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INCOMPLETE CONJUGATE ORTHOGONAL IDEMPOTENT LATIN SQUARES

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Let us denote by COILS(v) a (3, 2, 1)-conjugate orthogonal idempotent Latin square of order v, and by ICOILS(v, n) an incomplete COILS(v) missing a sub-COILS(n). We shall investigate the existence of ICOILS(v, n). The construction of an ICOILS(8, 2) has already been instrumental in the construction of a COILS(26), the existence of which was unknown for some time. A necessary condition for the existence of an ICOILS(v, n) is $v \geq 3n+1$. In this paper, it is shown that for all $n \geq 1$, an ICOILS(v, n) exists if $v = 3n+1$ or $v \geq 8n+42$. Moreover, for $2 \leq n \leq 6$, it is shown that an ICOILS(v, n) exists for all $v \geq 3n+1$ with very few possible exceptions.

1. Introduction

A *Latin square* of order v is a $v \times v$ matrix array (a_{ij}) based on the elements of a v-set, say $S = \{1, 2, \ldots, v\}$, such that in each row and in each column every element occurs exactly once. The Latin square (a_{ij}) is called *idempotent* if $a_{ii} = i$ for all i, $1 \leq i \leq v$. Two Latin squares (a_{ij}) and (b_{ij}), each based on the set S, are *orthogonal* if for each $(s, t) \in S \times S$ there exists a unique ordered pair (i, j) such that $a_{ij} = s$ and $b_{ij} = t$.

If $(Q, \otimes)$ is a quasigroup, we may define on the set Q six binary operations $\otimes(1,2,3)$, $\otimes(1,3,2)$, $\otimes(2,1,3)$, $\otimes(2,3,1)$, $\otimes(3,1,2)$ and $\otimes(3,2,1)$ as follows: $a \otimes b = c$ if and only if

$$a \otimes (1,2,3) b = c, \qquad a \otimes (1,3,2) c = b, \qquad b \otimes (2,1,3) a = c,$$
$$b \otimes (2,3,1) c = a, \qquad c \otimes (3,1,2) a = b, \qquad c \otimes (3,2,1) b = a.$$

These six (not necessarily distinct) quasigroups $(Q, \otimes(i, j, k))$ are called the *conjugates* of $(Q, \otimes)$ (see Stein [15]). If the multiplication table of a quasigroup $(Q, \otimes)$ defines a Latin square L, then the six Latin squares defined by the multiplication tables of its conjugates $(Q, \otimes(i, j, k))$ are called the conjugates of L. For more information on Latin squares and quasigroups, the interested reader may refer to the book of Dénes and Keedwell [5].

A Latin square which is orthogonal to its (i, j, k)-conjugate will be called (i, j, k)-*conjugate orthogonal*, where $\{i, j, k\} = \{1, 2, 3\}$. A $(2, 1, 3)$-conjugate orthogonal Latin square is usually called *self-orthogonal*, and it is well-known [2] that such squares exist for all orders $v \neq 2, 3, 6$. In [14], Phelps further proved that a $(3, 1, 2)$ (or $(2, 3, 1)$)-conjugate orthogonal Latin square exists for all orders $v \neq 2, 6$ and that a $(3, 2, 1)$ (or $(1, 3, 2)$)-conjugate orthogonal Latin square exists for all orders $v \neq 2, 6$ with the possible exception of $v = 14$ and $v = 26$. More recently, Bennett [1] has proved that a $(3, 2, 1)$ (or $(1, 3, 2)$)-conjugate orthogonal *idempotent* Latin square of order v, briefly COILS(v), exists for all orders $v \neq 2, 3, 6$ with the possible exception of $v = 12$, 14 and 18.

A pair of *incomplete orthogonal* Latin squares of order v missing a subsquare of order n, and each based on the set $S \cup T$, where $S = \{1, 2, \ldots, v - n\}$ and $T = \{v - n + 1, \ldots, v\}$, is a pair of order v Latin arrays which are orthogonal as Latin squares except for a common $n \times n$ subsquare which gives rise to the loss of all ordered pairs from $T \times T$. (For a more detailed description and resuls on incomplete orthogonal Latin squares, the reader may refer to [8–12, 17, 18]). An incomplete Latin square (a_{ij}), based on the set $S \cup T$ above, will be called idempotent provided that $a_{ii} = i$ for all i, $1 \leqslant i \leqslant v - n$. An incomplete (idempotent) Latin square which is orthogonal to its (i, j, k)-conjugate will be called an incomplete (i, j, k)-conjugate orthogonal (idempotent) Latin square, briefly (i, j, k)-ICOLS (or (i, j, k)-ICOILS).

Recently, some attention has been focused on the existence of incomplete self-orthogonal Latin squares and, in fact, it has been proved in [10] that a self-orthogonal Latin square of order v, containing or missing a self-orthogonal Latin subsquare of order n, exists if and only if $v \geqslant 3n + 1$ with the exception of $(v, n) = (6, 1)$ and $(8, 2)$ and the possible exception of $(v, n) \in \{(6m + i, 2m) \mid i = 2, 6\}$. In this paper, we shall be restricting our attention to incomplete $(3, 2, 1)$-conjugate orthogonal idempotent Latin squares of order v missing subsquares of order n, briefly ICOILS(v, n). A necessary condition for the existence of an ICOILS(v, n) is $v \geqslant 3n + 1$. We shall prove that for all $n \geqslant 1$, an ICOILS(v, n) exists if $v = 3n + 1$ or $v \geqslant 8n + 42$. Moreover, for $2 \leqslant n \leqslant 6$ it is shown that an ICOILS(v, n) exists if $v \geqslant 3n + 1$ with the following possible exceptions,

$n = 2$, $v \in \{11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 23, 24\}$,

$n = 3$, $v \in \{11, 12, 13, 14, 15, 17, 18, 20, 21, 24, 25, 26, 28, 29, 30, 32, 33\}$,

$n = 4$, $v \in \{14, 15, 18, 19, 23, 27\}$,

$n = 5$, $v \in \{18, 19, 22, 23, 28, 30, 34\}$,

$n = 6$, $v \in \{20, 21, 23, 24, 27, 28, 29, 32, 33, 35, 36, 39, 40, 44, 45, 47, 52, 53\}$.

