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Zero divisor graphs of semigroups

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Abstract

The zero divisor graph of a commutative semigroup with zero is a graph whose vertices are the nonzero zero divisors of the semigroup, with two distinct vertices joined by an edge in case their product in the semigroup is zero. We continue the study of this construction and its extension to a simplicial complex.

This article continues the study of the zero divisor graph of a commutative semigroup begun (implicitly) in [1,2] and in [4,5], though it is mostly self contained. Throughout S denotes a commutative semigroup with 0 whose operation is written multiplicatively. Associate to S a simple graph G whose vertices are the nonzero zero divisors of S with $x \neq y$ connected by an edge in case $xy = 0$. Since the zero divisors of S form an ideal in S, we usually assume S consists of zero divisors. Observe though, that an ideal in a zero divisor semigroup may not consist of zero divisors. For example, If $S = \{0, x, y \mid x^2 = x, y^2 = y, xy = 0\}$ then S consists of zero divisors but the ideal $\{0, x\}$ does not. Recall the semigroup S is nilpotent in case for each $x \in S$ there is a positive integer n with $x^n = 0$. Every subsemigroup of a nilpotent semigroup consists of zero divisors. Moreover, in order that G be non empty, we usually assume S always contains at least one nonzero

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zero divisor. While we are principally interested in the assignment $S \to G$, when helpful we extend the graph G to the graph G^0 which adds to G the vertex 0 connected to every other vertex of G, and say G^0 is the graph of S^0. In Theorem 1 we recall four conditions which are necessary for a graph to be the graph of a semigroup. Every graph with five or fewer vertices which satisfies these conditions is the graph of a semigroup, but we give an example of a graph with six vertices which is not the graph of a semigroup but does satisfy these conditions. In Theorem 2 we give examples of graphs which are the graph of a semigroup. For each positive integer k the set I_k of elements of S whose vertex degree in G is greater than or equal to k together with 0 forms a descending chain of ideals in S. If S is nilpotent then the subgraph of G spanned by vertices of degree greater than or equal to k is the graph of I_k. If S is nilpotent, then generalizing the corresponding result in [6], we show every edge of the core of G is the edge of a triangle in G.

The second purpose of this article is to introduce the association to S of a simplicial complex $K(S)$, where a simplex is a subset A of $S - \{0\}$ with the property that $x, y \in A$ with $x \neq y$ implies $xy = 0$. We give examples of semilattices whose associated complexes have non trivial homology in degree greater than 0 and show the complex associated to any finite nilpotent semigroup has trivial homology in degree greater than 0. As a result, we can give an example of a graph with the property that every edge in G is the edge of a triangle and G is the graph of a semilattice but G is not the graph of any nilpotent semigroup.

Section 1

If x is a vertex in a graph G, let $N(x)$ be the vertices in G adjacent to x (connected to x by a single edge) and $\overline{N(x)} = N(x) \cup \{x\}$.

A graph G is a star graph in case there is a vertex x in G such that every other vertex in G is an end, connected to x and no other vertex by an edge.

The core of G is the largest subgraph of G in which every edge is the edge of a cycle in G.

Theorem 1. *If G is the graph of a semigroup then G satisfies all of the following conditions.*

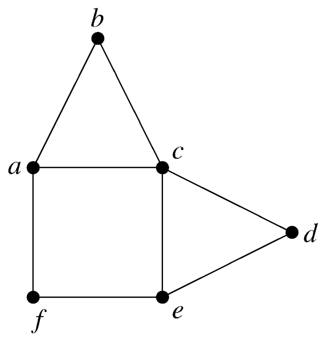
(1) *G is connected.*
(2) *Any two vertices of G are connected by a path with $\leqslant 3$ edges.*
(3) *If G contains a cycle then the core of G is a union of quadrilaterals and triangles, and any vertex not in the core of G is an end.*
(4) *For each pair x, y of nonadjacent vertices of G, there is a vertex z with $N(x) \cup N(y) \subset \overline{N(z)}$.*

