Jordan Algebras of Gelfand–Kirillov Dimension One

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INTRODUCTION

Narrow objects such as pro-p-groups and Lie algebras of finite coclass of finite width have been studied intensively since the paper [6] of Leedham-Green and Newman. Lie algebras of finite width are of Gelfand–Kirillov dimension 1. The structure of associative algebras of Gelfand–Kirillov dimension 1 has been clarified in the series of papers [12] and [13] by Small et al. which in turn depend on the work of Bergman (see [1]). For Lie algebras such a classification involving loop algebras and algebras of Cartan type remains an open problem.

However, under some natural conditions, Lie algebras of finite width have finite Z-grading and thus are related to Jordan systems via Tits–Kantor–Koecher construction. These Jordan systems are also of Gelfand–Kirillov dimension 1. In this paper we study Jordan algebras of Gelfand–Kirillov dimension 1 and prove theorems analogous to those of Small et al. and Bergman. The results of this paper will be used in a subsequent paper on pronipotent groups and Lie algebras of finite width.

Let $A$ be a finitely generated (not necessarily associative) algebra over a ground field $K$. Let $V$ be a finite dimensional $K$-vector space generating $A$ and let $V^n$ denote the linear span of all products of length $\leq n$ in elements of $V$. The Gelfand–Kirillov dimension of $A$ (denoted as

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GK dim(A) is defined as:

\[ \text{GK dim}(A) = \limsup_{n \to \infty} \frac{\ln(\dim V^n)}{\ln n}. \]

It is known that the above definition does not depend on the choice of a particular finite dimensional vector space generating \( A \) (see [1] and [5]).

Clearly \( \text{GK dim} A = 0 \) if and only if \( \dim_K A < \infty \) and, in fact, there are no algebras with \( 0 < \text{GK dim} A < 1 \) (see [5]). Bergman proved (see [1]) that there are no associative algebras with \( 1 < \text{GK dim} A < 2 \). However, there are algebras having dimension \( s \) for every \( 2 \leq s \).

In this paper we prove the following theorem that is similar to the result of Small \textit{et al.}

**Theorem.** Let \( J \) be a finitely generated linear Jordan algebra of Gelfand–Kirillov dimension 1. Then:

(a) if \( J \) is semiprime, then \( J \) is a finite module over a finitely generated central subalgebra of the associative center of \( J \),

(b) the radical of \( J \) is nilpotent.

In the process of the proof we show that there are no Jordan algebras of Gelfand–Kirillov dimension \( s, 1 < s < 2 \).

Some general properties of the Gelfand–Kirillov dimension in Jordan algebras were studied in [7]. We will essentially use the results of this paper.

### 1. SEMIPRIME JORDAN ALGEBRAS OF GK-DIMENSION < 2

From now on all algebras are considered over a ground field \( K \), \( \text{char } K \neq 2 \).

For elements \( a, b, c \) of a Jordan algebra \( J \), let \( \{a, b, c\} = (ab)c + a(bc) - (ac)b \) denote the Jordan triple product. We will use also the linear operators \( U_a : J \to J, U_a(x) = \{a, x, a\} \), and \( U_{a,b} : J \to J, U_{a,b}(c) = \{a, c, b\} \).

Recall that a Jordan algebra \( J \) is called nondegenerate if for an arbitrary element \( a \in J \), the equality \( \{a, J, a\} = (0) \) implies that \( a = 0 \).

A Jordan algebra \( J \) is called prime if for any non-zero ideals \( I, L \) of \( J \), the ideal \( (IL)L + IL^2 \) is also non-zero.

An algebra \( J \) is said to be semiprime if for an arbitrary ideal \( I \triangleleft J \), the equality \( I^3 = 0 \) implies \( I = 0 \). Clearly a non-generate Jordan algebra is semiprime.

A finitely generated Jordan algebra is non-degenerate if and only if it is semiprime (see [1]).
In [15] the classification of prime non-degenerate Jordan algebras is given.

Since in what follows we will speak about finitely generated algebras, we will apply the classification just mentioned to prime Jordan algebras.

We shall recall the definition of a Jordan algebra with a polynomial identity (see [4]). An element of the free Jordan algebra is said to be an $s$-identity if it is identically zero in all special Jordan algebras. A Jordan algebra $J$ is said to satisfy a polynomial identity if there exists an element $f$ of the free Jordan algebra that is not an $s$-identity such that $f$ is identically zero on $J$. In this case we say also that $J$ is a P.I. algebra.

**Proposition 1.1.** If $J$ is a finitely generated semiprime Jordan algebra and $\text{GK dim}(J) < 2$, then $J$ is P.I.

**Proof.** The algebra $J$ is a subdirect product of prime Jordan algebras. That is, there exists a set $I$ and a family of ideals $\{P_i\}_{i \in I}$ such that each algebra $J/P_i$ is prime and $\bigcap_{i \in I} P_i = 1$. Since the algebra $J$ is finitely generated each quotient $J/P_i$ is a prime non-degenerate algebra.

According to the results in [15], each quotient algebra $J/P_i$ is either a special or an Albert ring. Let $I_s = \{i \in I | J/P_i$ is special$\}$ and let $I_a = \{i \in I | J/P_i$ is an Albert ring$\}$.

Denote $P_s = \bigcap_{i \in I_s} P_i$ and $P_a = \bigcap_{i \in I_a} P_i$. Then $J_s = J/P_s$ is a subdirect product of the algebras $J/P_i$, $i \in I_s$, and $J_a = J/P_a$ is a subdirect product of the algebras $J/P_j$, $j \in I_a$. So:

(i) The algebra $J_s$ is a special Jordan algebra and $\text{GK dim}(J_s) \leq \text{GK dim}(J) < 2$. Let $A_s$ be an associative enveloping algebra of $J_s$.

Since $\text{GK dim}(A_s) = \text{GK dim}(J_s) < 2$, from the result of Bergman [1] it follows that $\text{GK dim}(A_s) = 0$ or 1. In view of [13], the algebra $A_s$ is P.I. and so is $J_s$.

(ii) Each Albert ring $J/P_i$, $i \in I_a$, is P.I. and all of them satisfy the same identity (they are central orders of a 27-dimensional Albert algebra), so the subdirect product $J_a$ is also P.I.

Let $\mathcal{L}(J, J)$ be the algebra of all linear transformations on the vector space $J$. The subalgebra of $\mathcal{L}(J, J)$ generated by all multiplications $R(a): J \rightarrow J$, $x \mapsto xa$ is called the multiplication algebra of $J$. We will denote it by $M(J)$.

**Lemma 1.1.** If $J$ is a finitely generated semiprime Jordan algebra with $\text{GK dim}(J) < 2$, then the multiplication algebra $M(J)$ of $J$ is a semiprime algebra.

**Proof.** (a) Let us suppose initially that $J$ is prime. It was proved in [15] that the associative center $Z(J)$ of $J$ is nonzero (since $J$ is P.I.) and every
element \( a \in Z^* = Z(J) \setminus \{0\} \) is regular and the algebra of fractions \( Z^*^{-1}J \) is either a simple finite dimensional Jordan algebra over \( Z^*^{-1}Z \) or a simple Jordan algebra of a bilinear form. Since Jordan algebras of bilinear forms are locally finite and our algebra \( J \) is finitely generated, the algebra \( Z^*^{-1}J \) is always finite dimensional over \( Z^*^{-1}Z \). So \( Z^*^{-1}M(J) \cong M(Z^*^{-1}J) \), that is, \( M(J) \) is a central order of a simple finite dimensional algebra. So it is prime.

(b) Let \( \mathcal{P} = \{ P | P \triangleleft J \} \) be the set of prime ideals in \( J \). If

\[
\varphi_p : M(J) \to M(J/P), \quad R(a) \to R(a + P)
\]

is the natural mapping, then (a) implies that the algebra \( M(J/P) \) is prime. So, to prove that \( M(J) \) is semiprime, it is enough to prove that \( \bigcap P \ker \varphi_p = 0 \).

Let \( \omega \in \bigcap \ker \varphi_p \). For an arbitrary element \( x \in J \) we have \( x \omega \in P \) (since \( \omega \in \ker \varphi_p \)), that is, \( x \omega \in \bigcap P = 0 \) (since \( J \) is semiprime). Thus \( \omega = 0 \).

**LEMMA 1.2.** (a) If \( J \) is a \( d \)-generated Jordan algebra of a bilinear form, then \( \dim_k J \leq d + 1 \).

(b) If \( J \) is a simple central finite-dimensional Jordan algebra that is not a Jordan algebra of a bilinear form and \( J \) satisfies a non-special identity of degree \( s \), then \( \dim_k J \leq \max(s^2/2, 27) \).

**Proof.** (a) Let \( J = K1 + V \) be a Jordan algebra of a bilinear form which is generated by elements \( a_1, \ldots, a_d \). Then for every generator \( a_i \in J \) we have \( a_i = a_i1 + m_i \), where \( a_i \in K, m_i \in V \). The elements \( m_1, \ldots, m_d \) span \( V \). Consequently, \( \dim J \leq 1 + \dim V \leq 1 + d \).

(b) We can assume that \( K \) is algebraically closed. So \( J \) is an algebra of one of the following types:

(i) \( \mathfrak{M}_n(K)^+ \) of dimension \( n^2 \),

(ii) \( H(\mathfrak{M}_n(K), t) \), where \( t \) is the transposition, that is, the Jordan algebra of symmetric matrices. Its dimension is \( n(n + 1)/2 \).

(iii) \( H(\mathfrak{M}_n(Q)) \), where \( Q = Q(K) \) is the quaternion algebra over \( K \). The dimension of this algebra is \( 2n^2 - n \).

(iv) the 27-dimensional Albert algebra.

If the algebra \( J \) satisfies an identity of degree \( s \), then it satisfies a multilinear identity of degree \( \leq s \) (see [17]).

Let \( e_{ij} \) be the matrix having 1 in the position \((i, j)\) and zeros elsewhere. If \( 2n - 1 \geq s \), then \( J \) contains \( s \) elements \( e_{11}, e_{12} + e_{21}, e_{22}, e_{23} + e_{32}, \ldots \).

It is easy to check (see [13]) that \( f(e_{11}, e_{12} + e_{21}, e_{22}, e_{23} + e_{32}, \ldots) \neq 0 \), hence \( 2n - 1 < s \), that is, \( n < (s + 1)/2 \).
In the three cases we have that \( \dim_K J \leq \frac{s^2}{2} \), which proves the lemma.

**Lemma 1.3.** Let \( J \) be a semiprime Jordan algebra generated by a subspace \( V \) of dimension \( n \). Suppose that \( J \) satisfies a P.I. of degree \( s \) and let

\[
N = \max \left( \frac{s^2}{2}, 27, n + 1 \right).
\]

If \( \omega \in M(J) \) and \( V^{(N)} \omega = 0 \), then \( \omega = 0 \).

