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# Legendre transform, Hessian conjecture and tree formula

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## Abstract

Let  $\varphi$  be a polynomial over  $K$  (a field of characteristic 0) such that the Hessian of  $\varphi$  is a nonzero constant. Let  $\bar{\varphi}$  be the formal Legendre transform of  $\varphi$ . Then  $\bar{\varphi}$  is well defined as a formal power series over  $K$ . The Hessian conjecture introduced here claims that  $\bar{\varphi}$  is actually a polynomial. This conjecture is shown to be true when  $K = \mathbb{R}$  and the Hessian matrix of  $\varphi$  is either positive or negative definite somewhere. It is also shown to be equivalent to the famous Jacobian conjecture. Finally, a tree formula for  $\bar{\varphi}$  is derived; as a consequence, the tree inversion formula of Gurja and Abyankar is obtained.

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## 1. Introduction

The Jacobian conjecture is one of the famous open fundamental problems in mathematics [1], and is very often stated as

**Conjecture 1.1** (*Jacobian Conjecture*). *Let  $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be a polynomial map whose Jacobian is a nonzero constant; then  $f$  is invertible and the inverse is also a polynomial.*

(In fact the field  $\mathbb{C}$  can be replaced by any field of characteristic zero. But the analogue for a field with characteristic  $p > 0$  is false. See Ref. [2].)

Originally called Keller's problem [3], the Jacobian conjecture has a few published faulty proofs [4–7]. Over a hundred papers have been published, but the conjecture is still open even in dimension two. Like many other famous conjectures, this conjecture is deceptively simple!

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Ref. [2] gives an excellent review on the Jacobian conjecture up to 1982. For a more recent review and references on the Jacobian conjecture, the reader may consult Ref. [8].

It is probably well known to people working on the Jacobian conjecture that there are many other conjectures which are equivalent to the Jacobian conjecture. Here we propose another equivalent conjecture—the Hessian conjecture. This conjecture grows out of the author’s failed attempt to settle the Jacobian conjecture and is interesting in its own right; and it looks simpler: instead of dealing with many polynomials, one just needs to deal with a single polynomial.

### 1.1. Hessian conjecture

Let  $K$  be a field of characteristic zero,  $\varphi$  a polynomial in  $n$  variables with coefficients in  $K$ , i.e.,  $\varphi \in K[x_1, \dots, x_n]$ . The Hessian matrix  $H_\varphi(x)$  is a symmetric matrix whose  $(i, j)$ -entry is  $\partial_i \partial_j \varphi(x)$ . By definition, the determinant of  $H_\varphi(x)$  is called the Hessian of  $\varphi$  at  $x$ , denoted by  $h_\varphi(x)$ .

Suppose that  $h_\varphi \neq 0$  at  $x = 0$ ; then  $y = \nabla \varphi(x) := (\partial_1 \varphi(x), \dots, \partial_n \varphi(x))$  has a formal inverse  $x = g(y)$ —a formal power series in  $y$ . Let  $\bar{\varphi}(y)$  be the (formal) Legendre transform of  $\varphi$ , i.e.,  $\bar{\varphi}(y)$  is a formal power series in  $y$  defined by the equation

$$\bar{\varphi}(y) = [xy - \varphi(x)]|_{x=g(y)}. \quad (1)$$

It is clear that  $x = \nabla \bar{\varphi}(y)$ , so  $\bar{\varphi}$  is a potential function for  $g$ . Obviously  $\bar{\varphi}$  is a formal power series in  $y$ ; however, we may consider the

**Conjecture 1.2** (*Hessian Conjecture*). *Let  $\varphi$  be a polynomial over  $K$  whose Hessian is a nonzero constant,  $\bar{\varphi}$  the formal Legendre transform of  $\varphi$ . Then  $\bar{\varphi}$  is also a polynomial.*

**Theorem 1.3.** *The Hessian conjecture is true when  $K = \mathbb{R}$  and the Hessian matrix is definite (either positive or negative) somewhere. Therefore, if*

$$\varphi(x) = \frac{1}{2}x^2 + \text{higher order terms}$$

*is a real polynomial with  $h_\varphi = 1$  everywhere, then  $\bar{\varphi}$  is also a polynomial.*

**Proof.** Let  $\varphi$  be a real polynomial function on  $\mathbb{R}^n$  whose Hessian is constant. Without the loss of generality we may assume  $h_\varphi = 1$  everywhere.