Proof. For semigroups, the first three conditions were proved in [4]. For the fourth, assume G is the graph of S and let x and y be nonadjacent vertices in G. Then $xy = z \neq 0$ in S. If $a \in N(x) \cup N(y)$ then either $ax = 0$ or $ay = 0$. In either case, $az = a(xy) = 0$ so $a \in \overline{N(z)}$. Thus $N(x) \cup N(y) \subset \overline{N(z)}$. $\square$

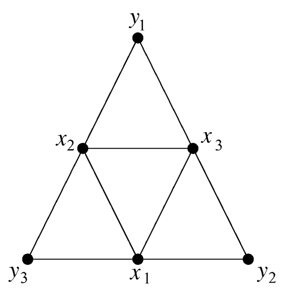
Theorem 2. *If G is a graph satisfying conditions* (1)–(4) *of Theorem* 1, *and there are five or fewer vertices in G then G is the graph of a semigroup.*

Proof. The proof is given by enumerating all the candidate graphs with fewer than six vertices and writing down a semigroup for each graph. $\square$

Example 1. The graph below gives a graph with six vertices which has diameter 2. The vertices a and d are not adjacent, so the product ad is nonzero. Observe that $\{b, c, e, f\} \subset N(a) \cup N(d)$. There does not exist a vertex z such that $\{b, c, e, f\} \subset \overline{N(z)}$, hence this graph fails to satisfy condition (4) of Theorem 1. Thus, this is not the graph of a semigroup.



Example 2. The graph G with six vertices pictured below satisfies the conditions of Theorem 1 and is not the graph of any semigroup.



Proof. Assume that the graph G pictured above is the graph of a semigroup S.

Step 1. First observe that since $\{x_2, x_3, y_2, y_3\} \subset \operatorname{ann}(x_1 y_1)$ and $\{x_2, x_3, y_2, y_3\} \subset \operatorname{ann}(x_1^2)$, $x_1 y_1, x_1^2 \in \{0, x_1\}$. Thus by symmetry, $\{0, x_i\}$ forms an ideal for $i = 1, 2, 3$. Thus, $x_i^2 = x_i$ or 0 for each i. Since $\{0, x_i\}$ forms an ideal and x_i is not adjacent to y_i for any i, we have that $x_i y_i \neq 0$, hence $x_i y_i = x_i$. Consider the triple product $x_k y_i y_j$ where $i \neq j$. By associativity, $x_k(y_i y_j) = (x_k y_i) y_j = y_i(x_k y_j)$. Since $i \neq j$, either $x_k y_j = 0$ or $x_k y_i = 0$. Therefore $x_k(y_i y_j) = 0$ for $k = 1, 2, 3$ and hence the product $y_i y_j \in \{x_1, x_2, x_3\}$.

Step 2. Consider the product $x_i y_i^2$. We have that $x_i y_i^2 = (x_i y_i) y_i = x_i y_i = x_i \neq 0$. Hence $y_i^2 \in \{x_i, y_i\}$ for any index $i \in \{1, 2, 3\}$.

Case 1. Suppose $y_\alpha^2 = y_\alpha$ for some fixed index α. Let j be any index not equal to α. Consider the triple product $y_\alpha y_\alpha y_j$. Then $y_\alpha^2 y_j = y_\alpha y_j \in \{x_1, x_2, x_3\}$ by Step 1. Also, $y_\alpha^2 y_j = y_\alpha(y_\alpha y_j) = y_\alpha \beta$ where $\beta \in \{x_1, x_2, x_3\}$. If $y_\alpha y_j = x_j$ or x_k for $j, k \neq \alpha$, we have $y_\alpha y_\alpha y_j = 0$, which gives a contradiction to $y_\alpha^2 y_j \in \{x_1, x_2, x_3\}$. This leaves the remaining case where $y_\alpha y_j = x_\alpha$.

The product $y_\alpha y_j^2 = (y_\alpha y_j) y_j = x_\alpha y_j = 0$ since $j \neq \alpha$, hence $y_j^2 \in \{0, x_j, x_k\}$ for j, k, α all distinct. In Step 2 it was shown that $y_j^2 \in \{x_j, y_j\}$ for all j, hence $y_j^2 = x_j$. Similarly, $y_k^2 = x_k$.