**Proof.** Let \( \omega \) be an operator from \( M(J) \) such that \( V^{(N)} \omega = 0 \). To prove that \( \omega = 0 \) it will be sufficient to prove that \( J \omega \subseteq P \) for every prime ideal \( P \) (since \( J \) is semiprime).

Let \( P \) be a prime ideal of \( J \) and let \( \bar{J} = J/P \). Then the algebra \( \bar{J} \) is prime. This implies that the associative center \( \bar{Z} \) of \( \bar{J} \) is nonzero, \( \bar{Z}^* = \bar{Z} \setminus \{0\} \) consists of regular elements, and the ring of fractions \( \bar{Z}^{s-1} \bar{J} \) is finite dimensional over \( \bar{Z}^{s-1} \) (see [15]). Since the algebra \( \bar{Z}^{s-1} \bar{J} \) satisfies a multilinear identity of degree \( \leq s \), we have that \( \dim \bar{Z}^{s-1} \bar{J} \leq N \) by the previous lemma.

By our assumption, \( V^{(N)} \omega = 0 \). Let \( \bar{V} = V + P/P \) and let \( \bar{\omega} \) be the image of \( \omega \) under the action of the homomorphism \( \varphi_p : M(J) \to M(J/P) \). Then \( \bar{V}^{(N)} \bar{\omega} = 0 \) in \( \bar{J} \). Now Lemmas 1.2 and 1.3 imply that \( \bar{J} \bar{\omega} = 0 \), which means that \( J \omega \subseteq P \).

**Theorem 1.1.** Let \( J \) be a finitely generated semiprime Jordan algebra and \( 1 \leq \text{GK dim}(J) < 2 \). Then \( \text{GK dim}(J) = \text{GK dim}(M(J)) \). Consequently \( \text{GK dim}(J) = 1 \).

**Proof.** Let \( J \) be generated by a subspace \( V \) of dimension \( d \). The multiplication algebra \( M(J) \) is generated by \( R(V) \) and \( R(V \cdot V) \) (see [17]). Let \( U \) be the linear span of \( V \) and \( V \cdot V \) and let \( W \) be the linear span of \( R(V) \) and \( R(V \cdot V) \).

Clearly \( V \subseteq U \) and \( V^{(N)}, W^{(m)} \subseteq U^{(N+m)} \). By Lemma 1, the algebra satisfies a nonspecial polynomial identity of degree \( s \). Let \( N = \max(s^2/2, 27, d + 1) \). By Lemma 2.3, every operator \( \varphi \) from \( W^{(m)} \) is determined by its restriction \( \varphi|_{V^{(N)}} \), that is, the mapping \( \varphi \to \varphi|_{V^{(N)}} \) is injective.

Just as in the proof of Proposition 1.1 we define an embedding \( \theta : W^{(m)} \to \mathcal{L}(V^{(N)}, U^{(N+m)}) \), where \( \mathcal{L}(V^{(N)}, U^{(N+m)}) \) denotes the vector space of all linear transformations of \( V^{(N)} \) in \( U^{(N+m)} \).

But \( \dim \mathcal{L}(V^{(N)}, U^{(N+m)}) = \dim(V^{(N)}) \cdot \dim(U^{(N+m)}) \), where \( c = \dim(V^{(N)}) \) is a constant, since \( N \) is fixed.
Thus
\[
\text{GK dim}(M(J)) = \limsup_{m \to \infty} \frac{\ln[\dim U^{(n)}]}{\ln m} \leq \limsup_{m \to \infty} \frac{\ln[c \cdot \dim U^{(N+m)}]}{\ln m}
\]
\[
= \limsup_{m \to \infty} \frac{\ln c + \ln[\dim U^{(N+m)}]}{\ln[(N + m)]} \cdot \frac{\ln[(N + m)]}{\ln m}
\]
\[
= \limsup_{n \to \infty} \frac{\ln[\dim U^{(n)}]}{\ln n} = \text{GK dim}(J).
\]

The theorem is proved.

By Lemma 1.1 the algebra $M(J)$ is semiprime. Hence, by the results of [13], there exist operators $\omega_1, \ldots, \omega_k \in M(J)$ and a finitely generated commutative subalgebra $C$ of the center of $M(J)$ such that $\text{GK dim}(C) = 1$ and $M(J) = \omega_1 C + \cdots + \omega_k C$. Since we assume that the algebra $J$ contains 1, we will identity $C$ with the subalgebra $1C$ of the associative center of $J$.

Now, if the algebra $J$ is generated by elements $b_1, \ldots, b_n$, then $J = b_1M(J) + \cdots + b_nM(J) = \sum b_i \omega_i C$.

So we have proved the following.

**Theorem 1.2.** If $J$ is a finitely generated semiprime Jordan algebra with 1 and $\text{GK dim}(J) = 1$, then $J$ is a finite module over a finitely generated subalgebra $C$ of the associative center of $J$.

2. GRADED JORDAN ALGEBRAS

Let $J$ be a graded algebra in the generators $x_1, \ldots, x_d$ and $\text{GK dim}(J) < 2$. Let $N = N(J)$ be the McCrimmon radical of $J$. According to what was proved in the first part of this paper, $J = J/N(J)$ is a finite module over $F[x_1, \ldots, x_d]$, a finitely generated subalgebra of the center $Z(J)$.

**Lemma 2.1.** Let $A$ be a graded finitely generated semiprime associative commutative ring, $A = \sum_{i=1}^d A_i$. Then $A$ contains a homogeneous regular element.

**Proof.** Let $d$ be the maximal number such that $A$ contains $d$ non-zero homogeneous elements $a^{(1)}, \ldots, a^{(d)}$ such that $a^{(i)} \cdot a^{(j)} = 0$ if $i \neq j$. (So $d \leq \text{Goldie dimension of } A$.)

If $b^{(1)} \in a^{(1)}A, \ldots, b^{(d)} \in a^{(d)}A$ are arbitrary nonzero elements, then their sum $b = b^{(1)} + \cdots + b^{(d)}$ is a regular element. Indeed, since $a^{(1)}A \oplus \cdots \oplus a^{(d)}A$ is a direct sum, it follows that for an arbitrary element
Let $c \in A$, we have $cb = 0$ if and only if $cb^{(1)} = \cdots = cb^{(d)} = 0$. The intersection $\bigcap_{i=1}^{d} \text{Ann}(b^{(i)})$ is a graded ideal. If this ideal is not zero, then it contains at least one nonzero homogeneous element $b^{(d+1)}$. So the system $b^{(1)}, \ldots, b^{(d+1)}$ contradicts the maximality of $d$. Consequently $\bigcap_{i=1}^{d} \text{Ann}(b^{(i)}) = 0$ and $b$ is a regular element.

Let $v_i = \text{deg}(a^{(i)})$ and $v = \prod_{i=1}^{d} v_i$. Then the degree of $a^{(i)(\nu/v)}$ is $v_i$. As we have seen before the element $a^{(1)(\nu/v)} + a^{(2)(\nu/v)} + \cdots + a^{(d)(\nu/v)}$ is regular. The lemma is proved.

**Lemma 2.2.** Let $A = \sum_{i=1}^{\infty} A_i$ be a graded finitely generated associative commutative ring and $\text{GK dim}(A) \leq 1$. Then $A$ is a finite module over $F[a]$, where $a$ is an arbitrary homogeneous regular element.

**Proof.** Without loss of generality we will assume that $A$ is generated by a finite collection of homogeneous elements. To prove that $A$ is finite over $F[a]$ it is sufficient to prove that every generator $b$ is integral over $F[a]$.

Let $b$ be a homogeneous element of $A$. Elements $b^{\text{deg}(a)}$ and $a^{\text{deg}(b)}$ have equal degrees and if $b^{\text{deg}(a)}$ is integral over $f[a^{\text{deg}(g)}]$, then $b$ is integral over $F[a]$. Thus we will assume that $\text{deg}(a) = \text{deg}(b)$.

As above let $V'(a, b)$ be the linear span of all products of elements $a$ and $b$ of length $\leq i$. Bergman (see [1]) proved that $\text{GK dim}(A) \leq 1$ implies that the dimensions of quotients $V^{i+1}(a, b)/V'(a, b)$, $1 \leq i < \infty$, are bounded from above. Let $n$ be the upper bound of all these dimensions. Then the system $b^n, b^{n-1}a, \ldots, ba^{n-1}, a^n$ is linearly dependent modulo the linear span of all products of $a, b$ of length $< n$. Since the algebra $A$ is graded, it follows that the elements $b^n, b^{n-1}a, \ldots, ba^{n-1}, a^n$ are linearly dependent. So, there exists scalars $\alpha_0, \alpha_1, \ldots, \alpha_n \in F$, not all zero, such that

$$a_0b^n + \alpha_1b^{n-1}a + \cdots + \alpha_{n-1}ba^{n-1} + \alpha_na^n = 0.$$ 

Let $i$ be the smallest number such that $\alpha_i \neq 0$. We have

$$(\alpha_ib^{n-i} + \alpha_{i+1}b^{n-i-1}a + \cdots + \alpha_na^{n-i})a^i = 0.$$ 

Since the element $a$ is regular it implies that

$$b^{n-i} + \alpha_i^{-1}\alpha_{i+1}b^{n-i-1}a + \cdots + \alpha_i^{-1}\alpha_na^{n-i} = 0,$$ 

so the element $b$ is integral over $F[a]$ and, consequently, $A$ is a finite module over $F[a]$. The lemma is proved.

Let $J$ be a finitely generated graded Jordan algebra with $\text{GK dim}(J) < 2$, $\overline{J} = J/N(J)$. We have seen in the semiprime part that $\overline{J}$ is a finite module over a finitely generated subring $A$ of $Z(\overline{J})$. By Lemmas 2.1 and 2.2, there
exists a homogeneous regular element \( \tilde{z} \) in \( A \) such that \( A \) is a finite modulo over \( F[\tilde{z}] \). Hence \( \tilde{J} \) is a finite module over \( F[\tilde{z}] \).

Let \( z \) be a homogeneous preimage of \( \tilde{z} \) under the homomorphism \( J \to \tilde{J} \).

**Lemma 2.3.** Let \( z \) be a fixed element of \( J \). For an arbitrary element \( a \in J \) there exists a number \( l = l(a) \) such that if \( i, j \geq l \) then \( \{ z^i, a, z^j \} = \sum \alpha_k \{ z^{i+k}, a, z^{j-k} \} + \sum W, \) where \( \alpha_k \) are coefficients from \( K \) and \( W's \) are products in \( z \) and \( a \), of length \( < i + j + 1 \).

**Proof.** We can consider the special Jordan algebra \( \langle a, z \rangle \) and its universal associative enveloping algebra \( \mathcal{A} \). We will denote by \( * : \mathcal{A} \to \mathcal{A} \) the involution defined by \( a^* = a, z^* = z \).

The \( \text{GK dim}(\mathcal{A}) = \text{GK dim}(\langle a, z \rangle) \leq \text{GK dim}(J) < 2 \) (see [7]). By the theorem of Bergman it follows that \( \text{GK dim}(\mathcal{A}) \leq 1 \).

