful (Cf. Footnote 4). Since $S({}_AA) \sim^w \sum_\lambda \oplus Ae_\lambda/Je_\lambda$ by hypothesis, $0 \neq e_\lambda(L_2/L_1)$ for some $\lambda \in \Lambda$, and so

$$0 \neq e_\lambda L_2 \subset e_\lambda Aa \subset Wa,$$

which implies that $W_A$ is faithful. Thus the proof is completed.

*Remark.* In case $\Lambda$ is a finite set, the equivalence (a) $\Leftrightarrow$ (c) was essentially obtained by Kato [12, Corollary].

As the equivalence (a) $\Leftrightarrow$ (b) in the above is useful, we shall restate it as follows:

COROLLARY 4.2. *A ring $A$ is right pseudo-perfect if and only if there exist pairwise non-isomorphic, local idempotents $\{e_\lambda \mid \lambda \in \Lambda\}$ of $A$ such that $S({}_AA) \sim^w \sum_{\lambda\in\Lambda} \oplus Ae_\lambda/Je_\lambda$ and $[\sum_{\lambda\in\Lambda} e_\lambda A]_A$ is faithful.*

In view of Corollary 4.2, the implication below is valid: right perfect rings $\Rightarrow$ semiperfect rings with essential left socle $\Rightarrow$ right pseudo-perfect rings (of finite type), which will justify the denomination of "right pseudo-perfect".

In case $A$ is a right (or left) dominant ring, a dominant module is called minimal dominant provided it is isomorphic to a direct summand of any dominant module.

Proposition 4.3. *Let $A$ be a right (or left) pseudo-perfect ring. Then $A$ has a minimal dominant module. Moreover a minimal dominant module is uniquely determined within isomorphism.*

*Proof.* Let $A$ be right pseudo-perfect and $W_A = \sum_\lambda \oplus e_\lambda A$ the dominant module stated in the definition. Assume $V_A$ is any dominant module. Then by Corollary 2.6 $V_A \sim^w W_A$, and of course $V_A$ is isomorphic to a direct summand of a direct sum $X_A$ of copies of $W_A$. Since each indecomposable direct summand of $X_A$ is cyclic and has a local endomorphism ring, by Warfield [34, Theorem 1] we have

$$V_A \simeq \sum_\lambda \oplus Ae_\lambda^{(I_\lambda)} \text{ with suitable index sets } I_\lambda \ (\lambda \in \Lambda),$$

where $Ae_\lambda^{(I_\lambda)}$ denotes the direct sum of $I_\lambda$ copies of $Ae_\lambda$.

On the other hand, since $W_A$ is isomorphic to a direct summand of a direct sum of copies of $V_A$, by Azumaya [1] we have $I_\lambda \neq \varnothing$ for each $\lambda \in \Lambda$, whence we can conclude that $W_A$ is minimal dominant.

The last assertion of the proposition may be proved similarly. Thus the proof is completed.

As for the endomorphism ring of a minimal dominant module we have the next.

Proposition 4.4. *Let $A$ be a right pseudo-perfect ring with the minimal dominant module $W_A = \sum_{\lambda \in \Lambda} \oplus e_\lambda A$ where $\{e_\lambda \mid \lambda \in \Lambda\}$ are the same as in the definition, and assume $S({}_AA) \sim^w \sum_{\lambda \in \Lambda} \oplus S(Ae_\lambda)$. Let us set $B = \mathrm{End}(W_A)$. Then $B$ is a right pseudo-perfect ring with a minimal dominant module $\sum_{\lambda \in \Lambda} \oplus E_\lambda B$ and with $S({}_BB) \sim^w \sum_{\lambda \in \Lambda} \oplus S(BE_\lambda)$,* where *$\{E_\lambda \mid \lambda \in \Lambda\}$ are pairwise non-isomorphic, local idempotents of $B$.*

*Proof.* Evidently $B$ consists of all column-finite $\Lambda \times \Lambda$ matrices $(b_{\lambda\mu})_{\lambda,\mu}$ with $b_{\lambda\mu} \in e_\lambda A e_\mu$ for each $(\lambda, \mu) \in \Lambda \times \Lambda$. Denote by $E_\lambda$ the $\Lambda \times \Lambda$ matrix such that its $(\lambda, \lambda)$-entry is $e_\lambda$ and 0 otherwise. Then by a routine calculation and by Corollary 4.2 we have our assertion. (However its details are ommited.)

