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Coloring graphs with no odd-K_4

Wenan Zang*,[1]

Department of Mathematics, University of Hong Kong, Hong Kong, China

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Abstract

The purpose of this note is to present a polynomial-time algorithm which, given an arbitrary graph G as its input, finds either a proper 3-coloring of G or an odd-K_4 that is a subgraph of G in time $O(mn)$, where m and n stand for the number of edges and the number of vertices of G, respectively.

1. Introduction

A *subdivision* of a graph G is any graph obtained from G by repeated applications of the following operation: introduce a new vertex w and replace an arbitrary edge uv by edges uw and wv. An *odd-K_4* is a subdivision of K_4 (the complete graph with four vertices) such that all four cycles corresponding to triangles in K_4 are odd (see Fig. 1, where the term *odd* in each face indicates that the bounding cycle is odd; each line stands for a path, nevertheless three straight lines in odd-K_3^2 may have length zero). The class of graphs with no odd-K_4 has been studied extensively in the past decade (e.g. [2–7]).

Catlin [2] proved that

Theorem 1. *Every graph containing no odd-K_4 can be vertex-colored with three colors.*

Let G_1 and G_2 be two undirected graphs. We call a map $\phi : V(G_1) \rightarrow V(G_2)$ a *homomorphism* from G_1 to G_2 if $\phi(u_1)\phi(u_2) \in E(G_2)$ for each $uv \in G_1$. Gerards [3] established the following:

* E-mail: wzang@maths.hku.hk.

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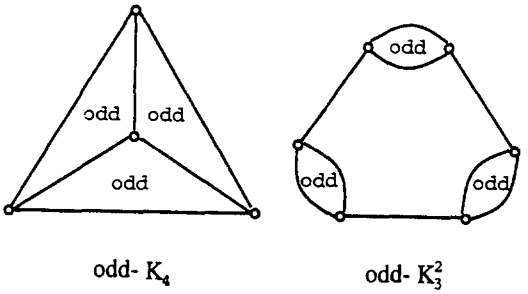

Fig. 1. Odd-K_4 and odd-K_3^2

Theorem 2. *Every graph with no odd-K_4 and no odd-K_3^2 (see Fig. 1) is homomorphic to a shortest odd cycle in the graph.*

Note that Gerards' theorem is stronger than Catlin's, because of the following theorem due to Gerards [5].

Theorem 3. *If a graph with no odd-K_4 contains an odd-K_3^2, then it has a cutset with at most two vertices.*

A slight modification of Catlin's proof yields a polynomial-time algorithm for coloring an arbitrary graph G containing no odd-K_4 with at most three colors, which can be implemented in time $O(mn^2)$, where m is the number of edges and n is the number of vertices in G. We point out that another polynomial-time algorithm can be built around two previous results.

The first of these results is due to Gerards [6].

Theorem 4. *Every graph G with no odd-K_4 and no odd-K_3^2 admits an orientation such that the number of forward arcs and the number of backward arcs in each cycle differ by at most one. Moreover, the above orientation can be found in polynomial time.*

The second of these results is due to Minty [8].

Theorem 5. *A graph G is vertex k-colorable if and only if G admits an orientation such that every cycle C contains at least $|C|/k$ arcs in each direction.*

Minty's proof of the 'if' part goes as follows: For each arc going from x to y, create a new arc going from y to x; assign length -1 to each old arc and assign length $k-1$ to each new arc. It is easy to see that the resulting directed graph contains no negative cycles and that, with $f(v)$ standing for the length of the shortest path from some fixed vertex v_0 to v, the values of f reduced mod k provide a proper coloring of G.

Let us point out that the values of f at all the vertices of G can be computed in time $O(mn)$ with n and m standing, as usual, for the number of vertices and the number of edges, respectively, of G (see, for instance, [10]).

It follows from Theorems 4, 5, and the remarks following Theorem 5 that the vertices of every 3-*connected* graph G with no odd-K_4 can be colored by three colors in polynomial time. Based on this subroutine, one can easily describe a recursive algorithm for coloring the vertices of an arbitrary graph G with no odd-K_4 with three colors.

The purpose of this note is to present a polynomial-time algorithm which, given an arbitrary graph G as its input, returns either a proper 3-coloring of G or an odd-K_4 that is a subgraph of G in time $O(mn)$ with m and n standing for the number of edges and the number of vertices of G, respectively. Our algorithm compares favorably with the two algorithms we stated above; it also contains a short proof of Catlin's theorem.