**Claim 1.**  *$H_\varphi$  is non-degenerate everywhere and has constant signature. Therefore, if  $H_\varphi$  is positive (negative) definite somewhere, it is positive (negative) definite everywhere.*

**Proof of the Claim 1.** Fix  $x \in \mathbb{R}^n$ . Define

$$O(t) := H_\varphi(tx).$$

Then  $O$  is a smooth path in the space of non-degenerate (because of the Hessian condition on  $\varphi$ ), real symmetric  $n \times n$  matrices; therefore we have a spectral flow from  $t = 0$  to  $t = 1$ . The Hessian condition on  $\varphi$  implies that the signature of  $O(1) = H_\varphi(x)$  must be equal to that of  $O(0)$ ; otherwise, there would be a zero eigenvalue somewhere along the path, say at  $t_0$  ( $0 < t_0 < 1$ ), but then we would have the following contradiction:

$$0 = \det O(t_0) = \det H_\varphi(t_0x).$$

**Claim 2.** *As a map from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ ,  $\nabla \varphi$  is one to one.*

**Proof of the Claim 2.** Suppose that  $\nabla\varphi(x_1) = \nabla\varphi(x_2)$  for some points  $x_1$  and  $x_2$  in  $\mathbb{R}^n$ . Set  $f(t) = (x_2 - x_1) \cdot \nabla\varphi(x_1 + t(x_2 - x_1))$  for  $0 \leq t \leq 1$ . Note that  $f(0) = f(1)$ , so there is a  $t_0 \in (0, 1)$  such that  $f'(t_0) = 0$ , i.e.,

$$(x_2 - x_1)^T H_\varphi(x_1 + t_0(x_2 - x_1))(x_2 - x_1) = 0.$$

By the assumption on  $\varphi$  and Claim 1 above, we know that  $H_\varphi(x_1 + t_0(x_2 - x_1))$  is definite, so  $x_2 - x_1 = 0$ , i.e.,  $x_2 = x_1$ . Since  $\nabla\varphi$  is one to one, by Theorem 2.1 of Ref. [2], we know  $\nabla\varphi$  has a polynomial inverse, so it is clear from Eq. (1) that  $\bar{\varphi}$  is also a polynomial.  $\square$

**Proposition 1.4.** *The Hessian conjecture is equivalent to the Jacobian conjecture.*

**Proof.** If the Jacobian conjecture is true, then Eq. (1) implies that the Hessian conjecture is also true. On the other hand, assume the Hessian conjecture is true; then the Jacobian conjecture is also true, and this can be proved by the following trick: Let  $f: K^n \rightarrow K^n$  be a polynomial map whose Jacobian is 1 everywhere. Let  $\varphi(v, x) = v \cdot f(x)$ ; then  $\varphi$  is a polynomial function on  $K^{2n}$  whose Hessian is  $(-1)^n$  everywhere. Then  $\bar{\varphi}$  is also a polynomial function by the assumption. Now  $\bar{\varphi}(w, y) = w \cdot f^{-1}(y)$  where  $f^{-1}(y)$  is the formal inverse of  $f$ , so  $f^{-1}(y)$  is also a polynomial.  $\square$

### 1.2. A reduction theorem

In view of the reduction theorem in [2] and the proof of Proposition 1.4, the following reduction theorem can be easily deduced.