Apparently we may send away local idempotents out of the definition of right pseudo-perfect rings, which will be done by using a notion of a projective cover. (Cf. [3])

Proposition 4.5. *A ring $A$ is right pseudo-perfect if and only if there are pairwise non-isomorphic, simple right $A$-modules $M_\lambda$ ($\lambda \in \Lambda$) such that each $M_\lambda$ has a projective cover $P_\lambda$ and $[\sum_{\lambda \in \Lambda} \oplus P_\lambda]_A$ is dominant. Accordingly, a property of a ring to be right pseudo-perfect is Morita-invariant.*

*Proof.* The only if part is obvious. To prove the if part, let $P_\lambda \to M_\lambda$ be a projective cover of the simple module $M_\lambda$. Then by [0, Proposition 17.19] there is a local idempotent $e_\lambda$ in $A$ with $P_\lambda \cong Ae_\lambda$ (and $M_\lambda \cong Ae_\lambda/Je_\lambda$), which prove the if part. Thus we obtain the former statement of the proposition.

Let now $A$ be a right pseudo-perfect ring and $W_A$ a minimal dominant module. Then, under the same notations as in Proposition 3.3 it was proved there that $F(W)_B$ is dominant. Moreover, noting that the equivalence between categories of modules preserves simples and projective covers, the latter statement follows immediately from the former. Thus the proof is completed.

PROPOSITION 4.6. *Let $A$ be a right pseudo-perfect ring with the minimal dominant module $\sum_{\lambda\in\Lambda} \oplus e_\lambda A$ stated in the definition. Denote by $A_0$ and $Q$ respectively the ring $\mathbb{Z}1_A + \sum_{\lambda\in\Lambda} e_\lambda A$ and the maximal left quotient ring of $A$. Then, any subring $C$ of $Q$ containing $A_0$ is right pseudo-perfect.*

Proof is trivial by virtue of Proposition 2.9.

In order to give a typical example of pseudo-perfect ring (of infinite type), for the present ${}_BW$ is assumed to be a cogenerator with $A = \mathrm{End}({}_BW)$, and let $\{V_\lambda \mid \lambda \in \Lambda\}$ be a complete representative set of isomorphism classes of all simple left $B$-modules. Then by [23, Lemma 1] we may regard as $E(V_\lambda) \subset {}_BW$ for each $\lambda \in \Lambda$ and $\sum_\lambda E(V_\lambda) = \sum_\lambda \oplus E(V_\lambda)$ in ${}_BW$.

Denoting by $p_\lambda$ the canonical projection of $\sum_\lambda \oplus E(V_\lambda)$ onto $E(V_\lambda)$, since $E(V_\lambda)$ is injective there is an element $e_\lambda$ in $A$ such that the diagram below commutes:

$$\begin{array}{ccc} 0 \to E(V_\lambda) \oplus \sum\limits_{\mu\neq\lambda} \oplus E(V_\mu) & \to & W \\ \quad p_\lambda \downarrow & \swarrow e_\lambda & \\ E(V_\lambda) & & \end{array}$$

Hence ${}_BWe_\lambda = E(V_\lambda)$ and $\{e_\lambda \mid \lambda \in \Lambda\}$ are pairwise orthogonal idempotents of $A$. Further noting that $\mathrm{End}({}_BWe_\lambda) = e_\lambda Ae_\lambda$ is local by [6, Proposition 5.8], every $e_\lambda$ is a local idempotent of $A$ and we have $Ae_\lambda \not\cong Ae_\mu$ for $\lambda \neq \mu$, because ${}_BWe_\lambda \not\cong {}_BWe_\mu$ for $\lambda \neq \mu$. Thus there exist pairwise non-isomorphic and pairwise orthogonal, local idempotents $\{e_\lambda \mid \lambda \in \Lambda\}$ of $A$ such that ${}_BWe_\lambda = E(V_\lambda)$ for each $\lambda \in \Lambda$.

Now following Azumaya [2], a left $B$-module $N$ is called upper distinguished if every simple left $B$-module is isomorphic to a simple factor module of ${}_BN$. Then we have the next:

THEOREM 4.7. *Let ${}_BW$ be an upper distinguished cogenerator with $A =$ $\mathrm{End}({}_BW)$. Then $A$ is a left pseudo-perfect ring.*

*Proof.* At first we want to show that ${}_A[\sum_{\lambda\in\Lambda} Ae_\lambda]$ is faithful. To do so, let $a$ be any non-zero element of $A$. Then since $W_A$ is faithful there is an element $u \in W$

such that $ua \neq 0$. As a simple factor module of $Bua$ is isomorphic to some $V_\lambda$ ($\lambda \in \Lambda$), we can find an element $a_0 e_\lambda \in Ae_\lambda$ so that the next diagram commutes:

$$\begin{array}{ccccc} 0 & \longrightarrow & Bua & \longrightarrow & W \\ & & \downarrow & \swarrow a_0 e_\lambda & \\ & & V_\lambda & & \\ & & \cap & & \\ & & We_\lambda & & \end{array}$$

because ${}_BWe_\lambda$ is injective. Hence $V_\lambda = Buaa_0e_\lambda$ and so $aa_0e_\lambda \neq 0$, whence ${}_A[\sum_\lambda Ae_\lambda]$ is faithful.

