Minimally 3-Connected Graphs*

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Communicated by the Editors

Received January 14, 1984

A constructive characterization of the class of minimally 3-connected graphs is presented. This yields a new characterization for the class of 3-connected graphs, which differs from the characterization provided by Tutte. Where Tutte's characterization requires the set of all wheels as a starting set, the new characterization requires only the graph $K_4$. The new characterization is based on the application of graph operations to appropriate vertex and edge sets in minimally 3-connected graphs.

1. Introduction

In this paper we present a construction which characterizes precisely the class of minimally 3-connected graphs. No such characterization has previously been available.

Definition. A $k$-connected graph $G$ is minimally $k$-connected (mkc) if it has no proper spanning $k$-connected subgraph.

Minimally $k$-connected graphs have applications in the field of cost-minimizing network design (e.g., [6, 7, 15]). For example, minimally $k$-connected graphs provide optimal solutions to network design problems which require maximal overall connectivity while using a minimal number of edges.

In fact, as Bollobas [3] observes, every $k$-connected graph can be obtained from a minimally $k$-connected one by the addition of edges. Thus the minimally $k$-connected graphs constitute in some sense an irreducible subset of the set of all $k$-connected graphs.

The study of mkc graphs seems to originate with Dirac, who first explored the class of m2c graphs in 1967 [5]. Further work on m2c graphs was done by Hedetniemi [10] and Plummer [13]. Halin [8, 9] and Mader

* Research supported in part by the Natural Sciences and Engineering Research Council of Canada.
[11, 12] derived many results on mkc graphs in general, and on m3c graphs in particular. However, neither Halin nor Mader give a complete characterization of the minimally 3-connected graphs.

The class of all 3-connected graphs has been characterized by Tutte in his well-known work on "wheels" [16], where he shows that the 3-connected graphs can be generated from the wheels by the operations of edge-addition and vertex-splitting. However, using Tutte's construction it is not possible to generate precisely the class of m3c graphs. To show this, we observe that if G is an m3c graph, adding an edge to G must produce a graph which is not minimally 3-connected. Hence, if Tutte's construction is to be used to generate precisely the class of m3c graphs, only vertex-splitting may be used. There are infinitely many m3c graphs which cannot be generated by applying vertex-splitting operations to any wheel (for example, $K_{3,3}$).

In addition, Slater [14] has characterized the class of all 4-connected graphs. This characterization, however, does not seem likely to yield a similar result for the m4c graphs.

Section 2 of this paper presents some definitions and basic lemmas that will be used in the proofs of the major theorems, which are presented in Section 3. Section 4 contains some conclusions that follow from the results in Section 3.

2. BASIC LEMMAS

We first introduce some minor results that pertain to k-connected and minimally k-connected graphs.

**Lemma 1** [3]. A k-connected graph G is minimally k-connected iff each pair of adjacent vertices in G is k-connected in G, and no pair of adjacent vertices is $(k + 1)$-connected.

This implies the following more general result.

**Lemma 2.** Let G be a k-connected graph containing an edge $e = (xy)$ where $x$ and $y$ are $(k + 1)$-connected vertices of G. Then $G - e$ is k-connected.

The following definition and lemma arise from the work of Halin [8].

**Definition.** In a graph G, a k-fan is a set of vertex-disjoint paths joining a vertex $x$ to a set of vertices $\{v_1, ..., v_k\}$.

**Lemma 3.** Let G be a k-connected graph on at least $k + 1$ vertices. Then
for any vertex \(x\) and vertex set \(\{v_1, \ldots, v_k\}\) where \(x \neq v_i\) and \(v_i \neq v_j, \forall i, j \leq k\), \(G\) contains a \(k\)-fan joining \(x\) to \(\{v_1, \ldots, v_k\}\).

**Definition.** A path \(p\) is a *chording path* if some edge \(e\) of \(p\) chords a cycle of \(G\) which has no intersection with \(p\) other that the end-vertices of \(e\).