Let \( V^k \) be the linear span of all products in \( \mathcal{A} \) of length \( \leq k \). Then we have the chain \( V^1 \subset V^2 \subset \cdots \) and (see [1]) there is a number \( l \) such that \( \dim V^{k+1}/V^k \leq l \) for every \( k \).

Let us consider the \( l + 1 \) elements \( z^i a z^j, z^{i-1} a z^j, \ldots, a z^j \) that have to be linearly dependent modulo \( V^l \). So, there exists a number \( q, 0 \leq q \leq l \), such that

\[
z^q a z^{l-q} = \sum_{k \geq 1} \alpha_k z^{q+k} a z^{l-q-k} + v_l, \quad v_l \in V^l.
\]

Applying the involution \( * \), we get

\[
z^{l-q} a^* z^q = \sum_{k \geq 1} \alpha_k z^{l-q-k} a^* z^{q+k} + v_l^*.
\]

If \( i, j \geq l \), then

\[
z^i a z^j = z^{i-q} z^q a z^{l-q} z^{j-l+q} = \sum_{k \geq 1} \alpha_k z^{i-q+k} a z^{l-q-k} z^{j-l+1+q} + v, \quad v \in V^{i+j}.
\]

And applying \( * \),

\[
z^i a^* z^j = \sum_{k \geq 1} \alpha_k z^{i-k} a z^{l+k} + v^*.
\]

Consequently,

\[
\{ z^i, a, z^j \} = \sum_{k \geq 1} \alpha_k \{ z^{i+k}, a, z^{j-k} \} + (v + v^*).
\]

By the theorem of Cohn (see [4]), the element \( v + v^* \) is a Jordan polynomial in \( a, z \) of degree \( < i + j \).
Lemma 2.4. Let \( z \) be a fixed element of \( J \). Then for an arbitrary element \( a \in J \) there exists a number \( l^a = l^a(a) \) such that if \( i, j \geq l^a \) then

\[
\{ z^i, a, z^j \} = \sum_{k \geq 1} \alpha_k \{ z^{i+k}, a, z^{j-k} \} + \sum_{\alpha + \beta < i + j} \gamma_{\alpha, \beta} \{ z^\alpha, a, z^\beta \}.
\]

In particular, if \( a \) and \( z \) are homogeneous elements, then the last summand does not appear.

Proof. By Lemma 2.3 it is enough to prove that Jordan products \( \omega \)'s appearing on the right-hand side (see Lemma 2.3) have degree 1 in \( a \) and they are of the type \( \{ z^\alpha, a, z^\beta \} \), \( \alpha + \beta < i + j \).

Let us consider the dual algebra of numbers \( K + Ke \), where \( e^2 = 0 \) and the algebra \( J \oplus K(K + Ke) \). We can apply the previous lemma to the \( K \)-elements \( a m e \) and \( z m e \).

So there exists a number \( l = l(a \otimes e) = \tilde{l}(a) \) such that for any \( i, j \) greater than or equal to \( l \) we have

\[
\{ z^i, a \otimes e, z^j \} = \sum_{k \geq 1} \alpha_k \{ z^{i+k}, a \otimes e, z^{j+k} \}
\]

\[
+ (\text{Jordan products of length } < i + j).
\]

If the element \( a \otimes e \) appears at least twice in a Jordan element, then the latter is equal to zero. So the right-hand side contains only products in which \( a \otimes e \) does not appear and products in which \( a \otimes e \) appears just once. The sum of all products in which \( a \otimes e \) does not appear has to be equal to zero. The lemma is proved.

Lemma 2.5. There exists a function \( f^*(n) \) such that if \( \omega \in M(J) \) is an operator of degree \( \leq n \), then \( z^i R(z^{i_1}) \cdots R(z^{i_r}) \in id(z^{i+1}) \) as soon as \( i_1 + \cdots + i_r \geq f^*(n) \) and \( i \geq f^*(n) \).

Proof. We will use induction on \( k \) to prove the existence of functions \( f(k, n) \) such that if \( \omega \) is a homogeneous multiplication operator of length \( \leq k \) and degree \( \leq n \), then \( z^i R(z^{i_1}) \cdots R(z^{i_r}) \in id(z^{i+1}) \) if \( i, j_1 + \cdots + j_r \geq f(k, n) \).

Let \( k = 1 \) and let \( a_1, \ldots, a_s \) be the set of all products in \( x_1, \ldots, x_d \) of length \( \leq n \). It is enough to take \( f(1, n) = \max_{1 \leq i \leq s} l(a_i) \), where the \( l(\cdot) \) is the function from Lemma 2.3.

(Note that \( z^i R(a) R(z^{j_1}) \cdots R(z^{j_r}) = \sum \{ z^{i+h}, a, z^j \} \), where \( j + h = j_1 + \cdots + j_r \).

Let us note that if \( a \) is an arbitrary element of degree \( \leq n \), then

\[
z^i R(a \cdot z^j) = \{ z^i, a, z^j \} + z^{i+j}, a \in id(z^{i+1}) \quad \text{if } i, j \geq f(1, n) \quad (1)
\]
Let \( k = 2 \) and let \( t = f(1, n + 2f(1, n) + 1)d_0 + 2f(1, n) + 1 \) where \( d_0 = d(z) \) is the degree of the element \( z \).

An arbitrary operator \( R(z^{j_1}) \cdots R(z^{j_r}) \) with \( j_1 + \cdots + j_r \geq t \) can be represented as a sum \( \sum R(z^{\alpha_i}) \cdots R(z^{\alpha_n}) \) where \( \alpha_i = 1 \) or 2 and \( \sum_{i=1}^n \alpha_i = \sum_{i=1}^r j_i \).

Let \( q \) be the smallest number such that \( \alpha_1 + \cdots + \alpha_q \geq 2f(1, n) \). Consequently, either \( \alpha_1 + \cdots + \alpha_{q-1} = 2f(1, n) \) or \( \alpha_1 + \cdots + \alpha_q = 2f(1, n) + 1 \).

We know that \( R(z^{\alpha_1}) \cdots R(z^{\alpha_q}) = \sum_{\beta_1 + \beta_2 = \sum_{i=1}^q R(z^{\beta_1})} R(z^{\beta_2}) \) and at least one of \( \beta_1 \) and \( \beta_2 \) will be \( \geq f(1, n) \). Since the operators \( R(z^{\beta_1}), R(z^{\beta_2}) \) commute, we can assume that \( \beta_1 \geq f(1, n) \).

So, the multiplication operator \( R(z^{j_1}) \cdots R(z^{j_r}) \) can be represented as a sum of operators \( \sum R(z^{\beta_1})R(z^{\gamma_1}) \cdots R(z^{\gamma_t}) \) such that in each summand we have \( f(1, n) \leq \gamma_1 \leq 2f(1, n) + 1 \) and \( \beta + \sum_{i=1}^t y_i \geq t \).

Let \( a_1 \) and \( a_2 \) be two elements such that \( d(a_1) + d(a_2) \leq n \). Then:

\[
R(a_1)R(a_2)R(z^\beta) = -R(z^\beta)R(a_2)R(a_1) - R((a_1 \cdot z^\beta) \cdot a_2) \\
+ R(a_1 \cdot a_2)R(z^\beta) \\
+ R(a_1 \cdot z^\beta)R(a_2) + R(a_2 \cdot z^\beta)R(a_1)
\]

Since \( \beta \geq f(1, n) \), it follows from (I) that the elements \( z^\beta R(z^{\gamma_1})R(a_1), z^\beta R(a_1 \cdot a_2)R(z^{\gamma_1}), z^\beta R(a_1 \cdot z^\beta)R(a_2), \) and \( z^\beta R(a_1 \cdot z^\beta)R(a_1) \) lie in the ideal \( id(z^{i+1}) \).

But the degree of \( (a_1 \cdot z^\beta) \cdot a_2 \) is not greater than \( n + 2f(1, n) + 1 \) and

\[
z^i F((a_1 \cdot z^\beta) \cdot a_2) R(z^{\gamma_1}) \cdots R(z^{\gamma_t}) \in id(z^{i+1})
\]

since \( \gamma_1 + \cdots + \gamma_t \geq t - \beta \geq t - (2f(1, n) + 1) \geq f(1, n + 2f(1, n) + 1)d_0 \). Let us suppose that \( k \geq 3 \) and \( \omega = R(a_1)R(a_2) \cdots R(a_k) \). Then

\[
\omega = \sum \omega' + \sum \omega'R(a_1 \cdot a_2), \text{ where each } \omega' \text{ is an operator of length } \leq k - 1 \text{ and each } \omega'' \text{ is an operator of length } k - 2.
\]

Let \( f(k, n) = f(k - 1, n + 2f(k - 2, n) + 1)d_0 + 2f(k - 2, n) + 1 \).

As we have already mentioned, every operator \( R(z^{j_1}) \cdots R(z^{j_r}) \) with \( j_1 + \cdots + j_r \geq f(k, n) \) can be represented as a linear combination of operators \( R(z^{\beta_1}) \cdots R(z^{\beta_t}) \) with \( f(k - 2, n) \leq \beta \leq 2f(k - 2, n) + 1 \).

Since \( f(k - 1, n) \leq f(k, n) \), by the induction assumption it follows that

\[
z^{i} \omega' R(z^{j_1}) \cdots R(z^{j_r}) \in id(z^{i+1}).
\]

Also \( \omega'' R(a_1 \cdot a_2) R(z^\beta) = \omega'' R(z^\beta) D(a_1, a_2) - \omega'' R(z^\beta D(a_1, a_2)). \)

Again, \( z^{i} \omega'' R(z^\beta) \in id(z^{i+1}) \). The operator \( \omega'' R(z^\beta D(a_1, a_2)) \) has length \( k - 1 \) and degree \( \leq n + \beta \cdot d_0 \leq n + (2f^*(k - 2, n) + 1)d_0 \). Since \( \alpha_i \)
+ \cdots + a_m \geq f^*(k-1,n) + (2f^*(k-2,n) + 1)d_0$, it is enough to apply the induction.

By the result of Skosirskii (see [11]) an arbitrary element of $M(J)$ is a product of not more than $2d + 1$ multiplication operators. Hence to finish the proof it is sufficient to take $f^*(n) = f(2d + 1, n)$.

**Lemma 2.6.** There exists a function $f(n)$ such that if $i, j \geq f(n)$ and $W$ and $W'$ are multiplication operators such that $d(W) + d(W') \leq n$, then $z'W'R(z/W') \in \text{id}(z^{i+1})$.

**Proof.** We know by the previous lemma that there exists a function $f^*(n)$ such that if $i, j \geq f^*(n)$ and $W$ is a homogeneous operator of degree $\leq n$, then $z'WR(z^l) \in \text{id}(z^{i+1})$.

Let $f(n) = 2f^*(n)$. Let $W = R(a_1) \cdots R(a_k)$. We will prove the assertion by induction on $k$.

