J. Math. Anal. Appl. 378 (2011) 734–740

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The convex point of continuity property in Banach spaces not containing ℓ_1 ☆

Ginés López Pérez *, José A. Soler Arias

Universidad de Granada, Facultad de Ciencias, Departamento de Análisis Matemático, 18071-Granada, Spain

ARTICLE INFO

Article history:
Received 15 July 2010
Available online 26 January 2011
Submitted by Richard M. Aron

Keywords:
Convex point of continuity property
Banach spaces not containing ℓ_1

ABSTRACT

We obtain a local characterization of the convex point of continuity property for every closed, bounded and convex subset not containing sequences equivalent to the standard unit basis of ℓ_1. As a consequence, we prove, in the setting of Banach spaces without ℓ_1-copies, that the convex point of continuity property is determined on subspaces with a Schauder basis, which is a partial answer to a well-known problem.

1. Introduction

Recall that a closed, bounded and convex subset C of Banach space is said to have the point of continuity property (PCP) (resp. the convex point of continuity property (CPCP)), provided every non-empty closed and bounded subset of C (resp. every non-empty closed, bounded and convex subset) admits a point of continuity of the identity map from the weak to norm topologies. A Banach space X satisfies PCP (resp. CPCP) if its closed unit ball has PCP (resp. CPCP). It is known that Banach spaces with Radon–Nikodým property (RNP), including separable dual spaces, satisfy PCP, which clearly implies CPCP, but the converses are false (see [1] and [4]). A well-known and long standing open problem is if PCP and CPCP are basically determined, that is, determined by the subspaces with a Schauder basis. In this sense, it is known that PCP and CPCP are separably determined [3], that is, determined by the separable subspaces. In fact, CPCP was introduced in [3] in order to prove that RNP is separably determined. PCP is basically determined for dual spaces [2] and for Banach spaces not containing subspaces isomorphic to ℓ_1 [5]. However, up to now, we don't know positive results on the basically determination for the CPCP case. The main goal of this note is to prove in Corollary 2.7 that CPCP is basically determined for Banach spaces not containing subspaces isomorphic to ℓ_1. For this, we obtain in Theorem 2.6 a local characterization of CPCP for closed, bounded and convex subsets in Banach spaces without ℓ_1-copies, and deduce from here our main goal. The above characterization is an extension of some results in [1], where CPCP is characterized for closed, bounded and convex subsets of c_0, by using some subsets in c_0 called P_0-simplex in [1]. Roughly speaking, it is proved in [1] that a closed, bounded and convex subset C in c_0 fails CPCP if, and only if, C contains isomorphically and affinely some P_0-simplex. A P_0-simplex in c_0 is the c_0-part of some weak-star compact and convex subset in ℓ_∞ which is affinely weak-star homeomorphic to the universal Poulsen simplex [14]. The P_0-simplex in c_0 was made to give an example of set failing CPCP and satisfying the strong regularity, a weaker property than CPCP.

The concept of P_0-simplex was generalized in [11], called there $P_{\{v_n\}}$-set, for general Banach spaces, and, as a consequence, it was shown in [11] that CPCP is basically determined for Asplund spaces. Then we improve the arguments in [1] and [11] to get that CPCP is in fact basically determined for Banach spaces without ℓ_1-copies. Furthermore, in the setting of

☆ Partially supported by MEC (Spain) Grant MTM2006-04837 and Junta de Andalucía Grants FQM-185 and Proyecto de Excelencia P06-FQM-01438.
* Corresponding author.
E-mail addresses: glopezp@ugr.es (G. López Pérez), jasoler@ugr.es (J.A. Soler Arias).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.01.052

Banach spaces X without ℓ_1-copies, the $P_{\{v_n\}}$-sets which we use to characterize CPCP are still the X-part of some weak-star compact and convex set in X^{**} which is weak-star affinely homeomorphic to the universal Poulsen simplex. So, the family of $P_{\{v_n\}}$-sets which characterize the CPCP in Banach spaces without ℓ_1-copies depends on Banach spaces considered, but its weak-star closures in the biduals agree with the universal Poulsen simplex in the affine and weak-star sense.

We begin with some notation and preliminaries (see [10]). Let X be a Banach space and let B_X, respectively S_X, be the closed unit ball, respectively sphere, of X. Given $\{e_n\}$ a basic sequence in X, $\{e_n\}$ is said to be semi-normalized if $0 < \inf_n \|e_n\| \leqslant \sup_n \|e_n\| < +\infty$, the closed linear span of $\{e_n\}$ is denoted by $[e_n]$ and the (non-closed) linear span of $\{e_n\}$ is denoted by $\operatorname{lin}\{e_n\}$. $\{e_n\}$ is called *shrinking* if $[e_n]^* = [e_n^*]$, where $\{e_n^*\}$ denotes the sequence of biorthogonal functionals associated to $\{e_n\}$, and, in this case, $[e_n]$ has a separable dual. $\{e_n\}$ is said to be *monotone* if for every $n \in \mathbb{N}$ and for every real numbers $\{t_i\}_{i=1}^{n+1}$ one has that $\|\sum_{i=1}^{n} t_i e_i\| \leqslant \|\sum_{i=1}^{n+1} t_i e_i\|$. Also, for every $x \in [e_n]$ and for every interval of integers numbers I, we do $x_{|I} = \sum_{n\in I} e_n^*(x) e_n$, whenever this sum exists, where $\{e_n^*\}$ denotes the sequence of biorthogonal functionals associated to $\{e_n\}$.