**Definition.** Let \(G\) be a graph. Then a set \(S\) of vertices and/or edges of \(G\) is *3-compatible* if it conforms to one of the following three types:

- **Type (1):** \(S = \{x, (ab)\}\) where \(x\) is a vertex of \(G\), \((ab)\) is an edge of \(G\), \(x \neq a\) and \(x \neq b\), and no \(xa\)-path or \(xb\)-path is a chording path of \(G - (ab)\).
- **Type (2):** \(S = \{(ab), (cd)\}\) where \((ab)\) and \((cd)\) are distinct edges of \(G\), and no \(ac\)-path, \(bc\)-path, \(ad\)-path, or \(bd\)-path is a chording path of \(G - (ab) - (cd)\).
- **Type (3):** \(S = \{x, y, z\}\) where \(x, y, \) and \(z\) are distinct vertices of \(G\) and no \(xy\)-path, \(xz\)-path, or \(yz\)-path is a chording path of \(G\). Observe that if \(G\) is a 3-connected graph, then \(x, y, \) and \(z\) must be mutually non-adjacent, since an edge between any two of them would constitute a chording path of the cycle formed by two other paths joining the vertices in question, and the existence of two such paths is guaranteed by the 3-connectivity of \(G\).

**Definition.** Let \(G\) be a graph. Then:

(a) If \(x\) and \((ab)\) are any non-incident vertex and edge of \(G\), respectively, we perform *Operation 1* or *Op 1* on \(\{x, (ab)\}\) by subdividing \((ab)\) with a new vertex \(y\), and then making \(x\) adjacent to \(y\). (See Fig. 1.)

(b) If \((ab)\) and \((cd)\) are any edges of \(G\), we perform *Operation 2* or *Op 2* by subdividing \((ab)\) with a new vertex \(x\), subdividing \((cd)\) with a new vertex \(y\), and making \(x\) adjacent to \(y\). Note that for the purposes of this definition, it is not required that \(a, b, c, \) and \(d\) distinct vertices. (See Fig. 2.)

(c) If \(x, y, \) and \(z\) are distinct vertices of \(G\), we perform *Operation 3* or *Op 3* on \(\{x, y, z\}\) by adding a new vertex \(w\) to \(G\), making \(w\) adjacent to \(x, y, \) and \(z\).

![Fig. 1. Operation 1.](image-url)
LEMMA 4. Let \( H \) be a 3-connected graph with 4-connected vertices \( w \) and \( z \). If \( G \) is derived by applying Op 1, Op 2, or Op 3 to \( H \), then \( w \) and \( z \) are 4-connected in \( G \).

The following well-known lemma will be used to eliminate some cases in later theorems.

LEMMA 5 [8]. Let \( G \) be an mkc graph with a \( K_k \) subgraph \( H \), where \( k \geq 3 \). Then at most one vertex of \( H \) has degree \( \geq k + 1 \) in \( G \).

We now introduce a lemma due to Barnette and Grunbaum, which will be vital in the characterization of m3c graphs which follows.

LEMMA 6 [1]. Let \( G \) be a 3-connected graph with six or more edges. Then \( G \) contains an edge \( e = (xy) \) such that if \( e \) is deleted from \( G \), and if either end-vertex of \( e \) is then of degree 2, that vertex is also deleted and an edge added between its neighbours, then the resultant graph, which may contain multiple edges, is 3-connected.

3. A Characterization of m3c Graphs

We are now prepared to give a new characterization of the m3c graphs. In particular, we will show that the class of m3c graphs is precisely the class of graphs that may be generated by starting with \( K_4 \) and repeatedly applying Operations 1, 2, and 3 to 3-compatible sets of vertices and edges.

THEOREM 7. Let \( H \) be a 3-connected graph and let \( G \) be constructed by applying Op 1, Op 2, or Op 3 to \( H \). Then \( G \) is 3-connected.

We will now show that 3-compatibility is necessary and sufficient to ensure that if any of these operations is applied to a 3-compatible set in an m3c graph, then the resultant graph is also m3c.
Theorem 8. Let $H$ be an m3c graph. Let $G$ be constructed by applying Op1, Op2, or Op3 to a set $S$ of edges and/or vertices of $H$. Then $G$ is an m3c graph iff $S$ is a 3-compatible set in $H$.

Proof. (IF) Assume that $S$ is a 3-compatible set in $H$, and that $G$ is not an m3c graph. Observe that by Theorem 7, $G$ is 3-connected.

Three cases arise (one for each operation). We will give a detailed proof only for Op1.

Assume $S = \{x, (ab)\}$, and Op1 is used to construct $G$ from $H$. By assumption, $G$ is 3-connected but not m3c. Then $G$ must contain a pair of adjacent vertices $u$ and $v$ with connectivity $\geq 4$. Neither of these vertices may be $p'$, because $d(y) = 3$. Since $u$ and $v$ are not 4-connected in $H$ (by Lemma 1), it must be true that for any set of four paths connecting $u$ and $v$ in $G$, one path must contain the edge $(xy)$. (See Fig. 3.)