Suppose at first that $k = 1$, that is, $W = R(a)$. Let us represent $j$ as a sum $j = j_1 + j_2$ such that $j_1, j_2 \geq f^*(n)$. We have

$$z'W'R(z^l \cdot a) = z'W'R((z^{j_1} \cdot z^{j_2}) \cdot a)$$

$$= z'W'(-R(z^{j_1})R(a)R(z^{j_2}) - R(z^{j_2})R(a)R(z^{j_1}))$$

$$+ R(z^{j_1})R(z^{j_2} \cdot a) + R(z^{j_2})R(z^{j_1} \cdot a) + R(a)R(z^{j_1 + j_2}).$$

But elements $z'W'R(z^{j_1})$, $z'W'R(z^{j_2})$, and $z'W'R(a)R(z^l)$ lie in $\text{id}(z^{i+1})$ by the previous lemma.

Let us suppose that $k \geq 2$ and the result is known for an operator $W$ having $\leq k - 1$ factors $R(\cdot)$. Let $W = W_1 R(a) R(b)$. We have

$$z'W'R(z/W_2 R(a) R(b)) = z'W'R((z/W_2) \cdot a \cdot b)$$

$$= z'W'[-R(z/W_2)R(b)R(a)$$

$$- R(a)R(b)R(z/W_2) + R(z/W_2 \cdot a) R(b)$$

$$+ R(z/W_2 \cdot b)R(a) + R(a \cdot b)R(z/W_2)].$$

The elements $z'W'R(z/W_2)$, $z'W'R(a)R(b)R(z/W_2)$, $z'W'(z/W_2 \cdot a)$, $z'W'R(z/W_2 \cdot b)$, $z'W'R(a \cdot b)R(z/W_2)$ lie in $\text{id}(z^{i+1})$ by the induction assumption. Indeed, lengths of the operators $W_2, W_2 R(a), \text{ and } W_2 R(b)$ are $k - 2, k - 1, \text{ and } k - 1$ respectively.

For an ideal $I$ of $J$, let $I^{(s)}$ denote the linear span of all products in $J$ containing at least $s$ factors from $I$. Clearly $I^{(s)}$ is an ideal of $J$.

**Definition 2.1.** We say that an ideal $I$ is strongly nilpotent if there exists a number $s \geq 1$ such that $I^{(s)} = (0)$. 

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It is known (see [13]) that any nilpotent ideal in a finitely generated Jordan algebra is strongly nilpotent.

**Proposition 2.1.** Let I be a strong nilpotent ideal. Let us suppose that there are finitely many multiplication operators \( \{V_i\} \) such that \( J = \sum_i F z^i V_i + I \). Then, there is another finite family of operators \( \{W_k\} \) such that \( J = \sum_{i,k} F z^i W_k \).

**Proof.** We will prove the following assertion (A):

There exist natural numbers \( N, t \) such that if \( W \) is an operator of degree \( \geq t \) and \( i \geq N \), then \( z^i W = z^{i+1} W_i \), where \( W_i \) is another multiplication operator.

Let us show that the assertion (A) implies the proposition. From the assertion (A) it follows that \( \text{id}(z^N) \subseteq \sum F z^i W_i \), where \( W_i \)’s run over the finite set of all homogeneous multiplication operators of degree \( < t \).

The ideal \( \text{id}(z^N) \) has finite co-dimension in \( J \). Indeed, the quotient algebra \( J/N(J) \) is a finite module over \( F[z] \), hence it is a finite module over \( F[z^N] \). This implies that there are finitely many elements \( \bar{u}_1, \ldots, \bar{u}_r \in J \) such that \( J \) is spanned by the elements \( \bar{u}_i, \bar{u}_i z^k N \), where \( k \geq 1 \). Let \( u_1, \ldots, u_r \) be preimages of the elements \( \bar{u}_1, \ldots, \bar{u}_r \) in \( J \). Then the algebra \( J \) is spanned by \( u_1, \ldots, u_r \) modulo \( N(J) + \text{id}(z^N) \).

Zhevlakov and Shestakov (see [16]) proved that an ideal of finite codimension in a finitely generated Jordan algebra is also finitely generated as an algebra. Hence the ideal \( N(J) + \text{id}(z^N) \) is a finitely generated algebra.

The quotient algebra \( N(J) + \text{id}(z^N)/\text{id}(z^N) \) is finitely generated and locally nilpotent (as a homomorphic image of \( N(J) \)), hence it has finite dimension. Thus

\[
|J : \text{id}(z^N)| = |J : N(J) + \text{id}(z^N)| + |N(J) + \text{id}(z^N) : \text{id}(z^N)| < \infty.
\]

If we consider operators \( \omega'_1, \ldots, \omega'_q \) such that the elements \( 1, \omega'_1, \ldots, \omega'_q \) generate \( J \) modulo \( \text{id}(z^N) \), then we will have \( J = \sum F z^i W_i + \sum F z^0 w'_k \) as stated in the proposition.

So we need to prove the assertion (A). To do that we will need some lemmas.

By the assumption, there exists a finite family of multiplication operators of \( J, V_1, \ldots, V_q \) such that \( J = \sum F z^i V_i + I \). Let \( d(v) = \max_{1 \leq i \leq q} d(V_i) \).

Let \( u_1, \ldots, u_r, \omega_1, \ldots, \omega_r \) be homogeneous elements of \( J \) and let \( f \) be the function whose existence was established in Lemma 2.6.

**Definition 2.2.** We say that an operator

\[
V = R(\omega_1) R(u_1) R(\omega_2) R(u_2) \cdots R(\omega_r) R(u_r) W,
\]
where $W$ is another multiplication operator, is in $s$-regular form if
\[ d(u_i) \geq d_0 f(d(\omega_1) + d(u_1) + \cdots + d(\omega_{i-1}) + d(u_{i-1}) + d(\omega_i) + d(v)) + d(v) \]
for each $i$, $1 \leq i \leq s$.

By the assumption of the Proposition 2.1, the ideal $I$ is strongly nilpotent. Let $I^{(s)} = 0$.

**Lemma 2.7.** Under the assumptions of Proposition 2.1, if $m \geq f(d(\omega_1) + d(u_1) + \cdots + d(\omega_{s-1}) + d(u_{s-1}) + d(\omega_s) + d(v))$, and $V$ is an operator in $s$-regular form, then $z^m \cdot V \in \text{id}(z^{m+1})$.

**Proof.** Let $u_i = \sum z^{k_i} V + a_i$ with $a_i \in I$ (recall that $J = \sum Fz' V_j + I$).

Then $d(u_i) = d(\sum z^{k_i} V_j) \leq d_0 k_i + d(v)$. According to the definition of an $s$-regular operator,

\[ k_{ij} \geq f(d(\omega_1) + d(u_1) + \cdots + d(\omega_{i-1}) + d(u_{i-1}) + d(\omega_i) + d(v)). \]

The induction on $h$ implies

\[ z^m R(\omega_1) R(u_1) \cdots R(u_h) = z^m R(\omega_1) R(a_1) R(\omega_2) R(a_2) \cdots R(\omega_h) R(a_h) \mod \text{id}(z^{m+1}). \]

Indeed, $z^m R(\omega_1) R(a_1) R(\omega_2) R(a_2) \cdots R(\omega_h) R(z^{k_i} V_j) \in \text{id}(z^{m+1})$ by Lemma 2.6.

But $I^{(s)} = 0$. Consequently, $z^m R(\omega_1) R(a_1) \cdots R(a_h) = 0$. Hence $z^m V \in \text{id}(z^{m+1})$.

**Remark.** In the previous lemma we have proved a particular case of the assertion (A). That is, we have proved that if $V$ is an operator in $s$-regular form and $m \geq f(d(\omega_1) + d(u_1) + \cdots + d(\omega_{s-1}) + d(u_{s-1}) + d(\omega_s) + d(v))$, then $z^m \cdot V \in \text{id}(z^{m+1})$.

**Lemma 2.8.** (a) There exists a function $g(l)$ such that every operator $W$ of degree $\geq g(l)$ can be represented as a linear combination of operators $R(\omega) \cdots$ and $R(x_1) R(\omega) \cdots$, where the $\omega$'s are words of degree $\geq l$.

(b) There exists a function $h(t)$ such that an arbitrary operator $W$ of degree $\geq h(t)$ can be represented as a linear combination of $t$-regular operators.

**Proof.** (a) Every operator can be represented as a product of $\leq 2d + 1$ right multiplications. Let $g(l) = (l - 1)(2d + 1) + 1$. Consider the operator $W = R(v_1) \cdots R(v_{2d+1})$ where the $v_i$'s are words in $x_1, \ldots, x_d$ and let us suppose that $d(W) \geq g(l)$. At least one of the summands $v_i$ has degree $\geq l$. If $i = 1$, $W$ has the wanted form with $\omega = v_1$. 


Let us suppose that \( i = 2 \). Then

\[
W = R(v_1)R(v_2) \cdots R(v_{2d+1}) = R(v_2)R(v_1) \cdots R(v_{2d+1}) \\
+ D(v_1,v_2)R(v_3) \cdots R(v_{2d+1}).
\]

Now in \( D(v_1,v_2) \) we can pass variables from \( v_2 \) to \( v_1 \) until we get \( D(v_1,v_2) = \sum D(x_i,v'_i) \) with degree \( (v'_i) \geq l \). So \( W \) has the wanted form.

Let us suppose that \( i \geq 3 \). Then

\[
R(v_{i-2})R(v_{i-1})R(v_i) = -R(v_i)R(v_{i-1})R(v_{i-2}) - R((v_i \cdot v_{i-2}) \cdot v_{i-1}) \\
+ R(v_{i-2} \cdot v_i)R(v_{i-1}) + R(v_{i-1} \cdot v_i)R(v_{i-2}) \\
+ R(v_{i-2} \cdot v_{i-1})R(v_i)
\]

and it remains to use the induction on \( i \).

(b) We will prove the assertion by induction on \( t \). If \( t = 1 \), the existence of \( h(1) \) has been established in part (a) of this lemma.

Let us suppose that \( h(t-1) \) exists and let us consider the finite-dimensional subspace of \( MJ \) spanned by all operators of degree \( h(t-1) \) or \( h(t-1) + 1 \).

By the induction assumption, there exists a basis of this subspace that consists of operators in \( (t-1) \)-regular form: \( R(\omega_1)R(u_1) \cdots R(\omega_{t-1})R(u_{t-1})W' \). For each operator in this basis we calculate the number:

\[
d_0 \cdot f \left( \sum_{i=1}^{t-1} d(\omega_i) + d(u_i) + d(W') + d(v) \right) + d(v)
\]

and let \( k \) be the biggest of these numbers.

Define \( h(t) = h(t-1) + 1 + g(k) \).

If \( W \) is an operator of degree \( \geq h(t) \), we can assume that \( W = R(a_1)R(a_2) \cdots \), where \( a_i \) is either a generator \( x_a \) or a product of two generators \( x_a \cdot x_b \).

Let \( i \) be minimal number such that \( R(a_1) \cdots R(a_i) \) has degree \( \geq h(t-1) \).