Therefore we have $0 = x_j y_k = (y_j y_j) y_k = y_j(y_j y_k)$ and similarly $0 = y_j x_k = y_j(y_k y_k) = (y_j y_k) y_k$. Since $y_j y_k \neq 0$, $y_j y_k$ is represented by a vertex which is adjacent to both y_j and y_k. Thus, $y_j y_k = x_\alpha$.

Hence the triple product $y_\alpha y_j y_k = y_\alpha(y_j y_k) = y_\alpha x_\alpha = x_\alpha$ and $y_\alpha y_j y_k = (y_\alpha y_j) y_k = x_\alpha y_k = 0$. This gives a contradiction since $x_\alpha \neq 0$.

Case 2. Suppose $y_i^2 = x_i$ for all i. Then for i, j distinct, $0 = x_i y_j = (y_i y_i) y_j = y_i(y_i y_j) = y_i \beta$, where $\beta \in \{x_1, x_2, x_3\}$. Hence $\beta \neq x_i$ and therefore $y_i y_j \in \{x_j, x_k\}$. Similarly, the same argument with $y_i y_j y_j$ shows that $y_i y_j \in \{x_i, x_k\}$. Therefore, we have $y_i y_j = x_k$ where i, j, k are all distinct. Then, we have $x_1 = y_1 x_1 = y_1(y_2 y_3) = (y_1 y_2) y_3 = x_3 y_3 = x_3$, which is a contradiction.

Hence, the above graph cannot be the graph of any semigroup. □

A graph G is a refinement of a graph H in case the vertex sets of G and H are the same and every edge in H is an edge in G.

Theorem 3. *The following graphs are the graph of a semigroup.*

(1) *A complete graph or a complete graph together with an end.*
(2) *A complete bipartite graph or a complete bipartite graph together with an end.*
(3) *A refinement of a star graph.*
(4) *A graph which has at least one end and has diameter $\leqslant 2$.*
(5) *A graph which is the union of two star graphs whose centers are connected by a single edge.*

Proof. (1) A complete graph G is the graph of the null semigroup on $G \cup \{0\}$. If x is an additional vertex connected only to $a \in G$ by an edge then beginning with the null semigroup on $G \cup \{0\}$ define $x^2 = x$, $xa = 0$ and $xb = b$ for all $b \neq a$ in $G \cup \{0\}$. It is easy to check the result is a semigroup whose graph is the complete graph G together with an end.

(2) Let G be the complete bipartite graph on $A \cup B$. Well order the elements in A and in B and let S be the semilattice with root 0 and two branches consisting of the vertices in A and B. The resulting semigroup has graph G. If x is an additional vertex connected only

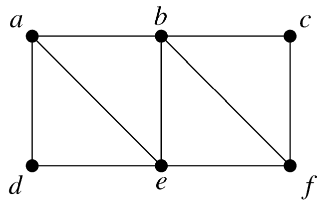
to $a \in A$ then let $a = a_1$ be the first element in A, $a_2 \in A$ the second element, and $b_1 \in B$ the first element in B. Then the semilattice defined above on $A \cup B$ with the additional relations $0 \leqslant x \leqslant a_2$ and $x \leqslant b_1$ defines a semilattice with the given graph.

(3) Let G be a refinement of a star graph with center z and end vertices $\{x_i \mid i \in I\}$. Define a semigroup S by: $z^2 = 0$, $x_i^2 = z$ $(i \in I)$, $x_i z = 0$ $(i \in I)$, $x_i x_j = 0$ if x_i and x_j are connected by an edge in G and $x_i x_j = z$ otherwise. Any triple product $a(bc)$ or $(ab)c$ is 0 in S so S is a semigroup with the given graph.

(4) Let G be a graph of diameter $\leqslant 2$ with an end vertex x. Let z be the vertex joined to x. Then because the diameter of G is $\leqslant 2$, every other vertex of G is connected to z by an edge, z is the center of a star graph which spans G, and G is the graph of a semigroup by (3).