It is perhaps worth mentioning that the construction of an ICOILS(8, 2) played an important role in the construction of a COILS(26) (see [1]), the existence of which was unknown for some time.

The construction of ICOILS(v, n) is not only of importance in providing solutions to some embedding problems, but is also useful in the construction of complete Latin squares with interesting properties which can be related to other combinatorial designs as illustrated in [1]. We shall use both recursive and direct methods in our constructions. The direct method of construction will be of the "starter-adder" type where the plan is to use the first row and last n elements of the first column of the ICOILS(v, n) to cyclically generate the entire array. The recursive type of construction will include the generalized (singular) direct product construction and the concept of pairwise balanced designs (PBDs).

2. Preliminaries

In this section we shall define some of the auxiliary designs and state some of the fundamental results which will be used later on. The reader is referred to [7, 16] for more general information on designs, in particular pairwise balanced designs (PBDs) and group divisible designs (GDDs).

Definition 2.1. Let K be a set of positive integers. A *pairwise balanced design* (PBD) of index unity $B(K, 1; v)$ is a pair (X, B), where X is a v-set (of *points*) and B is a collection of subsets of X (called *blocks*) with sizes in K such that every pair of distinct points of X is contained in eactly one block of B. The number $|X| = v$ is called the *order* of the PBD.

Definition 2.2. Let K and M be sets of positive integers. A *group divisible design* (GDD) (GD(K, 1, M; v) is a triple (X, G, B) where

(i) X is a v-set (of *points*),

(ii) G is a collection of non-empty subsets of X (called *groups*) with sizes in M and which partition X,

(iii) B is a collection of subsets of X (called *blocks*) each with size at least two in K,

(iv) no block meets a group in more than one point, and

(v) each pairset $\{x, y\}$ of points not contained in a group is contained in exactly one block.

We shall write $B(k, 1; v)$ for $B(\{k\}, 1; v)$ and similarly GD(k, 1, m; v) for GD($\{k\}$, 1, $\{m\}$; v). We observe that a *balanced incomplete block design* (BIBD) with parameters v, k and $\lambda = 1$ is a $B(k, 1; v)$. We shall also adapt the notation of Brouwer [3]: if $k \notin K$, then $B(K \cup \{k^*\}, 1; v)$ denotes a PBD $B(K \cup \{k\}, 1; v)$ which contains a unique block of size k; and if $k \in K$, then $B(K \cup \{k^*\}, 1; v)$ is a PBD $B(K, 1; v)$ containing at least one block of size k. For convenience, we shall denote by $B(k_1, k_2, \ldots, k_r)$ the set of all positive integers v such that there is a PBD $B(\{k_1, k_2, \ldots, k_r\}, 1; v)$.

Definition 2.3. A *transversal design* (TD) $T(k, 1; m)$ is a GDD $GD(k, 1, m; km)$, where each block is a transversal of the collection of groups.

Definition 2.4. Let (X, B) be a PBD $B(K, 1; v)$. A *parallel class* in (X, B) is a collection of disjoint blocks of B, the union of which is X. (X, B) is called *resolvable* if the blocks of B can be partitioned into parallel classes. A GDD $GD(K, 1, M; v)$ is resolvable if its associated PBD $B(K \cup M, 1; v)$ is resolvable with groups as a parallel class of the resolution.

We remark that the existence of a resolvable TD $RT(k, 1; m)$ is equivalent to the existence of a $T(k+1, 1; m)$ or $(k-1)$ mutually orthogonal Latin squares of order m. The following theorems are well-known and the appropriate references are cited.

Theorem 2.5. *For every prime power q, there exists a $T(q+1, 1; q)$ (see [7]).*

Theorem 2.6. *A $T(5, 1; m)$ exists for all positive integers m with the exception of $m = 2, 3, 6$ and possibly excepting $m = 10$ and $m = 14$ (see [4, 7]).*

Theorem 2.7. *A $B(4, 1; v)$ exists if and only if $v \equiv 1$ or $4 \pmod{12}$ (see [7]).*

Theorem 2.8. *If m is a positive integer, then $m \in B(4, 7^*)$ if and only if $m \equiv 7$ or $10 \pmod{12}$, $m \neq 10, 19$ (see [9]).*

3. Recursive constructions

In this section we shall state some recursive constructions for ICOILS(v, n), which are the standard constructions for Latin squares (see [5]).

Lemma 3.1. *If there is a PBD $B(K \cup \{n^*\}, 1; v)$ and there is a COILS(k) for every $k \in K$, then there is an ICOILS(v, n).*

Lemma 3.2. *If $m \neq 2, 3, 6, 10, 12, 14, 18$ and $0 < k < m$, then there is an ICOILS$(4m + k, s)$ for $s = k$ and for $s = 4, 5, m$ provided that $k \neq 2, 3, 6, 12, 14, 18$.*

Proof. From Theorem 2.6 we get a $GD(5, 1, m; 5m)$. Delete $m - k$ points from one group, we get a PBD $B(\{4, 5, m\} \cup \{k^*\}, 1; 4m + k)$. Then the conclusion follows from Lemma 3.1. □

Lemma 3.3. *If $m \neq 2, 3, 5, 6, 10, 11, 13, 14, 17$ and $0 < k < m$, then there is an*

ICOILS$(4m+k+1, s)$ *for* $s=k+1$ *and for* $s=4, 5, m+1$ *provided* $k \neq 1, 2, 5, 11, 13, 17$.

Proof. The PBD in the proof of Lemma 3.2 is also a GD$(\{4, 5\}, 1, \{k, m\}; 4m+k)$. We add one point to each group and get a PBD $B(\{4, 5, m+1\} \cup \{(k+1)^*\}, 1; 4m+k+1)$. Then the conclusion follows. □

The following lemma is a consequence of [6, Lemma 2.11] and Lemma 3.1.

Lemma 3.4. *If m is a prime power, $m \geqslant 5$, and $0 \leqslant k \leqslant m-3$, then there is an* ICOILS$(4m+k, s)$ *for* $s=4+k$, *for* $s=4, m$ *provided* $k \neq 2, 8, 10, 14$ *and for* $s=5$ *provided* $k \neq 0, 2, 8, 10, 14$.