The operator \( R(a_1) \cdots R(a_i) \) has degree \( h(t-1) \) or \( h(t-1) + 1 \). Hence \( R(a_1) \cdots R(a_i) \) is a linear combination of the basic operators \( R(\omega_1)R(u_1) \cdots R(u_{t-1})W' \) in \( (t-1) \)-regular form. The operator \( W'R(a_{i+1}) \cdots \) has degree \( \geq g(k) \). By part (a) of this lemma, \( W'R(a_{i+1}) \cdots \) is a linear combination of operators \( R(u) \cdots \) or \( R(x)R(u) \cdots \), where \( d(u) \geq k \).
Now it remains to note that operators
\[ R(\omega_1)R(u_1) \cdots R(u_{s-1})R(u) \cdots \]
and
\[ R(\omega_1)R(u_1) \cdots R(u_{s-1})R(x_j)R(u) \cdots \]
are in \( s \)-regular form. The lemma is proved.

Finally, we will prove the assertion (A). Let \( s \) be the number such that \( I(s) = 0 \) and \( m \geq f(h(s) + 1) + d(v) \). If \( V \) is an operator of degree \( \geq h(s) \), then by the part (b) of the previous lemma, \( V \) is a linear combination of operators in \( s \)-regular form. By Lemma 6, \( z^mV = \text{id} z^{m-1} \).

**COROLLARY 2.1.** If \( N(J) \) is strongly nilpotent, then there exists a finite family of homogeneous operators \( W_1, \ldots, W \) such that \( J = \sum Fz^i W_i \).

**Proof.** According to Proposition 2.1, it would be enough to prove that there exists a finite number of homogeneous multiplication operators \( W_1, \ldots, W \) such that \( J = \sum Fz^i W_i + N(J) \).

But \( \bar{J} = J/N(J) \) is a finite module over \( F[z] \), that is, there exist elements \( \bar{v}_1, \ldots, \bar{v}_h \) in \( \bar{J} \) such that \( \bar{J} = F[z]\bar{v}_1 + \cdots + F[z]\bar{v}_h \). Let \( W_i = R(v_i) \) where \( v_i \) is a preimage of \( \bar{v}_i \). Then \( \bar{J} = \sum_{i=0}^h Fz^i W_i + N(J) \).

Let us recall the construction of the Baer radical \( N(J) \) in a finitely generated Jordan algebra (which coincides with the McCrimmon radical, see [17]).

Corollary 2.1, there exist operators \( W_1, \ldots, W \) such that
\[ J = \sum Fz^i W_i + N(J)^{(r)}, \] (II)

Let us consider the function \( f \) from Lemma 6 and let \( d(\omega) = \max(d(W_i)) \),
\[ k = d_0 f(2d(\omega) + 1) + d(\omega). \] (III)

Zhevlakov proved that the power \( J^k \) of \( J \) is a finitely generated algebra. Hence, there exists a function \( Z \) such that for any \( k \), the \( J^k \) is...
generated by all words in \( x_1, \ldots, x_d \) of length \( l, k \leq l \leq \text{Zh}(k) \) (the number of generators of the algebra \( J \) is fixed).

Let \( U \) be the subspace of \( J \) generated by all words of degree \( \leq 2\text{Zh}(k) \). Then \( U \) is finite dimensional and \( U_0 = N(J)^0 \cap U \subseteq N_{\gamma_0} \) is a finite dimensional space.

If \( \gamma_0 = 0 \), then \( N(J) \) is strongly nilpotent. Let us suppose that \( \gamma_0 \neq 0 \) (so \( N(J) \) is not strong nilpotent), and let us consider the set \( \mathcal{R} = \{ \beta : U_0 \subseteq N_0 \} \). Then \( \gamma_0 \in \mathcal{R} \neq \emptyset \).

Let \( \beta_0 \) minimal element of \( \mathcal{R} \). Since \( U_0 \) is finite dimensional, \( \beta_0 - 1 \) does exist. Clearly \( \beta_0 \leq \gamma_0 \). Hence, \( N(J/N_{\beta_0 - 1}) \) is not strongly nilpotent.

Let \( I = \text{id}(U_0) \).

**Lemma 2.9.** We have \( J^k \subseteq \sum_{i \geq 2d(\omega) + 1} Fz_i^k \mathcal{W}_j + I \), where \( \{ \mathcal{W}_j \} \) are all operators of degree \( < g(k) \) and \( g \) is the function from Lemma 2.8.

**Proof.** By the result of Zhelavskov [16], an arbitrary element in \( J^k \) can be expressed by generators \( b_{\beta} \), each of them a word of length \( l \) with \( k \leq l \leq \text{Zh}(k) \). By (II) each of these generators can be expressed in the form \( \sum a_i z^i W_j + a \), with \( a \in N(J)^0 \).

Since \( d(a) = l \leq \text{Zh}(k) \) and \( a \in N(J)^0 \), it follows that \( a \in U_0 \). So \( a \in I \).

By the choice of \( k \), \( k = d_h f(2d(\omega) + 1) + d(\omega) \); it follows that \( i \geq f(2d(\omega) + 1) \). We will show that if \( i \geq f(2d(\omega) + 1) \) and \( V \) is a multiplication operator of degree \( \geq g(k) \), then \( z^i V \in \text{id}(z^{i+1}) + I \). Indeed, in view of Lemma 2.8 we only need to consider elements in the form \( z^i R(b) \) or \( z^i R(z^j) R(b) \) with \( b \in J^k \). The operator \( R(b) \) can be expressed by operators \( R(b_h), R(b_h \cdot b_h) \). Hence, without loss of generality we can assume that \( k \leq \text{deg}(b) \leq 2\text{Zh}(k) \).

Consequently \( b = \sum \beta_{pq} z^p W_q + a \), with \( a \in I \). The elements \( z^i R(z^p W_q) \) and \( z^i R(z^j) R(z^p W_q) \) lie in the ideal \( \text{id}(z^{i+1}) \), since \( i, p \geq f(2d(\omega) + 1) \) according to Lemma 2.6.

Now to prove the lemma it suffices to note that

\[
J^k = \sum b_j R(J^k) = \sum a_{ij} z^i W_j R(J^k)
\]

modulo \( I \), where \( i \geq f(2d(\omega) + 1) \). Now for an arbitrary operator \( V \in W_j R(J^k) \) with \( d(V) \geq g(k) \), we have that \( z^i V \in \text{id}(z^{i+1}) + I \). The lemma is proved.

The ideal \( N_{\beta_0} \) is a sum of ideals that are nilpotent modulo \( N_{\beta_0 - 1} \). Hence the finitely generated ideal \( I = \text{id}(U_0) \) is nilpotent modulo \( N_{\beta_0 - 1} \), so \( I \) is strongly nilpotent modulo \( N_{\beta_0 - 1} \). Now we will use the notation \( J \) for \( J/N_{\beta_0 - 1} \) and the notation \( I \) for \( I + N_{\beta_0 - 1}/N_{\beta_0 - 1} \). The ideal \( I \) is strongly nilpotent.

We have proved that \( J^k \subseteq \sum Fz_i^k \mathcal{W}_j + I \). Since the codimension of \( J^k \) in \( J \) is finite, there exists a finite family of multiplication operators \( \{ V_{\beta} \} \) such that \( J = \sum Fz_i^k \mathcal{W}_j + I \).
Now from Proposition 2.1 it follows that there exists a finite family of operators \( \{ W_j \} \) such that

\[
J = \sum Fz^j W_j.
\]  

(IV)

Remark that (IV) is an analog of the result of Bergman [1] concerning associative algebras of Gelfand-Kirillov dimension 1.

We will refer to (IV) as Bergman decomposition.

Now our strategy will be to prove that the Bergman decomposition (for a graded algebra \( J \)) implies that \( \text{GK dim}(M(J)) = 1 \).

Indeed, suppose that we have succeeded in doing this. Since for an arbitrary element \( a \in N(J) \) we have \( R(a) \in N(M(J)) \); it follows from [12] that the radical \( N(J) \) is nilpotent. This contradicts the minimality of \( \beta_0 \). Thus, we have proved (modulo our assumption) that the radical \( N(J) \) of an arbitrary graded Jordan algebra of Gelfand-Kirillov dimension 1 is nilpotent. Hence, it is strongly nilpotent. Hence, \( J \) has a Bergman decomposition. Hence \( \text{GK dim}(J) = 1 \).

Let \( J = \sum Fz^j W_j \) be a Bergman decomposition of \( J \) and let \( N' \) be the number of operators \( W_j \).

If \( J_\alpha \) is a homogeneous component of \( J \), then \( \dim(J_\alpha) \leq N' \). Indeed, \( J = \sum_{i \geq 0} Fz^i W_i \) with \( \{ W_1, \ldots, W_{N'} \} \) homogeneous multiplication operators.

If \( a \in J_\alpha \), then \( d(a) = i \cdot d_0 + \deg W_i = \alpha \), so \( i = (\alpha - \deg(W_i))/d_0 \). Consequently, \( \dim(J_\alpha) \leq N' \).

By the way, this implies that \( \text{GK dim}(J) = 1 \).

**Lemma 2.10.** There exists a function \( M(q) \) such that if \( d(a) \geq M(q) \), then \( a = \sum(b_i, z^i, c_i) \), with \( b_i, c_i \) homogeneous elements and \( q_i \geq q \).

*Proof.* Let \( \alpha = \sum a_{b_i} z^{i} W_i \), where \( a_{b_i} \in K \). Then \( d(a) = h d_0 + d(W_i) \leq h d_0 + d(\omega) \). So, \( h \geq (\deg(a) - d(\alpha))/d_0 \) (with \( d(\omega) = \max(d(W_i), \ldots, d(W_{N'})) \)).

We will find functions \( L_k(q) \) such that if \( i \geq L_k(q) \), then for arbitrary elements \( a_1, \ldots, a_k \in J \), \( z^i R(a_1) \cdots R(a_k) = \sum(b_j, z^i, c_j) \) with \( q_j \geq q \).

We define \( L_k(q) = q, L_2(q) = 2q, \ldots, L_k(q) = 2 L_{k-1}(q) = 2^{k-1}q \).

It is clear that if \( k = 1 \), then \( z^i R(a_1) = \{1, z^i, a_1\} \) is in the wanted form for every \( i \geq L_1(q) = q \).

If \( k = 2 \) and \( i \geq 2q \), then \( z^i = z^{i-q} \cdot z^q \) with \( i - q \geq q \). Then

\[
R(a_1) R(a_2) = \frac{1}{2} (R(a_1) R(a_2) - R(a_2) R(a_1)) + R(a_1) R(a_2) + R(a_2) R(a_1))
= \frac{1}{2} (D(a_1, a_2) + U(a_1, a_2) + R(a_1 \cdot a_2)).
\]
Consequently, \( z^iR(a_i)R(a_z) = \frac{1}{2}z^iD(a_1, a_2) + \frac{1}{2}(a_1, z^i, a_2) + \frac{1}{2}(1, z^i, a_1, a_2) \) and we only need to prove that \( z^iD(a_1, a_2) \) is in the wanted form.