A metrizable compact Choquet simplex K is a metrizable, convex and compact subset in some locally convex space such that every point in K is the barycenter of a unique Borel and regular measure on K supported on $\operatorname{Ext}(K)$, where $\operatorname{Ext}(K)$ denotes the set of extreme points in K. The Poulsen simplex, denoted by P, is the set in ℓ_2 constructed in [14]. P is a metrizable and compact Choquet simplex and, up to affine homeomorphism, P is the unique metrizable and compact Choquet simplex satisfying that the set of extreme points is dense [9]. We refer to [8] and [13] for the relation between simplexes, L_1-preduals and representing matrices, which we use for the proof of Theorem 2.6.

Recall that a triangular matrix of real numbers $A = (a_{i,n})$ defined for $1 \leqslant i \leqslant n$ and $n \in \mathbb{N}$ is called a *representing matrix* if $\sum_{i=1}^{n} |a_{i,n}| \leqslant 1$ for all $n \in \mathbb{N}$. When $a_{i,n} \geqslant 0$ for every $1 \leqslant i \leqslant n$ and $\sum_{i=1}^{n} a_{i,n} = 1$ for all n, we call such matrix A a *stochastic representing matrix*. In [8] a representing matrix is associated to every separable L_1-predual and, conversely, a separable L_1-predual in ℓ_∞ is constructed from every given representing matrix. Following Theorem 5.2 and its corollary in [8] joint to Lemma 1.1 in [13] we summarize in the following theorem the precise information which we use in the second part of Theorem 2.6.

Theorem 1.1. *Let $A = (a_{i,n})$ a stochastic representing matrix. Then there is a compact metrizable Choquet simplex Ω, a monotone basis $\{e_n\}$ of $A(\Omega)$, the space of continuous and affine functions on Ω, with $e_1 = \mathbf{1}$ and a triangular matrix $\{e_{i,n}\}$, $1 \leqslant i \leqslant n$, $n \in \mathbb{N}$, of vectors in $A(\Omega)$ satisfying for every n:*

(1) $e_{n,n} = e_n$.
(2) $\{e_{i,n}\}_i$ *is isometrically equivalent to the standard basis of* ℓ_n^∞.
(3) $e_{i,n} = e_{i,n+1} + a_{i,n} e_{n+1}$ *for every* $1 \leqslant i \leqslant n$.

Furthermore, it follows that $\Omega = \overline{\bigcup_n \Omega_n}$, where for every n, $\Omega_n = co\{e_1^*, \ldots, e_n^*\}$ and $\{e_n^*\}$ is the sequence of biorthogonal functionals of $\{e_n\}$ in $A(\Omega)^*$.

2. Main results

We begin with the definition of a $P_{\{v_n\}}$-set. When $\{v_n\}$ is the standard basis of c_0, the family of $P_{\{v_n\}}$-sets agrees with the family of P_0-simplexes constructed in [1]. Essentially the following definition appears in [11]. We include it here by sake of completeness.

Definition 2.1. Let X be a Banach space. Fix:

i) a null sequence $\{\varepsilon_n\}$ of strictly positive numbers,
ii) sequences $\{g_j\}_j$ and $\{a_j\}_j$ from X with $g_1 = a_1$,
iii) a strictly increasing sequence $\{l_n\}$ from $\mathbb{N}$ with $l_1 = 1$ and let $l_0 = 0$.

Define the (strictly increasing) sequence $\{m_n\}_n$ of integers by $m_1 = 1$ and

$$m_{n+1} = m_n + l_n$$

for every $n \in \mathbb{N}$ and let $m_0 = 0$.

Also, we define a sequence $\{v_j\}$ in X by $v_1 = a_1$ and $v_j = a_j - g_{j-m_{n-1}}$ if $j > 1$, where for every $j > 1$, $n \in \mathbb{N}$ is the unique natural number n such that $m_{n-1} < j \leqslant m_n$ (noting $1 \leqslant j - m_{n-1} \leqslant l_{n-1}$).

Let

$$K_n = co\big(\{a_j\colon\ 1 \leqslant j \leqslant m_n\}\big)$$

and

$$K = \overline{\bigcup_{n\in\mathbb{N}} K_n}.$$

Note that for each $n \in \mathbb{N}$:

a) $K = \overline{co}(\{a_j\})$,
b) K_n is a closed convex set,
c) K is a closed convex set, but not bounded in general.

We call such set K a $P_{\{v_n\}}$-set whenever for each $n \in \mathbb{N}$ the following holds:

d) $g_j \in K_n$ for $l_{n-1} < j \leqslant l_n$,
e) $\{g_j\}_{l_{n-1}<j\leqslant l_n}$ is an $\{\varepsilon_n\}$-net for K_n.

Also we call a $P_{\{v_n\}}$-set weakly null if, following the above definition, for every $j \in \mathbb{N}$ one has that $\{v_{m_n+j}\}_n$ is weakly null. (Note that there is some $n_0 \in \mathbb{N}$ such that the sequence $\{v_{m_n+j}\}_n$ is defined for $n \geqslant n_0$, since $\{l_n\}$ is strictly increasing.)

Note that, following this definition, one has:

f) $K_n \subset \operatorname{lin}\{v_k\colon 1 \leqslant k \leqslant m_n\}$ for each $n \in \mathbb{N}$,
g) $K = \overline{\{g_n\}_{n=n_0}^{\infty}}$ for each $n_0 \in \mathbb{N}$,
h) if K is norm bounded, then $\sup_{n\in\mathbb{N}}\{\|v_n\|\} < \infty$.