The next construction is a generalization of that in [13].

Lemma 3.5. *Suppose (X, B) is a* PBD $B(K, 1; v)$ *admitting t partitions of X $B_{i1}, B_{i2}, \ldots, B_{ik_i}$, $1 \leqslant i \leqslant t$, into disjoint blocks and all the blocks in these partitions are distinct. For every i, $1 \leqslant i \leqslant t$, suppose there is an integer n_i such that an* ICOILS$(|B_{ij}|+n_i, n_i)$ *exists for every j, $1 \leqslant j \leqslant k_i$. Let B^- be the collection of blocks not belonging to any partition. Suppose there is a* COILS(m) *for any block in B^- with size m. Then there is an* ICOILS$(v+n, n)$ *where $n = n_1+n_2+\cdots+n_t$. Moreover, if there is a* COILS(n), *then there is an* ICOILS$(v+n, m)$ *for any size m block in B^-.*

Apart from the above PBD recursive constructions, we have the following direct product and singular direct product constructions.

Lemma 3.6. *If there are* COILS(m) *and* COILS(n), *then there are* COILS(mn) *and* ICOILS(mn, k), $k=m$ *or* n.

Lemma 3.7. *If there are* COILS(v), COLS(m) *and* ICOILS$(m+k, k)$, *then there is an* ICOILS$(vm+k, p)$, $p=k$ *or* $m+k$. *Moreover, if there is a* COILS(p), $p=k$ *or* $m+k$, *then there is an* ICOILS$(vm+k, v)$.

4. Direct construction

Let $L_1=(a_{ij})_{m\times m}$, $a_{ij}=\lambda i+(1-\lambda)j$, $\lambda, i, j \in Z_m$. Since $a_{i+1,j+1}-a_{ij}=1$, L_1 can be generated cyclically from its 0th column. If $(\lambda, m)=(1-\lambda, m)=1$, then L_1 is a Latin square.

Let $k=\lambda i+(1-\lambda)j$, $i=\lambda^{-1}k+(1-\lambda^{-1})j$. The (3, 2, 1)-conjugate of L_1 will be $L_2=(b_{ij})$, $b_{ij}=\lambda^{-1}i+(1-\lambda^{-1})j$. L_2 can also be generated cyclically from its 0th column.

If the differences of the elements in the 0th column

$$a_{i0}-b_{i0}=\lambda i-\frac{1}{\lambda}i=\left(\lambda-\frac{1}{\lambda}\right)i,\quad i\in Z_m,$$

are all distinct, then L_1 and L_2 are orthogonal.

Lemma 4.1. *If there is an integer $\lambda\in Z_m$ such that $(\lambda(\lambda-1)(\lambda+1),m)=1$, then $L_1=(a_{ij})$, $a_{ij}=\lambda i+(1-\lambda)j$, $\lambda, i, j\in Z_m$ is a* COILS(m).

Corollary 4.2. *If $(m,6)=1$, then there is a* COILS(m).

Now we prolongate the square in Lemma 4.1. Let n be 3. For the entries $a_{i_10}=\lambda i_1$ and $a_{i_1+j,j}$ replace them by a new element x_1 and translate them into a new row and a new column, say x_1th row and x_1th column. Do it for i_2 and i_3. For the conjugate we get the corresponding translation to the entries in the 0th column i_1, i_2 and i_3. Suppose $K=(i_1,i_2,i_3)$. By relabelling the x_i's in the 0th column of the square, we get a reordering K' of i_1, i_2, i_3 in the 0th column of the conjugate, shown in Table 1, where $t=1/\lambda$.

Table 1

Row	0th column in square	0th column in conjugate
0	0	0
$\vdots$	$\vdots$	$\vdots$
i_1	λi_1 x_1	ti_1
i_2	λi_2 x_2	ti_2
i_3	λi_3 x_3	ti_3
$\vdots$	$\vdots$ relabelling	$\vdots$
λi_1	$\lambda^2 i_1$	i_1 x_1
λi_2	$\lambda^2 i_2$	i_2 x_2
λi_3	$\lambda^2 i_3$	i_3 x_3
$\vdots$	$\vdots$	$\vdots$
x_1	λi_1	i_1
x_2	λi_2	i_2 reordering
x_3	λi_3	i_3 K'

For the differences in the 0th column there are six ones missing, i.e.,

$$\left\{\lambda K-\frac{1}{\lambda}K,\lambda^2K-K\right\}.$$

From the x_tth rows we get three differences $\lambda K-K'$.

Consider the 0th row elements in the x_tth columns, they are $(\lambda-1)K$ in the square and $-(\lambda-1)K$ in the conjugate. So we get the other three differences $2(\lambda-1)K$.

It is obvious that the square is orthogonal to its conjugate if

$$\left\{\left(\lambda-\frac{1}{\lambda}\right)K,\ (\lambda^2-1)K\right\}=\{\lambda K-K',\ 2(\lambda-1)K\}.$$

In general we have

Lemma 4.3. *Suppose $\lambda \in Z_m$ such that $(\lambda(\lambda-1)(\lambda+1), m)=1$. Suppose $i_1, i_2, \ldots, i_n \in Z_m$ such that $i_1, i_2, \ldots, i_n, \lambda i_1, \lambda i_2, \ldots, \lambda i_n$ are all distinct and that*

$$\left\{\frac{\lambda^2-1}{\lambda}K,\ (\lambda^2-1)K\right\}=\{\lambda K-K',\ 2(\lambda-1)K\}, \tag{1}$$

where $K=(i_1, i_2, \ldots, i_n)$ and K' is a reordering of K. Then there is an ICOILS$(m+n, n)$.