But \( z^{i-q}D(a_1, a_2) = z^{i-q}D(a_1, a_2) + z^{q}R(z^{i-q}D(a_1, a_2)) = \{1, z^{i-1}, z^qD(a_1, a_2)\} + \{1, z^q, z^{i-q}D(a_1, a_2)\} \).

Let us suppose, by the induction assumption, that the result is true for \( k - 1 \), with \( k \geq 3 \).

If \( i \geq 2L_{k-1}(q) \), then \( z^iR(a_i) \cdots R(a_z) = z^i(\sum D(a_i, a_z)V_i + \sum \text{ operators of length } \leq k - 1) \). (Note that length \( V_i = k - 2 \).)

Now \( i = (i - L_{k-1}(q)) + L_{k-1}(q), \ i - L_{k-1}(q) \geq L_{k-1}(q) \) and \( z^iR(a_i) \cdots R(a_z) = \sum z^{i-L_{k-1}(q)}D(a_i, a_z)V_i + \sum z^iT_i \) (where operators \( T_i \) have length \( \leq k - 1 \) = \( \sum z^{i-L_{k-1}(q)}R(z^{L_{k-1}(q)}D(a_i, a_z))V_i + \sum z^iT_i \).)

Now it remains to apply the induction assumption.

To prove the lemma is enough to take \( M(q) = d_0L_{2d+1}(q) + d(\omega) \) since every operator \( W_j \) can be expressed as a product of \( \geq 2d + 1 \) right multiplication operators.

Let \( N = 4N' + 4 \).

**Lemma 2.11.** Let \( q \geq M = M(3N) \). Then \( \{z^qcz^{N}\} = c_jU(z^{N+2}) \) for some element \( c_j \).

**Proof.** Let us consider elements in the form \( \{z^i, c, z^j\} \) with \( i \) odd, \( j \geq i + 2 \), and \( i + j = 4N' + 4 \). There are \( N' + 1 \) of such elements, so they must be linearly dependent (we have seen that each homogeneous component has dimension \( \leq N' \)).

So, there is a number \( i_0 \) such that \( \{z^{i_0}, c, z^{N-i_0}\} = \sum_{i+j=N, j \geq i+2} \alpha_{ij}(z^i, c, z^j) \) and \( i \geq i_0 \) in each summand of the right-hand side.

Then, \( \{z^q, c, z^{N}\} = cU(z^q, z^{N}) = cU(z^{i_0}, z^{N-i_0})U(z^{N-i_0}, z^{q-N+i_0}) = cU(z^{2(N-i_0)}, z^{q-N+i_0}). \) But \( q = N + 2i_0 \geq 2N + 2i_0 \geq N + 2 \) if \( q \geq 2N \).

Furthermore, \( i_0 \leq (N - 2)/2 \), hence \( 2N - 2i_0 \geq N + 2 \). This implies that \( cU(z^{2(N-i_0)}, z^{q-N+i_0}) \in JU(z^{N+2}). \)

Now,

\[
\begin{align*}
cU(z^{i_0}, z^{N-i_0})U(z^{N-i_0}, z^{q-N+i_0}) &= \sum_{i+j=N} \alpha_{ij}cU(z^i, z^j)U(z^{N-i_0}, z^{q-N+i_0}) \\
&= \sum_{i,j} \alpha_{ij}cU(z^{N-i+j+i_0}, z^{q-N+j+i_0}) + \sum_{i,j} \alpha_{ij}cU(z^{q-N+i+i_0}, z^{N+j-i_0})
\end{align*}
\]

has also the wanted form, since \( q = N + j + i_0 \geq q = N + i + i_0 \geq 2N + i + i_0 \geq N + 2 \), and \( i - i_0 \geq 2, j - i_0 \geq 2 \).
Lemma 2.12. There exist numbers $M, N$ such that if $a$ is a homogeneous element of degree $\geq M$, then $aU(z^N) = (\alpha a' \cdot z + \beta a'' \cdot z^2)U(z^N)$, where $\alpha, \beta \in K$ and $a, a'$ are homogeneous elements.

Proof. Consider elements $(b, z^q, c)U(z^N)$, with $q \geq M = M(3N)$.

Then we have

$$\{b, z^q, c\}U(z^N) = z^qU(b, c)U(z^N) = \frac{1}{2}(bU(z^N, cU(z^q, z^N))$$

$$-z^N U(c, b)U(z^q, z^N) + cU(z^q, bU(z^q, z^N))).$$

By Lemma 2.11, $(z^N, b, \{z^q, c, z^N\})$ and $(z^N, c, \{z^q, b, z^N\})$ are in the wanted form. Indeed,

$$\{z^N, b, \{z^q, c, z^N\}\} = \{z^N, b, \{z^{N+2}, c_1, z^{N+2}\}\}$$

$$= (-\{b, z^{N+2}, c_1\}U(z^{N+1})$$

$$+ 2\{b, z^{N+1}, c_1U(z)\}R(z))U(z^N).$$

The verification of this identity in special Jordan algebras is straightforward. Since it depends on three variables and is linear in two of them, it remains to refer to the theorem of Glennie (see [14]).

Let $a_1, \ldots, a_t$ be all words in $x_1, \ldots, x_d$ of degree $< M$. By Lemma 2.4, for each element $a_i$ there exists a homogeneous multiplication operator $V_i$ involving only the element $z$, such that $a_iV_i = 0$. Let $V = V_1 \cdot V_2 \cdots V_t$.

Corollary 2.2. $U(z^N)V = 0$.

Proof. Let $v$ be a homogeneous element from $J$. If $d(v) < M$, then $v$ is a linear combination of $a_1, \ldots, a_t$ and then $vV = 0$.

If $d(v) \geq M$, then $uV(z^N) = (v'R(z) + v''R(z^2))U(z^N)$ (according to Lemma 2.12) and we have $vU(z^N)V = (v''U(z^N)VR(z) + v''U(z^N)VR(z^2)) = 0$, by the induction assumption.

The operator $V$ is a linear combination of operators $U(z^i, z^j)$ with $i + j$ fixed. Then we can represent $V$ in the form

$$V = U(z^{i_0}, z^{j_0}) + \sum_{i+j=i_0+j_0} \alpha_{ij}U(z^i, z^j)$$

where $i_0 \leq j_0$ and $i_0 < i \leq j$ for any $i, j$ in the second summand.
LEMMA 2.13. If $i \geq i_0 + N$, $j \geq j_0 + N$, then
\[ R(z^i)R(z^j) = \sum_{k \geq 1} \alpha_k R(z^{i+k})R(z^{j-k}). \]

Proof. We have $i_0 < j_0$. Otherwise $V = U(z^i)$ and $U(z^N)U(z^i) = U(z^{N+i_0}) = 0$, which contradicts semiprimes of $J = J/N(J)$.

Since $U(z^N)V = 0$, we have
\[ U(z^{N+i_0}, z^{N+j_0}) + \sum_{i_0 < k, l < j_0} \alpha_{kl} U(z^{k+N}, z^{l+N}) = 0. \]

We will use the equalities
\[ U(z^i, z^j)U(z^h, z^k) = U(z^{i+h}, z^{j+k}) + U(z^{i+k}, z^{j+h}) \]
and
\[ U(z^h, z^k) = 2R(z^h)R(z^k) - R(z^{h+k}). \]

Since $i \geq i_0 + N$, $j \geq j_0 + N$, we have
\[ U(z^i, z^j) = U(z^{i-i_0-N}, z^{j-j_0-N})U(z^{i_0+N}, z^{j_0+N}) - U(z^{i-i_0+j_0}, z^{j-j_0+i_0}) \]
\[ = - \sum_{i_0 < k, l < j_0} \alpha_{kl} U(z^{i-i_0-N}, z^{j-j_0-N})U(z^{k+N}, z^{l+N}) \]
\[ - U(z^{i+j_0-i_0}, z^{j-i_0-j_0}) \]
\[ = \sum_{i_0 < k, l < j_0} \alpha_{kl} \left( U(z^{i-i_0+k}, z^{j-j_0+l}) + U(z^{i+1-i_0}, z^{j+k-j_0}) \right) \]
\[ - U(z^{i+j_0-i_0}, z^{j-i_0-j_0}) \]
\[ = 2 \sum_{i_0 < k, l < j_0} \alpha_{kl} \left( R(z^{i-i_0+k})R(z^{j-j_0+l}) + R(z^{i+1-i_0})R(z^{j+k-j_0}) \right) \]
\[ - R(z^{i+j_0-i_0})R(z^{j-i_0-j_0}) - \left( 2\sum_{k,l} (\alpha_{kl} - 1)R(z^{i+j}) \right). \]

On the other hand,
\[ U(z^i, z^j) = 2R(z^i)R(z^j) - R(z^{i+j}). \]

(Note that $i + k - i_0 > i$, $i + l - i_0 > i$, $i + j_0 - i_0 > i + k - i_0 > i$, $i - j_0 < i + l - i_0 > i$, $i + j_0 < j$, $j + k - j_0 < j$, $i_0 - k - j_0 < j$, $i + l - j_0 < j$, $j + i_0 - j_0 < j + k - j_0 < j$.)
To finish the proof of the lemma it remains to note that \( R(z^i)R(z^j) = \frac{1}{2}(U(z^i, z^j) + R(z^{i+j})) \) and to substitute the above expression for \( U(z^i, z^j) \).

For a subalgebra \( B \) of \( J \), let \( M'(B) \) denote the subalgebra of \( M(J) \) generated by all multiplications \( R(b), b \in B \).

Recall that \( J = \sum_{i \geq 0, 1 \leq a \leq N'} F z^i W_a \) is a Bergman decomposition of \( J \), \( d(\omega) = \max_{1 \leq a \leq N'} d(W_a) \) and \( d \) is the number of generators of \( J \). Let \( L = 3^{d+1} \kappa \), where \( \kappa = \max(j_0 + N, d(\omega) + 1) \) and \( N = 4N' + 4 \).

**Lemma 2.14.** For any numbers \( i, j \geq L \), and any operators \( W_a, W_{\beta} \), we have \( R(z^i W_a)R(z^j W_\beta) \in M(J) \cdot M'(\text{id}(z^{i+j})) \cdot M(J) \).