(5) This is (2) of Theorem 1.3 of [4]. □

Example 3. By (3) and (5) of Theorem 3 we know that the refinement of a star graph and the union of two star graphs are each the graph of some semigroup.The graph drawn below is a refinement of the union of two star graphs with centers at vertex a and vertex b. However, this is not the graph of a semigroup. The vertices a and f do not satisfy condition (4) of Theorem 1 since vertex a is adjacent to d and vertex f is adjacent to c, but there is no vertex adjacent to both c and d.



A vertex x of a graph G has degree m in case $N(x)$ has m elements. For each positive integer k let G_k be the subgraph of G spanned by the vertices of G of degree $\geqslant k$.

Theorem 4. *Let S be a semigroup with graph G, and let I_k be the elements of G of degree $\geqslant k$ together with* 0. *Then $\{I_k\}$ is a descending chain of ideals in S.*

Proof. Let x be a vertex in G of degree $m \geqslant k$ and let y be a vertex in G. Assume x is connected by an edge exactly to the set $\{x_i \mid i \in I\}$. If $yx = 0$ then $yx \in I_k$. Otherwise $(yx)x_i = y(xx_i) = 0$ for all $i \in I$. If $yx \neq x_i$ for all $i \in I$ then $\deg(yx) \geqslant \deg(x)$. If $yx = x_i$ for some $i \in I$ then $\{x_j \mid j \neq i\} \cup \{x\}$ is a subset of $N(yx)$ so $\deg(yx) \geqslant \deg(x)$. The last statement of the theorem is obvious. □

Corollary 1. *If S is a semigroup with graph G then the core of G together with* $\{0\}$ *is an ideal in S whose graph is the core of G.*

Proof. The core of S together with 0 is the ideal I_2 of Theorem 4, and the graph of I_2 is the core of G. □

Corollary 2.

(1) *Let S be a nilpotent semigroup, and let G be the graph of S. Then G_k is the graph of I_k.*
(2) *Let S be a zero divisor semigroup, then $(G^0)_{k+1}$ is the graph whose vertices are the elements of the ideal I_k.*

Proof. (1) Since each element of I_k is nilpotent, each element of I_k is a zero divisor in I_k. Therefore the vertices of the graph of I_k correspond to the vertices of G_k. Two vertices in G_k are connected by an edge if and only if their corresponding product in I_k is 0, so the graph of I_k is G_k.

(2) Including 0 with the elements in the ideal I_k gives precisely the vertex set of the graph $(G^0)_{k+1}$. Note that there may be elements in the ideal I_k which are not zero divisors. These elements are adjacent only to the vertex 0. □

Corollary 3. *Let G be a graph and assume G_k is not the graph of a semigroup for some k. Then G is not the graph of a nilpotent semigroup.*

Proof. This is an immediate consequence of Corollary 2. □

Corollary 4. *If G is a graph equal to its core which is not the graph of a semigroup and H is a graph obtained from G by adding ends to G, then H is not the graph of a semigroup.*

Proof. This is an immediate consequence of Corollary 1. □

The following result for commutative rings is in [6].

Theorem 5. *Let S be a nilpotent semigroup.*

(1) *The diameter of the graph G of S is $\leqslant 2$.*
(2) *Every edge in the core of the graph G of S is the edge of a triangle in G.*

Proof. (1) Let a, b be vertices in G and assume a and b are not connected by an edge. Let n be the index of x and m the index of b. Then $ab \neq 0$ but $a^n b^m = 0$. There is a largest pair i, j in the lexicographic order with $a^i b^j \neq 0$. Then $a - a^i b^j - b$ is a path in G of length $= 2$.