**Theorem 1.5.** *The Hessian conjecture is true  $\Leftrightarrow$  for each integer  $n \geq 1$  and for each polynomial map  $\varphi: \mathbb{C}^{2n} \rightarrow \mathbb{C}$  of the form*

$$\varphi(x) = \frac{1}{2}x^2 + a \text{ homogeneous quartic polynomial in } x,$$

*if the Hessian of  $\varphi$  is constant, then  $\bar{\varphi}$  is a polynomial.*

In Section 2 we shall introduce and prove a tree formula for  $\bar{\varphi}$ ; as a consequence, we obtain the tree formula of Gurja and Abyankar [15,2].

## 2. A tree formula

Let  $x = (x^1, \dots, x^n)$  and

$$\varphi(x) = \sum_{2N \geq m \geq 2} \frac{1}{m!} T_m(x), \tag{2}$$

where  $N > 1$  is an integer and  $T_m(x)$  is a degree  $m$  homogeneous polynomial in  $x$ . Note that  $T_m(x)$  should be identified with  $T_m = [(T_m)_{i_1 \dots i_m}]$ —a symmetric tensor of  $m$  indices:

$$T_m(x) = (T_m)_{i_1 \dots i_m} x^{i_1} \dots x^{i_m}.$$

(Here the repeated indices are summed over.)

Assume that  $T_2$  is non-degenerate. Then we can introduce the symmetric tensor  $T_2^{-1}$ : by definition,  $[(T_2^{-1})^{ij}]$  is the inverse matrix of  $[(T_2)_{ij}]$ . Under the assumption, we can formally solve equation  $y = (\partial_1\varphi(x), \dots, \partial_n\varphi(x))$  for  $x$ , so the Legendre transformation (1) is well defined. We say  $\varphi$  is non-degenerate if its degree two homogeneous component is non-degenerate.

**Theorem 2.1 (Tree Formula).** Suppose that  $\varphi$  is non-degenerate; then the formal Legendre transform of  $\varphi$  has the following tree expansion formula:

$$\bar{\varphi}(y) = \sum_{\Gamma \in \{\text{connected tree diagrams}\}} w(\Gamma) \tag{3}$$

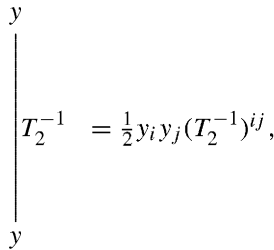
where  $w(\Gamma)$  is the contribution from tree diagram  $\Gamma$  and is given according to the following rules:

- (1) to each edge of  $\Gamma$ , assign  $T_2^{-1}$ ,
- (2) to each external vertex, assign  $y = (y_1, \dots, y_n)$ ,
- (3) to each internal vertex of degree  $n$ , assign  $-T_n$ ,
- (4) multiply all assignments in (1) through (3) and make all necessary contractions and then divide by  $|\text{Aut } \Gamma|$  to get  $w(\Gamma)$ .

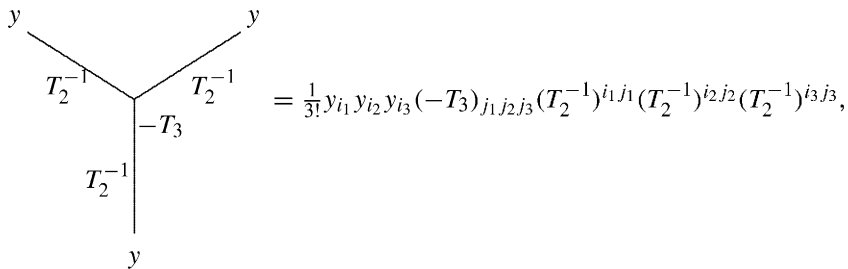
Here  $\text{Aut}(\Gamma)$  is the automorphism group of  $\Gamma$  (see [Appendix A](#) for its precise meaning) and  $|\text{Aut}(\Gamma)|$  is the order of  $\text{Aut}(\Gamma)$ .

To help readers understand the rules in the theorem, let us present two examples here:

**Example 1.**



**Example 2.**



where the repeated indices are summed over.