Remark 4.4. If $K'=K$, then (1) can be replaced by

$$\left\{\frac{\lambda+1}{\lambda},\ \lambda+1\right\}K=\{1,2\}K. \tag{2}$$

Example 4.5. Let $m=5$, $\lambda=2$, $i_1=2$, $i_2=3$, $K'=K$. So,

$$\{\tfrac{3}{2}, 3\}K=\{1,2,3,4\}, \qquad \{1,2\}K=\{1,2,3,4\}.$$

We get the ICOILS(7, 2) as follows, where $x=x_1$ and $y=x_2$,

0	4	y	x	1	3	2
2	1	0	y	x	4	3
x	3	2	1	y	0	4
y	x	4	3	2	1	0
3	y	x	0	4	2	1
4	0	1	2	3		
1	2	3	4	0		

and its conjugate

0	x	1	4	y	2	3
y	1	x	2	0	3	4
1	y	2	x	3	4	0
4	2	y	3	x	0	1
x	0	3	y	4	1	2
2	3	4	0	1		
3	4	0	1	2		

Corollary 4.6. *In Lemma* 4.3 *if m is an odd prime number having* 2 *as a quadratic non-residue and* $n = \frac{1}{2}(m-1)$, *then there is an* ICOILS$(m+n, n)$.

Proof. In Lemma 4.3, take $\lambda = 2$, $K = (i_1, i_2, \ldots, i_n)$ consisting of all the quadratic residues. Then

$$\left\{\frac{\lambda+1}{\lambda}, \lambda+1\right\}K = Z_m \setminus \{0\} = \{1, 2\}K. \quad \square$$

Corollary 4.7. *In Lemma* 4.3, *suppose K is a multiplicative subgroup of* Z_m^*, $2 \notin K$ *and* $\frac{2}{3} \in K$, $|K| = n$. *Then there is an* ICOILS$(m+n, n)$.

Proof. Let $\lambda = 2$, then $\lambda K \cap K = \emptyset$ since $2 \notin K$. On the other hand, $3K = 2K$ and $\frac{3}{2}K = K$ since $\frac{2}{3} \in K$. We then have

$$\left\{\frac{\lambda+1}{\lambda}, \lambda+1\right\}K = \{1, 2\}K. \quad \square$$

5. Main results

Theorem 5.1. *An* ICOILS$(3n+1, n)$ *exists if* $n \geqslant 1$.

Proof. From Bennett [1] there is a COILS(n) if $n > 3$ and $n \notin \{6, 12, 14, 18\}$. We also have COLS(3) and ICOILS(4, 1). In Lemma 3.7, take $v = n$, $m = 3$ and $k = 1$, we then get an ICOILS$(3n+1, n)$. For $n = 1$ the conclusion is obvious. For $n = 2$, we have an ICOILS(7, 2) in Example 4.5. For $n = 3$, an ICOILS(10, 3) can be derived from the example in [14]. In Corollary 4.6, let $n = 6, 14, 18$. Then $m = 2n+1$ is a prime number, in each case having 2 as a quadratic non-residue. Therefore an ICOILS$(3n+1, n)$ exists for $n = 6$, 14 and 18. For the last case $n = 12$, we first product COILS(4) with ICOLS(9, 3). Then we add a new row and column and using these replace the ICOLS(9, 3) on the main diagonal with ICOILS(10, 3), preserving the missing elements of the order 3 subarrays and their position. This gives the required ICOILS(37, 12). $\square$

Theorem 5.2. *If* $n = 2, 4, 5$, *an* ICOILS(v, n) *exists for all* $v \geqslant 3n+1$ *with the following possible exceptions,*

$$n = 2, \quad v \in \{11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 23, 24\},$$
$$n = 4, \quad v \in \{14, 15, 18, 19, 23, 27\},$$
$$n = 5, \quad v \in \{18, 19, 22, 23, 28, 30, 34\}.$$

Proof. Firstly, we have an ICOILS(7, 2) in Example 4.5. For $n = 2$, we have the other three consecutive cases as follows.

0	y	3	2	1	x	4	5	
x	1	y	4	3	2	5	0	
3	x	2	y	5	4	0	1	
5	4	x	3	y	0	1	2	
1	0	5	x	4	y	2	3	ICOILS(8, 2)
y	2	1	0	x	5	3	4	
2	3	4	5	0	1			
4	5	0	1	2	3			

0	6	5	y	x	2	1	3	4	
2	1	0	6	y	x	3	4	5	
4	3	2	1	0	y	x	5	6	
x	5	4	3	2	1	y	6	0	
y	x	6	5	4	3	2	0	1	
3	y	x	0	6	5	4	1	2	ICOILS(9, 2)
5	4	y	x	1	0	6	2	3	
1	2	3	4	5	6	0			
6	0	1	2	3	4	5			

0	y	6	4	1	3	5	x	2	7	
x	1	y	7	5	2	4	6	3	0	
7	x	2	y	0	6	3	5	4	1	
6	0	x	3	y	1	7	4	5	2	
5	7	1	x	4	y	2	0	6	3	
1	6	0	2	x	5	y	3	7	4	
4	2	7	1	3	x	6	y	0	5	ICOILS(10, 2)
y	5	3	0	2	4	x	7	1	6	
3	4	5	6	7	0	1	2			
2	3	4	5	6	7	0	1			

Secondly, for $v > 50$ we write $v = 4m + k$, $m > 10$ and $k = 7, 8, 9$ or 10. From Lemma 3.2 we get an ICOILS(v, s) for $s = 4, 5, k$ and then an ICOILS(v, 2) if $m \neq 12$, 14 and 18. Similarly, Lemma 3.3 takes care of the cases $m = 12$ and 18.

For $m=14$ and $n=2$, we write $v=8.7+k$. From Theorem 2.5 we have a resolvable GD(7, 1, 8; 56). Then we use Lemma 3.5 with ICOILS(8, 1) and ICOILS(9, 2) to get an ICOILS(v, k) and an ICOILS(v, 2). For $m=14$ and $n=4$, 5, Lemma 3.2 takes care of the following

$$4\cdot 14+7=4\cdot 13+11, \qquad 4\cdot 14+8=4\cdot 15+4, \qquad 4\cdot 14+9=4\cdot 15+5.$$

For the remaining case $v=4\cdot 14+10=7\cdot 7+17$, we use resolvable GD(7, 1, 7; 49) and Lemma 3.5 with ICOILS(9, 2) and ICOILS(10, 3) to get an ICOILS(66, 17). Since an ICOILS(17, n), $n=4$, 5, exists from Lemma 3.2 with $17=4\cdot 4+1$, we have an ICOILS(66, n) for $n=4$, 5.

Next, for $34<v\leqslant 50$, the orders 35, 36, 37, 39, 40, 41, 42, 47, 48, 49 and 50 can be taken care of by Lemma 3.4. The remaining orders can be done as follows, where ICOILS(v, 2) comes from ICOILS(v, k), $k=7$, 8, 9 or 10.