Next it will be shown that $e_\lambda A/e_\lambda J \subsetneq S(A_A)$ for every $\lambda \in \Lambda$. Since ${}_BW$ is an upper distinguished cogenerator, for a given $\lambda \in \Lambda$ we have $\mathrm{Hom}_B(W, V_\lambda) = (V_\lambda : W) \neq 0$ where $(V_\lambda : W) = \{a \in A \mid Wa \subset V_\lambda\}$. Hence, taking a non-zero element $ae_\lambda \in (V_\lambda : W)$ we have $V_\lambda = Wae_\lambda$ since $V_\lambda$ is a simple $B$-module. Noting $V_\lambda \subset S(W_A)$ by [15, Theorem 2], we have $Wae_\lambda J = 0$, i.e. $ae_\lambda J = 0$, whence $e_\lambda A/e_\lambda J \simeq ae_\lambda A \subset S(A_A)$.

Therefore ${}_A[\sum_\lambda Ae_\lambda]$ is dominant by Theorem 4.1, which proves the theorem.

## 5. ℵ-*QF* 3 RINGS

As was stated in the introduction, in order to establish an intrinsic characterization of the endomorphism ring of a generator-cogenerator we shall extend a notion of a right (resp. a left) QF 3 ring: A ring $A$ is defined to be right ℵ-QF 3 if there exist pairwise orthogonal and pairwise non-isomorphic, local idempotents $\{e_\lambda \mid \lambda \in \Lambda\}$ of $A$ such that each $e_\lambda A$ ($\lambda \in \Lambda$) is an injective module with a simple socle, and that $[\sum_{\lambda \in \Lambda} e_\lambda A]_A$ is faithful.[5] Similarly left ℵ-QF 3 rings will be defined.

As for the structure of a right ℵ-QF 3 ring we have the following:

THEOREM 5.1. *Let $A$ be a right* ℵ-QF 3 *ring. Then the next assertions are valid.*

(i) *A right* ℵ-QF 3 *ring is a right pseudo-perfect ring. More precisely, $[\sum_{\lambda \in \Lambda} e_\lambda A]_A$ stated in the definition is nothing else a minimal dominant module and is uniquely determined within isomorphism.*

(ii) $S({}_AA) \sim^w \sum_{\lambda \in \Lambda} \oplus Ae_\lambda/Je_\lambda$, $S(A_A) \sim^w \sum_{\lambda \in \Lambda} \oplus S(e_\lambda A) \subset r_A(J)$ *where* $J = J(A)$, *and* $S(e_\lambda A) \not\simeq S(e_\mu A)$ *if* $\lambda \neq \mu$.

(iii) *$E(S({}_AA))$ is faithful, $E(A_A)$ is torsionless, and $Q_r \subset Q_l$ where $Q_r$ (resp. $Q_l$) denotes the maximal right (resp. left) quotient ring of $A$.*

[5] ℵ represents the cardinal (finite or infinite) of $\Lambda$.

*Proof.* Let $\{e_\lambda \mid \lambda \in \Lambda\}$ be local idempotents of $A$ stated in the definition, and set $W_A = \sum_\lambda e_\lambda A$. At first we want to show that $JS(e_\lambda A) = 0$ for every $\lambda \in \Lambda$. If $JS(e_\lambda A) \neq 0$ then, since $W_A$ is faithful, there is an element $e_\mu n e_\lambda \in e_\mu J e_\lambda$ with some $\mu \in \Lambda$ such that

$$e_\mu n e_\lambda S(e_\lambda A) \neq 0.$$

Since $S(e_\mu A) \subset' e_\mu A$ we have $e_\mu n e_\lambda S(e_\lambda A) \cap S(e_\mu A) \neq 0$, which implies $e_\mu n e_\lambda S(e_\lambda A) = S(e_\mu A)$. Hence, noting $S(e_\lambda A) \subseteq' e_\lambda A$ we have

$$e_\lambda A \lesssim e_\mu J \subsetneqq e_\mu A \text{ via } e_\lambda a \mapsto e_\mu n e_\lambda a \qquad (a \in A),$$

and consequently $e_\mu A$ must be decomposable since $e_\lambda A$ is injective, a contradiction. Thus we obtain

$$JS(e_\lambda A) = 0 \qquad \text{for every} \quad \lambda \in \Lambda,$$

that is, $S(e_\lambda A) \subset e_\lambda r_A(J)$ and at the same time we see

$$Ae_\lambda / Je_\lambda \lesssim S({}_AA) \qquad \text{for every} \quad \lambda \in \Lambda.$$

Therefore $W_A$ is dominant by Theorem 4.1 (and $S({}_AA) \sim^w \sum_\lambda \oplus Ae_\lambda/Je_\lambda$), and further $W_A$ is minimal dominant by Proposition 4.3. Thus we have proved (i).

