The Volume of a Tetrahedron

E. C. CHO
Mathematics Department
Kentucky State University
Frankfort, KY 40601, U.S.A.

(Received March 1993; revised and accepted October 1994)

Abstract—A formula for the volume of a tetrahedron in terms of the length of edges is given. It may be viewed as a generalization of Heron’s formula to the volume of a tetrahedron.

Keywords—Heron, Simplex, Tetrahedron, Volume.

Heron’s formula represents the area of a triangle in terms of the length of its sides:

\[ A = \sqrt{p(p-a_1)(p-a_2)(p-a_3)}, \]

where \(a_1, a_2,\) and \(a_3\) are the length of the sides of the triangle and \(p = (1/2)(a_1 + a_2 + a_3)\).

It is natural to ask for such a formula for higher dimensional simplexes. Since the length of each edge (1-dimensional face) determines the \(n\)-simplex up to isometry, the volume of an \(n\)-simplex is determined by the length of its edges. The formula for a general 3-simplex is not very simple, but Theorem A gives a relatively simple formula of the volume of 3-simplexes which can be isometrically embedded as a face of a rectangular 4-simplex.

THEOREM A. Let \(w_1, w_2, w_3\) and \(w_4\) be vectors of \(E^3\) in a general position. Let \(T\) be the tetrahedron spanned by \(w_1, w_2, w_3\) and \(w_4\), and \(a_{ij}\) be the length of the edge \([w_i, w_j]\). If \(a_{ij}\) satisfy the conditions (3) and (4) given in Lemma 1, then

\[ \text{volume of } T = \frac{1}{6} \sqrt{A(4p(p-a_{12})(p-a_{23})(p-a_{31})) - q^2} \]  \hspace{1cm} (1)

where \(p = (1/2)(a_{12} + a_{23} + a_{31})\), \(q = a_{12} \cdot a_{23} \cdot a_{31}\), and \(A\) represents the common value in the condition (4), for example, \(A = a_{12}^2 + a_{34}^2\).

REMARK. Conditions (3) and (4) are those for \(T\) to be isometrically embedded as a 3-dimensional face of a 4-dimensional rectangular simplex.

Let \(E^n\) be the \(n\)-dimensional Euclidean space. Let \(v_0\) be the zero vector of \(E^n\). A rectangular \(n\)-simplex, by definition, is an \(n\)-simplex generated by \(v_0\) and a set of nonzero orthogonal vectors \(\{v_1, \ldots, v_n\}\) in \(E^n\). For more details on simplex, we refer to [1].

The following generalization of Pythagoras’ theorem holds on rectangular \(n\)-simplexes. (See [2] or [3] for details.)

THEOREM 1 OF [2]. If \(S\) is an \((n+1)\)-dimensional rectangular simplex generated by the zero vector \(v_0\) and a set of nonzero orthogonal vectors \(\{v_1, \ldots, v_{n+1}\}\), then

\[ |F_0|^2 = \sum_{i=1}^{n+1} |F_i|^2, \]
where $|F_i|$ is the volume of the face $F_i$ of $S$, the $n$-dimensional simplex generated by the set 
\{v_0, \ldots, \hat{v}_i, \ldots, v_{n+1}\}.

Since every $F_i$, $i \neq 0$, is a rectangular simplex, we have 
$|F_i| = \frac{1}{n!} |v_1| \cdots |v_{i-1}| |v_{i+1}| \cdots |v_{n+1}|$
and
\begin{equation}
|F_0| = \frac{1}{n!} \left( \sum_{i=1}^{n+1} |v_1| \cdots |v_{i-1}| |v_{i+1}| \cdots |v_{n+1}| \right)^{1/2}.
\end{equation}

For example, the area of a triangle which is the face $F_0$ of a rectangular tetrahedron generated 
by $v_0$ and a set of nonzero orthogonal vectors \{v_1, v_2, v_3\} is
\begin{equation}
|F_0| = \frac{1}{2} \left( \left( |v_1| |v_2| \right)^2 + \left( |v_2| |v_3| \right)^2 + \left( |v_3| |v_1| \right)^2 \right)^{1/2}.
\end{equation}

A regular $n$-simplex is a simplex generated by a set of equidistant vectors \{w_0, \ldots, w_n\} in $E^n$, 
i.e., $|w_i - w_j| = a \varepsilon_{ij}$ for some $a > 0$ where $\varepsilon_{ij} = 1$ for $i \neq j$ and $\varepsilon_{ii} = 0$ for every $i$. 
A regular $n$-simplex with edges of length $a$ is the face $F_0$ of the rectangular $(n+1)$-simplex generated 
by the zero vector $v_0$ and a set of nonzero orthogonal vectors \{v_1, \ldots, v_{n+1}\} with $|v_i| = a/\sqrt{2}$ for 
every $i$. So, we have the following corollary.