(2) Let $a - b$ be an edge in the core of G. By Theorem 1.5 of [4], $a - b$ is either the edge of a rectangle or triangle in G. In the first case let $a - b - c - d - a$ be a rectangle. Then $ac \neq 0$ so as in the proof of the first part, we get the triangle $a - a^i c^j - c - a$. □

Corollary 5. *If every element in S has finite order and some edge in the core of the graph G of S is the edge of a square but not a triangle then S contains a nonzero idempotent.*

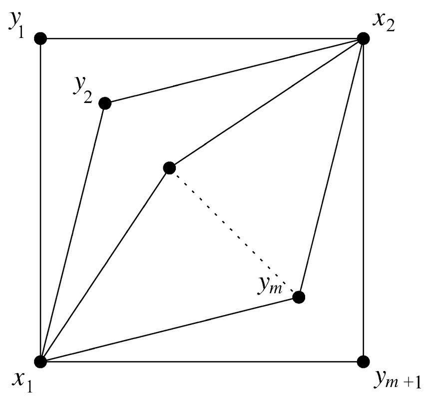
Proof. If G contains an edge in its core which is not the edge of a triangle then by part (2) of Theorem 5, S contains an element a which is not nilpotent. If $\langle a\rangle$ is finite then $\langle a\rangle$ contains a nonzero idempotent (see p. 19 of [3]). $\square$

Section 2

Associate to a commutative semigroup S with 0 a complex $K_0(S)$ by letting the simplices A in $K_0(S)$ be the finite subsets A of S such that $xy = 0$ for all $x \neq y \in A$. It is trivial to check $K_0(S)$ is an abstract complex [7, p. 15], and that the association $S \to K_0(S)$ defines as in [4] a covariant functor from the category whose objects are commutative semigroups with 0 and maps are semigroup homomorphisms taking 0 to 0 to the category whose objects are simplicial complexes and whose maps are simplicial maps. Since the complex $K_0(S)$ is a cone with vertex 0 over the nonzero elements of S the resulting complex has trivial homology and cohomology [7, p. 44]. As in the graph construction, the interesting object of study is the subcomplex whose vertices are the nonzero zero divisors in S. Let $K(S)$ denote this complex. Since $K(S)$ is connected (Theorem 1), $H_0(K, \mathbf{Z}) \cong \mathbf{Z}$ [7, 41].

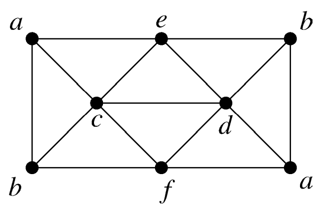
Example 4. (a) Let S be the semigroup with 0 generated by $\{x_i, y_j \mid 1 \leqslant i \leqslant 2, 1 \leqslant j \leqslant m+1\}$ where $m \geqslant 1$ and $x_i x_j = x_{\min(i,j)}$, $y_i y_j = y_{\min(i,j)}$ and $x_i y_j = 0$ for all i, j.

The complex $K(S)$ associated to S is the nest of quadrilaterals

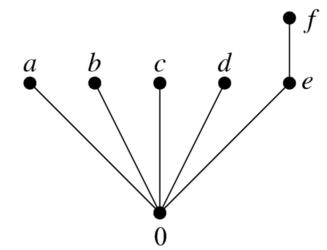


It is easy to see that if $m \geqslant 1$ then $H_1(K(S), \mathbf{Z}) \cong \mathbf{Z}^m$ and $H_n(K(S), \mathbf{Z}) \cong \{0\}$ for $n > 1$.

(b) The complex M drawn below gives a triangulation of the Mobius strip. Note that $H_1(M) = \mathbf{Z}$ and $H_n(M) = 0$ for $n > 1$.

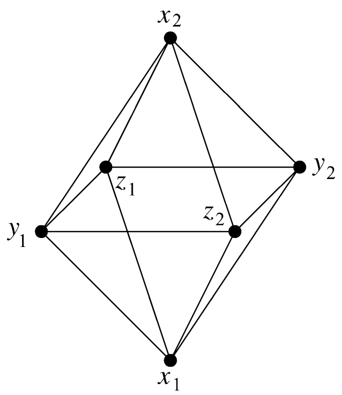


The semilattice below determines the complex M.



Example 5. Let S be the semigroup with 0 generated by $\{x_i, y_j, z_k \mid 1 \leqslant i, j, k \leqslant 2\}$ and relations $x_i x_j = x_{\min(i,j)}$, $y_i y_j = y_{\min(i,j)}$, $z_i z_j = z_{\min(i,j)}$ and $x_i y_j = y_j z_k = x_i z_k = 0$ for all i, j, k.