Next, since $W_A$ is faithful we have $W_A \subset A_A \lesssim W_A{}^W$ canonically and so $S(W_A) \subset S(A_A) \lesssim [S(W_A)]^W$, where $X_A{}^W$ denotes the direct product of $W$ copies of $X_A$. Consequently,

$$S(A_A) \overset{w}{\sim} S(W_A) = \sum_\lambda \oplus S(e_\lambda A).$$

Moreover, since $e_\lambda A = E(S(e_\lambda A))$ for each $\lambda \in \Lambda$, $S(e_\lambda A) \ncong S(e_\mu A)$ for $\lambda \neq \mu$; otherwise, by the uniqueness of its injective hulls we have $e_\lambda A \cong e_\mu A$, a contradiction. Thus we obtain (ii).

Finally, $E(S({}_AA))$ is faithful by Corollary 2.7 since $W_A$ is dominant. Also since $W_A$ is faithful we have

$$A_A \lesssim W_A{}^W \subset \Big[\prod_\lambda e_\lambda A\Big]^W,$$

whence $E(A_A) \lesssim [\prod_\lambda e_\lambda A]^W$ because $e_\lambda A$ is injective for each $\lambda \in \Lambda$. Hence $E(A_A)$ is torsionless, and so $Q_r \subset Q_l$ by [18, Proposition 2]. Thus we obtain (iii) and the theorem was proved.

*Remark.* In case $A$ is a right QF 3 ring, $Q_r \subset Q_l$ in (iii) was first obtained by Ringel and Tachikawa [24, Lemma 1.4].

Recall that a ring $A$ is called right QF 3 if it has a minimal faithful right $A$-module; that is, a faithful module which is isomorphic to a direct summand of every faithful module. Obviously a right $\aleph$-QF 3 ring of finite type is nothing else a right QF 3 ring (Cf. [5, Theorem 1]), and then a minimal dominant module coincides with a minimal faithful module. (Cf. [26, Corollary 1.2], [12, Example 1]).

The next means a "minimal faithfulness" of a minimal dominant module.

COROLLARY 5.2. *Let $A$ be a right* $\aleph$-QF 3 *ring and $W_A$ a minimal dominant module. Then,*

(i) *$W_A \subsetneq M_A$ for every faithful module $M_A$.*

(ii) *Any deletion of a non-zero direct summand out of $W_A$ amounts to a loss of its faithfulness.*

*Proof.* Without loss of generality we may set $W_A = \sum_{\lambda\in\Lambda} e_\lambda A$ stated in the definition. For any faithful module $M_A$ we have canonically

$$A_A \subsetneq M_A{}^M \text{ via } a \mapsto (ma)_{m\in M}.$$

For each $\lambda \in \Lambda$ there is an element $m_\lambda \in M$ such that $m_\lambda S(e_\lambda A) \neq 0$. Hence $m_\lambda e_\lambda A \cong e_\lambda A$ since $S(e_\lambda A) \subset' e_\lambda A$. Thus we have

$$W_A \cong \sum_\lambda \oplus\, m_\lambda e_\lambda A = \sum_\lambda m_\lambda e_\lambda A \subset M_A$$

since $S(e_\lambda A) \not\cong S(e_\mu A)$ for $\lambda \neq \mu$ by Theorem 5.1, which proves (i).

To prove (ii), by Warfield [34, Theorem 1] it may suffice to show that $M'_A = \sum_{\lambda\neq\mu} \oplus\, e_\lambda A$ $(\mu \in \Lambda)$ is not faithful. However, it is obvious since we have

$$M'S(e_\mu A) = \sum_{\lambda\neq\mu} e_\lambda A e_\mu S(e_\mu A) \subset JS(e_\mu A) = 0$$

by Theorem 5.1. Thus we obtain (ii) and the proof is completed.

The former half of the following is well known for artinian QF 3 rings.