Write  $\bar{\varphi}(y) = \sum_{m \geq 2} \frac{1}{m!} S_m(y)$  and the right-hand side of (3) as  $\sum_{m \geq 2} \frac{1}{m!} \tilde{S}_m(y)$ , where both  $S_m(y)$  and  $\tilde{S}_m(y)$  are degree  $m$  homogeneous polynomials in  $y$ . It is not hard to see that each coefficient  $C$  of  $S_m(y) - \tilde{S}_m(y)$  is a rational function (over the field of rational numbers) in the coefficients of  $T_2, \dots, T_{2N}$ . To prove (3), we need to show that each  $C$  is zero as a rational function in the coefficients of  $T_2, \dots, T_{2N}$ ; equivalently, we need to show that the zero set of each  $C$  contains an open subset. Therefore, without loss of generality, we may assume that  $K = \mathbb{R}$ ; moreover, we just need to show that each  $C$  has value zero for all  $\varphi$  in a nonempty open set of  $\mathcal{P}_N$ —the space all real polynomials without linear and constant terms and having degree at most  $2N$ , i.e., we just need to prove [Theorem 2.1](#) for all  $\varphi$  in an nonempty open set of  $\mathcal{P}_N$ . ( $\mathcal{P}_N$  is a vector space and we can put a metric on it: by definition, if  $f, g$  are in  $\mathcal{P}_N$ , then

the distance between  $f$  and  $g$  is defined to be the maximum of the absolute value of the coefficients of  $f - g$ .)

**Lemma 2.2.** *There is a nonempty open set  $U_N$  in  $\mathcal{P}_N$  such that for all  $\varphi \in U_N$  we have (1)  $\varphi(x) > \frac{1}{2}|x|^{2N}$  if  $|x|$  is sufficiently large; (2) 0 is the only critical point of  $\varphi$ ; (3) the quadratic component  $T_2$  of  $\varphi$  is positive definite.*

**Proof.** Let  $\varphi_0(x) = (|x|^2 + 1)^N - 1$ . Then  $\varphi_0$  satisfies conditions (1)–(3) in the lemma. It is not hard to see that if  $\varphi$  is sufficiently close to  $\varphi_0$ , then  $\varphi$  satisfies conditions (1)–(3) in the lemma, too. So we can take  $U_N$  to be a sufficiently small ball centered at  $\varphi_0$ .  $\square$

**Corollary 2.3.** *Theorem 2.1 is valid for all  $\varphi \in U_N$ .*

**Proof.** Assume  $\varphi \in U_N$ . Without loss of generality, we may assume  $T_2(x) = |x|^2$ —that amounts to a rotation of the coordinate system.

The proof is obtained by evaluating

$$\lim_{\hbar \rightarrow 0} \hbar \log \frac{\int dx \exp \frac{1}{\hbar} (yx - \varphi(x))}{\int dx \exp \left( -\frac{1}{2\hbar} T_2(x) \right)} \tag{4}$$

in two different ways. (The integrations are done over the whole space  $\mathbb{R}^n$ .)

On the one hand, assume  $|y|$  is sufficiently small; using the assumption on  $\varphi$ , by steepest descent [16], this limit becomes

$$yz - \varphi(z), \tag{5}$$

where  $z$  is the unique solution of equation

$$y - \nabla\varphi(x) = 0 \tag{6}$$

for  $x$ , i.e.,  $z = (\nabla\varphi)^{-1}(y)$ . Therefore, this limit is  $\varphi'(y)$ —the Legendre transform of  $\varphi$  as a function (not as a formal power series) in  $y$  and the coefficients of  $T_3, \dots, T_{2N}$ .

On the other hand, using the assumption on  $\varphi$ , we can calculate

$$\hbar \log \frac{\int dx \exp \frac{1}{\hbar} (yx - \varphi(x))}{\int dx \exp \left( -\frac{1}{2\hbar} T_2(x) \right)} \tag{7}$$

in terms of connected Feynman diagrams to get its asymptotic series expansion<sup>1</sup> in  $\hbar$ ,  $y$  and the coefficients of  $T_3, \dots, T_{2N}$ ; see Appendix A for more details. Note that the contribution from a connected Feynman diagram with  $m$  loops is proportional to  $\hbar^m$ , so only the contributions from the tree diagrams survive in the limit (4). Since the contributions from the tree diagrams are exactly given by the rules specified in Theorem 2.1, we have the right-hand side of (3) which can be seen to be an asymptotic series expansion for  $\varphi'$  in  $y$  and the coefficients of  $T_3, \dots, T_{2N}$ .