$38=(5\cdot 8-4)+2$	Lemma 3.5 (delete four points from one block in $T(5, 1; 8)$, then add two points to the groups),
$43=4\cdot 9+7$	Lemma 3.2,
$44=4\cdot 9+8$	Lemma 3.2,
$45=4\cdot 9+8+1$	Lemma 3.3,
$46\in B(4, 5, 7, 8)$	Hanani [7, p. 277].

Finally, the conclusion is obvious for $v<17$ and we will deal with the case $17\leqslant v\leqslant 34$. The orders 19 and 23 are possible exceptions and the orders 29, 31, 32 and 33 can be done by Lemma 3.4. Order 26 comes from the construction of the COILS(26) in [1] which contains some subsquares of size 8, 4 and 5. The remaining orders can be done as follows, shown in Table 2, where "?" indicates the possible exceptions. □

Table 2

v	$n=2$	$n=4$	$n=5$
$17=4\cdot 4+1$	?	Lemma 3.7	Lemma 3.7
$18=4\cdot 4+2$	Lemma 3.2	?	?
$20=4\cdot 5$	?	Lemma 3.6	Lemma 3.6
$21=4\cdot 5+1$	?	Lemma 3.7	Lemma 3.7
$22=4\cdot 5+2$	Lemma 3.7	Lemma 3.7	?
$24=4\cdot 5+4$	?	Lemma 3.2	Lemma 3.2
$25=8\cdot 3+1$	Lemma 3.7	Lemma 3.7	Lemma 3.6 ($25=5\cdot 5$)
$27=5\cdot 5+2$	Lemma 3.7	?	Lemma 3.7
$28=4\cdot 7$	Lemma 3.6	Lemma 3.6	?
$30=4\cdot 7+2$	Lemma 3.7	Lemma 3.7	?
$34\in B(4, 7)$	Theorem 2.8	Theorem 2.8	?

Theorem 5.3. *An* ICOILS(v, 3) *exists if* $v \geq 10$ *and* $v \notin \{11, 12, 13, 14, 15, 17, 18, 20, 21, 24, 25, 26, 28, 29, 30, 32, 33\}$.

Proof. The case $v = 10$ is done in Theorem 5.1. For $v = 16$, we use Lemma 4.3 with $m = 13$, $\lambda = 8$ and $K' = K = (1, 3, 9)$. For $v = 22$, we use Corollary 4.7 with $m = 19$ and $n = 3$.

If $v \equiv 0 \pmod 4$, write $v = 4m + 16$. When $m > 16$ we can get from Lemma 3.2 and Lemma 3.3 an ICOILS(v, 16) and then an ICOILS(v, 3). Lemmas 3.6 and 3.7 will take care of the following

$$\begin{aligned} m = 6, \quad & v = 4 \cdot 6 + 16 = 4 \cdot 10, \\ 9, \quad & 4 \cdot 9 + 16 = 7 \cdot 7 + 3, \\ 11, \quad & 4 \cdot 11 + 16 = 5 \cdot 11 + 5, \\ 12, \quad & 4 \cdot 12 + 16 = 4 \cdot 16, \\ 13, \quad & 4 \cdot 13 + 16 = 5 \cdot 13 + 3, \\ 15, \quad & 4 \cdot 15 + 16 = 5 \cdot 15 + 1, \\ 16, \quad & 4 \cdot 16 + 16 = 5 \cdot 16. \end{aligned}$$

For $m = 7$, $v = 4 \cdot 7 + 16 = 4 \cdot 9 + 7 + 1$, we can use Lemma 3.3 to get an ICOILS(44, 10) and then an ICOILS(44, 3). For $m = 8$, $v = 4 \cdot 8 + 16 = 5 \cdot 8 + 8$, we have a resolvable GD(5, 1, 8; 40) from Theorem 2.5. In Lemma 3.5 we use four partitions one of which is the groups. Since there are ICOILS(u, 2) for $u = 7$, 8, 10, there is an ICOILS(48, 10) and then an ICOILS(48, 3). Similarly, for $v = 4 \cdot 10 + 16 = 8 \cdot 5 + 16$ and $v = 4 \cdot 14 + 16 = 8 \cdot 7 + 16$, we can also use Lemma 3.5 with ICOILS(7, 2) and ICOILS(9, 2) to get an ICOILS(v, 16) and then an ICOILS(v, 3). Finally, for $v = 4 \cdot 5 + 16 = 5 \cdot 7 + 1$, we first product COILS(5) with ICOILS(7, 2) like we did in the proof of Theorem 5.1. Then we use an ICOILS(8, 2) to get an ICOILS(36, 10) and an ICOILS(36, 3).

If $v \equiv 2 \pmod 4$, write $v = 4m + 10$. When $m > 10$ and $m \neq 14$, we use Lemmas 3.2 and 3.3 to get an ICOILS(v, 10) and an ICOILS(v, 3). The remaining cases can be done by Lemmas 3.6 and 3.7 as follows

$$\begin{aligned} &34 = 4 \cdot 8 + 2, \qquad 38 = 5 \cdot 7 + 3, \qquad 42 = 5 \cdot 8 + 2, \\ &46 = 5 \cdot 9 + 1, \qquad 50 = 5 \cdot 10, \qquad 66 = 9 \cdot 7 + 3. \end{aligned}$$

If $v \equiv 3 \pmod 4$, write $v = 4m + 3$. When $m > 3$ and $m \neq 6, 10, 14$, an ICOILS(v, 3) comes from Lemmas 3.2 and 3.3. The cases $43 = 4 \cdot 9 + 7$ and $59 = 8 \cdot 7 + 3$ can be done by Lemmas 3.3 and 3.7 respectively. Finally, an ICOILS(27, 3) follows from Lemma 3.1 and the following PBD $B(\{5, 4, 3^*\}, 1; 27)$.