COROLLARY 5.3. *Let $A$ be both a right and a left* $\aleph$-QF 3 *ring* (*i.e.* $\aleph$-QF 3 *ring*), *and let $\sum_{\lambda\in\Lambda} e_\lambda A$ and $\sum_{\gamma\in\Gamma} Af_\gamma$ be respectively the minimal dominant module stated in the definition. Then there is a bijection $\pi$ of $\Lambda$ onto $\Gamma$ such that*

$$S(e_\lambda A) \cong f_{\pi(\lambda)}A/f_{\pi(\lambda)}J \qquad \text{and} \qquad S(Af_{\pi(\lambda)}) \cong Ae_\lambda/Je_\lambda,$$

*and $Q_r = Q_l$ where $Q_r$* (*resp. $Q_l$*) *denotes the maximal right* (*resp. left*) *quotient ring of $A$.*

*Proof.* Since $A$ is both right and left $\aleph$-QF 3, by Theorem 5.1 we have

$$\sum_{\lambda\in\Lambda} \oplus\, Ae_\lambda/Je_\lambda \sim^w S({}_AA) \sim^w \sum_{\gamma\in\Gamma} \oplus\, S(Af_\gamma),$$

whence it follows that there is a bijection $\pi$ of $\Lambda$ onto $\Gamma$ such that $S(Af_{\pi(\lambda)}) \cong Ae_\lambda/Je_\lambda$ for each $\lambda \in \Lambda$; because, the decomposition of $S({}_AA)$ into the direct sum of homogeneous components is unique.

Further $S(Af_{\pi(\lambda)}) \subset l_A(J)$ by Theorem 5.1, and hence we can find a non-zero element $x$ in $e_\lambda[l_A(J) \cap r_A(J)]f_{\pi(\lambda)}$ such that $S(Af_{\pi(\lambda)}) = Ae_\lambda x f_{\pi(\lambda)}$, and hence

$$S(e_\lambda A) = e_\lambda x f_{\pi(\lambda)} A \cong f_{\pi(\lambda)} A / f_{\pi(\lambda)} J.$$

The last assertion is a direct consequence of (iii) in Theorem 5.1.

The following gives a criterion on a right $\aleph$-QF 3 ring, which is an analogue of Colby and Rutter [5, Theorem 1].

PROPOSITION 5.4. *Let $A$ be a ring. Then $A$ is a right $\aleph$-QF 3 ring if and only if there are pairwise non-isomorphic, simple right $A$-modules $\{M_\lambda \mid \lambda \in \Lambda\}$ such that $[\sum_{\lambda \in \Lambda} \oplus E(M_\lambda)]_A$ is faithful and projective. Furthermore, in this case $S(A_A) \overset{w}{\sim} \sum_{\lambda \in \Lambda} \oplus M_\lambda$ holds.*

*Proof.* By the definition and Theorem 5.1, the only if part is obvious. Hence we have only to prove the if part. Assume $\sum_\lambda \oplus E(M_\lambda)$ is faithful and projective. Since each $E(M_\lambda)$ $(\lambda \in \Lambda)$ is indecomposable projective and injective, by [29, Lemma 5] $E(M_\lambda)$ is isomorphic to a direct summand $E_\lambda$ of $A_A$, and then it is readily seen that $\sum_\lambda E_\lambda = \sum_\lambda \oplus E_\lambda$ in $A$.

Now denoting by $p_\lambda$ the canonical projection of $\sum_\lambda \oplus E_\lambda$ onto $E_\lambda$, since $E_\lambda$ is injective there is an element $e_\lambda$ of $A$ such that the diagram below commutes:

$$\begin{array}{ccc} 0 \to E_\lambda \oplus \sum_{\mu \neq \lambda} \oplus E_\mu & \to & A \\ \downarrow p_\lambda & \swarrow e_\lambda & \\ E_\lambda & & \end{array}$$

Hence we have $E_\lambda = e_\lambda A$ for each $\lambda \in \Lambda$ and, as is easily seen, $\{e_\lambda \mid \lambda \in \Lambda\}$ are local idempotents of $A$ which are pairwise orthogonal and pairwise non-isomorphic, and $\sum_\lambda e_\lambda A$ is faithful by hypothesis. This implies that $A$ is a right $\aleph$-QF 3 ring, and

$$S(A_A) \overset{w}{\sim} \sum_\lambda \oplus S(e_\lambda A) \cong \sum_\lambda \oplus M_\lambda$$

by Theorem 5.1. Thus the proof is completed.

COROLLARY 5.5. *A property of a ring to be right (resp. left) $\aleph$-QF 3 is Morita-invariant.*

Proof is trivial by virtue of Proposition 5.4; because, the equivalence between categories of modules preserves respectively faithfulness, simples, indecomposables, projectives, injectives and injective hulls.

The next is a slight extension of a portion of Tachikawa [32, Proposition 4.3].

PROPOSITION 5.6. *Let $A$ be a right $\aleph$-QF 3 ring and $Q_r$ the maximal right quotient ring of $A$. Then any subring $C$ of $Q_r$ containing $A$ is a right $\aleph$-QF 3 ring.*

*Proof.* Analogous to Tachikawa [32, Proposition 4.3].

*Remark.* Compare this with Proposition 4.6. The distinction between them will imply a peculiarity of $\aleph$-QF 3 rings, which will be illustrated by the example below.