Also we remark that the set $\{a_j\}_{m_0<j\leqslant m_1}$ has one vector, namely:

$$a_1 = v_1 = g_1.$$

The set $\{a_j\}_{m_1<j\leqslant m_2}$ has one vector, namely:

$$a_2 = g_1 + v_2.$$

In general, if $n \geqslant 2$ then the set $\{a_j\}_{m_{n-1}<j\leqslant m_n}$ has l_{n-1} vector, namely:

$$\begin{aligned}
&a_{m_{n-1}+1} = g_1 + v_{m_{n-1}+1},\\
&a_{m_{n-1}+2} = g_2 + v_{m_{n-1}+2},\\
&\vdots\\
&a_{m_n} = a_{m_{n-1}+l_{n-1}} = g_{l_{n-1}} + v_{m_{n-1}+l_{n-1}} = g_{l_{n-1}} + v_{m_n}.
\end{aligned}$$

Whenever we speak of a $P_{\{v_n\}}$-set, we will use the notation just set forth in the above definition. Observe that a $P_{\{v_n\}}$-set is constructed from fixed sequences $\{\varepsilon_n\}$, $\{a_j\}$, $\{g_j\}$ and $\{l_n\}$ where $\{\varepsilon_n\}$ is a null sequence of positive real numbers, $\{a_j\}$ and $\{g_j\}$ are sequences in X, with $a_1 = g_1$, and $\{l_n\}$ is a strictly increasing sequence of natural numbers with $l_1 = 1$. Then, following the above definition, we build the sequences $\{m_n\}$ in $\mathbb{N}$ and $\{v_n\}$ in X which are determined, in a unique way, by the initially fixed sequences.

In general, if K is a $P_{\{v_n\}}$-set, the sequence $\{v_n\}$ is not necessarily a basic sequence. One of the key points in Theorem 2.6 will be constructing, under appropriate assumptions, a bounded $P_{\{v_n\}}$-set, where $\{v_n\}$ is a seminormalized basic sequence.

For convenience we show now that a $P_{\{v_n\}}$-set can be seen in other way.

Lemma 2.2. *Let X be a Banach space and let $\{v_n\}$ be a basic sequence in X such that K is a $P_{\{v_n\}}$-set in X. Then there exists $\bar{x} = \{x_n\} \subset [v_n]$ satisfying*

$$x_1 = v_1, \qquad x_{n+1} \in co\{x_1, x_1 + v_2, \ldots, x_n + v_{n+1}\} \quad \forall n \in \mathbb{N},$$

such that $K = L_{\bar{x}}$, where $L_{\bar{x}} = \overline{\bigcup_{n=1}^{+\infty} L_n}$ and

$$L_1 = \{x_1\}, \qquad L_{n+1} = co\big\{L_n \cup \{x_n + v_{n+1}\}\big\} \quad \forall n \in \mathbb{N}.$$

Proof. Assume that K is a $P_{\{v_n\}}$-set in X. For $n \geqslant 1$ and $m_n \leqslant j < m_{n+1}$ put $x_j = g_{j+1-m_n}$. We show that $K_n = L_{m_n}$ for every n and then $K = \overline{\bigcup_{n=1}^{+\infty} L_{m_n}} = L_{\bar{x}}$.

Indeed, it is clear that $K_1 = L_{m_1}$, since $m_1 = 1$ and $g_1 = v_1 = x_1$. Assume inductively that $n \geqslant 1$ and $K_n = L_{m_n}$. Now we prove inductively on i, $1 \leqslant i \leqslant l_n$, that

$$L_{m_n+i} = co\big\{K_n \cup \{g_j + v_{m_n+j}\colon 1 \leqslant j \leqslant i\}\big\}. \tag{1}$$

This is clear for $i = 1$ since $x_{m_n} = g_1$. Assuming that (1) is true for $i < l_n$. Then, from the definition, we have that

$$L_{m_n+i+1} = co\big\{L_{m_n+i} \cup \{x_{m_n+i} + v_{m_n+i+1}\}\big\} = co\big\{K_n \cup \{g_j + v_{m_n+j}\colon 1 \leqslant j \leqslant i\} \cup \{g_{i+1} + v_{m_n+i+1}\}\big\},$$

since $x_{m_n+i} = g_{i+1}$. This proves (1). The case $i = l_n$ is just to say that $L_{m_{n+1}} = K_{n+1}$. $\square$

The next lemma shows that a $P_{\{v_n\}}$-set with additional properties fails the CPCP.

Lemma 2.3. *If K is a weakly null, bounded $P_{\{v_n\}}$-set with*

$$\inf_{n\in\mathbb{N}}\{\|v_n\|\} > 0$$

then K fails the CPCP.

Proof. Pick $j \in \mathbb{N}$, then there exists $n_0 \in \mathbb{N}$ such that $m_n + j < m_{n+1}$ for every $n \geqslant n_0$, since $m_{n+1} = m_n + l_n$ and $\{l_n\}$ is strictly increasing. Furthermore, from the definition, we can choose n_0 satisfying also that $g_j + v_{m_n+j} = a_{m_n+j} \in K$ for every $n \geqslant n_0$. Hence g_j is not a point of continuity of K, since K is weakly null and $\inf_{n\in\mathbb{N}}\{\|v_n\|\} > 0$. Now, from the definition, $\{g_j\}$ is weakly dense in K, and so K fails CPCP. $\square$

As we mention in the introduction CPCP is separably determined. Much more was essentially proved in [2], where it is showed the PCP case. This is also consequence of a stronger result in [7]. However, we need the separable determination of CPCP in a local way. The following lemma shows that CPCP is separably determined for closed, bounded and convex subsets.