(1, 7, 13, 19, 26)	(1, 5, 9, 25)	(3, 5, 6, 22)	(7, 9, 10, 14)	(2, 3, 12, 19)
(2, 8, 14, 20, 26)	(13, 17, 21, 25)	(4, 8, 12, 25)	(1, 8, 22, 23)	(4, 15, 17, 24)
(3, 9, 15, 21, 26)	(1, 6, 15, 16)	(16, 20, 24, 25)	(8, 10, 11, 15)	(25, 26, 27)
(4, 10, 16, 22, 26)	(1, 3, 4, 20)	(4, 9, 18, 19)	(12, 13, 20, 23)	
(5, 11, 17, 23, 26)	(5, 13, 16, 18)	(4, 6, 7, 23)	(2, 9, 23, 24)	
(6, 12, 18, 24, 26)	(2, 6, 10, 25)	(8, 16, 19, 21)	(9, 11, 12, 16)	
(1, 10, 12, 17, 27)	(14, 18, 22, 25)	(5, 10, 19, 20)	(1, 14, 21, 24)	
(2, 4, 5, 21, 27)	(2, 7, 16, 17)	(5, 7, 8, 24)	(3, 10, 13, 24)	
(3, 14, 16, 23, 27)	(6, 14, 17, 19)	(9, 17, 20, 22)	(2, 13, 15, 22)	
(6, 8, 9, 13, 27)	(3, 7, 11, 25)	(6, 11, 20, 21)	(4, 11, 13, 14)	
(7, 15, 18, 20, 27)	(15, 19, 23, 25)	(10, 18, 21, 23)	(1, 2, 11, 18)	
(11, 19, 22, 24, 27)	(3, 8, 17, 18)	(7, 12, 21, 22)	(5, 12, 14, 15)	

If $v \equiv 1 \pmod 4$, $37 = 4 \cdot 9 + 1$ is the smallest known case from the product construction. For $v > 185$ we use Lemma 3.2 with $v = 4m + 37$ to get an ICOILS(v, 37) and then an ICOILS(v, 3). For $65 \leqslant v \leqslant 185$ we again use Lemma 3.2 with $v = 4m + k$ to get an ICOILS(v, m) and an ICOILS(v, 3), where $m \in \{16, 19, 23, 27, 31, 39\}$ and $0 < k < m$, $k \equiv 1 \pmod 4$. For the remaining cases we have

$41 = 10 \cdot 4 + 1$, Lemma 3.7, $45 = 4 \cdot 9 + 8 + 1$ Lemma 3.3,

$49 = 16 \cdot 3 + 1$, Lemma 3.7, $61 = 4 \cdot 15 + 1$ Lemma 3.7,

$53 = 5 \cdot 11 - 2$, Lemma 3.1, (delete two points from one block in GD(5, 1, 11; 55) and get $B(\{11, 10, 5, 4, 3^*\}, 1; 53)$).

Finally, for $v = 57$ we have a RGD(7, 1, 7; 49) from Theorem 2.5. Delete two points from one group we get a RGD($\{6, 7\}$, 1, $\{7, 5\}$; 47). In Lemma 3.5, add one point or two points to the blocks in each parallel class except the groups, we get an ICOILS(57, 10) since $10 = 4 \cdot 1 + 3 \cdot 2$. An ICOILS(57, 3) then follows. □

Theorem 5.4. *An* ICOILS(v, 6) *exists for all* $v \geqslant 19$ *with the possible exception of* $v \in \{20, 21, 23, 24, 27, 28, 29, 32, 33, 35, 36, 39, 40, 44, 45, 47, 52, 53\}$.

Proof. The proof of this theorem will be divided into three parts.

Case 1. We consider $v \equiv 2 \pmod 4$

Here we shall construct an ICOILS(v, 6) for all $v \geqslant 22$. First of all, if $v = 4m + 6$ and $m \geqslant 7$, $m \neq 10$, 14, the result follows from Lemmas 3.2 and 3.3. For the remaining values of v, namely, $v \in \{22, 26, 30, 46, 62\}$ we have the following constructions

$v = 22 = 4 \cdot 5 + 2$,	Lemma 3.4,
$v = 26$,	Lemma 3.5 (delete one point from a group in RGD(5, 1, 5; 25), add two points to the groups and use ICOILS(7, 2)),
$v = 30 = 4 \cdot 7 + 2$,	Lemma 3.4,
$v = 46 = 4 \cdot 11 + 2$,	Lemma 3.4,
$v = 62 = 7 \cdot 8 + 6$,	Lemma 3.1 (delete two points from a group in a GD(8, 1, 8; 64)).

Case 2. Here we consider $v \equiv 0, 1, 3 \pmod 4$ and construct an ICOILS(v, 6) for all $v \geqslant 88$, $v \neq 91$. The construction is based on the existence of an ICOILS(v, 6) for $v = 4k + 2$, $k \geqslant 5$, in Case 1 above. If $v \equiv 0, 1, 3 \pmod 4$ and $v \geqslant 88$, $v \neq 91$, then we can express v as $v = 4(4k + 2) + s$, where $s \in \{0, 1, 4, 5, 7, 8, 9, 11, 13, 15, 19\}$ or $v = 4(4k + 1) + 16$, $k \geqslant 5$. From Lemmas 3.2 and 3.3, respectively, we have an ICOILS(v, $4k + 2$) and then an ICOILS(v, 6).

Case 3. Here we deal with the remaining values of v which are not listed as possible exceptions in the statement of the theorem. First of all, we have an ICOILS(19, 6) from Theorem 5.1. By direct construction we have ICOILS(v, 6) for $v = 25$, 37, 43 and 49 as follows

$v = 25$, $m = 19$, $n = 6$, Corollary 4.7,

$v = 37$, $m = 31$, $n = 6$, $\lambda = 28$, $\zeta = 25$, $K' = \zeta K$, $K = (1, 26, 25, 30, 5, 6)$, Lemma 4.3,

$v = 43$, $m = 37$, $n = 6$, $\lambda = 19$, $\zeta = 36$, $K' = \zeta K$, $K = (1, 27, 26, 36, 10, 11)$, Lemma 4.3,

$v = 49$, $m = 43$, $n = 6$, $\lambda = 40$, $\zeta = 7$, $K' = \zeta K$, $K = (1, 37, 36, 42, 6, 7)$, Lemma 4.3.