2. The algorithm

Our algorithm, `COLOR(`G`)`, given an arbitrary graph G as its input, finds either a proper 3-coloring of G or an odd-K_4 that is a subgraph of G. It begins by preprocessing G in order to find

(1) a vertex of G that belongs to at most one triangle,

(2) or a diamond (graph obtained from K_4 by deleting an edge) that is a subgraph of G,

(3) or triangles $a_1a_2b_1$, $a_2a_3b_2$, $a_3a_4b_3$ such that $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ are seven distinct vertices.

It is not difficult to see that at least one of (1)–(3) can always be found in linear time, since for each vertex, either itself or one of its neighbors is in such a configuration.

In case (1), let u denote the vertex of G that belongs to at most one triangle and let $N(u)$ denote the set of neighbors of u. If u belongs to no triangle then set $A = N(u)$; if u belongs to a triangle then let uvw denote this triangle and set $A = N(u) - \{v\}$. Note that, in either case, A is a stable set. Construct a graph G^* by deleting u from G and shrinking A into a single new vertex a. Then call `COLOR(`G^*`)`.

If `COLOR(`G^*`)` finds a proper 3-coloring φ^* of G^* then set

$$\varphi(x) = \begin{cases} \varphi^*(x) & \text{if } x \text{ is a vertex of } G - (A \cup \{u\}), \\ \varphi^*(a) & \text{if } x \in A; \end{cases}$$

note that φ is a proper 3-coloring of $G - \{u\}$ and that its values in $N(u)$ are restricted to at most two colors; hence, φ can be extended into a proper 3-coloring of G.

If `COLOR(`G^*`)` finds a subgraph F of G^* such that F is an odd-K_4 then either F is a subgraph of G or else a is a vertex of F; in the latter case, it is easy to find an odd-K_4 in $(F - \{a\}) \cup A \cup \{u\}$.

In case (2), label the vertices of the diamond as a, b, c, d so that a and d are nonadjacent, and then distinguish between two subcases:
(2.1) $G - \{b, c\}$ is disconnected,
(2.2) $G - \{b, c\}$ is connected.

In case (2.1), construct proper subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$ and the overlap of G_1, G_2 consists of the edge bc; then call COLOR(G_1) and COLOR(G_2). If each COLOR(G_i) finds a proper 3-coloring of G_i then, after renaming colors if necessary, the union of these two colorings constitutes a proper 3-coloring of G; else at least one COLOR(G_i) finds an odd-K_4 that is a subgraph of G_i, and therefore of G.

In case (2.2), construct the union Q of all the paths from a to d in $G - \{b, c\}$. Then let Q^* denote the graph obtained from Q by shrinking $\{a, d\}$ into a single new vertex u and distinguish between two subcases:
(2.2.1) Q^* is not bipartite,
(2.2.2) Q^* is bipartite.

In case (2.2.1), we propose to find an odd path P from a to d in $G - \{b, c\}$; this path and the diamond will form an odd-K_4 in G. To find P, we first find an odd cycle in Q^*. In Q, this cycle yields either the desired P or an odd cycle C. In the former case, we are done; in the latter case, we shall find vertex-disjoint paths P_a, P_d such that P_a links a to C and P_d links d to C. (If C passes through a or d then one of these paths has length zero.) Clearly, the union of P_a, P_d, and C contains the desired P.

In case (2.2.2), the bipartition of Q^* yields a bipartition (A, B) of Q with $\{a, d\} \subseteq A$; we distinguish between two subcases:
(2.2.2.1) b and c share a neighbor in B,
(2.2.2.2) b and c share no neighbor in B.

In case (2.2.2.1), we find an odd-K_4 in G: if w is a common neighbor of b and c in B then the odd-K_4 is formed by the diamond on $\{w, b, c, d\}$ and a path of odd length from w to d in Q.