Lemma 2.4. *Let X be a Banach space and C a closed, bounded and convex subset of X failing CPCP. Then there exits a closed, bounded, convex and separable subset of C failing CPCP.*

Proof. Assume that C fails CPCP. Then there is A a convex subset of C and a $\delta > 0$ such that every relatively weak open subset of A has diameter at least δ. Then $a \in \overline{A \setminus B(a, \delta/2)}^w$ for every $a \in A$. From Lemma 5.5 in [2], we have that for every $a \in A$ there is C_a a countable subset of $A \setminus B(a, \delta/2)$ such that $a \in \overline{C_a}^w$.

Pick $a_0 \in A$ and define inductively $D_1 = \{a_0\} \cup co(C_{a_0})$ and $D_{n+1} = D_n \cup co(\bigcup_{a\in D_n} C_a)$ for every $n \in \mathbb{N}$. Now we do $D = \bigcup_{n=1}^{+\infty} D_n$, so $\overline{D}$ is a separable, closed, bounded and convex subset of C because A is convex and $A \subset C$. Observe that D is convex since, for every $n \in \mathbb{N}$, one has that $D_n \subset \overline{co(\cup_{a\in D_n} C_a)}$ and $D_{n+1} = D_n \cup co(\bigcup_{a\in D_n} C_a)$.

We claim that every relatively open subset of D has diameter, at least, $\delta/2$. Indeed, if U is a relatively weak open subset of D then there is $x \in U \cap D_n$ for some $n \in \mathbb{N}$ and $x \in \overline{C_x}^w$. Hence there is $y \in C_x \cap U$ and $y \in D \cap U$. But $\|x - y\| > \delta/2$, so $\operatorname{diam}(U) \geqslant \delta/2$ and the claim is proved.

Finally, we obtain that every relatively open subset of $\overline{D}$ has diameter, at least, $\delta/2$ since D is obviously weakly dense in $\overline{D}$ and so $\overline{D}$ is a separable, closed, bounded and convex subset of C failing CPCP. $\square$

Our main result, Theorem 2.6 is a local characterization of CPCP in terms of $P_{\{v_n\}}$-sets. In particular, starting with a closed, bounded, convex and nonempty subset C of a Banach space and assuming that C fails CPCP and does not contain sequences equivalent to the standard unit basis of ℓ_1, it is showed that it is possible to construct a bounded and weakly null $P_{\{v_n\}}$-set in C, where $\{v_n\}$ is a seminormalized basic sequence. This is also proved in [11], by assuming that X is an Asplund space, which is a stronger hypothesis than assuming that C does not contain sequences equivalent to the standard unit basis of ℓ_1. Moreover the assumption in [11] is global, that is on X, and our new weaker assumption is local, that is on C. To do this, Lemma 2.4 is necessary. The construction in the following theorem shares the same scheme that the one done in [11], but there is one more important difference between [11] and the next theorem: the $P_{\{v_n\}}$-set constructed in [11] is obtained, because one assumes an Asplund setting, with $\{v_n\}$ being a seminormalized shrinking basic sequence and then $\{v_n\}$ is weakly null. This proof does not work if one even only assumes that the infinite-dimensional Banach space does not contain ℓ_1-copies, since there are Banach spaces without ℓ_1-copies and not containing shrinking basic sequences. Indeed, in [6] a Banach space G without ℓ_1-copies is constructed such that every infinite-dimensional separable subspace has a non-separable dual and then there are not shrinking basic sequences in G. Recall that the subspace spanned by a shrinking basic sequence has always a separable dual.

In the next theorem a $P_{\{v_n\}}$-set is constructed, where $\{v_n\}$ is a basic sequence but we cannot expect that $\{v_n\}$ is weakly null. We get a weaker condition which is the definition of weakly null $P_{\{v_n\}}$-set and this is enough to attain our result. For this, we use the following fact [12].

Lemma 2.5. *Let X be a Banach space, A a subset in X. Assume that A does not contain sequences equivalent to the standard unit basis of ℓ_1. If $a \in \overline{A}^w$, the weak closure of A, then there is a sequence in A weakly converging to a.*

We show now the aforementioned local characterization of CPCP for subsets without ℓ_1-sequences in terms of $P_{\{v_n\}}$-sets, where it is also proved that, roughly speaking, Banach spaces with a separable dual failing CPCP contain a special Poulsen simplex in its biduals.

Theorem 2.6. *Let C be a nonempty closed, bounded and convex subset of a Banach space X. Assume that C does not contain sequences equivalent to the standard unit basis of ℓ_1. Then the following assertions are equivalent:*

i) *C fails CPCP.*
ii) *There exists a seminormalized basic sequence $\{v_n\}$ in X such that C contains a weakly null $P_{\{v_n\}}$-set.*

Furthermore, if X^* is separable, then $\overline{P_{\{v_n\}}}^{w^*}$, the weak-star closure of $P_{\{v_n\}}$ in X^{**}, is affinely weak-star homeomorphic to the Poulsen simplex.