Next, for $v \in \{76, 77, 79, 80, 81, 83, 84, 85, 87, 91\}$ we may express v in the form $v = 4m + k$, where $m \in \{18, 19\}$ and $0 < k < m$, so that Lemmas 3.2, 3.3 and 3.4 can be applied to obtain an ICOILS(v, 19) and consequently an ICOILS(v, 6).

For $v \in \{31, 41, 51, 55, 61, 63, 65, 67, 68, 69, 71, 72, 73, 75\}$, we may express v in the form $v = kq + r + 6$, where $q \in \{5, 7, 8, 9, 11, 13\}$, $5 \leqslant k \leqslant q$, $0 \leqslant r < q$ and a resolvable GD($\{k, k+1, q\}$, 1, $\{q, r\}$; $kq + r$) exists from a (truncated) TD $T(q + 1, 1; q)$. We can then apply Lemma 3.5, where the six points are appropriately adjoined to the parallel classes of blocks (or to the groups) of this resolvable GDD, to obtain an ICOILS(v, 6) using an ICOILS(v, 2), $v =$

7, 8, 9, 10 where necessary. Specifically, we have the following, where "*" indicates that one of the points must be adjoined to the groups, but in other cases we use $6 = 2 + 2 + 2$.

$$31 = 5 \cdot 5 + 6, \quad 41 = 5 \cdot 7 + 6, \quad 51 = 5 \cdot 9 + 6,$$
$$55 = 7 \cdot 7 + 6, \quad 61 = 5 \cdot 11 + 6, \quad 63 = 7 \cdot 8 + 1 + 6,$$
$$^*65 = 7 \cdot 8 + 3 + 6, \quad 67 = 7 \cdot 8 + 5 + 6, \quad ^*68 = 7 \cdot 8 + 6 + 6,$$
$$69 = 7 \cdot 9 + 6, \quad 71 = 5 \cdot 13 + 6, \quad ^*72 = 7 \cdot 9 + 3 + 6,$$
$$73 = 7 \cdot 9 + 4 + 6, \quad ^*75 = 7 \cdot 9 + 6 + 6.$$

For $v \in \{56, 57, 59\}$, we adjoin additional points to the seven parallel classes of blocks of a resolvable GD($\{6, 7\}, 1, \{7, 6^*\}; 6 \cdot 7 + 6$) so that Lemma 3.5 can be applied to obtain an ICOILS(v, 6) using an ICOILS(v, 2), $v = 8, 9$ where necessary:

$$56 = (6 \cdot 7 + 6) + 8, \quad 8 = 2 + 1 + 1 + 1 + 1 + 1 + 1,$$
$$57 = (6 \cdot 7 + 6) + 9, \quad 9 = 2 + 2 + 1 + 1 + 1 + 1 + 1,$$
$$59 = (6 \cdot 7 + 6) + 11, \quad 11 = 2 + 2 + 2 + 2 + 1 + 1 + 1.$$

Finally, for $v \in \{48, 60, 64\}$ we have the following constructions for ICOILS(v, 6).

$v = 48$. We take the PBD $B(\{4^*, 5, 7, 8\}, 1; 46)$ of Hanani [7, p. 277] and appropriately delete one of the points from the unique block of size 4 so as to form a GDD GD($\{5, 7\}, 1, \{7, 3^*\}; 45$). Using an ICOILS(10, 3) we may then adjoin three additional points to the groups of the GDD and obtain an ICOILS(48, 6).

$v = 60 = (6 \cdot 7 + 5) + 13$, and $13 = 2 \cdot 5 + 1 \cdot 3$. We can construct by Lemma 3.5 an ICOILS(60, 6) by adding 13 points to a resolvable GD($\{6, 7\}, 1, \{7, 5\}; 47$) with one of the points adjoined to the groups and the remaining 12 points added to the seven parallel classes of blocks using an ICOILS(v, 2), $v = 8, 9$ as required.

$v = 64 = 5 \cdot 9 + 19$ and $19 = 9 \cdot 2 + 1$. So we can construct an ICOILS(64, 19), and consequently an ICOILS(64, 6), by using an ICOILS(7, 2) where two points are adjoined to each of the nine parallel classes of blocks of a resolvable GD(5, 1, 9; 45) and a single point to the groups of this GDD. The proof is then complete. □

Let A be a series of positive integers such that a GD($9, 1, m; 9m$) exists for any $m \in A$. From the list in Brouwer [4], we have a subseries of A as follows:

$$\{a_n\} = \{11, 13, 17, 19, 23, 25, 31, 37, 43, 49, 53, 59, 65, 71, 73, 79, 83, 89, 91, 97, 99, \ldots\},$$

such that $a_{i+1} - a_i \leqslant 6$ for any $a_i \in \{a_n\}$.

Lemma 5.5. *If* $m \in \{a_n\}$, *then an* ICOILS(v, n) *exists for any* $4 \leq n \leq m$ *and* $7(m+1)+n \leq v \leq 9m+n$.

Proof. First, we suppose $m > 17$. We write $v = 7m+n+k$, $7 \leq k \leq 2m$. Since $m \in \{a_n\}$, we have a GD$(9, 1, m; 9m)$ and then a RGD$(8, 1, m; 8m)$. Delete $m-n$ points from one group, we get a RGD$(\{8, 7\}, 1, \{m, n\}; 7m+n)$. Add d points to each parallel class of blocks, where $d = 0$, 1 or 2, we then apply Lemma 3.5 to get an ICOILS(v, n), $v = 7m+n+k$, $7 \leq k \leq 2m$ and $k \neq 12$, 14, 18. For $k = 12$, 14, 18, we write $v = 7m+(n-1)+(k+1)$. Then the same Lemma can be applied to get the required ICOILS(v, n), provided that one of the $(k+1)$ points must be added to the groups.