**Corollary.** Let $S$ be a regular $n$-simplex with edges of length $a$. Then the volume of $S$ is given 
by
\begin{equation}
|S| = \frac{a^n}{n!} \sqrt{\frac{n+1}{2^n}}.
\end{equation}

For example, the above formula shows the area of a unit equilateral triangle is $\sqrt{3}/4$ and the 
volume of a unit regular tetrahedron is $\sqrt{2}/12$.

We will apply Theorem A to find the volume formula for the tetrahedra which are faces of 
rectangular 4-simplexes. Any acute triangle (a triangle whose angles are less than or equal to 
the right angle) is a face of a rectangular tetrahedron. A triangle is acute if and only if the length 
of its edges $a_1, a_2, \text{ and } a_3$ satisfy the inequalities
\begin{equation}
a_i^2 \leq a_j^2 + a_k^2, \text{ for distinct } i, j, k \in \{1, 2, 3\}.
\end{equation}

Lemma 1 gives a condition for a tetrahedron $T$ to be a face of a rectangular 4-simplex. More 
precisely, it gives a condition that there is a rectangular 4-simplex $S$ in higher dimensional 
Euclidean space such that $T$ can be isometrically embedded as a face of $S$.

**Lemma 1.** Let $T = [w_1, w_2, w_3, w_4]$, i.e., $T$ is the 3-simplex (tetrahedron) spanned by $w_1, w_2, w_3, \text{ and } w_4$. 
If the lengths $a_{ij}$ of the edges $[w_i, w_j]$ satisfy
\begin{equation}
a_{12}^2 + a_{34}^2 = a_{13}^2 + a_{24}^2 = a_{14}^2 + a_{23}^2
\end{equation}
and if every 2-dimensional face of $T$ is an acute triangle or, equivalently, the condition (3) is 
satisfied on each 2-dimensional face of $T$, then $T$ is a face of a rectangular 4-simplex.

**Remark.** The condition (4), imposed on all the three pairs of opposite edges of $T$, means that 
the sum of squares of the length of the two opposite edges in the pair are the same for all pairs.

**Proof.** Consider the following system of simultaneous quadratic equations with four variables 
$x_1, x_2, x_3, \text{ and } x_4$.
\begin{equation}
x_i^2 + x_j^2 = a_{ij}^2, \text{ for every } i, j \in \{1, 2, 3, 4\} \text{ with } i < j.
\end{equation}

This system of equations has a unique solution if the conditions of the lemma are satisfied and 
if we require $x_i > 0$ for $i = 1, 2, 3, 4$. Let $S$ be the rectangular 4-simplex $[v_0, v_1, v_2, v_3, v_4]$ where
\(v_0\) is the zero vector, \(\{e_1, e_2, e_3, e_4\}\) is the standard basis of \(E^4\), and \(v_i = x_i e_i\) for \(i = 1, 2, 3, 4\). The correspondence between \(w_i\) and \(v_i\) can be extended piecewise linearly to define an isometric embedding of \(T\) as the face \(F_0\) of \(S\). This proves the lemma.

**Proof of Theorem A.** Suppose \(T\) is isometrically embedded as the face \(F_0\) of a rectangular 4-simplex \([v_0, v_1, v_2, v_3, v_4]\) by mapping \(w_i\) to \(v_i\) for \(i = 1, 2, 3, 4\). Then we have

\[
a_{ij}^2 = |v_i|^2 + |v_j|^2, \quad \text{for } 1 \leq i \neq j \leq 4, \tag{5}
\]

and

\[
\text{(volume of } T)^2 = \sum_{i=1}^{4} |F_i|^2 = \sum_{i=1}^{4} \left(\frac{1}{6} |v_1| \cdots |v_i| \cdots |v_4|\right)^2. \tag{6}
\]

By substituting the relation (5) into (6) and simplifying, the conclusion (1) of the theorem follows.

Theorem A applies to regular tetrahedra, for example. If \(T\) is a regular tetrahedron with sides of length \(a\), then the volume of \(T\) is \(a^3/(6\sqrt{2})\).

**References**