However, it may be seen that the $xa$-path $x \cdots p \cdots u - v \cdots p' \cdots a$ is a chording path in $H - (ab)$, which contradicts the 3-compatibility of $\{x, (ab)\}$ in $H$. Observe that if $u = x$ (and/or $v = a$), then $p$ (and/or $p'$) is a path of length 0. Therefore $G$ is minimally 3-connected.

Similar analyses for Op2 and Op3 show that, as for Op1, the existence of a pair of adjacent 4-connected vertices in $G$ implies the existence of a chording path which contradicts the 3-compatibility of $S$ in $H$. Therefore $G$ is minimally 3-connected if $S$ is a 3-compatible set in $H$.

(ONLY IF) Assume $G$ is a m3c graph, but that $S$ is not a 3-compatible set in $H$. Again, we must consider the three possible forms of $S$ separately, but will only discuss the first case in any detail. Assume $S = \{x, (ab)\}$. (See Fig 4.) Since $S$ is not 3-compatible in $H$, there must exist a chording path joining $x$ to $a$ in $H - (ab)$. However, in $G$, the endvertices of the distinguished edge of that chording path are 4-connected, which contradicts the minimal 3-connectivity of $G$. Therefore $S$ must be 3-compatible in $H$.

Figure 3. $u$ and $v$ are 4-connected.
Again, the analyses for Op2 and Op3 are similar to that for Op1. If \( S \) is not 3-compatible in \( H \), then \( H \) must contain an appropriate chording path. This leads in both cases to a contradiction of the minimal 3-connectivity of \( G \). Therefore \( S \) is 3-compatible if \( G \) is minimally 3-connected. \( \qed \)

We are now prepared to show that the minimally 3-connected graphs, with the exception of \( K_4 \), are precisely the graphs obtained by applying Operations 1, 2, and 3 to smaller m3c graphs.

**Theorem 9.** Let \( G \) be a graph without loops or multiple edges. \( G \) is minimally 3-connected iff

(a) \( G \cong K_4 \), or

(b) there exists an m3c graph \( G' \), \(|G'| < |G|\) such that \( G \) can be constructed by applying one of Op1, Op2, or Op3 to a 3-compatible set in \( G' \).

**Proof.** (IF) follows directly from Theorem 8.

(ONLY IF) Assume \( G \) is an m3c graph. We will show that there exists an appropriate m3c graph \( G' \).

The proof examines two mutually exclusive cases, based on the presence or absence of a \( K_3 \) subgraph in \( G \).

**Case 1.** \( G \) contains a \( K_3 \) subgraph on some vertex set \( \{x, y, z\} \).

Three subcases arise, based on the degree of \( x, y, \) and \( z \).

Subcase (a). \( d(x) = d(y) = d(z) = 3 \). (See Fig. 5.)

Observe that since \( G \) is 3-connected, either \( G \cong K_4 \), or \( a, b, \) and \( c \) are distinct vertices (otherwise, if \( a = b \), for example, the set \( \{a, z\} \) separates \( G \)). Consider the graph \( G' \), formed by deleting vertices \( y \) and \( z \) from \( G \), and adding the edges \((xb)\) and \((xc)\).

A straightforward argument shows that \( G' \) is 3-connected. We therefore show that \( G' \) is minimally 3-connected. If \( G' \) is not minimally 3-connected,
then it contains an edge which joins a pair of 4-connected vertices (by Lemma 1). Since these vertices must have degree at least 4, neither of these vertices can be $x$. Hence these vertices must also be 4-connected and adjacent in $G$ (by Lemma 4), as $(bx)$ and $(cx)$ are the only edges of $G'$ which are not also in $G$. This contradicts the minimal 3-connectivity of $G$. Therefore $G'$ is a minimally 3-connected graph. Since we can construct $G$ by applying $\text{Op2}$ to $S = \{(xb), (xc)\}$, we can conclude by Theorem 8 that $S$ is a 3-compatible set in $G'$.

Subcase (b). $d(x) \geq 4$, $d(y) = d(z) = 3$.

The proof of this case is similar to the proof of subcase (a).

Subcase (c). $d(x) \geq 4$, $d(y) \geq 4$, and $d(z) \geq 3$.

By Lemma 5, this subcase cannot occur. Therefore, the theorem holds for all m3c graphs with a $K_3$ subgraph.

Case 2. $G$ does not contain a $K_3$ subgraph.