EXAMPLE. (Cf. Tachikawa [32, p. 78]). Let $B$ be a Cozzens's domain (Cf. [7, p. 362]), i.e. $B$ satisfy the next conditions:

(i) $B$ is a principal right and left ideal domain.

(ii) $B$ is a simple ring, but is no division ring.

(iii) $B$ has a simple, injective cogenerator $V_B$.

Set now $C = \mathrm{End}(V_B)$. Then $C$ is a division ring and by the hypotheses (i)–(iii) ${}_CV$ is a free module of infinite dimension. Then, Tachikawa showed that the matrix ring

$$A = \begin{pmatrix} C & {}_CV_B \\ 0 & B \end{pmatrix}$$

is a right QF 3 ring; more precisely, $c_{11}A$ is a minimal faithful module with a simple socle $c_{12}V$ where $c_{ij}$ $(1 \leqslant i, j \leqslant 2)$ denote the matrix units, and further that $A$ coincides with the maximal right quotient ring of $A$ itself.

Moreover set respectively $D = \mathrm{End}({}_CV)$, ${}_DV_C^* = \mathrm{Hom}_C({}_CV_D, {}_CC)$ and $Q = \mathrm{Biend}([c_{11}A]_A)$. Then $Q$ is the maximal left quotient ring of $A$ by Corollary 2.5, and a routine calculation shows that $Q$ may be regarded as the matrix ring

$$\begin{pmatrix} C & {}_CV_D \\ {}_DV_C^* & D \end{pmatrix},$$

under the multiplications defined by

$$v \cdot f = (v, f) \quad \text{and} \quad f \cdot v = [f, v] \quad \text{for} \quad v \in V,\ f \in V^*,$$

where $(C, D, V^*, V, (\ ,\ ), [\ ,\ ])$ denotes the Morita context derived from ${}_CV$, i.e.

$$(\ ,\ ) \colon {}_CV \otimes {}_DV_C^* \to C, \qquad \text{via} \quad v \otimes f \mapsto (v, f) = f(v)$$
$$[\ ,\ ] \colon {}_DV^* \otimes {}_CV_D \to D, \qquad \text{via} \quad f \otimes v \mapsto [f, v]$$

defined by $u[f, v] = (u, f)v$ $(u, v \in V, f \in V^*)$.

Clearly $c_{11}Q = c_{11}A$ and is a simple, dominant right $Q$-module. But we want to show that $[c_{11}Q]_Q$ is not injective. To do so, let $\{v_i \mid i \in I\}$ be a free basis of $_CV$ and $\{f_i \mid i \in I\}$ the elements of $V^*$ defined by $f_i(v_j) = \delta_{ij}1_C$ $(i, j \in I)$, where $\delta_{ij}$ denotes the Kronecker's delta. Then, by using the Morita context it is readily seen that

$$\sum_{i \in I} f_i C = \sum_{i \in I} \oplus f_i C \qquad \text{in } V_C^*$$

and

$$\sum_{i \in I} [f_i, V] = \sum_{i \in I} \oplus [f_i, V] \qquad \text{in } D_D,$$

together with $V_D \cong [f_i, V]$ via $v \mapsto [f_i, v]$ $(v \in V)$.

Set now

$$N = \begin{pmatrix} 0 & 0 \\ \sum_{i \in I} \oplus f_i C & \sum_{i \in I} \oplus [f_i, V] \end{pmatrix}$$

and define a map

$$\varphi\colon N \to c_{11}Q, \text{ via } \begin{pmatrix} 0 & 0 \\ \sum f_i c_i & \sum [f_i, u_i] \end{pmatrix} \mapsto \begin{pmatrix} \sum c_i & \sum u_i \\ 0 & 0 \end{pmatrix}.$$

Then $N$ is a right ideal of $Q$ and $\varphi$ is a well-defined $Q$-homomorphism. But $\varphi$ can not be extended to a $Q$-homomorphism: $Q \to c_{11}Q$; otherwise, there is a matrix

$$\begin{pmatrix} c & v \\ 0 & 0 \end{pmatrix}$$

in $c_{11}Q$ such that

$$\begin{pmatrix} c & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \sum f_i c_i & \sum [f_i, u_i] \end{pmatrix} = \begin{pmatrix} \sum c_i & \sum u_i \\ 0 & 0 \end{pmatrix}$$

for every $c_i \in C$ and $u_i \in V$, which is impossible since $v$ is a $C$-linear combination of a finite number of $v_i$'s. This shows $c_{11}Q$ is no injective module.

At last we shall establish a structure theorem on endomorphism rings of generator-cogenerators, which is a natural generalization of Ringel and Tachikawa [24, Theorem 2.1], and which will supply us many a example of left (or right) $\aleph$-QF 3 rings.