In case (2.2.2.2), we first partition B into disjoint sets B_b, B_c so that no x in B_b is adjacent to b and no x in B_c is adjacent to c. Then we construct a proper 3-coloring of the subgraph of G induced by $Q \cup \{b, c\}$ by setting

$$\begin{aligned}
&\varphi(b) = 2,\\
&\varphi(c) = 3,\\
&\varphi(x) = 1 \quad \text{whenever } x \in A,\\
&\varphi(x) = 2 \quad \text{whenever } x \in B_b,\\
&\varphi(x) = 3 \quad \text{whenever } x \in B_c.
\end{aligned}$$

Then we distinguish between two subcases:
(2.2.2.2.1) G has no vertex outside $Q \cup \{b, c\}$,
(2.2.2.2.2) G has a vertex outside $Q \cup \{b, c\}$.

In case (2.2.2.2.1), we are done: φ is a proper 3-coloring of G.

In case (2.2.2.2.2), find the set X of all the vertices x in Q that are cutpoints of $G - \{b, c\}$; for each vertex x in X, construct the connected subgraph G_x of G induced by x and all the components of $G-\{b,c\}$ which contain no vertex in Q. Clearly, every vertex of $G-Q$ belongs to some G_x with $x \in X$. For each $x \in X$, construct a graph G_x^* as follows:

- if $x \in A$ then
  identify a and x in the subgraph of G induced by $G_x \cup \{a,b,c\}$,
- if $x \in B_b$ then
  identify b and x in the subgraph of G induced by $G_x \cup \{b,c\}$,
- if $x \in B_c$ then
  identify c and x in the subgraph of G induced by $G_x \cup \{b,c\}$.

Then call COLOR(G_x^*) for every x in X.

If every COLOR(G_x^*) finds a proper 3-coloring φ_x of G_x^* then we may rename the three colors in the range of φ_x so that $\varphi_x(x)=\varphi(x)$, $\varphi_x(b)=\varphi(b)$, $\varphi_x(c)=\varphi(c)$; clearly, the union of φ and all φ_x is a 3-coloring of G.

If at least one COLOR(G_x^*) finds a subgraph F^* of G_x^* that is an odd-K_4 then we shall find a subgraph F of G that is an odd-K_4. More specifically,

- if $x \in A$
  then there is a path of an even length in Q between a and x;
- if $x \in B_b$
  then there is a path of an even length in $Q \cup \{b\}$ between b and x;
- if $x \in B_c$
  then there is a path of an even length in $Q \cup \{c\}$ between c and x.

No matter which of the three sets x belongs to, F^* either is a subgraph of G or can be transformed into the desired F by means of the appropriate path.

In case (3), distinguish between two subcases:
(3.1) $G-\{a_2,a_3,b_2\}$ is disconnected,
(3.2) $G-\{a_2,a_3,b_2\}$ is connected.

In case (3.1), construct proper subgraphs G_1, G_2 of G such that $G=G_1 \cup G_2$ and the overlap of G_1, G_2 consists of the triangle $a_2a_3b_2$; then call COLOR(G_1) and COLOR(G_2). If each COLOR(G_i) finds a proper 3-coloring of G_i then, after renaming colors if necessary, the union of these two colorings constitutes a proper 3-coloring of G; else at least one COLOR(G_i) finds an odd-K_4 that is a subgraph of G_i, and therefore of G.

In case (3.2), distinguish between two subcases:
(3.2.1) $G-\{a_2,a_3\}$ is disconnected,
(3.2.2) $G-\{a_2,a_3\}$ is connected.

In case (3.2.1), it is easy to see that b_2 is of degree 2. Let $G_1=\{a_2,a_3,b_2\}$ and $G_2=G-\{b_2\}$. Call COLOR(G_1) and COLOR(G_2). If COLOR(G_2) finds a proper 3-coloring of G_2 then, after renaming colors if necessary, the union of these two colorings constitutes a proper 3-coloring of G; else COLOR(G_2) finds an odd-K_4 that is a subgraph of G_2, and therefore of G.

In case (3.2.2), we propose to find an odd-K_4 that is a subgraph of G. First, we find a path P from $\{a_1,b_1\}$ to $\{b_3,a_4\}$ in $G-\{a_2,a_3,b_2\}$ such that all internal vertices of P are outside $\{a_1, b_1, b_3, a_4\}$ and we rename vertices (if necessary) so that P goes from a_1 to a_4. Then we find a path Q from b_2 to $\{a_1, b_1, b_3, a_4\} \cup P$ in $G - \{a_2, a_3\}$. If Q goes from b_2 to a vertex x of P then let P_1 denote the segment of P that goes from x to a_1 and let P_2 denote the segment of P that goes from x to a_4; an odd-K_4 in G is formed by the triangle $a_2a_3b_2$ and the three paths

- Q,
- one of P_1a_2 and $P_1b_1a_2$ (whichever of the two has the parity of Q),
- one of P_2a_3 and $P_2b_3a_3$ (whichever of the two has the parity of Q).