Let $G$ be an m3c graph, and let $e = (xy)$ be the edge specified in Lemma 6. Let $G'$ be the graph remaining when $e$ is “deleted” as described in that lemma. Three subcases arise, depending on the degree of $x$ and $y$ in $G$.

Subcase (a). $d(x) \geq 4$ and $d(y) \geq 4$.

By Lemma 6, $G'$ is 3-connected. But then $G'$ is a proper spanning 3-connected subgraph of $G$, which contradicts the minimal 3-connectivity of $G$. Hence this subcase can never arise.

Subcase (b). $d(x) \geq 4$ and $d(y) = 3$. (See Fig. 6.)

Observe that since $G$ is a simple graph, $G'$ will contain multiple edges only if $(ab)$ is an edge of $G$. However, if $(ab)$ is an edge of $G$, then $G$ con-
FIG. 6. $G'$ is isomorphic to $G$ except for the illustrated subgraph.

contains a $K_3$ subgraph on $\{y, a, b\}$. Therefore $G'$ is a simple graph. Lemma 6 ensures that $G'$ is 3-connected. Two sub-subcases must be considered.

sub-subcase (i). $G'$ is minimally 3-connected.

Then $G$ can be constructed from the m3c graph $G'$ by applying Op1 to $S = \{x, (ab)\}$, which is therefore a 3-compatible set in $G'$ by Theorem 8.

Sub-subcase (ii). $G'$ is not minimally 3-connected.

$G'$ must contain an edge which joins two vertices which are 4-connected in $G'$. If this edge is not $(ab)$, then its end-vertices are also adjacent and 4-connected in $G$. Thus the only edge of $G'$ which joins 4-connected vertices is $(ab)$. Hence $G'' = G' - (ab)$ is minimally 3-connected, by Lemma 1 and Lemma 2.

Then we can construct $G$ from the m3c graph $G''$ by applying Op3 to $S = \{x, a, b\}$, which is therefore a 3-compatible set in $G''$, by Theorem 8.

Subcase (c). $d(x) = d(y) = 3$. The proof is similar to that of the previous case.

Therefore the result holds for all m3c graphs not containing a $K_3$ subgraph.

4. Corollaries and Conclusions

The new characterization of the class of 3-connected graphs may now be stated. The 3-connected graphs are precisely the graphs which may be generated by constructing the minimally 3-connected graphs and adding edges.

As a final corollary, we present a result which guarantees that in an iterative generation process for the m3c graphs, no "dead-end" can occur.

Corollary 10. Let $G$ be an m3c graph on $n \geq 4$ vertices. Then $G$ contains a 3-compatible set.

Proof. If $G$ is $K_4$, any pair of non-incident edges forms a 3-compatible set. If $G$ is not $K_4$, $G$ can be constructed from a smaller m3c graph $G'$.
Straightforward arguments for each of the three construction operations show that a 3-compatible set in $G$ can be derived from the 3-compatible set used to construct $G$ from $G'$. 

Finding a 3-compatible set in an m3c graph for which the sequence of generating operations is known is trivial. If the generating sequence is not known, finding a 3-compatible set is more difficult, but may still be achieved in polynomial time with respect to the size of the graph.

As mentioned in the Introduction, there have been previous characterizations of some classes of $k$-connected graphs. These have been unrelated to one another. However, it is possible to define operations similar to Op1, Op2, and Op3 that allow new characterizations of the 2-connected and 1-connected graphs. Details of this generalization are contained in [4].

We also observe that the set of planar m3c graphs may be characterized as follows.

**Corollary 11.** Let $G$ be a planar graph. $G$ is an m3c graph iff

(a) $G \cong K_4$, or

(b) there exists a planar m3c graph $G'$ such that $G$ can be constructed by applying one of Op1, Op2, or Op3 to a 3-compatible set on a face of $G'$.

**Proof.** We observe that none of the operations will construct a planar graph when applied to a non-planar graph. Furthermore, when any of the operations are applied to a 3-compatible set not lying on some face of a planar graph, the resulting graph is non-planar. The corollary then follows from Theorem 9.

**Acknowledgments**

First and foremost I must gratefully acknowledge the efforts of Professor D. Corneil, both in inspiring the research and in proofreading multiple drafts of this paper. I would also like to express my appreciation to Professor R. Mathon for his careful reading of the original proofs of the theorems in this paper and for suggesting the inclusion of Corollary 10. I would like to thank the referee for suggesting the inclusion of Corollary 11. Financial support during the preparation of this paper was gratefully received from the Natural Sciences and Engineering Research Council of Canada.

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