The complex $K(S)$ associated to S is the following surface.



It is easy to see $H_2(K(S), \mathbf{Z}) \cong \mathbf{Z}$ and for $n > 0$, $n \neq 2$, $H_n(K(S), \mathbf{Z}) \cong \{0\}$.

As a result of Example 5, the following can be considered an improvement of Theorem 5 in the case where S is finite.

Theorem 6. *Let S be a finite commutative nilpotent semigroup with* 0. *Let $K(S)$ be the associated complex. Then $H_n(K(S), \mathbf{Z}) \cong \{0\}$ for all $n > 0$.*

Proof. Since S is finite, S contains a maximal ideal I. Since every element in S is nilpotent, every element in S/I is nilpotent, where S/I is the semigroup obtained by identifying each element in I with 0 and leaving all other elements alone. Then S/I is a 0-simple nilpotent semigroup with $|S| - |I| + 1$ elements. Let $T = S/I$ and let $b \in T$, $b \neq 0$. Then $Tb = T$ or $Tb = 0$. The first case is impossible since T is finite and b is a zero divisor, so $Tb = 0$. Therefore, $\{0, b\}$ is an ideal in T, so $T = \{0, b\}$ and $b^2 = 0$. Thus $|I| = |S| - 1$.

Let $J = \{x \in S - \{0\} \mid bx = 0\}$ and let $V = J \cup \langle b\rangle$. Note that V is a subsemigroup of S. Let $K_0 = K(V)$, $K_1 = K(I)$ and $K = K(S)$. If $A \in K(S)$ and if $b \in A$ then $A \in K(V) =$

K_0 and if $b \notin A$ then $A \in K(I) = K_1$. Thus $K = K_0 \cup K_1$. If $A \in K_0 \cap K_1$ then $A \subset V$ and $A \subset I$ so $A \subset V \cap I$. Thus $K_0 \cap K_1 \subset K(V \cap I)$. Conversely, if $A \in K(V \cap I)$ then $A \in K_0 \cap K_1$.

Let n be the index of b, i.e., $b^n = 0$ but $b^{n-1} \neq 0$, $n > 1$. Then V is a cone with vertex b^{n-1}, so by Theorem 8.2, p. 45 of [7], $H_p(K_0) = 0$ for $p > 0$. Part of the Mayer–Vietoris sequence (p. 142 of [7]) gives $\cdots \to H_p(K_0) \oplus H_p(K_1) \to H_p(K) \to H_{p-1}(K_0 \cap K_1) \to H_{p-1}(K_0) \oplus H_{p-1}(K_1) \cdots$.

Proceed by induction on $|S|$. Clearly if $|S| = 2$ then $H_p(K) = 0$ for $p > 0$. By induction, $H_p(K_1) = H_p(K(I)) = 0$ for $p > 0$. Observe that $V \cap I$ is a nilpotent semigroup of order strictly less than the order of S since the element $b \notin V \cap I$. By induction, $H_p(K_0 \cap K_1) = 0$ for $p > 0$, hence $H_p(K) = 0$ for $p > 1$.

If $p = 1$ then we have $\cdots \to H_1(K_0) \oplus H_1(K_1) \to H_1(K) \to H_0(K_0 \cap K_1) \to H_0(K_0) \oplus H_0(K_1) \to H_0(K) \to 0$ from Mayer–Vietoris. In the case $p = 1$ this sequence gives us $0 \to H_1(K) \to Z \to Z \oplus Z \to Z \to 0$. By exactness, we have that $H_1(K) = 0$. □

Corollary 6. *Let R be a finite commutative local ring. Let $K(S)$ be the associated complex. Then $H_n(K(S), \mathbf{Z}) \cong \{0\}$ for all $n > 0$.*

Proof. The complement of the maximal ideal of R consists of units, and thus the set of zero divisors of R coincides with the maximal ideal of R, which is nilpotent. Thus the corollary follows from Theorem 6. □

Question. Is there a simplicial decomposition of the Klein bottle or the real projective plane which is the complex of a semigroup?

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