Now, we consider the cases $m = 11$, 13 and 17. From Theorem 2.5, we have a $T(11, 1; m)$. We may obtain, by truncating groups, a GDD GD$(\{7, 8, 9, 11\}, 1, \{m, n^*, r, s, t\}; v)$ where $v = 7m+n+r+s+t$, $4 \leq n \leq m$ and $0 \leq r, s, t \leq m$. Obviously, a COILS(m) exists, and appropriate choices of r, s, t can be made to produce an ICOILS(v, n) from this GDD where $7m+n \leq v \leq 10m+n$, except where $m = 13$ and $r+s+t = 38$. If $m = 13$ and $r+s+t = 38$, we may write, for $4 \leq n \leq m$, $v = 7 \cdot 13 + n + 13 + 13 + 12 = 9 \cdot 13 + (n-4) + 12 + 4$. We then adjoin four points to the groups of a GDD GD$(\{9, 10, 11\}, 1, \{12, 13, (n-4)^*\}; v-4)$ using ICOILS$(16, 4)$ and ICOILS$(17, 4)$ to form an ICOILS(v, n). The proof is then complete. □

Theorem 5.6. *For all* $n \geq 1$, *an* ICOILS(v, n) *exists if* $v \geq 8n+42$.

Proof. The conclusion is true for $n = 1$ from [1]. It is also true for $2 \leq n \leq 6$ from Theorems 5.2, 5.3 and 5.4. In what follows, we suppose $n \geq 7$.

For $n \geq 7$, let $\hat{n}$ be the first integer in $\{a_n\}$ such that $\hat{n} \geq n$. The property of $\{a_n\}$ implies that $\hat{n} \leq n+5$.

On the other hand, Lemma 5.5 and its proof imply the existence of an ICOILS(v, n) for the following cases:

$$
\begin{aligned}
&7 \leq n \leq 11, && 84+n \leq v \leq 99+n,\\
&7 \leq n \leq 13, && 98+n \leq v \leq 130+n,\\
&7 \leq n \leq 17, && 126+n \leq v \leq 153+n,\\
&7 \leq n \leq 19, && 140+n \leq v \leq 171+n,\\
&7 \leq n \leq 23, && 168+n \leq v \leq 207+n,\\
&7 \leq n \leq m, && 7m+n+7 \leq v \leq 7m+n+49,
\end{aligned}
$$

where $m \geq 25$ and $m \in \{a_n\}$. We then know that for $n \geq 7$, an ICOILS(v, n) exists if $v \geq 7\hat{n}+n+7$. Since $v \geq 8n+42 = 7(n+5)+n+7 \geq 7\hat{n}+n+7$, an ICOILS$(v, n)$ exists for all $n \geq 7$ and $v \geq 8n+42$. The proof is now complete. □

6. Conclusion

The existence result of ICOILS(v, n) for $2 \leqslant n \leqslant 6$ seems to suggest that the general bound $v \geqslant 8n + 42$ for all n could be further improved. Compared with the self-orthogonal Latin square case it seems that the bound should be closer to the necessary condition $v \geqslant 3n + 1$. It is certainly possible to improve the bound by imposing conditions on the order n of the missing subsquare. For example, one can easily improve the bound to $v \geqslant 7n$ if $n \geqslant 63$. However, much more work needs to be done if such restrictions are to be avoided.

Note added in proof

Since this paper was first submitted for publication, special constructions have been found to eliminate some of the possible exceptions cited in Theorems 5.2–5.4. In particular an ICOILS(v, n) exists for $(v, n) \in \{(33, 3), (14, 4), (18, 5), (28,5), (34, 5), (32, 6), (35, 6), (36, 6), (39, 6), (40, 6), (44, 6), (45, 6), (47, 6), (52, 6), (53, 6)\}$. This implies that a (3, 2, 1) (or (1, 3, 2))-conjugate orthogonal Latin square exists for all orders $v \neq 2, 6$, which completely solves the problem of Phelps [14]. It also follows that a (3, 2, 1) (or (1, 3, 2))-COILS(v) exists for all $v \neq 2, 3, 6$ and possibly excepting $v = 12$, eliminating two of the possible exceptions in [1]. Details of these and other constructions will appear in a subsequent paper.

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References

[1] F.E. Bennett, Conjugate orthogonal Latin squares and Mendelsohn designs, Ars Combin. 19 (1985) 51–62.
[2] R.K. Brayton, D. Coppersmith and A.J. Hoffman, Self-orthogonal Latin squares of all orders $n \neq 2$, 3 or 6, Bull. Amer. Math. Soc. 80 (1974) 116–118.
[3] A.E. Brouwer, Optimal packings of K_4's into a K_n, J. Combin. Theory Ser. A 26 (1976) 278–297.
[4] A.E. Brouwer, The number of mutually orthogonal Latin squares—a table up to order 10 000, Report ZW123, Math. Centrum, Amsterdam, June (1979).
[5] J. Dénes and A.D. Keedwell, Latin squares and their applications, (Academic Press, New York, 1974).
[6] D.A. Drake and J.A. Larson, Pairwise balanced designs whose line sizes do not divide six, J. Combin. Theory Ser. A 34 (1983) 286–300.

[7] H. Hanani, Balanced incomplete block designs and related designs, Discrete Math. 11 (1975) 255–369.
[8] K. Heinrich, Near-orthogonal Latin squares, Utilitas Math. 12 (1977) 145–155.
[9] K. Heinrich and L. Zhu, Existence of orthogonal Latin squares with aligned subsquares, Discrete Math. 59 (1986) 69–78.
[10] K. Heinrich and L. Zhu, Incomplete self-orthogonal Latin squares, J. Austral. Math. Soc. Ser. A, to appear.
[11] J.D. Horton, Sub-Latin squares and incomplete orthogonal arrays, J. Combin. Theory Ser. A 16 (1974) 23–33.
[12] R.C. Mullin, A generalization of the singular direct product with applications to skew Room squares, J. Combin. Theory Ser. A 29 (1980) 306–318.
[13] M.J. Pelling and D.G. Rogers, Stein quasigroups I: Combinatorial aspects, Bull. Austral. Math. Soc. 18 (1978) 221–236.
[14] K.T. Phelps, Conjugate orthogonal quasigroups, J. Combin. Theory Ser. A 25 (1978) 117–127.
[15] S.K. Stein, On the foundations of quasigroups, Trans. Amer. Math. Soc. 85 (1957) 228–256.
[16] R.M. Wilson, Constructions and uses of pairwise balanced designs, Math. Centre Tracts 55 (1974) 18–41.
[17] L. Zhu, Orthogonal Latin squares with subsquares, Discrete Math. 48 (1984) 315–321.
[18] L. Zhu, Some results on orthogonal Latin squares with orthogonal subsquares, Utilitas Math. 25 (1984) 241–248.