Proof. i) $\Rightarrow$ ii) Assume that C fails CPCP. From Lemma 2.4, we can assume that C is a closed, bounded, convex and separable subset in X failing CPCP and not containing sequences equivalent to the standard unit basis of ℓ_1. Now, as the closed linear span of C is a separable subspace of X, we can assume that X itself is separable. So X embeds isometrically in a Banach space Z with a monotone a normalized basis $\{e_n\}$ with biorthogonal functionals $\{f_n\}$. Take, for example, $Z = C[0,1]$ the space of continuous functions on $[0,1]$ with the sup norm. Since C fails CPCP, we can find a convex subset A of C and $\delta > 0$ such that every relative weak open subset of A has diameter grater than 2δ and so $a \in \overline{A \setminus B(a,\delta)}^w$ for every $a \in A$, where $B(a,\delta)$ denotes the open ball with center a and radius δ. Now, from Lemma 2.5, for every $a \in A$ we fix a sequence $\{y_j^a\}$ in $A \setminus B(a,\delta)$ weakly converging to a. Fix also a sequence $\{\delta_j\}$ of strictly positive numbers such that

$$2\sum_{j=1}^{+\infty} \delta_j < 1$$

and let $\{\varepsilon_n\}$ be a decreasing null sequence of positive numbers.

Fixed the sequence $\{\varepsilon_n\}$, our goal is constructing a weakly null $P_{\{v_n\}}$-set contained in C, for some seminormalized basic sequence $\{v_n\}$. For this, we shall construct inductively:

i) a sequence $\{l_n\}$ of positive integers numbers with $l_1 = 1$ and put $l_0 = 0$,
ii) sets $\{r_j\}_{m_{n-1}<j\leqslant m_n}$ such that $\{r_j\}$ is a strictly increasing sequence of integers and put $r_0 = 0$,
iii) sets $\{g_j\}_{m_{n-1}<j\leqslant m_n}$ such that $\{g_j\}$ is a sequence in A,
iv) sets $\{u_j\}_{m_{n-1}<j\leqslant m_n}$ such that $\{u_j\}$ is a sequence in Z,
v) sets $\{a_j\}_{m_{n-1}<j\leqslant m_n}$ such that $\{a_j\}$ is a sequence in A with $a_1 = g_1$,

such that, if, following Definition 2.1, we define from i), iii) and v) the corresponding

vi) strictly increasing sequence of integers $\{m_n\}$,
vii) subsets $\{v_j\}_{m_{n-1}<j\leqslant m_n}$ in X,
viii) sequence $\{K_n\}$ of subsets in X,

then one has that:

ix) $u_j \in [e_i\colon\ r_{j-1} < i \leqslant r_j]$,
x) $\|u_j - v_j\| < \delta_j$,
xi) $\|v_j\| > \frac{\delta}{2}$,
xii) $g_j \in K_n$ whenever $l_{n-1} < j \leqslant l_n$,
xiii) $\{g_j\}_{l_{n-1}<j\leqslant l_n}$ is an ε-net for K_n,
xiv) for every j there exist some $g \in A$ and $p_j \in \mathbb{N}$ such that $a_j = y_{p_j}^g$ and $\{p_j\}$ is strictly increasing.

Let us start the induction with $n = 1$ and note that $l_0 = 0 = m_0$ and $l_1 = 1 = m_1$. It is clear that $\operatorname{diam}(A) \geqslant 2\delta$, then there is $a_0 \in A$ such that $\|a_0\| > \delta$. Now, since $a_0 \in \overline{A \setminus B(a_0,\delta)}^w$, we can choose $p_1 \in \mathbb{N}$ such that $\|y_{p_1}^{a_0}\| > \frac{\delta}{2}$. Put $v_1 = a_1 = g_1 = y_{p_1}^{a_0}$ and $K_1 = \{a_1\}$.

Pick $r_1 > r_0$ such that $\|v_1|_{(r_1,+\infty)}\| < \delta_1$ and do $u_1 = v_1|_{[1,r_1]}$. This finishes the first step of the inductive construction.

Let $n \geqslant 1$ and assume that we have constructed

$$\{l_j\}_{1\leqslant j\leqslant n}, \qquad \{r_j\}_{1\leqslant j\leqslant m_n}, \qquad \{p_j\}_{1\leqslant j\leqslant m_n}\{a_j\}_{1\leqslant j\leqslant m_n}, \qquad \{u_j\}_{1\leqslant j\leqslant m_n}, \qquad \{g_j\}_{1\leqslant j\leqslant l_n}.$$

In fact, $m_{n+1} = 1 + \sum_{j=1}^{n} l_j$. Now we have to construct

$$\{r_j\}_{m_n<j\leqslant m_{n+1}}, \qquad \{p_j\}_{m_n<j\leqslant m_{n+1}}, \qquad \{u_j\}_{m_n<j\leqslant m_{n+1}}.$$

Take $j \in \{m_n+1, \ldots, m_{n+1}\}$, so $1 \leqslant j - m_n \leqslant l_n$. Then r_{j-1} and p_{j-1} have been already constructed. Define

$$V_j^n = \bigcap_{p=1}^{r_{j-1}} \left\{ a \in A\colon\ f_p(a - g_{j-m_n}) < \frac{\delta_j}{2r_{j-1}} \right\}.$$