THEOREM 5.7. *Let $A$ be a ring. Then $A$ is an endomorphism ring of a generator-cogenerator, say $_BW$, if and only if $A$ satisfies the next three conditions*:

(i) *$A = Q_l$, the maximal left quotient ring of $A$ itself.*

(ii) *$A$ is a right dominant ring of finite type.*

(iii) *$A$ is a left $\aleph$-QF 3 ring.*

*Moreover, $B$ has only finite many isomorphism classes of simple left $B$-modules if and only if $A$ becomes a left* QF 3 *ring in* (iii) *above mentioned.*

Proof of the only if part: Let ${}_BW$ be a generator-cogenerator with $A = \mathrm{End}({}_BW)$. Then $A$ satisfies (i) and (ii), which was obtained by Kato [11, Example 3] as follows: Since ${}_BW$ is a generator, by the Morita theorem $W_A$ is faithful, finitely generated projective and $B = \mathrm{End}(W_A)$. On the other hand, since ${}_BW$ is a cogenerator and so contains a copy of every simple left $B$-module, $W_A$ is a finitely generated dominant module; i.e. $A$ satisfies (ii). Noting $A = \mathrm{Biend}(W_A)$, we have $A = Q_l$ by Corollary 2.5.

Next we want to show the validity of (iii). Denote by $\{V_\lambda \mid \lambda \in \Lambda\}$ the complete representative set of isomorphism classes of all simple left $B$-modules. Then by Theorem 4.7 there are pairwise orthogonal and pairwise non-isomorphic, local idempotents $\{e_\lambda \mid \lambda \in \Lambda\}$ of $A$ such that $E(V_\lambda) = We_\lambda$ for each $\lambda \in \Lambda$ and that ${}_A[\sum_{\lambda \in \Lambda} Ae_\lambda]$ is faithful.

Since $W_A$ is flat and ${}_BWe_\lambda$ is injective, from the isomorphism

$$Ae_\lambda \cong {}_A[\mathrm{Hom}_B({}_BW_A, {}_BWe_\lambda)]$$

it follows that each $Ae_\lambda$ $(\lambda \in \Lambda)$ is injective by [21, Lemma 1.3]. Moreover it will be shown that

$$S(Ae_\lambda) = (W^*, V_\lambda) \text{ and is simple,}$$

where $W^* = \mathrm{Hom}_A(W_A, A_A)$. To show this, setting $T = \mathrm{Tr}(W_A)$, it should be noted that $TX = X$ for every simple left ideal $X$ of $A$, because $T_A$ is faithful. Hence the correspondence in (i) of Lemma 1.1 induces a bijection between simple submodules of ${}_BWe_\lambda$ and simple subideals of $Ae_\lambda$, and consequently $S(Ae_\lambda) = (W^*, V_\lambda)$ and is simple because $V_\lambda$ is the unique simple submodule of ${}_BWe_\lambda$. (And $S(Ae_\lambda) \not\cong S(Ae_\mu)$ if $\lambda \neq \mu$; because, ${}_BW \otimes {}_AS(Ae_\lambda) \cong WS(Ae_\lambda) = V_\lambda$ for each $\lambda \in \Lambda$ and $V_\lambda \not\cong V_\mu$ for $\lambda \neq \mu$.) Thus $A$ becomes a left $\aleph$-QF 3 ring (and $S({}_AA) \sim^{ir} \sum_{\lambda \in \Lambda} \oplus S(Ae_\lambda)$ by Theorem 5.1), which proves the only if part.

Proof of the if part: Assume (i), (ii) and (iii) of Theorem 5.7. Then there is a finitely generated dominant module, say $W_A$, by (ii). Since $W_A$ is finitely generated projective, setting $B = \mathrm{End}(W_A)$, ${}_BW$ is a generator by the Morita theorem. Also since $W_A$ is dominant we have $\mathrm{End}({}_BW) = Q_l$ by Corollary 2.5 and so $\mathrm{End}({}_BW) = A$ by (i).

Further we want to show that ${}_BW$ is a cogenerator. By (iii) there are pairwise orthogonal and pairwise non-isomorphic, local idempotents $\{e_\lambda \mid \lambda \in \Lambda\}$ of $A$ such that each $Ae_\lambda$ $(\lambda \in \Lambda)$ is an injective module with a simple socle.