By the definition of  $\tilde{\varphi}$  and  $\varphi'$ , one can see that  $\tilde{\varphi}$  is a convergent power series expansion of  $\varphi'$ ; hence it is also an asymptotical series expansion for  $\varphi'$  in  $y$  and the coefficients of  $T_3, \dots, T_{2N}$ . By the uniqueness of asymptotical series expansion, we have a proof of Theorem 2.1 for  $\varphi \in U_N$ .  $\square$

<sup>1</sup> For a definition of the asymptotic series expansion, see Ref. [16].

**Proof of the Theorem 2.1.** The proof in the general case follows from the above corollary and the discussion preceding to Lemma 2.2.  $\square$

**Remark 2.4.** Strictly speaking, we should obtain some estimates to fully justify some of the arguments in the above proof of Lemma 2.2 and its corollary. These estimates are not hard to obtain; however, they would make the work lengthy and also make the main ideas behind the proof a little bit obscure.

**Remark 2.5.** Using the trick involved in the proof of Proposition 1.4, it is not hard to see that the tree formula given in this work and the tree formula of Gurja and Abyankar actually imply each other. While the original proof of the tree formula of Gurja and Abyankar is purely algebraic, the proof given here for our tree formula is both algebraic and analytic.

## Acknowledgements

Our thanks to A. Voronov for introducing us to the one-dimensional tree inversion formula and Ref. [2]. A discussion of the tree inversion formula of Gurja and Abyankar in terms of Feynman diagrams has recently appeared in [9,10], but our discussion has a different perspective and our proof and derivation of the tree formula use somewhat different ideas. Significant work has appeared pertaining to polynomial maps with symmetric Jacobian matrices [11–13]. In [13] de Bondt and van den Essen describe a Hessian conjecture virtually identical to the one we formulate, but which does not involve the Legendre transform. They also show its equivalence to the Jacobian conjecture and prove the reduction theorem of Section 1.2 in this work. We learned of that paper only after this work was originally written. Although the Hessian conjecture has been articulated as such only recently, the first result in this area—the case  $n = 2$ —was proved in 1991 [14]. This work was supported by the Hong Kong Research Grants Council under the RGC project HKUST6161/97P.

## Appendix A. Feynman diagrams for Lebesgue integrals

A very good reference for the discussion below is [17]. Let

$$Y(\lambda) = \int dx \exp\left(-\frac{a}{2}x^2 - \frac{\lambda}{4!}x^4\right) \equiv \left\langle \exp\left(-\frac{\lambda}{4!}x^4\right) \right\rangle \quad (8)$$

where  $a > 0$  and  $\lambda > 0$  are parameters and the integration is done over  $\mathbb{R}$  and the integration measure is normalized so that

$$\int dx \exp\left(-\frac{a}{2}x^2\right) = 1. \quad (9)$$

We are interested in the perturbative computation of  $Y(\lambda)$ . Formally, we have

$$Y(\lambda) \sim \sum_n \frac{1}{n!(4!)^n} (-\lambda)^n \left\langle \underbrace{x^4 \cdots x^4}_n \right\rangle, \quad (10)$$

where symbol  $\sim$  means the asymptotic series expansion of  $Y(\lambda)$  as  $\lambda \rightarrow 0$ . We would like to compute

$$\frac{1}{n!(4!)^n} (-\lambda)^n \left\langle \underbrace{x^4 \cdots x^4}_n \right\rangle,$$