It is clear that V_j^n is a relatively weak open subset of A containing g_{j-m_n} and so $\operatorname{diam}(V_j^n) > 2\delta$. But we have fixed a sequence $y_p^{g_{j-m_n}}$ in $A \setminus B(g_{j-m_n}, \delta)$ weakly convergent to g_{j-m_n}. Then there exists $p_j > p_{j-1}$ such that $y_{p_j}^{g_{j-m_n}} \in V_j^n \subset A$. Put $a_j = y_{p_j}^{g_{j-m_n}}$ and $v_j = a_j - g_{j-m_n}$. Then

$$\|v_j|_{[1,r_{j-1}]}\| = \left\| \sum_{p=1}^{r_{j-1}} f_p(v_j) e_p \right\| \leqslant \sum_{p=1}^{r_{j-1}} \left|_p(v_j)\right| < \delta_j/2.$$

Pick $r_j > r_{j-1}$ such that $\|v_j|_{(r_j,+\infty)}\| < \delta_j/2$ and put $u_j = v_j|_{(r_{j-1},r_j]}$. This completes the construction of sets

$$\{r_j\}_{m_n<j\leqslant m_{n+1}}, \qquad \{p_j\}_{m_n<j\leqslant m_{n+1}}, \qquad \{u_j\}_{m_n<j\leqslant m_{n+1}}.$$

Define $K_{n+1} = co\{a_k\colon 1 \leqslant k \leqslant m_{n+1}\}$. Then there exist $l_{n+1} > l_n$ and $\{g_j\}_{l_n<j\leqslant l_{n+1}}$ in K_{n+1} such that $\{g_j\}_{l_n<j\leqslant l_{n+1}}$ is an ε_{n+1}-net in K_{n+1}. This completes the inductive construction.

Now doing $K = \overline{co}\{a_n\colon n \in \mathbb{N}\}$ it is clear that K is a $P_{\{v_n\}}$-set contained in C, where $\{v_n\}$ is a seminormalized basic sequence in X equivalent to the basic block $\{u_n\}$ of the basis $\{e_n\}$ from ix), x) and xi).

Let us see that K is a weakly null $P_{\{v_n\}}$-set failing CPCP. Pick $j \in \mathbb{N}$, then there exists $n_0 \in \mathbb{N}$ such that $m_n + j < m_{n+1}$ for every $n \geqslant n_0$, since $m_{n+1} = m_n + l_n$ and $\{l_n\}$ is strictly increasing. Furthermore, from the definition, we can choose n_0 satisfying also that $g_j + v_{m_n+j} = a_{m_n+j} \in K$ for every $n \geqslant n_0$. From the construction of the sequence $\{a_n\}$ it is now clear that $a_{m_n+j} = y_{p_{m_n+j}}^{g_j}$ whenever $n \geqslant n_0$, being $\{p_j\}$ strictly increasing. Thus, for $n \geqslant n_0$, we have that $v_{m_n+j} = y_{p_{m_n+j}}^{g_j} - g_j$ and $\{y_{p_{m_n+j}}^{g_j}\}$ converges weakly to g_j, so we have proved that K is a weakly null $P_{\{v_n\}}$-set contained in C.

ii) $\Rightarrow$ i) If K is a weakly null $P_{\{v_n\}}$-set contained in C, being $\{v_n\}$ seminormalized, then, by Lemma 2.3 we get that K fails CPCP and so C fails also CPCP. This finishes the proof of ii) $\Rightarrow$ i).

Assume now that X^* is separable. In order to prove that $\overline{K}^{w^*}$ is affinely weak-star homeomorphic to the Poulsen simplex, we introduce some notation. From Lemma 2.2, since K is a $P_{\{v_n\}}$-set, there exists a sequence $\{x_n\} \subset [v_n] \subset X$ satisfying

$$x_1 = v_1, \qquad x_{n+1} \in co\{x_1, x_1 + v_2, \ldots, x_n + v_{n+1}\} \quad \forall n \in \mathbb{N},$$

such that $K = L_{\bar{x}}$, where $L_{\bar{x}} = \overline{\cup_{n=1}^{+\infty} L_n}$ and

$$L_1 = \{x_1\}, \qquad L_{n+1} = co\big\{L_n \cup \{x_n + v_{n+1}\}\big\} \quad \forall n \in \mathbb{N}.$$

We define a norm $\|.\|_L$ in $[v_n]^*$ given by $\|x^*\|_L = \sup_{l \in L_{\bar{x}}} |x^*(l)|$ for every $x^* \in [v_n]^*$. $\|.\|_L$ is a norm on $[v_n]^*$ because, from Lemma 2.2, $[L_{\bar{x}}] = [v_n]_{n\geqslant 1}$. Furthermore, from the proof of i) $\Rightarrow$ ii), $\{v_n\}$ is equivalent to a basic block of the basis in Z. Since X^* is separable, Z can be chosen a Banach space with a shrinking basis by [15], then we can assume that $\{v_n\}$ is shrinking and so $[v_n]^* = [v_n^*]$, where $\{v_n^*\}$ is the sequence of biorthogonal functionals of $\{v_n\}$. We put $Y = (\operatorname{lin}\{v_n^*\}, \|.\|_L)$ and $i\colon \operatorname{lin}\{v_n^*\} \to Y$ the canonical injection. Also we do $z_1 = v_1$ and $z_{n+1} = x_n + v_{n+1}$ for every $n \in \mathbb{N}$. Thus we have that $L_n = co\{z_i\colon 1 \leqslant i \leqslant n\}$ for every n and so L_n is a simplex for every n. Thus, since $x_n \in L_n$ for every n, we have that for every $n \in \mathbb{N}$ there exists a unique finite positive scalars sequence $\{a_{1,n}, \ldots, a_{n,n}\}$ satisfying $\sum_{i=1}^n a_{i,n} = 1$ and $x_n = \sum_{i=1}^n a_{i,n} z_i$. Now from Theorem 1.1 there are a unique compact metrizable Choquet simplex Λ and a monotone basis $\{e_n\}$ of $A(\Lambda)$, the space of affine and continuous functions on Λ, with $e_1 = \mathbf{1}$, the affine function constant equal 1, and a triangular matrix $\{e_{i,n}\}$ in $A(\Lambda)$ satisfying for all $n \in \mathbb{N}$:

I) $e_{n,n} = e_n$.
II) $\{e_{i,n}\}_{1\leqslant i\leqslant n}$ is isometrically equivalent to the standard basis of ℓ_n^∞.
III) $e_{i,n} = e_{i,n+1} + a_{i,n} e_{n+1}$ $\forall 1 \leqslant i \leqslant n$.

Consider $T : Y \to \operatorname{lin}\{e_n\}$ the unique linear map satisfying that $T(v_n^*) = e_n$ for every $n \in \mathbb{N}$.

First, we show that T is an isometry. Given $n \in \mathbb{N}$ we define a finite sequence $\{u_{i,n}\}_{1\leqslant i\leqslant n} \subset Y$ by $T(u_{i,n}) = e_{i,n}$ and for $x \in \operatorname{lin}\{e_{i,n}\colon 1 \leqslant i \leqslant n\}$ given define $y \in Y$ by $T(y) = x$. Now there are unique scalars $\{c_1, \ldots, c_n\}$ such that $x = \sum_{i=1}^n c_i e_{i,n}$ and so $\|x\| = \max\{|c_i\|\colon 1 \leqslant i \leqslant n\}$, from II). Furthermore it is easy to see that $u_{i,n}(z_j) = \delta_{ij}$ $1 \leqslant i, j \leqslant n$. Hence $\|y\|_L \geqslant \max_{l \in L_n} |y(l)| = \max_{1\leqslant j\leqslant n} |y(z_j)| = \max_{1\leqslant j\leqslant n} |c_j|$, since $y = \sum_{i=1}^n c_i u_{i,n}$. Thus we have proved that $\|y\|_L \geqslant \|x\|$.

For the other inequality, consider $\{P_n\}$ the sequence of natural projections of the basis $\{v_n\}$ from $[v_n]$. It is clear that $P_n^*(y) = y$. Now, if $m > n$ and $l \in L_m$, $P_n(l) \in L_n$ and so $|y(l)| = |y(P_n(l))|$. From the equality $\|y\|_L = \sup\{|y(l)|\colon l \in \cup_{m=1}^\infty L_m\}$ we deduce that $\|y\|_L \leqslant \|x\|$ and so T is an isometry.

In order to show that $\overline{K}^{w^*}$ is affinely weak-star homeomorphic to the Poulsen simplex, we prove that $(iT)^*|_\Lambda : \Lambda \to \overline{L_{\bar{x}}}^{w^*}$ is an onto affinely weak-star homeomorphism.

First we show that $(iT)^*(\Lambda) = \overline{L_{\bar{x}}}^{w^*}$. Consider for every $n \in \mathbb{N}$, $\Lambda_n = co\{e_1^*, \ldots, e_n^*\}$, where $\{e_n^*\}$ is the sequence of biorthogonal functionals of the basis $\{e_n\}$. Following Theorem 1.1, $\Lambda = \overline{\cup_{n=1}^\infty \Lambda_n}$. Now we have that $(iT)^*(e_n^*) = v_n$ for every $n \in \mathbb{N}$, from it follows immediately that $(iT)^*(\Lambda_n) = L_n$ for every $n \in \mathbb{N}$. The definition and weak-star continuity of $(iT)^*$ give that $(iT)^*(\Lambda) = \overline{\cup_{n=1}^\infty L_n}^{w^*} = \overline{L_{\bar{x}}}^{w^*}$.

To see that $(iT)^*|_\Lambda$ is one to one, pick $\lambda_1 \neq \lambda_2 \in \Lambda$. Then there is $x \in [e_i]_{1\leqslant i\leqslant n}$ for some $n \in \mathbb{N}$ such that $x(\lambda_1) \neq x(\lambda_2)$. Do $y = T^{-1}(x)$. Hence $(iT)^*(\lambda_1)(y) = x(\lambda_1)$ and $(iT)^*(\lambda_2)(y) = x(\lambda_2)$, so $(iT)^*(\lambda_1) \neq (iT)^*(\lambda_2)$ and $(iT)^*|_\Lambda$ is an onto affinely weak-star homeomorphism. This proves that $\overline{K}^{w^*}$ is affinely weak-star homeomorphic to a compact metrizable Choquet simplex.

Finally, following [9], it is enough to see that the set of extreme points of $\overline{K}^{w^*}$ is weak-star dense in $\overline{K}^{w^*}$ to deduce that $\overline{K}^{w^*}$ is affinely weak-star homeomorphic to the Poulsen simplex.