Setting now $T = \mathrm{Tr}(W_A)$, we have $r_{Ae_\lambda}(T) = 0$ since $T_A$ is faithful. As $Ae_\lambda$ is indecomposable, injective and $r_{Ae_\lambda}(T) = 0$, by [25, Lemma 2.3] (or by Theorem 2.4 together with $\mathscr{G}({}_BW) = {}_B\mathscr{M}$) we have

$${}_BW \otimes {}_AAe_\lambda \cong {}_BWe_\lambda \text{ and is indecomposable, injective.}$$

By the same reason as in the last of the proof of the only if part, the correspondence in (i) of Lemma 1.1 induces a bijection between simple submodules of ${}_BWe_\lambda$ (resp. ${}_BW$) and simple subideals of $Ae_\lambda$ (resp. ${}_AA$). Therefore, setting $V_\lambda = WS(Ae_\lambda)$, $V_\lambda$ is a (unique) simple submodule of ${}_BWe_\lambda$ and so

$$E(V_\lambda) = We_\lambda \text{ for each } \lambda \in \Lambda,$$

because ${}_BWe_\lambda$ is indecomposable and injective. At the same time we have

$$S({}_BW) = \sum_X WX = W \cdot \sum_X X = WS({}_AA) \cong {}_BW \otimes {}_AS({}_AA)$$

where $X$ ranges over all simple left ideals of $A$. However, since $S({}_AA) \sim^w \sum_{\lambda\in\Lambda} \oplus S(Ae_\lambda)$ by Theorem 5.1 we have

$$S({}_BW) \cong {}_BW \otimes {}_AS({}_AA) \overset{w}{\sim} {}_BW \otimes_A \left[\sum_{\lambda\in\Lambda} \oplus S(Ae_\lambda)\right] \cong \sum_{\lambda\in\Lambda} \oplus V_\lambda ,$$

where $V_\lambda \not\cong V_\mu$ if $\lambda \neq \mu$; because, by Theorem 1.4 we have $S(Ae_\lambda) \cong {}_AW^* \otimes {}_BV_\lambda/J({}_AW^* \otimes {}_BV_\lambda)$ for each $\lambda \in \Lambda$ and by Theorem 5.1 we see $S(Ae_\lambda) \not\cong S(Ae_\mu)$ for $\lambda \neq \mu$.

As $W_A$ is finitely generated dominant, this implies that $\{V_\lambda \mid \lambda \in \Lambda\}$ is a complete representative set of isomorphism classes of all simple left $B$-modules. Therefore ${}_BW$ is a cogenerator by [23, Lemma 1] since $E(V_\lambda) = We_\lambda \subset W$ for every $\lambda \in \Lambda$. Thus the proof (of the if part) of Theorem 5.7 is completed.

*Remark.* Another characterization on endomorphism rings of generator-cogenerators was obtained by Tachikawa [30, Theorem 4], Kato [11, Example 3] and by Morita [21, Corollary 8.4] respectively. Their characterizations are rather categorical than ours.

In Theorem 5.7 ${}_BW$ is not uniquely determined in view of Corollary 2.6. However the next holds:

COROLLARY 5.8. *Let ${}_{B'}W'$ as well as ${}_BW$ be a generator-cogenerator with $A = \mathrm{End}({}_BW) \cong \mathrm{End}({}_{B'}W')$. Then there is an equivalence $F$: ${}_B\mathscr{M} \sim {}_{B'}\mathscr{M}$ with $F(W) = W'$.*

*Proof.* From the proof of the only if part of Theorem 5.7 it follows that $W'_A$ as well as $W_A$ is finitely generated dominant, and so by [11, Theorem 1] (or by Theorem 2.4) we have respectively

$${}_B\mathscr{M} = \mathscr{G}({}_BW) \sim \mathscr{D}(E({}_AA)) \text{ via } \mathrm{Hom}_B({}_BW_A , -)$$

and

$$\mathscr{D}(E({}_AA)) \sim \mathscr{G}({}_{B'}W') = {}_{B'}\mathscr{M} \text{ via } {}_{B'}W' \otimes_A -.$$

Set now $F = {}_{B'}W' \otimes_A \mathrm{Hom}_B({}_BW_A, -)$. Then we have

$$ {}_B\mathcal{M} \sim {}_{B'}\mathcal{M} \text{ via } F \text{ and } F(W) = {}_{B'}W' \otimes {}_AA = {}_{B'}W', $$

which proves the corollary.

The following are direct consequences of Theorem 5.7, and Corollary 5.9 has been observed by Kato, too.

COROLLARY 5.9. *Let $B$ be a semiperfect ring and ${}_BW$ a generator-cogenerator. Then $A = \mathrm{End}({}_BW)$ is a left* QF 3 *ring.*

COROLLARY 5.10 (Cf. Sugano [29]). *Let $A$ be a ring. Then ${}_AA$ is a cogenerator if and only if $A$ is a left $\aleph$-*QF 3 *ring with a lower distinguished, minimal dominant module. Moreover, in this case $A = Q_l$, the maximal left quotient ring of $A$ itself.*

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