If Q goes from b_2 to $\{b_1, b_3\}$ then we rename vertices (if necessary) so that Q goes from b_2 to b_3; if Q is odd then an odd-K_4 in G is formed by the triangle $a_2a_3b_2$ and the three paths

- Q,
- b_3a_3,
- one of b_3Pa_2 and $b_3Pb_1a_2$ (whichever of the two is odd);

if Q is even then an odd-K_4 in G is formed by the triangle $a_2a_3b_2$ and the three paths

- Qa_4,
- a_3a_4,
- one of Pa_2 and Pb_1a_2 (whichever of the two is odd).

3. Implementation

Case (2.2): To construct the union Q of all the paths from a to d in $G - \{b, c\}$, let us shrink $\{a, d\}$ into a single vertex u and let H^* denote the resulting graph. By applying *the biconnectivity algorithm* in [9], we can find all the biconnected components of H^*. Let B be the union of all the biconnected components of H^* that contain at least one vertex in $N(a)$ and contain at least one vertex in $N(d)$, where $N(w)$ is the neighbor of w in G for $w = a$ and d. Now let us first split u into a and d in B and then connect a to each vertex in $N(a) \cap B$ and connect d to each vertex in $N(d) \cap B$. The resulting graph is Q.

Case (2.2.1): To find an odd cycle in Q^*, we may apply the depth-first search method (see [9]).

To find the vertex-disjoint paths P_a, P_b in linear time, we may appeal to network flow theory (see, for instance, [1]).

Case (2.2.2.2.2): Apply *the biconnectivity algorithm* in [9] for finding all the cut-points and biconnected components in a graph.

Let us assume that no recursive call will be applied to a graph with at most four vertices. The following lemma bounds the number of recursive calls in our algorithm:

Lemma. The number of recursive calls is at most $n - 4$, where n is the number of vertices in G and $n \geqslant 4$.

Proof. Let $f(n)$ stand for the largest possible number of recursive calls when our algorithm is applied to a graph with n vertices. Then $f(4)=0$. Now let us show by induction method that $f(n)\leqslant n-4$. Since the statement holds for $n=4$, we proceed to the induction step.

In case (1), we have $f(n)\leqslant f(n-1)+1$. Hence, $f(n)\leqslant (n-1)-4+1=n-4$.

In cases (2.1), (3.1), (3.2.1), and (3.2.2), let n_i denote the number of vertices in G_i for each $i=1,2$. Then we have $f(n)\leqslant f(n_1)+f(n_2)+1$. Hence, $f(n)\leqslant (n_1-4)+(n_2-4)+1=(n_1+n_2-3)-4\leqslant n-4$.

In case (2.2.2.2.2), let n_x stand for the number of vertices in G_x^* for each x in X. Then $f(n)\leqslant \sum_{x\in X} f(n_x)+1$. Hence, $f(n)\leqslant \sum_{x\in X}(n_x-4)+1=\sum_{x\in X} n_x-4|X|+1$. Note that when $|X|\geqslant 2$, we have $\sum_{x\in X} n_x\leqslant n+2(|X|-1)$. Thus, $f(n)\leqslant n-2|X|-1\leqslant n-4$; when $|X|=1$, we have $n_x\leqslant n-1$. Hence, $f(n)\leqslant n_x-4+1\leqslant n-4$, as desired. □

4. Running time

In our algorithm, we reduce the problem on the input graph to corresponding problems on a hierarchy of components. Let us call each component at the bottom of the hierarchy an *atom*. Then each atom is either a graph with at most four vertices or a graph as stated in case (2.2.2.2.1), on which the solution can be obtained in linear time (recall case (2.2.2.2)). It follows from our lemma that the total number of atoms is $O(n)$, so the total time taken on atoms is $O(n(n+m))$. Note that the implementation of each case stated in the last section takes no more than $O(n+m)$ time. By our lemma the total time taken in all recursive calls is $O(n(n+m))$. Hence, the time complexity of our algorithm is $O(n(n+m))+O(n(n+m))=O(nm)$.

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