Indeed, following Definition 2.1, it is clear that $\{a_j\}_{1\leqslant j\leqslant m_n} \subset \mathrm{Ext}(K_n)$ for every $n \in \mathbb{N}$. Now we see that K_n is a face, extremal subset, of $\overline{K}^{w^*}$. If $\alpha, \beta \in \overline{K}^{w^*}$ and $\frac{\alpha+\beta}{2} \in K_n$ then $v_i^*(\alpha) = v_i^*(\beta) = 0$ whenever $m_n < i$. Taking into account that $v_i^*(K) \subset \mathbb{R}_0^+$ we deduce that $\alpha, \beta \in K_n$. Then $\{a_j\}_{1\leqslant j\leqslant m_n} \subset \mathrm{Ext}(K_n) \subset \mathrm{Ext}(\overline{K}^{w^*})$ for every $n \in \mathbb{N}$. But K is a weakly null $P_{\{v_n\}}$-set and so $\{g_i\} \subset \overline{\bigcup_{n=1}^{\infty} \mathrm{Ext}(K_n)}^{w^*} \subset \overline{\mathrm{Ext}(\overline{K}^{w^*})}^{w^*}$. Since $\{g_i\}$ is dense in K, we deduce that the set of extreme points of $\overline{K}^{w^*}$ is weak-star dense in $\overline{K}^{w^*}$ and we are done. $\square$

Remark.

(1) In order to possible future applications of Theorem 2.6 we claim that every subsequence of $\{v_n^*\}$ has a further subsequence $\{w_n\}$ which is pointwise convergent on $\overline{P_{\{v_n\}}}^{w^*}$. Indeed, it have been proved that T is an isometry with $T(v_n^*) = e_n$ for every $n \in \mathbb{N}$, being $\{e_n\}$ a natural basis of some L_1-predual. Hence $\{e_n\}$ is weakly precompact by Lemma 1.4 in [1] and the claim follows.

(2) We don't know if the above characterization is true for general Banach spaces without the weakly null condition. This is the case for i) $\Rightarrow$ ii).

(3) The fact that $\overline{P_{\{v_n\}}}^{w^*}$ is affinely weak-star homeomorphic to the Poulsen simplex can easily be verified for Asplund spaces, since every separable subspace of an Asplund space has a separable dual and from Lemma 2.4 CPCP is locally separably determined. That is, if C is a nonempty closed, bounded and convex subset failing CPCP in an Asplund space then, by Lemma 2.4, we can assume that C is separable and so $[C]$ is a Banach space with a separable dual. Now the second part of the above theorem can be applied. However we don't know if the same is true for Banach spaces without ℓ_1-copies.

As we announced in the introduction, we can deduce now that, for Banach spaces not containing ℓ_1 isomorphically, CPCP is determined on subspaces with a Schauder basis.

Corollary 2.7. *Let X be a Banach space not containing ℓ_1 isomorphically. Then the following assertions are equivalent*:

i) *X has CPCP.*

ii) *Every subspace of X with a Schauder basis has CPCP.*

Proof. The implication i) $\Rightarrow$ ii) is clear. For ii) $\Rightarrow$ i) it is enough to apply Theorem 2.6 being C the unit closed ball of X and note that the $P_{\{v_n\}}$-set K is a subset of the subspace $[v_n]$, failing CPCP by Lemma 2.3.

References

[1] S. Argyros, E. Odell, H. Rosenthal, On certain convex subsets of c_0, in: Functional Analysis, in: Lecture Notes in Math., vol. 1332, Springer, Berlin, 1988, pp. 80–111.

[2] J. Bourgain, La propriété de Radon–Nikodým, Publications de l'Université Pierre et Marie Curie, vol. 36, Université Pierre et Marie Curie, Paris, 1979.

[3] J. Bourgain, Dentability and finite-dimensional decompositions, Studia Math. 67 (1980) 135–148.

[4] J. Bourgain, H.P. Rosenthal, Geometrical implications of certain finite dimensional decompositions, Bull. Belg. Math. Soc. Simons Stevin 32 (1980) 54–75.

[5] N. Ghoussoub, B. Maurey, G_δ-embeddings in Hilbert spaces II, J. Funct. Anal. 78 (2) (1988) 271–305.

[6] W.T. Gowers, A Banach space not containing c_0, ℓ_1 or a reflexive subspace, Trans. Amer. Math. Soc. 344 (1) (1994) 407–420.

[7] D.E.G. Hare, An extension of a Structure Theorem of Bourgain, J. Math. Anal. Appl. 147 (1990) 599–603.

[8] A. Lazar, J. Lindenstrauss, Banach spaces whose duals are L_1 spaces and their representing matrices, Acta Math. 120 (1971) 165–193.

[9] L. Lindenstrauss, G. Olson, Y. Sternfeld, The Poulsen simplex, Ann. Inst. Fourier (Grenoble) 28 (1978) 91–114.

[10] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces I, Springer Verlag, Berlin, 1977.

[11] G. López, The convex point of continuity property in Asplund spaces, J. Math. Anal. Appl. 239 (1999) 264–271.

[12] M. López-Pellicer, V. Montesinos, Some remarks on ℓ_1-sequences, Arch. Math. 79 (2002) 119–124.

[13] W. Lusky, On separable Lindenstrauss spaces, J. Funct. Anal. 26 (1977) 103–120.

[14] E.T. Poulsen, A simplex with dense extreme points, Ann. Inst. Fourier (Grenoble) 11 (1961) 83–87.

[15] M. Zippin, Banach spaces with separable duals, Trans. Amer. Math. Soc. 1 (1988) 371–379.