and for that purpose we observe that

- (1)  $\langle e^{Jx} \rangle = e^{\frac{J^2}{2a}}$ ,
- (2)  $\langle x^{2m+1} \rangle = 0$  for any integer  $m \geq 0$ ,
- (3)  $\langle xx \rangle = \frac{\partial^2}{\partial J^2} \langle e^{Jx} \rangle|_{J=0} = \frac{1}{a}$ ,
- (4)  $\left\langle \underbrace{x \cdots x}_{2m} \right\rangle = \frac{\partial^{2m}}{\partial J^{2m}} \langle e^{Jx} \rangle|_{J=0}$  which is equal to the number of complete pairings of  $x$ 's in  $\underbrace{x \cdots x}_{2m}$  times  $\left(\frac{1}{a}\right)^m$ .

Viewing  $\underbrace{x^4 \cdots x^4}_n$  as  $\mathcal{X}_n$ —a collection of  $n$  identical copies of  $x$ -cross (here  $x$ -cross means a cross with each of its four legs attached to an  $x$ ). The topological symmetry group of  $\mathcal{X}_n$  is  $G_n = (S_4)^n \times S_n$ —the semi-product of  $(S_4)^n$  with  $S_n$ . Let  $\mathcal{P}_n$  be the set of all possible complete pairings of  $x$ 's in  $\mathcal{X}_n$ . Then  $G_n$  acts on  $\mathcal{P}_n$ . Note that an orbit of this action can be identified with a graph obtained by pairing the  $x$ 's in  $\mathcal{X}_n$  according to any complete pairing in the orbit. Now, if  $\Gamma$  is such an orbit or graph (called a Feynman diagram), then

$$|\Gamma| = \frac{|G_n|}{|\text{Aut}(\Gamma)|} \tag{11}$$

where  $|S|$  denotes the number of elements in set  $S$  and  $\text{Aut}(\Gamma)$  means the subgroup of  $G_n$  that fixes an element in  $\Gamma$ , called the symmetry group of the Feynman diagram  $\Gamma$ . Therefore,

$$\frac{1}{n!(4!)^n} (-\lambda)^n \left\langle \underbrace{x^4 \cdots x^4}_n \right\rangle = \sum_{\Gamma \in \left\{ \begin{smallmatrix} 4\text{-valent closed graphs} \\ \text{with } n\text{-vertices} \end{smallmatrix} \right\}} \frac{1}{|\text{Aut}(\Gamma)|} (-\lambda)^n \left(\frac{1}{a}\right)^{2n},$$

where  $\frac{1}{|\text{Aut}(\Gamma)|} (-\lambda)^n \left(\frac{1}{a}\right)^{2n}$  is the contribution from Feynman diagram  $\Gamma$  according to the following Feynman rules:

- (1) To each vertex of  $\Gamma$  we assign  $-\lambda$ .
- (2) To each 1-simplex of  $\Gamma$  we assign  $\frac{1}{a}$  (called the propagator).
- (3) We multiply the contributions from all vertices and all 1-simplexes and then divide by the order of the symmetry group of  $\Gamma$ .

In summary, we have

$$Y(\lambda) \sim \sum_{\Gamma \in \{4\text{-valent closed graphs}\}} \frac{1}{|\text{Aut}(\Gamma)|} (-\lambda)^{v_\Gamma} \left(\frac{1}{a}\right)^{e_\Gamma},$$

where  $v_\Gamma$  and  $e_\Gamma$  are the number of vertices and 1-simplexes of  $\Gamma$ . Note that the contribution from the empty graph is set to 1 by convention. And it is tautological that

$$\log Y(\lambda) \sim \sum_{\Gamma \in \left\{ \begin{smallmatrix} \text{connected nonempty} \\ 4\text{-valent closed graphs} \end{smallmatrix} \right\}} \frac{1}{|\text{Aut}(\Gamma)|} (-\lambda)^{v_\Gamma} \left(\frac{1}{a}\right)^{e_\Gamma}. \tag{12}$$

It is not hard to see how to generalize all the above discussion to the general case when other types of versus (such as 3-valent, 5-valent, . . . , ones) may also appear.

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