**Proof.** Without loss of generality, we will assume that \( W = W' R(a) \), where \( d(W) < d(W_\beta) \).

We have:

\[
R(z^i W_a)R(z^j W_\beta) = R(z^i W_a) R((z^j W_\beta') \cdot a)
\]

\[
= -R(z^i W_\beta') R((z^j W_a) \cdot a)
\]

\[
- R(a) R(z^i W_a \cdot z^j W_\beta') + R(a) R(z^i W_a) R(z^j W_\beta')
\]

\[
+ R(z^i W_\beta') R(z^j W_a) R(a) + R((z^i W_\beta') \cdot a) (z^j W_a).
\]

Since \( i, j \geq d(\omega) + 1 \), it follows that \( z^i W_a \cdot z^j W_\beta = \sum \alpha_{h, \gamma} z^h W_\gamma \in \text{id}(z^{i+j}) \), because \( d_0 h + d(W_\gamma) = d_0 (i + j) + \deg(W_a) + \deg(W_\beta) \geq d_0 i + d(\omega) + 1 \) and consequently \( h \geq i + 1 \). In the same way, \( z^i W_a \cdot z^j W_\beta = \sum \beta_{l, \mu} z^l W_\mu \), with \( l \geq i + 1 \), that is, \( z^i W_a \cdot z^j W_\beta \in \text{id}(z^{i+j}) \).

If we keep reducing the degree of \( W_\beta \) (at most \( 2d + 1 \) times, since each \( W_\beta \) is a product of not more than \( 2d + 1 \) right multiplications), we get finally:

\[
R(z^i W_a)R(z^j W_\beta) = \sum \cdots R(z^i W_a) R(z^j) + \cdots + \sum \cdots R(z^j) R(z^i W_\beta) \cdots
\]

mod \( R(J)M'(\text{id}(z^{i+j}))R(I) \).

Let us represent \( j \) as a sum \( j = j_1 + j_2 + j_3 \), where each of \( j_1, j_2, j_3 \) is greater than or equal to the integer part of \( j/3 \). We have

\[
R(z^j) = R((z^{j_1} \cdot z^{j_2}) z^{j_3})
\]

\[
= -R(z^{j_1}) R(z^{j_2}) R(z^{j_3}) - R(z^{j_1}) R(z^{j_2}) R(z^{j_3})
\]

\[
+ R(z^{j_1} \cdot z^{j_2}) R(z^{j_3}) + R(z^{j_1} \cdot z^{j_3}) R(z^{j_2}) + R(z^{j_2} \cdot z^{j_3}) R(z^{j_1}).
\]

So, \( R(z^j) = \sum R(z^h) R(z^l) \) with \( h, l \geq [j/3] \).
Let $V = V' \cdot R(b)$ where $d(V') < d(V)$. We have:

$$R(z'V' \cdot b) R(z^h) R(z^i)$$

$$= -R(z'V' \cdot z^h) R(b) R(z') - R(z^h \cdot b) R(z'V') R(z')$$

$$+ R(b) R(z^h) R(z'V') R(z') + R(z'V') R(z^h) R(b) R(z')$$

$$+ R(((z'V') \cdot a) \cdot z^h) R(z').$$

In the first and the last summands $h \geq \kappa$, so $z'V' \cdot z^h (z'V') \cdot a) \cdot z^h \in \text{id}(z^{i+1})$.

In the other three summands, factors $R(z'V') R(z^j)$ or $R(z'V') R(z^j)$ appear. If we repeat the process at most $2d + 1$ times, we will totally get rid of the operator $V$ and arrive to the situation where every summand contains factors $R(z^j) R(z^h)$ with $h \geq \kappa$. Now it remains to refer to Lemma 2.13.

The lemma is proved.

Finally, we will see that Lemma 2.14 implies that $\text{GK dim}(R(J)) = 1$.

Zhevlakov proved that a finitely generated soluble Jordan algebra is nilpotent. This result implies the existence of a function $g(r)$ such that if $J$ is an $r$-generated Jordan algebra then $J^{g(r)} \subseteq (J^r \cdot J^r) \cdot J^r \subseteq J^r \cdot J^r$.

**Lemma 2.15.** Let $V_1, V_2$ be two homogeneous operators from $M(J)$. Let $W_\alpha$ be an operator from the Bergman decomposition of $J$ and let $i \geq L$. If $d(V_1)$ or $d(V_2)$ is greater than or equal to $(2d + 1) g(d_0 L + d(\omega))$ then

$$V_1 R(z^i W_\alpha) V_2 \in M(J) M^i (\text{id}(z^{i+1})) M(J).$$

**Proof.** Suppose that $d(V_1) \geq (2d + 1) g(d_0 L + d(\omega))$. Without loss of generality, we can assume that

$$V = R(a_1) \cdots R(a_{2d+1}),$$

where $a_i$ are homogeneous elements from $J$. At least one element $a_i$ has degree greater than or equal to $g(d_0 L + d(\omega))$. Moreover, arguing as in the proof of Lemma 2.8 (a), we can assume that $V_1$ has one of the following forms:

(i) $V_1 = \cdots R(b)$ or

(ii) $V_1 = \cdots R(b) R(x_i)$, where $d(b) \geq g(d_0 L + d(\omega))$.

In the case (i), $R(b) R(z^i W_\alpha) \in M(J) \cdot M^i (\text{id}(z^{i+1})) \cdot M(J)$ by Lemma 2.14.

Let us consider the case (ii). We have $b \in (J^{d_0 L + d(\omega)})^2$, so let us assume that $b = b_1 b_2$, $b_1, b_2 \in J^{d_0 L + d(\omega)}$. 

Then \( R(b)R(x_k) = R(b_1 b_2)R(x_k) = -R(b_1 x_k)R(b_2) - R(b_2 x_k)R(b_1) + R(x_k)R(b_1 b_2) + R(b_1)R(b_2 x_k) + R(b_2)R(b_1 x_k) \) and we can apply Lemma 2.14 to:

- \( R(b_2)R(z^k W_\alpha) \)
- \( R(b_1)R(z^k W_\alpha) \)
- \( R(b_1 b_2)R(z^k W_\alpha) \)
- \( R(x_1 b_1)R(z^k W_\alpha) \)
- \( R(x_1 b_2)R(z^k W_\alpha) \)

respectively.

The Lemma is proved.

**PROPOSITION 2.2.** \( M(J) \) is the linear span of all operators in the form: \( W'R(z^i W')W \), with \( d(W') \leq d(\omega) \) and \( d(W'), d(W) < (2d + 1)g(d_0 L + d(\omega)) \) (where \( L \) is the number from Lemma 2.14 and \( d_0 = d(z) \)).

**Proof.** Consider an expression of \( V \) in the form \( V = R(a_1) \cdots R(a_{2d+1}) \), where \( a_j \)'s are homogeneous elements. At least one element \( a_i \) has degree greater than or equal to \( g(d_0 L + d(\omega)) \). Without loss of generality we can assume that \( a_i = z^i W_\omega \). Since \( g(r) \geq r \) for any \( r \), it follows that \( i \geq L \).

Let \( V = V_1 R(z^i W_\alpha) W_2 \). If \( d(V_1), d(V_2) < (2d + 1)g(d_0 L + d(\omega)) \) then there is nothing to prove. In the other case, the previous lemma implies that

\[ V = \sum V_{ij}R(z^{i+1} W_\alpha) V_{2j}. \]

If \( d(W_{\alpha_j}) \leq d(\omega) \), then \( W_{\alpha_j} \) is a linear combination of the operators involved in the Bergman decomposition. If \( d(W_{\alpha_j}) > d(\omega) \), then we have \( z^{i+1} W_{\alpha_j} = \sum \alpha_\mu z^{i_\mu} W_\mu \), where each \( i_\mu \) is greater than \( i + 1 \). Thus, if \( d(V_1) \) or \( d(V_2) \) is again greater than \( (2d + 1)g(d_0 L + d(\omega)) \), then we can again apply Lemma 2.14. After a finite number of steps we will arrive at the expression we have been looking for. The proposition is proved.

Now Proposition 2 implies the result:

**THEOREM 2.1.** Let \( J \) be a finitely generated graded Jordan algebra and \( 1 \leq \text{GK dim}(J) < 2 \). Then \( \text{GK dim}(M(J)) = 1 \). Consequently \( \text{GK dim}(J) = 1 \). Furthermore, \( N(J) \), the Baer radical of \( J \), is nilpotent.

### 3. GENERAL CASE

In this chapter we will extend the results of the previous section to nongraded algebras. In doing so we will closely follow the methods of Small, et al. in [13].
Let $J$ be a finitely generated Jordan algebra (not necessarily graded) and GK dim$(J) < 2$. Let $x_1, \ldots, x_d$ be a finite system of generators of $J$ and let $V$ be the linear span of $x_1, \ldots, x_d$. As usual, $V^k$ is the linear span of all products of elements of length $\leq k$.

Then we have a chain of subspaces \( (0) = V^0 \subseteq V = V^1 \subseteq V^2 \subseteq V^3 \subseteq \cdots \) and we can consider $\text{gr}(J) = V^1 \oplus V^2 \oplus V^3 \oplus V^2 \oplus \cdots$, the $m$-generated graded algebra associated to $J$. Clearly GK dim$(J) = \text{GK dim}(\text{gr}(J))$.

So, results in Section 2 imply

**Corollary 3.1.** Let $J$ be a finitely generated Jordan algebra and $1 \leq \text{GK dim}(J) < 2$. Then $\text{GK dim}(J) = 1$.

By the result of Section 2, the algebra $\text{gr}(J)$ contains a homogeneous element $\bar{z}$ and operators $\bar{W}_1, \ldots, \bar{W}_s$ in $M(\text{gr}(J))$ such that $\text{gr}(J) = \sum_{i \geq 0} F\bar{z}^i \bar{W}_i$.

Let us suppose that $\bar{z} \in V^k/V^{k-1}$ and let $z$ be a representative from the coset $\bar{z}$. Similarly we can find operators $W_1, \ldots, W_s$ in $M(J)$ associated to $\bar{W}_j$ (if $\bar{W}_j = R(a_{ij}) \cdots R(a_{i1})$, then $W_j = R(a_{ij}) \cdots R(a_{i1})$, where $a_{ij}$’s are preimages of $\bar{a}_{ij}$’s).

We claim that $J = \sum_{i \geq 0, 1 \leq j \leq s} Fz^i W_j$. Indeed, let us use induction on $l$ to prove that $V^l \subseteq \sum Fz^i W_j$. For $l = 0$ it is clear. Let $c \in V^l$. There are coefficients $a_{ij} \in K$ such that $c + V^{l-1} = \sum a_{ij} z^i W_j$. Consequently $c - \sum a_{ij} z^i W_j \in V^{l-1}$ and it remains to use the induction assumption.

That is, if $J$ is a Jordan algebra generated by elements $x_1, \ldots, x_d$ and with $\text{GK dim}(J) < 2$, then there is a homogeneous expression $z = z(x_1, \ldots, x_d)$ and a finite family of operators $W_1, \ldots, W_s \in M(J)$ such that $J = \sum_{i \geq 0, 1 \leq j \leq s} Fz^i W_j$.

**Lemma 3.1 (See 13).** Let $J$ be a Jordan algebra generated by elements $x_1, \ldots, x_d$, such that $\text{GK dim}(J) < 2$. Let $N(J)$ be the McCrimmon radical of $J$. Suppose that $\bigcap_{i=1}^\infty N(J)^{(i)} = 0$ and let $I$ be a finitely generated (as an ideal) ideal of $J$ that lies in $N(J)$. Then $I$ is nilpotent.

**Proof.** Let $J_0 = J/N(J)$, $J_i = N^{(i)}/N^{(i+1)}$. Consider the graded algebra $J_0 + J_1 + J_2 + \cdots$.

Suppose that the ideal $I$ is generated (as an ideal) by elements $a_1, \ldots, a_r$.

Let $\tilde{x}_1, \ldots, \tilde{x}_d$ be the images of the elements $x_1, \ldots, x_d$ in $J_0$ (that is, $\tilde{x}_i = x_i + N(J)$) and let $\tilde{a}_1, \ldots, \tilde{a}_r$ be images of the elements $a_1, \ldots, a_r$ in $J_1$ (that is, $\tilde{a}_i = a_i + N(J)^{(2)}$).

Let $\tilde{J}$ be the subalgebra of $\sum_{i=0} r J_i$ generated by the elements $\tilde{x}_i, \tilde{a}_j$, $1 \leq i \leq d, 1 \leq j \leq r$.

We will prove that the elements $\tilde{a}_1, \ldots, \tilde{a}_r$ generate a nilpotent ideal in $\tilde{J}$. This will be enough to assure that the ideal $I$ is nilpotent in $J$. 


Indeed, let \( \text{id}_J(\tilde{a}_1, \ldots, \tilde{a}_r)^{(s)} = 0 \). This equally means that an arbitrary product \( v \) of elements \( x_i, a_j \) containing \( k \geq s \) factors \( a_j \) can be represented as a linear combination of products \( v = \sum \alpha_i v_i \), where \( \alpha_i \)’s are scalars and each \( v_i \) is a product of elements \( x_i, a_j \) containing at least \( k + 1 \) factors \( a_j \). Applying this process to each of the elements \( v_i \) and repeating it on and on, we get finally that \( v \in \bigcap_{k=1}^{\infty} I^{(k)} = (0) \). That is, \( I \) is a nilpotent ideal. So we only need to prove that the ideal \( \text{id}_J(\tilde{a}_1, \ldots, \tilde{a}_r) \) is nilpotent.

According to what was proved about the Bergman decomposition, there exists a homogeneous expression \( z \) in \( \tilde{x}_1, \ldots, \tilde{x}_d \) and a finite family of homogeneous operators \( W_1, \ldots, W_m \in M(J) \) such that \( J = \sum Fz^j W_j \).

Remark. The element \( z \) can be assumed to depend only on \( \tilde{x}_1, \ldots, \tilde{x}_m \), not on \( \tilde{a}_1, \ldots, \tilde{a}_r \). It follows from the way in which it was constructed, since \( \tilde{a}_j \in N(J) \).

For each operator \( W_j \) let \( a(W_j) \) be the number of times that the elements \( \tilde{a}_1, \ldots, \tilde{a}_r \) occur in \( W_j \). Since generators \( \tilde{x}_i \) have degree 0 in \( J = J_0 + J_1 + \cdots \) and generators \( \tilde{a}_j \) have degree 1, the \( a(W_j) \) is the degree of \( W_j \).

We have \( Fz^j W_j \subseteq \tilde{J}^{a(W_j)} \) and therefore \( \tilde{J} = \sum_{j=1}^{n} \tilde{J}^{a(W_j)} \).

Now, if \( s > a(W_j) \) for any \( j \), then \( \text{id}_J(\tilde{a}_1, \ldots, \tilde{a}_r)^{(s)} \subseteq \tilde{J}_s + \tilde{J}_{s+1} + \cdots = (0) \).

This proves the lemma.

**THEOREM 3.1.** If \( J \) is a finitely generated Jordan algebra, and \( \text{GK dim}(J) < 2 \), then its McCrimmon radical \( N_J \) is finitely generated as an ideal.

**Proof.** Let \( N = N(J) \).

Then the quotient algebra \( J = J/N \) is a semiprime Jordan finitely generated algebra and \( \text{GK dim}(J/N) \leq 1 \). So there is a subalgebra \( \overline{C} \subseteq Z(J) \) such that \( \text{GK dim}(\overline{C}) = 1 \) and \( J/N \) is finite over \( \overline{C} : J/N = \overline{u}_1 \overline{C} + \cdots + \overline{u}_r \overline{C} \). Let \( u_1, \ldots, u_r \) be preimages of the elements \( \overline{u}_1, \ldots, \overline{u}_r \) respectively. Without loss of generality we will assume that the set \( \{u_1, \ldots, u_r\} \) includes all generators of the algebra \( J \).

We have \( \overline{u}_i \overline{u}_j = \sum_{k=1}^{r} \overline{u}_k \overline{c}_{ij}^k, \overline{c}_{ij}^k \in \overline{C} \).

That is,

\[
u_i u_j - \sum_k u_k c^k_{ij} \in N.
\]

\((*)\)

Let \( D(a, b) = R(a)R(b) - R(b)R(a) \) be the inner derivation defined by the elements \( a, b \). So we can consider the finite set of elements from \( N \):

\[
u_h D\left( c^k_{ij}, u_l \right) \quad \text{and} \quad \nu_h D\left( c^k_{ij}, u_l \cdot u_t \right)
\]

\(**\)**
Let $I$ be the ideal of $J$ generated by the elements in (*) and (**) (so $I$ is finitely generated as an ideal) and let us consider the quotient $J/I$.

The images of $c_{ij}^k$ belong to the center of $J/I$. Indeed, relations (**) assure that $D(c, J) = 0$ modulo $I$, since $M(J)$ is generated by $R(u_i)$ and $R(u_i \cdot u_j)$.

In a Jordan algebra the equality $D(c, J) = 0$ implies that $c \in Z(J)$. So the subalgebra $\mathcal{A}$ of $J/I$ finitely generated by the elements $\{c_{ij}^k + I\}$ is contained in its center and $J/I = \Sigma(u_i + I)\mathcal{A}$, that is, $J/I$ is generated as an $\mathcal{A}$-module by $\{u_1 + I, \ldots, u_r + I\}$.

Consequently, every submodule of $J/I$ (in particular, $N/I$) is finitely generated. But $I$ is finitely generated as an ideal. Hence $N$ is finitely generated as an ideal.

Now, the two previous lemmas imply

**Corollary 3.2.** If $J$ is a Jordan algebra generated by elements $x_1, \ldots, x_d$ such that $\text{GK dim}(J) < 2$ and $\bigcap_{k=1}^n N(J)^{(k)} = 0$, then $N(J)$ is nilpotent.

Let $J$ be a Jordan algebra generated by a finite dimensional subspace $V$. As usual, we consider the filtration $V = V^1 \subset V^2 \subset \ldots$, where $V^n$ denote the linear span of all products of elements from $V$ of length $\leq n$.

If $I$ is the ideal of $J$ generated by $Y$, a finite dimensional subspace of $J$, and $Y \subset V^{(n)}$, we can consider the homomorphism $- : J \to J/I = J/V$ the image of $V$, and the filtration $\mathcal{V} = \mathcal{V}^1 \subseteq \mathcal{V}^2 \subseteq \ldots$.

**Lemma 3.2.** If the ideal $I$ is infinite dimensional, then for any $n \geq s$ we have $\dim(\mathcal{V}^{n+2}/\mathcal{V}^n) < \dim(\mathcal{V}^{n+2}/\mathcal{V}^n)$.

**Proof.** The mapping $J \to J$ induces another mapping $V^{n+2}/V^n \to \mathcal{V}^{n+2}/\mathcal{V}^n$.

The kernel of the second mapping is $V^{n+2} \cap (I + V^n)/V^n$. If this kernel is zero, then $V^{n+2} \cap (I + V^n) = V^n$ and, in particular, $V^{n+2} \cap I = V^n \cap I$.

Let us show that this equality implies that

$$V^n \cap I = V^{n+1} \cap I = V^{n+2} \cap I = V^{n+3} \cap I = \ldots$$

and, thus, $I$ lies in $V^n$ and is finite dimensional.

The ideal $I$ is generated by the subspace $Y \subseteq V^* \subseteq V$. Since the multiplication algebra $M(J)$ is generated by $R(V)$ and $R(V^2)$, it follows that $I$ is spanned by elements of the type $aR(u_i) \cdots R(u_i)$, where $a \in I \cap V^n$ and each element $u_i$ lies either in $V$ or in $V^2$.

We have $aR(u_i) \in V^{n+2} \cap I = V^n \cap I$. Hence, $aR(u_i)R(u_j) \in (V^n \cap I)R(u_j) \leq V^{n+2} \cap I = V^n \cap I$ and so on. Finally, we get that $aR(u_i) \cdots R(u_i) \in V^n \cap I$ and $I \leq V^n$. The lemma is proved.
THEOREM 3.2. Let $J$ be a Jordan algebra generated by a finite dimensional subspace $V$ with $\text{GK dim}(J) < 2$. Then $N(J)$ is nilpotent.

Proof. Let $H = \cap_{k=1}^\infty N(J)^{(k)}$. By Corollary 3.1 the radical of $J/H$ is nilpotent. Hence, there exists $i \geq 1$ such that $N(J)^{(i)} = H$.

Let $I$ be an infinite dimensional finitely generated ideal of $J$. The induction on the upper limit, $\limsup_{n \to \infty} \dim(V^{n+2}/V^n)$, shows that the radical of $J/I$ is nilpotent, according to the previous lemma. Hence, there exists $q \geq 1$ such that $N(J)^{(q)} \subseteq I$. But $H \subseteq N(J)^{(q)}$. Hence, if $I \subseteq H$, then $I = H$.

We have proved that if $I$ is a finitely generated ideal of $J$ that lies in $H$, then either $I = H$ or $\text{dim}_F I < \infty$.

Let $h$ be an arbitrary element from $H$ and $I = \text{id}_J(h^2)$.

If $I = H$, then $h \in \text{id}_J(h^2)$. Let us show that this is impossible. Indeed, if $h \in \text{id}_J(h^2)$, then there exists a multiplication operator $W \in M(J)$ such that $h = hR(h)W$.

Iterating, we get $h = h(R(h)W)^n$ for any $n \geq 1$. But $R(h)$ lies in the locally nilpotent radical of $M(J)$ (see [11]), hence $R(h)W$ is nilpotent, the contradiction.

Hence, $\dim_F I < \infty$. Then $I$ is a finite dimensional bimodule over $J$ and a finite dimensional module over $M(J)$. If $k \geq \dim_F I$ and $\text{Jac}(M(J))$ is the Jacobson radical of $M(J)$, then $I \cdot \text{Jac}(M(J))^k = (0)$.

Now, $R(N(J)^{(2k)}) \subseteq \text{Jac}(M(J))^k$.

Hence, $I \cdot N(J)^{(2k)} = (0)$ and consequently $I \cdot H = (0)$.

Hence $h^2 \cdot H = 0$ for an arbitrary element $h \in H$ and so $H^3 = (0)$. Hence, $N(J)$ is solvable and $J$ is P.I. By the theorem of Medvedev (see [8], $N(J)$ is nilpotent.

REFERENCES